A refinement of Nesterenko's linear independence criterion with applications to zeta values

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Abstract

We refine (and give a new proof of) Nesterenko's famous linear independence criterion from 1985, by making use of the fact that some coefficients of linear forms may have large common divisors. This is a typical situation appearing in the context of hypergeometric constructions of $\mathbb{Q}$-linear forms involving zeta values or their q-analogues. We apply our criterion to sharpen previously known results in this direction.

1 Introduction

1.1 Nesterenko's criterion

In this text, we refine Nesterenko's linear indepence criterion by taking into account the existence of common divisors to the coefficients of the linear forms. Consider the following situation:

(N) Let $\xi_0, \ldots, \xi_r$ be real numbers, with $r \geq 1$. Let $0 < \alpha < 1$ and $\beta > 1$. For any $n \geq 1$, let $\ell_{0,n}, \ldots, \ell_{r,n}$ be integers such that

$$\lim_{n\to\infty} \left| \sum_{i=0}^{r} \ell_{i,n}\xi_i \right|^{1/n} = \alpha \quad \text{and} \quad \limsup_{n\to\infty} |\ell_{i,n}|^{1/n} \leq \beta \quad \text{for any } i \in \{0, \ldots, r\}.$$

Let us recall a special case of Nesterenko's criterion [14].

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Theorem A (Yu. Nesterenko). *Assume that hypothesis* **(N)** *holds. Then we have*

$$\dim_{\mathbb{Q}} \operatorname{Span}_{\mathbb{Q}}(\xi_0, \ldots, \xi_r) \geq 1 - \frac{\log \alpha}{\log \beta}.$$

Hypothesis **(N)** implies $\dim_{\mathbb{Q}} \operatorname{Span}_{\mathbb{Q}}(\xi_0, \ldots, \xi_r) \geq 2$, because otherwise $\xi_0, \ldots, \xi_r$ would be integer multiples of a (possibly zero) real number ξ, so that all linear forms $\sum_{i=0}^{r} \ell_{i,n}\xi_i$ would be integer multiples of ξ. This is impossible since these linear forms tend to 0 without vanishing (for n sufficiently large). This remark shows that the first interesting case is trying to get a dimension greater than or equal to three. This special case of Theorem A reads as follows.

Theorem B. *Assume that hypothesis* **(N)** *holds. If $\alpha\beta < 1$, then*

$$\dim_{\mathbb{Q}} \operatorname{Span}_{\mathbb{Q}}(\xi_0, \ldots, \xi_r) \geq 3.$$

In other words, among $\xi_0, \ldots, \xi_r$ there are at least three numbers that are linearly independent over the rationals.

1.2 A refinement

We obtain the following improvement of Nesterenko's criterion, the proof of which relies on Minkowski's convex body theorem and yields a new proof of Nesterenko's Theorem A.

Theorem 1. *Assume that hypothesis* **(N)** *holds. For any $n \geq 1$ and any $i \in \{1, \ldots, r\}$, let $\delta_{i,n}$ be a positive divisor of $\ell_{i,n}$. Assume that*

(i) *$\delta_{i,n}$ divides $\delta_{i+1,n}$ for any $n \geq 1$ and any $i \in \{1, \ldots, r-1\}$, and*

(ii) *$\dfrac{\delta_{j,n}}{\delta_{i,n}}$ divides $\dfrac{\delta_{j,n+1}}{\delta_{i,n+1}}$ for any $n \geq 1$ and any $0 \leq i < j \leq r$, with $\delta_{0,n} = 1$.*

Furthermore, assume that for any $i \in \{1, \ldots, r\}$ the limit of $\delta_{i,n}^{1/n}$ as $n \to \infty$ exists. Let $s = \dim_{\mathbb{Q}} \operatorname{Span}_{\mathbb{Q}}(\xi_0, \ldots, \xi_r) - 1$. Then we have $s \geq 1$ and

$$\alpha\beta^s \geq \prod_{i=1}^{s} \lim_{n\to\infty} \delta_{i,n}^{1/n}.$$

The conclusion of this theorem has to be understood as a lower bound for s, namely,

$$s \geq -\frac{\log\bigl(\alpha / \prod_{i=1}^{s} \lim_{n\to\infty} \delta_{i,n}^{1/n}\bigr)}{\log \beta};$$

but is should be noted that the product contains s factors.

The following special case of Theorem 1, in which we let $d_n = \operatorname{lcm}(1, 2, \ldots, n)$, is useful when studying linear independence of zeta values (see, for example, the proofs of Theorems 3 and 5 below).

Corollary 1. *Assume that hypothesis* **(N)** *holds. Let* $e_1 \le \dots \le e_r$ *be non-negative integers such that* $d_n^{e_i}$ *divides* $\ell_{i,n}$ *for any* $n \ge 1$ *and any* $i \in \{1, \dots, r\}$.

Let $s = \dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}(\xi_0, \dots, \xi_r) - 1$. *Then we have* $s \ge 1$ *and*

$$s \ge \frac{e_1 + \dots + e_s - \log \alpha}{\log \beta}.$$

Again the lower bound we obtain for s in this corollary actually depends on s itself.

Although Theorem 1 comes as a special case of a more general statement (see Theorem 6 below), it is already interesting to see what happens when we just try to prove that $\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}(\xi_0, \dots, \xi_r) \ge 3$, as in Theorem B. Then we may assume, without loss of generality, that $\delta_{1,n} = \dots = \delta_{r,n}$ for any n. In this case, the assumption of Theorem 1 is that $\delta_{1,n}$ divides $\delta_{1,n+1}$. Actually, we obtain the following stronger improvement of Theorem B, in which this assumption is replaced with a lower bound on the greatest common divisor of $\delta_{1,n}$ and $\delta_{1,n+1}$.

Theorem 2. *Assume that hypothesis* **(N)** *holds. For any* $n \ge 1$, *let* δ_n *be a common positive divisor of* $\ell_{1,n}, \dots, \ell_{r,n}$. *Assume that*

$$\alpha\beta < \liminf_{n \to \infty} \big(\gcd(\delta_n, \delta_{n+1})\big)^{1/n}.$$

Then

$$\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}(\xi_0, \dots, \xi_r) \ge 3.$$

The main interest of Theorem 2 is actually its proof, which is simpler than that of Theorem 1 (cf. Sections 2.4 and 2.5 below).

1.3 Applications

A typical situation when our refinement becomes useful, refers to an arithmetic problem for the so-called odd zeta values — the values of Riemann's zeta function

$$\zeta(l) = \sum_{k=1}^{\infty} \frac{1}{k^l}$$

at odd integers $l > 1$; see [1], [2], [7], [11], [20] and [22] for history and known results in this arithmetic direction. The following theorem improves on previous bounds ($i_1 \le 145$ and $i_2 \le 1971$) from [20, Theorem 0.3].

Theorem 3. *There exist odd integers* $i_1 \le 139$ *and* $i_2 \le 1961$ *such that the numbers*

$$1,\ \zeta(3),\ \zeta(i_1),\ \text{and}\ \zeta(i_2)$$

are linearly independent over $\mathbb{Q}$.

Another but related application is devoted to arithmetic properties of the following q-analogue of Riemann's zeta function ($|q| < 1$):

$$\zeta_q(l) = \sum_{k=1}^{\infty} \frac{k^{l-1} q^k}{1 - q^k} = \sum_{k=1}^{\infty} \sigma_{l-1}(k) q^k,$$

where $\sigma_{l-1}(k) = \sum_{d|k} d^{l-1}$. A usual setup for a number q is to be of the form $1/p$, where $p \in \mathbb{Z} \setminus \{0, \pm 1\}$. Although the irrationality of $\zeta_q(1)$ and even the transcendence of $\zeta_q(l)$ for any even positive integer l are known, not so much is obtained for $\zeta_q(l)$ with $l > 1$ odd; we refer the reader to the works [10] and [12] for details. For example, F. Jouhet and E. Mosaki show in [10] that at least one of the four numbers $\zeta_q(3)$, $\zeta_q(5)$, $\zeta_q(7)$, $\zeta_q(9)$ is irrational and give further results for the odd q-zeta values in the spirit of Theorem 3 above. Our next theorem sharpens the corresponding bounds from [10].

Theorem 4. *Let q be a rational of the form $1/p$, where $p \in \mathbb{Z} \setminus \{0, \pm 1\}$. There exist odd integers $1 < i_0 < i_1 < i_2 < i_3$ such that $i_0 \le 9$, $i_1 \le 37$, $i_2 \le 83$, $i_3 \le 145$ and the numbers*

$$1,\ \zeta_q(i_0),\ \zeta_q(i_1),\ \zeta_q(i_2),\ \text{and}\ \zeta(i_3)$$

are linearly independent over $\mathbb{Q}$.

Our third application also appeals to arithmetic of the odd zeta values, but this time we add $\log 2$ to the set.

Theorem 5. *There exist odd integers $i_1 \le 93$ and $i_2 \le 1151$ such that the numbers*

$$1,\ \log 2,\ \zeta(i_1),\ \text{and}\ \zeta(i_2)$$

are linearly independent over $\mathbb{Q}$.

In our proof of Theorem 5 we use a (seemingly) new hypergeometric construction of linear forms in 1, $\log 2$ and odd zeta values. We find rather curious that a 'degenerate' case of our construction, when odd zeta values do not occur at all, resembles well the rational approximations [1] from Apéry's proof of the irrationality of $\zeta(3)$ (cf., for example, [7]); this is the subject of our final Section 3.4.

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2 The linear independence criterion

In this part, we state (Section 2.1) and prove (Sections 2.2 and 2.3) our main linear independence criterion, as well as its special case (Section 2.4) corresponding to Theorem 2. We gather some remarks in Section 2.5.

2.1 Statement

Our main result is the following statement, which contains Theorem 1 as a special case (by taking $Q_n = \beta^n$ and $\tau = -(\log \alpha)/(\log \beta)$).

Theorem 6. *Let $\xi_0, \dots, \xi_r$ be real numbers, with $r \geq 1$. Let $\tau, \gamma_1, \dots, \gamma_r > 0$. For any $n \geq 1$ and any $i \in \{0, \dots, r\}$, let $\ell_{i,n} \in \mathbb{Z}$. For $n \geq 1$ and $i \in \{1, \dots, r\}$, let $\delta_{i,n}$ be a positive divisor of $\ell_{i,n}$ such that*

(i) *$\delta_{i,n}$ divides $\delta_{i+1,n}$ for any $n \geq 1$ and any $i \in \{1, \dots, r-1\}$, and*

(ii) *$\dfrac{\delta_{j,n}}{\delta_{i,n}}$ divides $\dfrac{\delta_{j,n+1}}{\delta_{i,n+1}}$ for any $n \geq 1$ and any $0 \leq i < j \leq r$, with $\delta_{0,n} = 1$.*

Assume that there exists an increasing sequence $(Q_n)_{n \geq 1}$ of integers such that, as $n \to \infty$, the following conditions are met:

$$Q_{n+1} = Q_n^{1+o(1)},$$
$$\max_{0 \leq i \leq r} |\ell_{i,n}| \leq Q_n^{1+o(1)},$$
$$\left| \sum_{i=0}^{r} \ell_{i,n} \xi_i \right| = Q_n^{-\tau + o(1)},$$
$$\delta_{i,n} = Q_n^{\gamma_i + o(1)} \quad \text{for any } i \in \{1, \dots, r\}.$$

Let $s = \dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}(\xi_0, \dots, \xi_r) - 1$. Then we have

$$s \geq \tau + \gamma_1 + \dots + \gamma_s.$$

Remark 1. The existence of arbitrarily small non-zero linear combinations of $\xi_0, \dots, \xi_r$ with integer coefficients implies $\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}(\xi_0, \dots, \xi_r) \geq 2$, that is, $s \geq 1$.

Remark 2. In the statement of Theorem 6 and in all other linear independence criteria we prove in this text, no assumption is made on whether ξ_0 vanishes or not. Actually, we can always assume that $\xi_0 \neq 0$, because if $\xi_0 = 0$ then Remark 1 provides an integer i such that $\xi_i \neq 0$, and we can consider the linear forms $0\xi_i + \ell_{1,n}\xi_1 + \dots + \ell_{r,n}\xi_r$ in $(\xi_i, \xi_1, \dots, \xi_r)$.

The proof of Theorem 6 splits into two parts. First we prove this result (in Section 2.2) under the assumption that $\xi_0, \dots, \xi_r$ are linearly independent over the rationals. Next we deduce the general case (in Section 2.3).

2.2 Proof in the linear independence case

In this section, we prove Theorem 6 under the assumption of the $\mathbb{Q}$-linear independence of $\xi_0, \dots, \xi_r$ (that is, $s = r$).

Denote by $\boldsymbol{\xi}$ the point $(\xi_0, \dots, \xi_r) \in \mathbb{R}^{r+1}$, and by L_n the linear form $\ell_{0,n}X_0 + \dots + \ell_{r,n}X_r$, so that $L_n(\boldsymbol{\xi}) = \sum_{i=0}^{r} \ell_{i,n}\xi_i$.

Thanks to Remark 2, we may assume that $\xi_0 \neq 0$ and even $\xi_0 = 1$ (dividing all ξ_i by ξ_0 if necessary).

Let n be a sufficiently large integer. In what follows, $o(1)$ stands for any sequence that tends to 0 as n tends to infinity.

We take

$$R_n = \frac{\delta_{r,n}}{2|L_n(\boldsymbol{\xi})|} \quad \text{and} \quad \varepsilon_n = \delta_{r,n}\left(\frac{3|L_n(\boldsymbol{\xi})|}{\prod_{i=1}^r \delta_{i,n}}\right)^{1/r},$$

so that

$$\varepsilon_n = Q_n^{\gamma_r-(\tau+\gamma_1+\cdots+\gamma_r)/r+o(1)}.$$

Arguing by contradiction, assume that $\tau + \gamma_1 + \cdots + \gamma_r > r$. Now $\gamma_r \leq 1$ because $\delta_{r,n}$ divides $\ell_{r,n}$, hence $\lim_{n\to\infty} \varepsilon_n = 0$.

Consider the set

$$\mathcal{C}_n = \left\{(p'_0, \dots, p'_r) \in \mathbb{R}^{r+1} : |p'_0| \leq \frac{R_n}{\delta_{r,n}} \text{ and, for any } i \in \{1, \dots, r\},\ |\delta_{i,n}p'_0\xi_i - p'_i| \leq \frac{\delta_{i,n}}{\delta_{r,n}}\varepsilon_n\right\}.$$

The volume of $\mathcal{C}_n$ is

$$\frac{2^{r+1}R_n\varepsilon_n^r\prod_{i=1}^r \delta_{i,n}}{\delta_{r,n}^{r+1}} = \frac{3}{2}\cdot 2^{r+1} > 2^{r+1}.$$

Since $\mathcal{C}_n$ is a convex body, symmetric with respect to the origin, there is a non-zero integer point $(p'_0, \dots, p'_r)$ in $\mathcal{C}_n$. Of course, $(p'_0, \dots, p'_r)$ also depends on n, but we do not write it down explicitly. Then rescaling

$$p_0 = \delta_{r,n}p'_0 \quad \text{and} \quad p_i = \frac{\delta_{r,n}}{\delta_{i,n}}p'_i \in \mathbb{Z} \quad \text{for any } i \in \{1, \dots, r\}$$

we have

$$|p_0| \leq R_n \quad \text{and} \quad |p_0\xi_i - p_i| \leq \varepsilon_n \quad \text{for any } i \in \{1, \dots, r\}. \tag{1}$$

Let k_n denote the least positive integer such that

$$|p_0| \leq \frac{\delta_{r,k_n}}{2|L_{k_n}(\boldsymbol{\xi})|}.$$

By definition of R_n, we have $k_n \leq n$ since $|p_0| \leq R_n$. Moreover, k_n tends to infinity with n thanks to (1), since $\varepsilon_n \to 0$ and $(1, \xi_1, \dots, \xi_r)$ are linearly independent over $\mathbb{Q}$. By minimality of k_n, we have

$$|p_0| = Q_{k_n}^{\gamma_r+\tau+o(1)}. \tag{2}$$

Now we can write

$$\sum_{i=0}^r \ell_{i,k_n}p_i = p_0\sum_{i=0}^r \ell_{i,k_n}\xi_i + \sum_{i=0}^r \ell_{i,k_n}(p_i - p_0\xi_i).$$

On the right-hand side, the first term has absolute value equal to $|p_0L_{k_n}(\boldsymbol{\xi})|$, therefore less than or equal to $\frac{1}{2}\delta_{r,k_n}$ by definition of k_n. If the second term has absolute value less than the first one, then the absolute value of the right-hand side is less than δ_{r,k_n}. But it is equal to the left-hand side, which is an integer multiple of δ_{r,k_n}, since

$$\ell_{i,k_n} p_i = \ell_{i,k_n} \frac{\delta_{r,n}}{\delta_{i,n}} p'_i \quad \text{is a multiple of} \quad \delta_{i,k_n} \frac{\delta_{r,k_n}}{\delta_{i,k_n}} = \delta_{r,k_n}$$

(by condition (ii) and $n \geq k_n$): it has to be zero. But then both terms on the right-hand side would have the same absolute value, in contradiction with the assumption.

Therefore, using (1) and (2) we have

$$Q_{k_n}^{\gamma_r + o(1)} = |p_0 L_{k_n}(\boldsymbol{\xi})| = \left| p_0 \sum_{i=0}^{r} \ell_{i,k_n} \xi_i \right| \leq \left| \sum_{i=0}^{r} \ell_{i,k_n} (p_i - p_0 \xi_i) \right| \leq Q_{k_n}^{1+o(1)} \varepsilon_n.$$

Since $k_n \leq n$ and $\gamma_r \leq 1$, this implies

$$1 \leq Q_{k_n}^{1-\gamma_r+o(1)} \varepsilon_n \leq Q_n^{1-\gamma_r+o(1)} \varepsilon_n = Q_n^{1-(\tau+\gamma_1+\cdots+\gamma_r)/r+o(1)},$$

which contradicts the assumption $\tau+\gamma_1+\cdots+\gamma_r > r$ for n sufficiently large and completes the proof of Theorem 6 under the assumption that $\xi_0, \ldots, \xi_r$ are linearly independent over the rationals.

2.3 Proof in the general case

In this section, we deduce Theorem 6 from the special case proved in Section 2.2.

Thanks to Remarks 1 and 2, we have $s \geq 1$ and we may assume that $\xi_0 \neq 0$. Take $i_0 = 0$, and let i_1 be the least positive integer such that ξ_0 and ξ_{i_1} are linearly independent over the rationals. Define inductively i_k, for $k \in \{0, \ldots, s\}$, to be the least integer such that $\xi_{i_0}, \xi_{i_1}, \ldots, \xi_{i_k}$ are linearly independent over $\mathbb{Q}$. Clearly, we have $0 = i_0 < i_1 < \cdots < i_s$ and, for any $i \in \{0, \ldots, r\}$, we can write $\xi_i = \sum_{j=0}^{k} c_{i,j} \xi_{i_j}$ with $c_{i,j} \in \mathbb{Q}$ and $k \in \{0, \ldots, s\}$ defined by $i_k \leq i < i_{k+1}$ (with $i_{s+1} = r+1$). For any n, this gives

$$\sum_{i=0}^{r} \ell_{i,n} \xi_i = \sum_{j=0}^{s} \ell'_{j,n} \xi'_j$$

by letting $\xi'_j = \xi_{i_j}$ and $\ell'_{j,n} = \sum_{i=i_j}^{r} \ell_{i,n} c_{i,j}$. Let d denote a common denominator of the rational numbers $c_{i,j}$; note that d is independent of n. For any n and any $j \in \{0, \ldots, s\}$, $d\ell'_{j,n}$ is an integer and, moreover, a multiple of $\delta_{i_j,n}$, since $\delta_{i_j,n}$ divides $\delta_{i,n}$ for any i between i_j and r.

Since $\xi'_0, \ldots, \xi'_s$ are linearly independent over the rationals, we can apply to these numbers the special case of Theorem 6 proved in Section 2.2, with the linear forms $\sum_{j=0}^{s} d\ell'_{j,n} \xi'_j$, the same sequence $(Q_n)_{n\geq 1}$ and the same τ, with divisors $\delta'_{j,n} = \delta_{i_j,n}$ for $j \in \{1, \ldots, s\}$ and exponents γ'_j which satisfy $\gamma'_j = \gamma_{i_j} \geq \gamma_j$, because $\delta'_{j,n} = \delta_{i_j,n} \geq \delta_{j,n}$ for any j and any n. This completes the proof of Theorem 6.

2.4 Proof of Theorem 2

The following statement implies Theorem 2.

Proposition 1. *Let $\xi_0, \ldots, \xi_r$ be real numbers, with $r \geq 1$. For any $n \geq 1$ and any $i \in \{0, \ldots, r\}$, let $\ell_{i,n} \in \mathbb{Z}$. Let $(\delta_n)_{n\geq 1}$ be a sequence of positive integers, such that δ_n is a common divisor of $\ell_{1,n}, \ldots, \ell_{r,n}$ for any $n \geq 1$. For any $n \geq 1$, let H_n and ε_n be positive real numbers such that*

$$\max_{0\leq i\leq r} |\ell_{i,n}| \leq H_n \quad \text{and} \quad \left|\sum_{i=0}^{r} \ell_{i,n}\xi_i\right| \leq \varepsilon_n.$$

Assume that $\sum_{i=0}^{r} \ell_{i,n}\xi_i \neq 0$ for infinitely many n and that

$$\lim_{n\to\infty} \frac{H_n\varepsilon_{n+1} + H_{n+1}\varepsilon_n}{\gcd(\delta_n, \delta_{n+1})} = 0.$$

Then we have

$$\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}(\xi_0, \ldots, \xi_r) \geq 3.$$

Proof. Thanks to Remarks 1 and 2, we have $\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}(\xi_0, \ldots, \xi_r) \geq 2$ and we may assume that $\xi_0 \neq 0$, and even that $\xi_0 = 1$. Since $\xi_1, \ldots, \xi_r$ play symmetric roles, we may assume that ξ_1 is irrational. Let us argue by contradiction, assuming on the contrary that $\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}(\xi_0, \ldots, \xi_r) = 2$. Then $\xi_2, \ldots, \xi_r$ are rational linear combinations of $\xi_0 = 1$ and ξ_1. Repeating the argument of Section 2.3, we obtain a positive integer d independent of n and rational numbers $\ell'_{0,n}$ and $\ell'_{1,n}$ such that

$$\sum_{i=0}^{r} \ell_{i,n}\xi_i = \ell'_{0,n} + \ell'_{1,n}\xi_1$$

with $d\ell'_{0,n}, d\ell'_{1,n} \in \mathbb{Z}$ of absolute value less than $d'H_n$ for some constant d' independent of n; moreover, δ_n divides $d\ell'_{1,n}$. Now consider the determinant

$$\Delta_n = \begin{vmatrix} d\ell'_{0,n} & d\ell'_{1,n} \\ d\ell'_{0,n+1} & d\ell'_{1,n+1} \end{vmatrix} = d^2\ell'_{1,n+1}(\ell'_{0,n} + \ell'_{1,n}\xi) - d^2\ell'_{1,n}(\ell'_{0,n+1} + \ell'_{1,n+1}\xi)$$

which satisfies

$$|\Delta_n| \leq dd'(H_n\varepsilon_{n+1} + H_{n+1}\varepsilon_n) < \gcd(\delta_n, \delta_{n+1})$$

if $n \geq N$ for some integer N. Now Δ_n is the determinant of a matrix in which all entries in the second column, namely, $d\ell'_{1,n}$ and $d\ell'_{1,n+1}$, are integer multiples of $\gcd(\delta_n, \delta_{n+1})$. Therefore, $\Delta_n = 0$ for any $n \geq N$ and for any such n the vector $(d\ell'_{0,n}, d\ell'_{1,n})$ is proportional to $(d\ell'_{0,N}, d\ell'_{1,N})$. This means that for any $n \geq N$ there exists an integer c_n such that

$$d\ell'_{0,n} = c_n \frac{d\ell'_{0,N}}{\gcd(d\ell'_{0,N}, d\ell'_{1,N})} \quad \text{and} \quad d\ell'_{1,n} = c_n \frac{d\ell'_{1,N}}{\gcd(d\ell'_{0,N}, d\ell'_{1,N})}.$$

This implies

$$\ell'_{0,n} + \ell'_{1,n}\xi_1 = c_n \frac{\ell'_{0,N} + \ell'_{1,N}\xi_1}{\gcd(d\ell'_{0,N}, d\ell'_{1,N})}$$

with $c_n \in \mathbb{Z}$, in contradiction with the fact that $\ell'_{0,n} + \ell'_{1,n}\xi_1$ tends to 0 without being identically equal to 0 for n sufficiently large, and Proposition 1 follows. □

2.5 Remarks

In this section, we make some comments on the proofs given above.

In the case where all divisors $\delta_{j,n}$ are equal to 1, the proof of Theorem 6 gives a new proof of Theorem A, while that of Proposition 1 yet another one in the special case of Theorem B. Nesterenko's general result in [14] is exactly Theorem 6 in the special case $\delta_{j,n} = 1$, except for one point: Nesterenko assumes that $Q_n^{-\tau_1+o(1)} \leq \left|\sum_{i=0}^{r} \ell_{i,n}\xi_i\right| \leq Q_n^{-\tau_2+o(1)}$, whereas we treat the case $\tau_1 = \tau_2$ only. Our method should generalize easily to the situation where $\tau_1 \neq \tau_2$, but we do not write it down because the equality holds in all the applications we have in mind. For the same reason, we did not try to replace $\mathbb{Q}$ with another number field, though Nesterenko's criterion can be generalized to this setting (see [3] and [18]).

Nesterenko's proof consists in obtaining a lower bound for the distance of $\boldsymbol{\xi} = (\xi_0, \ldots, \xi_r)$ to any linear subspace of $\mathbb{R}^{r+1}$, defined over $\mathbb{Q}$, of dimension $t < \tau + 1$. He proceeds by induction on t, whereas we use in Section 2.2 only the first step (namely $t = 1$, see below) of his induction. P. Colmez [4] writes down Nesterenko's proof in another way (from notes by F. Amoroso). Assume for simplicity that $\xi_0, \ldots, \xi_r$ are $\mathbb{Q}$-linearly independent (the general case follows from this special case as in Section 2.3). For any sufficiently large integer n_0, one constructs by an analogous induction procedure a decreasing sequence $n_0 > n_1 > \cdots > n_r$ of positive integers such that the determinant Δ of the matrix $[\ell_{i,n_j}]_{0 \leq i,j \leq r}$ is not zero. The easy case is when $n_0, \ldots, n_r$ are, roughly speaking, of same size (for instance, if they are consecutive integers). Then replacing the first line with the linear combination of the lines which is given by $(\xi_0, \ldots, \xi_r)$, we obtain $|\Delta| \leq Q_n^{r-\tau+o(1)}$. Since Δ is a non-zero integer, this gives $r \geq \tau$ and completes the proof in this case. The difficult part of this proof is to obtain some control upon $n_0, \ldots, n_r$. In Amoroso–Colmez's version of Nesterenko's proof, the sequence $n_0 > n_1 > \cdots > n_r$ is constructed, and yields the result $r \geq \tau$, but there might be huge gaps between successive n_j and n_{j+1}. It would be very interesting to know whether such a sequence can always be constructed with n_r 'nearly as large' as n_0. This is what we do (in the case $r = 1$) in the proof of Proposition 1 (Section 2.4): for infinitely many integers n_0, we prove that $n_1 = n_0 - 1$ implies $\Delta \neq 0$. This kind of method is similar to the ones used by H. Davenport and W. Schmidt (see, for instance, [5] and [6]).

On the other hand, our proof of Theorem 6 in Section 2.2 relies on a completely different idea. In the case when all divisors $\delta_{j,n}$ are equal to 1, it can be summarized as follows (see [9] for a translation in terms of exponents of Diophantine approximation). Dirichlet's box principle yields (under the assumption that $\xi_0, \ldots, \xi_r$ are linearly independent over $\mathbb{Q}$) very good simultaneous approximants $p_1/p_0, \ldots, p_r/p_0$ to $\xi_1/\xi_0, \ldots, \xi_r/\xi_0$ with the same denominator p_0. This means that $(\xi_0, \dots, \xi_r)$ is sufficiently close to the line generated by $(p_0, \dots, p_r)$ in $\mathbb{R}^{r+1}$, and contradicts (if $r < \tau$) the lower bound proved in the first induction step of Nesterenko's proof (see above). Actually, this first step is very easy to prove directly (without Nesterenko's machinery for controlling the intersection of a linear subspace with a hyperplane). Indeed, for some n (denoted by k_n in Section 2.2) the hyperplane H_n defined by $\ell_{0,n}X_0 + \dots + \ell_{r,n}X_r = 0$ has comparatively small height and is very close to $(\xi_0, \dots, \xi_r)$, hence to $(p_0, \dots, p_r)$, so that $(p_0, \dots, p_r)$ has to belong to H_n. But then the distance from $(\xi_0, \dots, \xi_r)$ to H_n is less than, or equal to, the distance of $(\xi_0, \dots, \xi_r)$ to $(p_0, \dots, p_r)$; this is too small, in contradiction with the lower bound for $|L_n(\boldsymbol{\xi})|$.

At last, let us comment briefly on the optimality of our criterion. It is likely that the conclusion $s \geq \tau + \gamma_1 + \dots + \gamma_s$ of Theorem 6 cannot be improved (see [8] and [9] for related results when $s = 1$), so that another strategy has to be used for refining the lower bound

$$\dim_{\mathbb{Q}} \operatorname{Span}_{\mathbb{Q}}\big(1, \zeta(3), \zeta(5), \zeta(7), \dots, \zeta(a)\big) \geq \frac{1 + o(1)}{1 + \log 2} \log a.$$

of [2], [16]. However, the assumptions of Theorem 6 can perhaps be weakened (even though they are already weak enough to be met in all applications we have in mind). Assumption (i) is used in Section 2.3, whereas Assumption (ii) is used (with $j = r$) in Section 2.2 (and also with $j = i_s$ in Section 2.3). Such a refinement might come from a different approach, like in Proposition 1 where the assumptions of Theorem 6 are weakened (for instance, the fact that δ_n should divide δ_{n+1}). It is interesting to point out that in Proposition 1 we do not need to assume $Q_{n+1} = Q_n^{1+o(1)}$, nor to have a positive lower bound for $\big|\sum_{i=0}^{r} \ell_{i,n}\xi_i\big|$.

3 Applications of the criterion

3.1 First application: Odd zeta values

For a pair of positive integers s and t with $t < s$, consider the (very-well-poised) hypergeometric series

$$h_n = 2n!^{2(s-t)} \sum_{k=1}^{\infty} \Big(k + \frac{n}{2}\Big) \frac{\prod_{j=1}^{tn}(k - j) \cdot \prod_{j=1}^{tn}(k + n + j)}{\prod_{j=0}^{n}(k + j)^{2s}}. \tag{3}$$

It is known [2], [16], [20] that, for some $a_{i,n} \in \mathbb{Q}$,

$$h_n = a_{0,n} + \sum_{i=1}^{s-1} a_{i,n}\zeta(2i + 1). \tag{4}$$

First of all, we would like to summarize the auxiliary results from [20] (namely, Propositions 2.1, 3.1, and 4.1 with Lemma 4.5 there) and translate them for the linear forms (3), (4).

Denote by x_0 the maximal real zero of the polynomial

$$\bigl(x+t+\tfrac12\bigr)\bigl(x-\tfrac12\bigr)^{2s+1}-\bigl(x-t-\tfrac12\bigr)\bigl(x+\tfrac12\bigr)^{2s+1}; \tag{5}$$

it belongs to the interval $\bigl]t+\frac12,+\infty\bigr[$. Introduce the function

$$\begin{aligned} f(x)=\bigl(t+\tfrac12\bigr)\log\bigl(x+t+\tfrac12\bigr)+\bigl(t+\tfrac12\bigr)\log\bigl(x-t-\tfrac12\bigr) \\ -\bigl(s+\tfrac12\bigr)\log\bigl(x+\tfrac12\bigr)-\bigl(s+\tfrac12\bigr)\log\bigl(x-\tfrac12\bigr). \end{aligned} \tag{6}$$

Consider the following product over primes:

$$\Pi_n=\Pi_n^{(t)}=\prod_{l=1}^{2t-1}\prod_{\substack{\sqrt{(t+1)n}<p\le n\\ \{n/p\}\in E_l}}p^l, \tag{7}$$

where

$$\begin{aligned} E_{2l}&=\Bigl[\frac{l}{t},\frac{l+1}{t+1/2}\Bigr[\quad \text{for } l=0,1,\dots,t-1, \\ E_{2l-1}&=\Bigl[\frac{l}{t+1/2},\frac{l}{t}\Bigr[\quad \text{for } l=1,2,\dots,t, \end{aligned} \tag{8}$$

and $\{\,\cdot\,\}$ denotes the fractional part of a number.

Proposition 2. *In the above notation,*

$$\lim_{n\to\infty}\frac{\log|h_n|}{n}=f(x_0)$$

and

$$\limsup_{n\to\infty}\frac{\log|a_{i,n}|}{n}\le \operatorname{Re} f(0)=2(s-t)\log 2+(2t+1)\log(2t+1) \quad \textit{for } i=0,1,\dots,s-1.$$

Moreover, the rational coefficients of the forms (4) *satisfy*

$$d_n^{2s}\Pi_n^{-1}a_{0,n}\in\mathbb{Z} \qquad \textit{and} \qquad d_n^{2(s-i)-1}\Pi_n^{-1}a_{i,n}\in\mathbb{Z} \quad \textit{for } i=1,\dots,s-1,$$

while the asymptotic behavior of (7) *is determined by*

$$\begin{aligned} \lim_{n\to\infty}\frac{\log\Pi_n}{n}=\varpi_t&=2t(1-\gamma)-\Bigl(2t+\frac12\Bigr)\sum_{l=1}^{t}\frac1l-\sum_{l=1}^{t}\Bigl(\psi\Bigl(\frac{l}{t}\Bigr)+\psi\Bigl(\frac{l}{t+1/2}\Bigr)\Bigr) \\ &=2t\psi(2)-\sum_{l=1}^{t}\Bigl(\psi\Bigl(1+\frac{l}{t}\Bigr)+\psi\Bigl(1+\frac{l}{t+1/2}\Bigr)\Bigr), \end{aligned} \tag{9}$$

where $\psi(x)$ is the digamma function (that is, the logarithmic derivative of the Gamma function) and $\gamma=-\psi(1)$ is Euler's constant.

The essential news settled after the work [20] is the proof of the so-called 'denominator conjecture' by C. Krattenthaler and T. Rivoal in [11]. They show that

$$d_n^{2s-1}a_{0,n}\in\mathbb{Z} \qquad\text{and}\qquad d_n^{2(s-i)-2}a_{i,n}\in\mathbb{Z} \quad \text{for } i=1,\dots,s-1,$$

in other words, they get rid of the extra d_n. Note that for primes p from the interval $\sqrt{(t+1)n}<p\le n$, we have $\operatorname{ord}_p d_n=1$ and $\operatorname{ord}_p \Pi_n\ge 1$ unless $\{n/p\}\in E_0=[0,1/(t+1/2)[$. The proportion of the latter primes is characterized by the quantity

$$\lim_{n\to\infty}\frac{1}{n}\log\prod_{\substack{\sqrt{(t+1)n}<p\le n\\ \{n/p\}\in E_0}} p=\psi\biggl(1+\frac{1}{t+1/2}\biggr)-\psi(1)$$

(cf. [20, Lemma 4.4]), but even this tiny improvement can be taken into account to sharpen the arithmetic part of Proposition 2. Taking

$$\widehat{\Pi}_n=\widehat{\Pi}_n^{(t)}=\prod_{l=2}^{2t-1}\prod_{\substack{\sqrt{(t+1)n}<p\le n\\ \{n/p\}\in E_l}} p^{l-1}, \tag{10}$$

we obtain the following result.

Proposition 3. *In the above notation, the rational coefficients of the forms* (4) *satisfy*

$$d_n^{2s-1}\widehat{\Pi}_n^{-1}a_{0,n}\in\mathbb{Z} \qquad\textit{and}\qquad d_n^{2(s-i)-2}\widehat{\Pi}_n^{-1}a_{i,n}\in\mathbb{Z} \quad \textit{for } i=1,\dots,s-1, \tag{11}$$

and the asymptotic behavior of (10) *is determined by*

$$\begin{aligned}\lim_{n\to\infty}\frac{\log\widehat{\Pi}_n}{n}=\widehat{\varpi}_t&=\varpi_t-1+\psi\biggl(1+\frac{1}{t+1/2}\biggr)-\psi(1)\\ &=(2t-1)\psi(2)-\sum_{l=1}^{t}\psi\biggl(1+\frac{l}{t}\biggr)-\sum_{l=2}^{t}\psi\biggl(1+\frac{l}{t+1/2}\biggr).\end{aligned} \tag{12}$$

Proof (Theorem 3). The collection of numbers under consideration is

$$(\xi_0,\xi_1,\xi_2,\dots,\xi_{s-1})=\bigl(1,\zeta(3),\zeta(5),\dots,\zeta(2s-1)\bigr).$$

Setting

$$\ell_{i,n}=d_n^{2s-1}\widehat{\Pi}_n^{-1}a_{i,n}\in\mathbb{Z} \quad \text{for } i=0,\dots,s-1,$$

from (11) we see that

$$d_n^{2i+1}\mid \ell_{i,n} \quad \text{for } i=1,\dots,s-1,$$

hence in the notation of Theorem 1 we have $r=s-1$, $\delta_{i,n}=d_n^{2i+1}$,

$$\log\alpha=f(x_0)+2s-1-\widehat{\varpi}_t$$

and

$$\log\beta = 2(s-t)\log 2 + (2t+1)\log(2t+1) + 2s - 1 - \widehat{\varpi}_t.$$

Using standard formulas for the digamma function we can write the quantity in (12) by means of elementary functions only:

$$\begin{aligned}\widehat{\varpi}_t = 3t - \frac{1}{2} - \left(2t+\frac{1}{2}\right)\sum_{l=1}^{t}\frac{1}{l} + \frac{\pi}{2}\sum_{l=2}^{t}\cot\frac{2\pi l}{2t+1} + 2\sum_{l=1}^{t}\cos\frac{4\pi l}{2t+1}\,\log\sin\frac{\pi l}{2t+1} \\ - \log 2 + t\log t + \left(t-\frac{1}{2}\right)\log(2t+1).\end{aligned}$$

We now apply Theorem 1. With the choice $s = 70$, $t = 10$ we obtain

$$1 - \frac{\log\alpha - 3}{\log\beta} = 2.0004232415\ldots > 2,$$

hence

$$\dim_{\mathbb{Q}} \operatorname{Span}_{\mathbb{Q}}\big(1, \zeta(3), \zeta(5), \ldots, \zeta(139)\big) \ge 3;$$

in the same way, taking $s = 981$, $t = 65$ we get

$$1 - \frac{\log\alpha - (3+5)}{\log\beta} = 3.0003426048\ldots > 3$$

yielding

$$\dim_{\mathbb{Q}} \operatorname{Span}_{\mathbb{Q}}\big(1, \zeta(3), \zeta(5), \ldots, \zeta(1961)\big) \ge 4.$$

This computation implies Theorem 3. □

3.2 Second application: Odd q-zeta values

We now fix a number q of the form $1/p$, where $p \in \mathbb{Z} \setminus \{0, \pm 1\}$. As in the previous section, we take a pair of positive integers s and t satisfying $t < s$. With the help of the basic hypergeometric series

$$\begin{aligned} h_n(q) &= (q)_n^{2(s-t)} \sum_{k=1}^{\infty} (1 - q^{2k+n}) \frac{(q^{k-tn})_{tn} \cdot (q^{k+n+1})_{tn}}{(q^k)_{n+1}^{2s}} q^{k(s-t)n+ks-k} \\ &= a_{0,n}(q) + \sum_{i=1}^{s-1} a_{i,n}(q)\zeta_q(2i+1), \end{aligned} \tag{13}$$

where $(b)_n = (b;q)_n = \prod_{k=1}^{n}(1 - q^{k-1}b)$ is the q-Pochhammer symbol, it was shown in [12] (see also [10], where the 'q-denominator conjecture' is proved) that

$$\dim_{\mathbb{Q}} \operatorname{Span}_{\mathbb{Q}}\big(1, \zeta_q(3), \zeta_q(5), \ldots, \zeta_q(2s-1)\big) \ge \frac{\pi + o(1)}{2\sqrt{\pi^2+12}}\sqrt{2s}$$

as $s \to \infty$. The coefficients $a_{i,n}(q)$ are, in fact, rational functions of the variable $p = 1/q$, whose denominators involve only powers of p and of the cyclotomic polynomials

$$\Phi_j(p) = \prod_{\substack{k=1 \\ (k,j)=1}}^{j} (p - e^{2\pi\sqrt{-1}k/j}) \in \mathbb{Z}[p], \quad \deg_p \Phi_j(p) = \varphi(j), \qquad j = 1, 2, \dots . \tag{14}$$

In these settings, the q-analogue of the quantity d_n is the least common multiple of the polynomials $p - 1, p^2 - 1, \dots, p^n - 1$, which equals

$$d_n(p) = \prod_{j=1}^{n} \Phi_j(p);$$

Mertens' theorem asserts that, for a real number p with $|p| > 1$,

$$\lim_{n\to\infty} \frac{\log |d_n(p)|}{n^2 \log |p|} = \frac{3}{\pi^2}.$$

The following statement summarizes the analytic and arithmetic results of [10], [12] for the linear forms in the odd q-zeta values.

Proposition 4. *In the above notation,*

$$\lim_{n\to\infty} \frac{\log |h_n(q)|}{n^2 \log |p|} = -t(s - t),$$

and

$$\limsup_{n\to\infty} \frac{\log |a_{i,n}(q)|}{n^2 \log |p|} \le \frac{s + 2t^2}{4} \quad \text{for } i = 0, 1, \dots, s - 1.$$

Moreover, the coefficients of the forms (13) *satisfy*

$$\begin{gathered} (2s)!p^M d_n(p)^{2s-1} a_{0,n}(q) \in \mathbb{Z}[p] \qquad \text{and} \\ (2s)!p^M d_n(p)^{2(s-i)-2} a_{i,n}(q) \in \mathbb{Z}[p] \quad \text{for } i = 1, \dots, s - 1, \end{gathered} \tag{15}$$

where

$$M = \left\lceil \frac{s(n + 1)^2}{4} \right\rceil + \frac{tn(tn - 1)}{2} + (2s + 1)n - \left\lfloor \frac{(s - t)n}{2} \right\rfloor.$$

The arithmetic conclusion (15) may be significantly sharpened using the argument of [20, Section 4]: one just replaces primes by cyclotomic polynomials (14) (cf. [21, Section 1]). In order to state the resulting improvement of Proposition 4, we introduce the p-polynomials

$$\widehat{\Pi}_n(p) = \widehat{\Pi}_n^{(t)}(p) = \prod_{l=2}^{2t-1} \prod_{\substack{\sqrt{(t+1)n} < j \le n \\ \{n/j\} \in E_l}} \Phi_j(p)^{l-1}, \tag{16}$$

where the sets E_l are defined in (8).

Proposition 5. *In the above notation, the coefficients of the linear forms* (13) *satisfy*

$$\begin{gathered}(2s)!p^M d_n(p)^{2s-1}\widehat\Pi_n(p)^{-1}a_{0,n}(q)\in\mathbb{Z}[p] \qquad \text{and} \\ (2s)!p^M d_n(p)^{2(s-i)-2}\widehat\Pi_n(p)^{-1}a_{i,n}(q)\in\mathbb{Z}[p] \quad \text{for } i=1,\dots,s-1,\end{gathered} \tag{17}$$

and the asymptotic behavior of (16) *is determined by*

$$\begin{aligned}\lim_{n\to\infty}\frac{\log\widehat\Pi_n(p)}{n^2\log|p|}=\widehat\varpi'_t&=(2t-1)\psi_1(2)-\sum_{l=1}^t\psi_1\Big(1+\frac{l}{t}\Big)-\sum_{l=2}^t\psi_1\Big(1+\frac{l}{t+1/2}\Big)\\ &=\frac{(t-1)^2}{2}+\frac{3}{\pi^2}\Big(2t-1-t^2\sum_{l=1}^t\frac{1}{l^2}\Big)-\sum_{l=2}^t\psi_1\Big(1+\frac{l}{t+1/2}\Big),\end{aligned}$$

where

$$\psi_1(x)=\frac{3}{\pi^2}\frac{\mathrm{d}\psi(x)}{\mathrm{d}x}=-\frac{3}{\pi^2}\sum_{k=0}^\infty\frac{1}{(x+k)^2}$$

denotes the (*normalized*) *trigamma function.*

Proof (Theorem 4). In the notation $p=1/q\in\mathbb{Z}\setminus\{0,\pm1\}$, set

$$\ell_{i,n}=p^M d_n(p)^{2s-1}\widehat\Pi_n(p)^{-1}a_{i,n}(q)\in\mathbb{Z} \quad \text{for } i=0,\dots,s-1.$$

To these linear forms in

$$(\xi_0,\xi_1,\xi_2,\dots,\xi_{s-1})=\big(1,\zeta_q(3),\zeta_q(5),\dots,\zeta_q(2s-1)\big),$$

we apply Theorem 6 taking $Q_n=\beta^{n^2\log|p|}$ and $\tau=-(\log\alpha)/(\log\beta)$, where

$$\log\alpha=-t(s-t)+\frac{s+2t^2}{4}+\frac{3(2s-1)}{\pi^2}-\widehat\varpi'_t$$

and

$$\log\beta=\frac{s+2t^2}{2}+\frac{3(2s-1)}{\pi^2}-\widehat\varpi'_t.$$

From (17) we see that

$$d_n(p)^{2i+1}\mid\ell_{i,n} \quad \text{for } i=1,\dots,s-1,$$

hence we may take $\delta_{i,n}=d_n(p)^{2i+1}$ to meet the required conditions of Theorem 6. The existence of an odd integer $3\le i_0\le 9$, for which $\zeta_q(i_0)$ is irrational, is already shown in [10]. The following choices of s and t and Theorem 6 ensure the truth of Theorem 4:

$$\begin{aligned}s&=19,\ \ t=4: \quad 1-\frac{\log\alpha-3\cdot3/\pi^2}{\log\beta}=2.0300573456\ldots>2,\\ s&=42,\ \ t=6: \quad 1-\frac{\log\alpha-(3+5)\cdot3/\pi^2}{\log\beta}=3.0344397971\ldots>3,\\ s&=73,\ \ t=8: \quad 1-\frac{\log\alpha-(3+5+7)\cdot3/\pi^2}{\log\beta}=4.0108485236\ldots>4. \qquad \square\end{aligned}$$

3.3 Third application: $\log 2$ and odd zeta values

As in the two previous sections, we take a pair of positive integers s and t with $t < s$, but this time we assume s to be *even*. Consider the hypergeometric series

$$\widetilde{h}_n = 2\frac{(2^{-2n}(2n)!)^s}{n!^{2t}} \sum_{k=1}^{\infty}\Big(k + \frac{n}{2}\Big)\frac{\prod_{j=1}^{tn}(k-j)\cdot\prod_{j=1}^{tn}(k+n+j)}{\prod_{j=0}^{2n}(k+j/2)^s}. \tag{18}$$

Its k-rational summand

$$H_n(k) = (2k+n)\frac{\prod_{j=1}^{tn}(k-j)}{n!^t}\,\frac{\prod_{j=1}^{tn}(k+n+j)}{n!^t}\bigg(\frac{2^{-2n}(2n)!}{\prod_{j=0}^{2n}(k+j/2)}\bigg)^s \tag{19}$$

differs from the corresponding one in (3) a little: the s products $\big(n!/\prod_{j=0}^{n}(k+j)\big)^2$ in (3) are replaced by $2^{-2n}(2n)!/\prod_{j=0}^{2n}(k+j/2)$ in (18), and these two have similar asymptotics as $n \to \infty$. This similarity allows us to compute, like in [2] or [20], the asymptotic behavior of (18) and of the coefficients in the 'zeta' decomposition of (18) which we are going to describe in the next statement.

Lemma 1. *In the above notation, we have*

$$\widetilde{h}_n = \widetilde{a}_{0,n} + \widetilde{a}_{1,n}\log 2 + \sum_{i=2}^{s/2}\widetilde{a}_{i,n}\zeta(2i-1), \tag{20}$$

where

$$2^{4tn}d_{2n}^{s}\widetilde{a}_{0,n} \in \mathbb{Z} \quad \textit{and} \quad 2^{4tn}d_{2n}^{s-2i+1}\widetilde{a}_{i,n} \in \mathbb{Z} \quad \textit{for } i = 1, 2, \ldots, s/2. \tag{21}$$

Proof. The function (19) is the product of integer-valued polynomials $2k+n$,

$$\frac{\prod_{j=1}^{n}(k-ln-j)}{n!}, \quad l = 0, 1, \ldots, t-1, \qquad \text{and} \qquad \frac{\prod_{j=1}^{n}(k+ln+j)}{n!}, \quad l = 1, 2, \ldots, t,$$

and of s copies of the rational function

$$\frac{2^{-2n}(2n)!}{\prod_{j=0}^{2n}(k+j/2)} = \frac{2\cdot(2n)!}{\prod_{j=0}(2k+j)} = \sum_{j=0}^{2n}\frac{(-1)^j\binom{2n}{j}}{k+j/2}. \tag{22}$$

It follows from the Leibniz rule for differentiating a product (cf. [20, Lemmas 1.2–1.4] and the formula (32) below) that

$$H_n(k) = \sum_{i=1}^{s}\sum_{j=0}^{2n}\frac{A_{i,j}}{(k+j/2)^i}$$

with the property

$$2^{4tn}d_{2n}^{s-i}A_{i,k} \in \mathbb{Z}. \tag{23}$$

For a variable x in the unit circle $|x|<1$, we now perform the summation

$$\begin{aligned}
\sum_{k=1}^{\infty} H_n(k) x^{2k} &= \sum_{k=1}^{\infty} x^{2k} \sum_{i=1}^{s} \sum_{j=0}^{2n} \frac{A_{i,j}}{(k+j/2)^i} = \sum_{i=1}^{s} \sum_{j=0}^{2n} A_{i,j} x^{-j} \sum_{k=1}^{\infty} \frac{x^{2k+j}}{(k+j/2)^i} \\
&= \sum_{i=1}^{s} \sum_{j=0}^{n} A_{i,2j} x^{-2j} \biggl(\sum_{k=1}^{\infty} - \sum_{k=1}^{j} \biggr) \frac{x^{2k}}{k^i} \\
&\qquad + \sum_{i=1}^{s} \sum_{j=1}^{n} A_{i,2j-1} x^{-2j+1} \biggl(\sum_{k=1}^{\infty} - \sum_{k=1}^{j} \biggr) \frac{x^{2k-1}}{(k-1/2)^i} \\
&= \sum_{i=1}^{s} \mathrm{Li}_i(x^2) \sum_{j=0}^{n} A_{i,2j} x^{-2j} + \sum_{i=1}^{s} \bigl(2^i \mathrm{Li}_i(x) - \mathrm{Li}_i(x^2) \bigr) \sum_{j=1}^{n} A_{i,2j-1} x^{-2j+1} \\
&\qquad - \sum_{i=1}^{s} \sum_{j=0}^{n} A_{i,2j} \sum_{k=1}^{j} \frac{x^{2k-2j}}{k^i} - \sum_{i=1}^{s} \sum_{j=1}^{n} A_{i,2j-1} \sum_{k=1}^{j} \frac{x^{2k-2j}}{(k-1/2)^i},
\end{aligned} \tag{24}$$

where

$$\mathrm{Li}_i(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^i}$$

denotes the ith polylogarithm function. To compute the limit $x \to 1^-$ in (24), we use Abel's theorem for power series, the sum residue theorem in the form

$$\sum_{j=0}^{n} A_{1,2j} + \sum_{j=1}^{n} A_{1,2j-1} = \sum_{j=0}^{2n} \mathrm{Res}_{k=-j/2} H_n(k) = -\mathrm{Res}_{k=\infty} H_n(k) = 0,$$

and the identity $\mathrm{Li}_1(x) - \mathrm{Li}_1(x^2) = -\mathrm{Li}_1(-x)$. Therefore,

$$\begin{aligned}
\widetilde{h}_n &= \log 2 \cdot 2 \sum_{j=1}^{n} A_{1,2j-1} + \sum_{i=2}^{s} \zeta(i) \biggl(\sum_{j=0}^{n} A_{i,2j} + (2^i - 1) \sum_{j=1}^{n} A_{i,2j-1} \biggr) \\
&\qquad - \sum_{i=1}^{s} \biggl(\sum_{j=0}^{n} A_{i,2j} \sum_{k=1}^{j} \frac{1}{k^i} + \sum_{j=1}^{n} A_{i,2j-1} \sum_{k=1}^{j} \frac{1}{(k-1/2)^i} \biggr),
\end{aligned} \tag{25}$$

where we used the evaluations $-\mathrm{Li}_1(-1) = \log 2$ and $\mathrm{Li}_i(1) = \zeta(i)$ for $i = 2, \dots, s$. Finally, note that the parity of s implies from (19) that

$$H_n(-k-n) = -H_n(k),$$

hence $A_{i,j} = (-1)^{i-1} A_{i,2n-j}$ and $\sum_{j=0}^{n} A_{i,2j} = \sum_{j=1}^{n} A_{i,2j-1} = 0$ for $i = 2, 4, \dots, s$. This implies the required decomposition (20) with

$$\begin{gathered}\widetilde{a}_{0,n}=-\sum_{i=1}^{s}\biggl(\sum_{j=0}^{n}A_{i,2j}\sum_{k=1}^{j}\frac{1}{k^i}+\sum_{j=1}^{n}A_{i,2j-1}\sum_{k=1}^{j}\frac{1}{(k-1/2)^i}\biggr),\\ \widetilde{a}_{1,n}=2\sum_{j=1}^{n}A_{1,2j-1},\quad\text{and}\\ \widetilde{a}_{i,n}=\sum_{j=0}^{n}A_{2i-1,2j}+(2^{2i-1}-1)\sum_{j=1}^{n}A_{2i-1,2j-1}\quad\text{for } i=2,\dots,s/2.\end{gathered}\tag{26}$$

Using (23) we arrive at the inclusions (21). □

We are now in power to sharpen the inclusions (21) in the way we already did in Propositions 3 and 5. Note that for an integer $N>2$ and a prime $p>\sqrt{2N}$ we have

$$\operatorname{ord}_p\Gamma(N+1)=\operatorname{ord}_p N!=\left\lfloor\frac{N}{p}\right\rfloor\quad\text{and}\quad\operatorname{ord}_p\frac{\Gamma(N+1/2)}{\Gamma(1/2)}=\operatorname{ord}_p\frac{(2N)!}{2^{2N}N!}=\left\lfloor\!\left\lfloor\frac{N}{p}\right\rfloor\!\right\rfloor,\tag{27}$$

where

$$\lfloor\!\lfloor x\rfloor\!\rfloor=\lfloor 2x\rfloor-\lfloor x\rfloor\tag{28}$$

(see the proof of Lemma 3 below for another expression of $\lfloor\!\lfloor x\rfloor\!\rfloor$).

Lemma 2. *For the coefficients in the decomposition* (20)*, we have the inclusions*

$$2^{4tn}d_{2n}^{s}\widetilde{\Pi}_n^{-1}\widetilde{a}_{0,n}\in\mathbb{Z}\quad\textit{and}\quad 2^{4tn}d_{2n}^{s-2i+1}\widetilde{\Pi}_n^{-1}\widetilde{a}_{i,n}\in\mathbb{Z}\quad\textit{for } i=1,2,\dots,s/2,\tag{29}$$

where

$$\widetilde{\Pi}_n=\widetilde{\Pi}_n^{(t)}=\prod_{\sqrt{2(t+1)n}<p\le 2n}p^{\tau(n/p)}\tag{30}$$

and the function $\tau(\,\cdot\,)$ is defined as follows:

$$\tau(x)=\tau_t(x)=\min_{y\in\mathbb{R}}\{\tau_1(x,y),\tau_2(x,y)\},\tag{31}$$

$$\begin{aligned}\tau_1(x,y)&=\Bigl\lfloor\Bigl(t+\frac12\Bigr)x+\frac y2\Bigr\rfloor+\Bigl\lfloor\Bigl(t+\frac12\Bigr)x-\frac y2\Bigr\rfloor-\Bigl\lfloor\frac x2+\frac y2\Bigr\rfloor-\Bigl\lfloor\frac x2-\frac y2\Bigr\rfloor-2t\lfloor x\rfloor,\\ \tau_2(x,y)&=\Bigl\lfloor\!\Bigl\lfloor\Bigl(t+\frac12\Bigr)x+\frac y2\Bigr\rfloor\!\Bigr\rfloor+\Bigl\lfloor\!\Bigl\lfloor\Bigl(t+\frac12\Bigr)x-\frac y2\Bigr\rfloor\!\Bigr\rfloor-\Bigl\lfloor\!\Bigl\lfloor\frac x2+\frac y2\Bigr\rfloor\!\Bigr\rfloor-\Bigl\lfloor\!\Bigl\lfloor\frac x2-\frac y2\Bigr\rfloor\!\Bigr\rfloor-2t\lfloor x\rfloor.\end{aligned}$$

Proof. In the notation of the above proof of Lemma 1, we can write

$$A_{i,j}=\frac{1}{(s-i)!}\frac{\mathrm{d}^{s-i}}{\mathrm{d}k^{s-i}}\Bigl(H_n(k)\Bigl(k+\frac j2\Bigr)^s\Bigr)\Big|_{k=-j/2}\quad\text{for } i=1,\dots,s\text{ and } j=0,1,\dots,2n.\tag{32}$$

Taking into account the partial fraction decomposition (22) and

$$\frac{\prod_{l=1}^{tn}(k-l)}{n!^t}\,\frac{\prod_{l=1}^{tn}(k+n+l)}{n!^t}\bigg|_{k=-j/2}$$
$$= (-1)^{tn}\frac{\Gamma((2t+1)n/2+(j-n)/2+1)}{n!^t\Gamma(n/2+(j-n)/2+1)}\,\frac{\Gamma((2t+1)n/2+(n-j)/2+1)}{n!^t\Gamma(n/2+(n-j)/2+1)} \qquad \text{for } j=0,1,\dots,2n, \tag{33}$$

with the help of [20, Lemma 4.1] we conclude that any common multiple Π of the numbers in (33), involving primes $p\le 2n$ only, can be used in sharpening the inclusions (23):

$$2^{2tn}d_{2n}^{s-i}\Pi^{-1}A_{i,j}\in\mathbb{Z} \quad \text{for } i=1,\dots,s \text{ and } j=0,1,\dots,2n.$$

In view of (26), it is enough to show that $\widetilde{\Pi}_n$ defined in (30) is such a multiple. From (27) we see that

$$\operatorname{ord}_p\frac{\Gamma((2t+1)n/2+(j-n)/2+1)}{n!^t\Gamma(n/2+(j-n)/2+1)}\,\frac{\Gamma((2t+1)n/2+(n-j)/2+1)}{n!^t\Gamma(n/2+(n-j)/2+1)}$$
$$=\begin{cases}\tau_1(n/p,(j-n)/p) & \text{for } j \text{ even},\\ \tau_2(n/p,(j-n)/p) & \text{for } j \text{ odd},\end{cases} \qquad j=0,1,\dots,2n,$$

and this implies the desired result. □

Lemma 3. *The quantity* (30) *can be written as follows:*

$$\widetilde{\Pi}_n=\widetilde{\Pi}_n^{(t)}=\prod_{l=1}^{2t-1}\prod_{\substack{\sqrt{2(t+1)n}<p\le 2n\\ \{n/p\}\in E_l}}p^l, \tag{34}$$

where the sets E_l are given in (8)*. In addition,*

$$\lim_{n\to\infty}\frac{\log\widetilde{\Pi}_n}{n}=\widetilde{\varpi}_t=4\left\lfloor\frac{t}{2}\right\rfloor-\left(2t+\frac12\right)\sum_{l=1}^{\lfloor t/2\rfloor}\frac1l+\frac{\pi}{2}\sum_{l=1}^{t}\cot\frac{2\pi l}{2t+1}+t\log t+\left(t+\frac12\right)\log(2t+1). \tag{35}$$

Proof. Using a simple identity $\lfloor 2x\rfloor=\lfloor x\rfloor+\lfloor x+1/2\rfloor$, we see that the function (28) is nothing else but $\lfloor x+1/2\rfloor$. This implies that $\tau_2(x,y)=\tau_1(x,y+1)$, hence

$$\tau(x)=\min_{y\in\mathbb{R}}\{\tau_1(x,y)\}. \tag{36}$$

Furthermore, it follows from (31) that $\tau_1(x+1,y)=\tau_1(x,y+1)$, hence the function $\tau(x)$ is 1-periodic:

$$\tau(x)=\tau(\{x\}).$$

Moreover, we have $\tau_1(x, y+2) = \tau_1(x, y)$ and $\tau_1(x, -y) = \tau_1(x, y)$ implying that the minimum in (36) can be performed for $y \in [0, 1[$ only:

$$\tau(x) = \min_{0 \le y < 1} \{\tau_1(x, y)\}.$$

It remains to use the results of [20, Section 4] (already taken into account in Sections 3.1 and 3.2):

$$\min_{0 \le y < 1} \{\tau_1(x, y)\} = l \quad \text{for } x \in E_l, \quad l = 0, 1, \dots, 2t-1,$$

where the sets $E_0, E_1, \dots, E_{2t-1}$ are defined in (8); this gives us the desired form (34) of the quantity (30).

To compute the asymptotics in (35) we apply [20, Lemma 4.4]:

$$\begin{aligned}
\lim_{n\to\infty} \frac{\log \widetilde{\Pi}_n}{n} &= \sum_{l=1}^{2t-1} l \biggl(\int_{E_l \cap [0,1/2)} \mathrm{d}\psi(1+x) + \int_{E_l \cap [1/2,1)} \mathrm{d}\psi(x) \biggr) \\
&= \sum_{l=1}^{2t-1} l \biggl(\int_{E_l} \mathrm{d}\psi(1+x) + \int_{E_l \cap [1/2,1)} \mathrm{d}\Bigl(-\frac{1}{x}\Bigr) \biggr) \\
&= \varpi_t + \sum_{l=\lfloor t/2 \rfloor}^{t-1} 2l \int_{E_{2l}} \mathrm{d}\Bigl(-\frac{1}{x}\Bigr) - 2\Bigl\lfloor \frac{t}{2} \Bigr\rfloor \int_{\lfloor t/2 \rfloor / t}^{1/2} \mathrm{d}\Bigl(-\frac{1}{x}\Bigr) \\
&\qquad + \sum_{l=\lfloor t/2 \rfloor + 1}^{t} (2l-1) \int_{E_{2l-1}} \mathrm{d}\Bigl(-\frac{1}{x}\Bigr) \\
&= \varpi_t + \sum_{l=\lfloor t/2 \rfloor}^{t-1} 2l \Bigl(\frac{t}{l} - \frac{t+1/2}{l+1} \Bigr) - 2\Bigl\lfloor \frac{t}{2} \Bigr\rfloor \Bigl(\frac{t}{\lfloor t/2 \rfloor} - 2 \Bigr) \\
&\qquad + \sum_{l=\lfloor t/2 \rfloor + 1}^{t} (2l-1) \Bigl(\frac{t+1/2}{l} - \frac{t}{l} \Bigr) \\
&= \varpi_t + \Bigl(2t + \frac{1}{2} \Bigr) \sum_{l=\lfloor t/2 \rfloor + 1}^{t} \frac{1}{l} - 2\Bigl(t - 2\Bigl\lfloor \frac{t}{2} \Bigr\rfloor \Bigr),
\end{aligned}$$

where ϖ_t is defined in (9). It remains to apply identities for the digamma function, and the lemma follows. ☐

The following statement summarizes our findings in this section.

Proposition 6. *For positive integers s and t with s even and $t < s$, the linear forms* (18), (20) *and their coefficients admit the asymptotics*

$$\lim_{n\to\infty} \frac{\log |\widetilde{h}_n|}{n} = f(x_0)$$

and

$$\limsup_{n\to\infty}\frac{\log|\widetilde{a}_{i,n}|}{n}\le \operatorname{Re} f(0)=2(s-t)\log 2+(2t+1)\log(2t+1) \quad \text{for } i=0,1,\dots,s/2,$$

where $x_0 \in \left]t+\frac{1}{2},+\infty\right[$ *is the maximal real zero of the polynomial* (5) *and the function* $f(x)$ *is defined in* (6). *Moreover, the coefficients in the decomposition* (20) *satisfy* (29) *and the asymptotics of the quantities* (30), (34) *is determined in* (35).

Proof (Theorem 5). This time we have $\mathbb{Q}$-linear forms in

$$(\xi_0,\xi_1,\xi_2,\dots,\xi_{s/2})=\bigl(1,\log 2,\zeta(3),\zeta(5),\dots,\zeta(s-1)\bigr),$$

whose coefficients are

$$\ell_{i,n}=2^{4tn}d_{2n}^{s}\widetilde{\Pi}_n^{-1}\widetilde{a}_{i,n}\in\mathbb{Z} \quad \text{for } i=0,1,\dots,s/2.$$

Then (29) implies

$$d_{2n}^{2i-1} \mid \ell_{i,n} \quad \text{for } i=1,2,\dots,s/2,$$

hence in the notation of Theorem 1 we have $r=s/2$, $\delta_{i,n}=d_{2n}^{2i-1}$,

$$\log\alpha=f(x_0)+4t\log 2+2s-\widetilde{\varpi}_t$$

and

$$\log\beta=2(s+t)\log 2+(2t+1)\log(2t+1)+2s-\widetilde{\varpi}_t.$$

Applying the theorem with the choice $s=94$, $t=11$ we obtain

$$1-\frac{\log\alpha-2}{\log\beta}=2.0064440535\ldots>2,$$

while the choice $s=1152$, $t=67$ results in

$$1-\frac{\log\alpha-(2+6)}{\log\beta}=3.0004493689\ldots>3.$$

This implies the required independence result. □

3.4 Triple integrals for rational approximations to $\log 2$

The particular case $s=2$, $t=1$ of our construction in Section 3.3 is of independent interest, since the corresponding series

$$\widetilde{h}_n=2\left(\frac{2^{-2n}(2n)!}{n!}\right)^2\sum_{k=1}^{\infty}\left(k+\frac{n}{2}\right)\frac{\prod_{j=1}^{n}(k-j)\cdot\prod_{j=1}^{n}(k+n+j)}{\prod_{j=0}^{2n}(k+j/2)^2} \tag{37}$$

goes in parallel with Ball's series

$$h_n = 2n!^2 \sum_{k=1}^{\infty} \Big(k + \frac{n}{2}\Big) \frac{\prod_{j=1}^{n}(k-j) \cdot \prod_{j=1}^{n}(k+n+j)}{\prod_{j=0}^{n}(k+j)^4}$$

for Apéry's approximations to $\zeta(3)$. Switching to the classical hypergeometric notation [17] and using

$$\frac{2^{-2n}(2n)!}{n!} = \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})} = \frac{\Gamma(n+1/2)}{\sqrt{\pi}},$$

we can write the series in (37) as the following very-well-poised hypergeometric series:

$$\begin{aligned} \widetilde{h}_n &= \frac{\Gamma(3n+3)\Gamma(n+1)^3\Gamma(n+\frac{1}{2})^2\Gamma(n+\frac{3}{2})^2}{\pi\Gamma(2n+2)^3\Gamma(2n+\frac{3}{2})^2} \\ &\qquad \times {}_7F_6\left(\begin{matrix} 3n+2, \ \frac{3n+2}{2}+1, \ n+1, \ n+1, \ n+\frac{3}{2}, \ n+1, \ n+\frac{3}{2} \\ \frac{3n+2}{2}, \ 2n+2, \ 2n+2, \ 2n+\frac{3}{2}, \ 2n+2, \ 2n+\frac{3}{2} \end{matrix} \,\middle|\, 1 \right). \end{aligned}$$

Applying now Bailey's transformation (see [17, Eq. (4.7.1.3)] or [22, Proposition 2]), after a little reduction of the gamma factors we get the Barnes-type integral

$$\widetilde{h}_n = \frac{2n+1}{2\pi} \cdot \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(n+1+\xi)^2\Gamma(n+\frac{1}{2}+\xi)\Gamma(n+\frac{3}{2}+\xi)\Gamma(-\xi)\Gamma(-\frac{1}{2}-\xi)}{\Gamma(2n+2+\xi)\Gamma(2n+\frac{3}{2}+\xi)} \, \mathrm{d}\xi, \quad (38)$$

where the path separates the decreasing sequence of poles $\xi = -n-\frac{1}{2}, -n-1, -n-\frac{3}{2}, \dots$ and the increasing sequence of poles $\xi = 0, \frac{1}{2}, 1, \dots$ of the integrand. The result can be expressed as a triple real integral thanks to a theorem of Nesterenko [22, Proposition 1]:

$$\widetilde{h}_n = \frac{2n+1}{2\pi} \iiint\limits_{[0,1]^3} \frac{x^n(1-x)^n y^{n-1/2}(1-y)^n z^{n+1/2}(1-z)^{n-1/2}}{(1-(1-xy)z)^{n+1}} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z. \quad (39)$$

On the other hand, using the duplication formula $\Gamma(z)\Gamma(z+\frac{1}{2}) = \sqrt{\pi}\, 2^{1-2z}\Gamma(2z)$ and the Barnes-type and Euler integrals for the Gauss hypergeometric function (see [17, Sections 1.6.1 and 4.1]) we can transform (38) further:

$$\begin{aligned} \widetilde{h}_n &= 4(2n+1) \cdot \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(2n+1+2\xi)\Gamma(2n+2+2\xi)\Gamma(-1-2\xi)}{\Gamma(4n+3+2\xi)} \, \mathrm{d}\xi \\ &= 2(2n+1) \cdot \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(2n+\xi)\Gamma(2n+1+\xi)\Gamma(-\xi)}{\Gamma(4n+2+\xi)} \, \mathrm{d}\xi \\ &= 2(2n+1) \frac{\Gamma(2n)\Gamma(2n+1)}{\Gamma(4n+2)} \, {}_2F_1\left(\begin{matrix} 2n, \ 2n+1 \\ 4n+2 \end{matrix} \,\middle|\, -1\right) = 2\int_0^1 \frac{x^{2n-1}(1-x)^{2n+1}}{(1+x)^{2n+1}} \, \mathrm{d}x, \end{aligned}$$

and for the latter integral the decomposition

$$\widetilde{h}_n = \widetilde{a}_{0,n} + \widetilde{a}_{1,n} \log 2 \qquad \text{with} \quad d_{2n}\widetilde{a}_{0,n} \in \mathbb{Z} \quad \text{and} \quad \widetilde{a}_{1,n} \in \mathbb{Z} \quad (40)$$

is known (see, for example, [19]). The arithmetic inclusions in (40) are much better than the ones we have from Proposition 6; this suggests the existence of a 'power denominator conjecture' for the linear forms constructed in Section 3.3. In addition, a more general form of the triple integral in (39) could be of use in study of the quality of rational approximations to $\log 2$; it is due to a remarkable resemblance of such integrals with the ones used by G. Rhin and C. Viola [15] in proving the record irrationality measure for $\zeta(3)$. For a different construction of rational approximations to $\log 2$ using the Rhin–Viola method, we refer the reader to the paper [13], where R. Marcovecchio obtains a new irrationality measure for this constant.

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