Virtual cohomology of the moduli space of curves in the unstable range

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0. — Introduction

Let $\mathcal{M}_g$ denote the moduli space of smooth, projective curves of genus $g \geq 2$. The cohomology space $H^i(\mathcal{M}_g)$, for $i < \frac{g}{2}$, is independent of g; according to conjectures of Mumford [5] it should be represented by tautological classes which, in particular, are Tate classes (for the natural action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, taking étale cohomology). On the other hand, it is known that for $g >> 0$, $\mathcal{M}_g$ is of general type and, in particular, carries many holomorphic sections of the pluricanonical bundle.

Harris and Mumford [6] have asked whether (for large g) $\mathcal{M}_g$ carried holomorphic forms of degree g, $2g-1$ or $3g-3$: these degrees are suggested by the allowed degrees for holomorphic forms on the space $\mathcal{A}_g$ of principally polarized Abelian varieties ([9]) and its coverings. In this paper we will answer the question, but only, unfortunately, in a virtual fashion.

Write $\mathcal{M}_g = \Gamma_g \backslash \mathfrak{T}_g$, where $\mathfrak{T}_g$ is the Teichmüller space, and Γ_g the Teichmüller group. There is a natural map $\Gamma_g \to \mathrm{Sp}(g, \mathbb{Z})$ given by the action of Γ_g on the cohomology of the "universal" curve of genus g. Let $\Gamma_g(N)$ be the inverse image in Γ_g of the full level N subgroup $\Gamma(N)$ in $\Gamma = \mathrm{Sp}(g, \mathbb{Z})$. Thus $\Gamma_g/\Gamma_g(N) \cong \mathrm{Sp}(g, \mathbb{Z}/N\mathbb{Z})$ since $\Gamma_g \to \Gamma$ is surjective.

Denote by $\mathcal{M}_g(N)$ the quotient $\Gamma_g(N)\backslash\mathfrak{T}_g$, a Galois covering of $\mathcal{M}_g$ with group $\mathrm{Sp}(g, \mathbb{Z}/N\mathbb{Z})/(\pm 1)$. We will prove :

THEOREM 1. — *For fixed g, and N sufficiently large,*

$$H^0(\mathcal{M}_g(N), \Omega^i) \neq 0 \text{ for } i = g,\ 2g-1,\ 3g-3,$$

assuming moreover that $g > 3$ (if $i = 2g-1$) and $g > 5$ (if $i = 3g-3$).

Our proof relies on a method developed in an earlier paper [2] and applied there to the restriction of holomorphic cohomology classes to subvarieties of Shimura varieties. We use it here to study the restriction to $\mathcal{M}_g$ (via the Torelli embedding) of holomorphic cohomology classes on $\mathcal{A}_g$. A simple differential computation implies that this restriction is (virtually) injective. The theorem follows from existence results for holomorphic forms on $\mathcal{A}_g$; the precise theorem we use is due to Li [4].

Note that according to Weissauer [9], that are no holomorphic forms on $\mathcal{A}_g$ in degrees g, $2g-1$, $3g-3$, at least for $g >> 0$. Thus it may be natural to expect the same of $\mathcal{M}_g$ (rather than its coverings!).

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1. — Differential calculus

Let $\mathcal{M}_g$ denote the moduli space of smooth, projective curves of genus $g \geq 2$. We use the transcendental realization of $\mathcal{M}_g$ as $\Gamma_g \backslash \mathfrak{T}_g$, where $\mathfrak{T}_g$, the Teichmüller space, is a bounded, contractible, holomorphically convex domain in $\mathbb{C}^{3g-3}$. The Torelli map t which to a curve C associates its Jacobian is an injection of $\mathcal{M}_g$ into $\mathcal{A}_g$, the space of principally polarized Abelian varieties of genus g.

The associated map $\Gamma_g \to \Gamma := \mathrm{Sp}(g, \mathbb{Z})$ is surjective, and we define $\Gamma_g(N)$ as the inverse image in Γ_g of the full level N subgroup

$$\Gamma(N) = \{\gamma \in \Gamma : \gamma \equiv 1[N]\} \tag{1.1}$$

in Γ. We will consider the associated map

$$t(N) : \mathcal{M}_g(N) = \Gamma_g(N) \backslash \mathfrak{T}_g \to \mathcal{A}_g(N) \tag{1.2}$$

with $\mathcal{A}_g(N)$ the space of principally polarized Abelian varieties with full level N structure. We view $\mathcal{A}_g(N)$ as the quotient $\Gamma(N) \backslash \mathcal{H}_g$, where $\mathcal{H}_g$ is the Siegel upper-half space. We will denote by G the $\mathbb{Q}$-group $\mathrm{Sp}(g)$; thus $G(\mathbb{R})$ acts on $\mathcal{H}_g$.

Let ω be a holomorphic i-form on $\mathcal{A}_g(N)$, which we view as a form on $\mathcal{H}_g$ invariant under $\Gamma(N)$. If $\gamma \in G(\mathbb{Q})$ is seen as acting by (left) translations on $\mathcal{H}_g$, $\gamma^*\omega$ is then invariant under $\Gamma(1) \cap \gamma\Gamma(N)\gamma^{-1}$, a congruence subgroup of $\Gamma(1)$ which contains a subgroup $\Gamma(M)$. Thus $\gamma^*\omega$ is a i-form on $\mathcal{A}_g(M)$ for some M.

We will say that ω is **virtually non-zero along** $\mathfrak{T}_g$ if there exists $\gamma \in G(\mathbb{Q})$ such that

$$t(M)^*\gamma^*\omega \neq 0 \tag{1.3}$$

M being of course determined as above by γ. (*)

We denote by Ω^i or Ω^i_X the sheaf of holomorphic i-forms on a variety X. On $\mathcal{A}_g(N)$ we have an invariant measure, and we can consider the corresponding spaces of square-integrable forms.

PROPOSITION 1. — *Assume $\omega \in H^0(\mathcal{A}_g(N), \Omega^i)$ is square-integrable and non-zero $(i = g, 2g-1, 3g-3)$. Then ω is virtually non-zero along $\mathfrak{T}_g$.*

Proof : Suppose, on the contrary, that $t(M)^*\gamma^*\omega = 0$ for all γ and all M such that $\Gamma(1) \cap \gamma\Gamma(N)\gamma^{-1} \supset \Gamma(M)$. In particular, consider the lift $\tilde{t}$ to $\mathfrak{T}_g$ of the Torelli map :

$$\tilde{t} : \mathfrak{T}_g \to \mathcal{H}_g\ . \tag{1.4}$$

(*) In [2] we would have termed ω "stably non-vanishing along $\mathfrak{T}_g$", but this would be confusing in the present context.

If we view ω as a form on $\mathcal{H}_g$, $\gamma^*\omega$ then must vanish on $\widetilde{t}(\mathfrak{T}_g)$, i.e. : $\widetilde{t}^*(\gamma^*\omega) = 0$. For $\gamma \in G(\mathbb{R})$, $\gamma^*\omega$ is a holomorphic i-form on $\mathcal{H}_g$ that depends continuously on γ. By continuity we deduce that

$$\widetilde{t}^*(\gamma^*\omega) = 0\ ,\ \gamma \in G(\mathbb{R})\ . \tag{1.5}$$

Now fix a point $\gamma \in \mathcal{M}_g$ and let $\widetilde{C} \in \mathfrak{T}_g$ be a base point above C. Let K be the isotropy subgroup of $\widetilde{t}(\widetilde{C})$ in $G(\mathbb{R})$, a group conjugate to $U(g) \subset \mathrm{Sp}(g,\mathbb{R})$. Then we have in particular

$$\widetilde{t}^*(k^*\omega_{\widetilde{J}}) = 0 \text{ for all } k \in K \tag{1.6}$$

where $\widetilde{J} = \widetilde{t}(\widetilde{C})$ lifts the Abelian variety $J = t(C)$, $\omega_{\widetilde{J}}$ is the form ω at the point $\widetilde{J}$, and $\widetilde{t}^*$ is the obvious map between exterior powers of the cotangent spaces at $\widetilde{C}$ and $\widetilde{J}$.

Denote by $V = V(\omega, \widetilde{J})$ the K-span of the vector $\omega_{\widetilde{J}} \in \Lambda^i T^*_{\widetilde{J}}(\mathcal{H}_g)$: we then have

LEMMA 1. — $\widetilde{t}^*_{\widetilde{C}}(V) = 0$.

Note that $T^*_{\widetilde{J}}(\mathcal{H}_g) \cong T^*_J(\mathcal{A}_g)$ and $T^*_{\widetilde{C}}(\mathfrak{T}_g) \cong T^*_C(\mathcal{M}_g)$. We now describe the map $T^*_{\widetilde{C}}(\mathfrak{T}_g) \to T^*_{\widetilde{J}}(\mathcal{H}_g)$ through these identifications. Thus we are interested in the natural map

$$t^* : T^*_J(\mathcal{A}_g) \to T^*_C(\mathcal{M}_g) \tag{1.7}$$

where J is the Jacobian variety of C.

Now both tangent spaces are described by deformation theory; for the Abelian variety we have

$$T^*_J(\mathcal{A}_g) \cong \mathrm{Sym}^2 H^0(J,\Omega). \tag{1.8}$$

Assume that C is not hyperelliptic : then $t(\mathcal{M}_g)$ is non-singular at $t(C)$ and its cotangent space is canonically described as

$$T^*_C(\mathcal{M}g) = H^0(C, \otimes^2\Omega^1_C)\ , \tag{1.9}$$

the space of quadratic differentials on C. We have a canonical isomorphism $H^0(C,\Omega_C) = H^0(J,\Omega_J)$ and we want to describe t^* using these isomorphisms. Thus we get a map $t^* : \mathrm{Sym}^2 H^0(C,\Omega) \to H^0(C,\otimes^2\Omega^1)$ which, according to Andreotti and Mayer, [1] (see also Mumford [6, p. 88] is simply obtained by associating to symmetric tensors the corresponding quadratic differentials.

We now turn to the representation-theoretic interpretation of Lemma 1. Recall that holomorphic, L^2 g-forms on $\Gamma\backslash\mathcal{H}_g$, $\Gamma \subset \Gamma(1)$ being a congruence subgroup, correspond bijectively to submodules of $L^2_{\mathrm{dis}}(\Gamma\backslash G(\mathbb{R}))$ isomorphic to a certain representation $A_{\mathfrak{q}}$, where $\mathfrak{q} \subset \operatorname{Lie} G(\mathbb{R}) \otimes \mathbb{C} = \mathfrak{sp}(g, \mathbb{C})$ is a parabolic subalgebra stable by a Cartan involution. This is due to Parthasarathy, Kumaresan and Vogan-Zuckerman; for a precise description of the correspondence in our context see [2], especially § 3C. We will use the notions contained in this paper without further comment.

We can realize $\mathrm{Sp}(g)$ as the group

$$\left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(g,g) : {}^t g \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix} \right\} . \tag{1.10}$$

Then $K = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} : D = {}^t A^{-1}, A \in U(g) \right\}$. We have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} + \oplus \mathfrak{p}_-$, $\mathfrak{p}_+$ being the holomorphic tangent space at the fixed point (here $\widetilde{J}$) associated to K. Then

$$\mathfrak{p}_+ = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} : {}^t B = B, B \in M_n(\mathbb{C}) \right\} . \tag{1.11}$$

Thus $\mathfrak{p}_+$ is naturally identified to $\mathrm{Sym}^2(\mathbb{C}^g)$. The form $\omega_{\widetilde{J}}$ is an elemtn of $\Lambda^i \mathfrak{p}_+^*$. We now assume temporarily that $i = g$. Then $\mathfrak{q}$ is the parabolic subalgebra associated to

$$x = \sqrt{-1} \begin{pmatrix} x_1 & & & & & & \\ & 0 & & & & & \\ & & \ddots & & & & \\ & & & 0 & & & \\ & & & & -x_1 & & \\ & & & & & 0 & \\ & & & & & & \ddots \\ & & & & & & & 0 \end{pmatrix} \in \operatorname{Lie}_{\mathbb{R}} K, x_1 \neq 0 . \tag{1.12}$$

Let $(e_1, \dots e_g)$ be the natural basis of $\mathbb{C}^g$, and let $V(\mathfrak{q})$ be the K-span of the vector

$$e(\mathfrak{q}) = e_1^2 \wedge e_1 e_2 \wedge \cdots \wedge e_1 e_g \in \Lambda^g \mathfrak{p}_+ . \tag{1.13}$$

This space is irreducible, and occurs exactly once in $\Lambda^g \mathfrak{p}_+$. If $\omega_{\widetilde{J}} \neq 0$, the K-span of $\omega_{\widetilde{J}}$ is then the dual space of $V(\mathfrak{q})$ in $\Lambda^g \mathfrak{p}_+^*$ (see [2,§ 2]). Let (e_i^*) be the dual basis.

Lemma 2. — *Assume $\omega_{\widetilde{J}} \neq 0$. Then $V = V(\omega, \widetilde{J})$ contains $e^*(\mathfrak{q}) = (e_1^*)^2 \wedge e_1^* e_2^* \wedge \cdots \wedge e_1^* e_g^*$.*

This is clear by multiplicity one, since $e^*(\mathfrak{q})$ is dual to $e(\mathfrak{q})$.

We now consider the restriction map $\widetilde{t}^*_{\widetilde{C}}$. Note that in our identifications the space $\mathbb{C}^g$ used to describe $\mathfrak{p}_+$ is naturally identified with the holomorphic tangent space $T_0(J)$; the vectors e_i^* are then differential forms, on J, which form an orthonormal basis of $H^0(J, \Omega_J)$

for the scalar product given by the canonical polarization. By lemmas 1 and 2, if $\omega_{\widetilde{J}} \neq 0$, we must have $\widetilde{t}^*_{\widetilde{C}}(e^*(\mathfrak{g})) = 0$. By (1.8), (1.9) and the description of the restriction map following (1.9), we deduce that

(1.14) $$(e_1^*)^2 \wedge \cdots \wedge e_1^* e_g^* = 0 \ ,$$

an identity in $\Lambda^g H^0(C, \otimes^2 \Omega^1)$. Consequently the quadratic differentials $(e_1^*)^2, (e_1^* e_2^*), \ldots, e_1^* e_g^*$ are linearly dependent. This yields a relation

(1.15) $$e_1^* \big(\sum_{i=1}^{g} \alpha_i e_i^* \big) = 0$$

between differentials on C, which clearly implies that $\sum_{i=1}^{g} \alpha_i e_i^* = 0$, a contradiction.

We have proved that if $\gamma^* \omega$ vanishes for all $\gamma \in \mathcal{A}(\mathbb{Q})$ when restricted to (the suitable covering of) $\mathcal{M}_g$, then $\omega_{\widetilde{J}} = 0$ if $\widetilde{J}$ is the lift of the Jacobian variety of a non-hyperelliptic curve : thus by density $\omega_{\widetilde{J}} = 0$ for all Jacobian lifts. We many apply this conclusion to $\delta^* \omega$ for $\delta \in G(\mathbb{Q})$: the conclusion is that $\delta^* \omega$ vanishes on $\widetilde{t(\mathcal{M}_g)}$ for any $\delta \in G(\mathbb{Q})$. This is then true, by continuity, for any $\delta \in G(\mathbb{R})$ and ω must vanish : this proves Proposition 1 for $i = g$.

We now extend the proof to the other allowable degrees. Note that $\mathcal{A}_g$ can have holomorphic cohomology only in degrees $i = \frac{g(g+1)}{2} - \frac{x(x+1)}{2} = hg - \frac{h(h-1)}{2}$, where $h + x = g$, $0 \geq h \geq g$. (See Weissauer [9], as well as Parthasarathy [7] or [2, § 3C] for L^2-holomorphic forms). In particular the only relevant degrees for restriction to $\mathcal{M}_g$ are g, $2g-1$ and $3g-3$. We consider the cases where $i = 2g-1$ or $3g-3$. According to [2, § 3C] the corresponding L^2-forms on $\mathcal{A}_g$ are associated to representations $A_{\mathfrak{q}}$ of $G(\mathbb{R})$ of the following types. We keep the notations recalled above (1.10) for Lie algebras, and refer the reader to [2] for the description of the Vogan-Zuckerman theory in this context.

(1.16) $$i = 2g - 1$$

$$x = \sqrt{-1} \begin{pmatrix} x_1 & & & & & & & & & \\ & x_1 & & & & & & & & \\ & & 0 & & & & & & & \\ & & & \ddots & & & & & & \\ & & & & 0 & & & & & \\ & & & & & -x_1 & & & & \\ & & & & & & -x_1 & & & \\ & & & & & & & 0 & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & 0 \end{pmatrix}$$

$$\mathfrak{q} \cap \mathfrak{p}_+ = < e_1^2, e_1e_2, \ldots, e_1e_g, e_2^2, e_2e_3, \ldots e_2e_g > \ .$$

$$e(\mathfrak{q}) = e_1^2 \wedge e_1e_2 \wedge \cdots \wedge e_1e_g \wedge e_2^2 \wedge e_2e_3 \wedge \cdots \wedge e_2e_g \ .$$

We now resume the previous analysis. As in the case of $i = g$ we want to check that $\widetilde{t}^*_{\widetilde{C}}(e^*(\mathfrak{q})) \neq 0$, where now $e(\mathfrak{q}) = (e_1^*)^2 \wedge e_1^*e_2^* \wedge \cdots \wedge e_2^*e_g^*$. To prove this we must show that the associated quadratic differentials on C are linearly independent. This amounts to :

LEMMA 3. — *Assume C sufficiently general. Then there exists an orthonormal basis (ω_i) of $H^0(C,\Omega)$ such that the quadratic differentials $\omega_1^2, \omega_1\omega_2, \ldots, \omega_1\omega_3, \ldots, \omega_2\omega_g$ are linearly independent.*

This follows from Petri's 1922 paper [8]; we rely on Mumford's exposition in [6, Lecture 1]. Assume C is not hyperelliptic. Choose g points $x_1, \ldots x_g \in C$ in general position. Then we can take a dual basis (ω_i), with $(\omega_i)_{x_i} \neq 0$ and $(\omega_i)_{x_j} = 0$ $(j \neq i)$. Petri (and Mumford) then show that the quadratic differentials $\omega_1^2, \omega_1\omega_2, \ldots, \omega_1\omega_g, \omega_2^2, \ldots \omega_2\omega_g$ are linearly independent.

To complete the argument, we have to show that this can be ensured with the ω_i an orthonormal basis. However this now follows from Gram-Schmid orthonormalization. (We need an orthonormal basis to ensure that the ω_i are a dual basis of the basis e_i used in Lemma 2, without further linear algebra).

Now assume

$$i = 3g-3 \ , \ g \geq 3 \tag{1.17}$$

$$x = \sqrt{-1}\begin{pmatrix} x_1 &&&&&&&&&&& \\ & x_1 &&&&&&&&&& \\ && x_1 &&&&&&&&& \\ &&& 0 &&&&&&&& \\ &&&& \ddots &&&&&&& \\ &&&&& 0 &&&&&& \\ &&&&&& -x_1 &&&&& \\ &&&&&&& -x_1 &&&& \\ &&&&&&&& -x_1 &&& \\ &&&&&&&&& 0 && \\ &&&&&&&&&& \ddots & \\ &&&&&&&&&&& 0 \end{pmatrix}$$

$$\mathfrak{q} \cap \mathfrak{p}_+ = < e_1^2, e_1e_2, \ldots, e_1e_g, e_2^2, e_2e_3, \ldots e_2e_g, e_3^2, e_3e_4, \ldots e_3e_g >$$

$$e(\mathfrak{q}) = e_1^2 \wedge e_1e_2 \wedge \cdots \wedge e_3e_g \ .$$

The same argument now shows that proposition 1 follows from

LEMMA 4. — *For C sufficiently general in $\mathcal{M}_g$ and $\omega_1,\dots\omega_g$ a suitable orthonormal basis, the quadratic differentials $\omega_1^2, \omega_1\omega_2, \dots, \omega_1\omega_g, \omega_2^2, \omega_2\omega_3, \dots \omega_2\omega_g, \omega_3^2, \dots \omega_3\omega_g$ are linearly independent.*

Proof (*) : We may forget the orthogonality condition since it can be ensured by orthonormalization. Thus we want to show that for a generic basis of $H = H^0(C,\Omega)$ the indicated quadratic differentials are independent. Start with differentials $\omega_1,\dots,\omega_g$ satisfying Petri's conditions (cf. after lemma 3, and [6, p. 18]). Then [6, p. 18-19] the differentials

$$\omega_1^2, \omega_1\omega_2, \dots, \omega_1\omega_g, \omega_2^2, \dots, \omega_2\omega_g, \omega_3^2, \omega_4^2, \dots, \omega_g^2 \tag{1.18}$$

are linearly independent. On the other hand, the differentials $\omega_i\omega_j$ $(i \neq j\ ,\ i,j \geq 3)$ are then linear combinations of the $\omega_1\omega_i$ and $\omega_2\omega_i$ (ibid., p. 19). Now take the new basis obtained by replacing ω_3 by $\omega_3' = \omega_3 + \lambda_4\omega_4 + \cdots + \lambda_g\omega_g$. The space V generated by the $(2g-1)$ first differentials in (1.18) does not change. Modulo V, we now have

$$(\omega_3')^2 = \omega_3^2 + \lambda_4^2\omega_4 + \cdots + \lambda_g^2\omega_g^2 \tag{1.19}$$

$$\omega_3'\omega_4 = \lambda_4\omega_4$$
$$\vdots$$
$$\omega_3'\omega_g = \lambda_g\omega_g^2\ .$$

For $\lambda = (\lambda_4,\dots,\lambda_g)$ nearly 0 and $\lambda_i \neq 0$, $(\omega_1, \omega_2, \omega_3', \dots \omega_g)$ is indeed a basis of H while (1.19) shows that $(\omega_3')^2, \dots, \omega_3'\omega_g$ is a basis for thhe quadratic differentials mod V. This implies the lemma, and the proof of Proposition 1.

We conclude this paragraph with the remark that the square-integrability condition in Proposition 1 is very likely superfluous. We explain how it could be removed. Let ω be a differential form on $\mathcal{A}_g$, invariant under a subgroup $\Gamma(N)$, and consider the lifted differential $\tilde{\omega}$ on $\mathcal{H}_g$. If $x \in \mathcal{H}_g$ and $K = K_x$ is the corresponding isotropy subgroup, we may view ω_x as an element of $\mathrm{Hom}_K(\Lambda^i \mathfrak{p}_x^+, \mathbb{C})$ where $\mathfrak{p}_x^+$ is the holomorphic tangent space at x. Then in degrees $i = g$, $2g-1$, $3g-3$, ω_x should lie in the irreducible K_x-module specified by $A_\mathfrak{q}$, where $\mathfrak{q}$ is the parabolic subalgebra associated to the degree. This is strongly suggested by Weissauer's result [9] according to which $\mathcal{A}_g$ can have holomorphic cohomology only in the degrees $hg - \dfrac{h(h-1)}{2}$ $(0 \leq h \leq g)$ allowed by the holomorphic parabolic subalgebras $A_\mathfrak{q}$, cf. before Lemma 3. Then the previous arguments apply to prove Proposition 1. A stronger statement (which should also be true) is that the space generated by $\tilde{\omega}$ under $G(\mathbb{R})$ is of type $A_\mathfrak{q}$. We leave this to the interested reader.

(*) We thank D. Perrin and A. Beauville for indicating to us the proof of this lemma.

2. — Existence of cohomology on $\mathcal{A}_g(N)$

In order to apply proposition 1, we still need to show the existence of the corresponding classes on $\mathcal{A}_g(N)$. If we did not impose an L^2-condition, (see the discussion at the end of the previous paragraph), we could, in a lot of cases, simply quote a result of Weissauer [10] :

THEOREM 2 (Weissauer). —
 (i) $H^0(\mathcal{A}_g(4), \Omega^g) \neq 0$ *and* $H^0(\mathcal{A}_g(4), \Omega^{3g-3}) \neq 0$ *if* g *is* **even**
 (ii) $H^0(\mathcal{A}_g(4), \Omega^{2g-1}) \neq 0$ *if* g *is* **odd**.

However these differential forms are not cuspidal, and there seems to be no reason to assume that they are square-integrable. We want to prove the existence of L^2-forms of type $A_{\mathfrak{q}}$, for the representations $A_{\mathfrak{q}}$ described in § 1 (associated to $i = g$, $2g-1$, $3g-3$). For this we simply rely on a recent theorem of J.-S. Li. Using the theory of thêta-series he proves the following result.

Denote by A_i the irreducible representation of G with holomorphic cohomology in degree i ($i = g, 2g-1, 3g-3$). We denote by $\mathrm{mult}(A_i, L^2(\Gamma\backslash G))$ the multiplicity of A_i in the discrete part of the L^2-space.

THEOREM 3 (Li [4]). — *For any sufficiently deep congruence subgroup* Γ *of* $\mathrm{Sp}(g, \mathbb{Z})$,
 (i) $\mathrm{mult}(A_g, L^2(\Gamma\backslash G)) > 0$ (*for any* $g \geq 1$)
 (ii) $\mathrm{mult}(A_{2g-1}, L^2(\Gamma\backslash G)) > 0$ ($g > 3$)
 (iii) $\mathrm{mult}(A_{3g-3}, L^2(\Gamma\backslash G)) > 0$ ($g > 5$).

This is theorem 5.8 of [4, p. 209], once the requisite notations are taken into account; note that in [4, formula (54)] the algebra $\mathfrak{l}$, the reductive part of the parabolic subalgebra $\mathfrak{q}$ defining $A_{\mathfrak{q}}$, is isomorphic to $u(\alpha) \times sp(g-\alpha)$ for each of our modules A_i, as follows easily from the description in [2, § 3C]. (Here $i = \alpha g - \dfrac{\alpha(\alpha-1)}{2}$, so $\alpha = 1, 2, 3$).

This concludes the proof of theorem 1.

References

1. A. Andreotti, A. Mayer, *On period relations for abelian integrals on algberaic curves*, Amm. Sc. Num. Sup. Pisa **21** (1967), 189-238.
2. L. Clozel, T.N. Venkataramana, *Restriction of holomorphic cohomology of a Shimura variety to a smaller Shimura variety*, preprint.
3. J. Harris, D. Mumford, *On the Kodaira dimension of the moduli space of curves*, Inv. Math. **67** (1982), 23-86.
4. J.-S. Li, *Non-Vanishing theorems for the cohomology of certain arithmetic quotients*, J. reine angew. Math. **428** (1992), 177-217.
5. D. Mumford, *Towards an enumerative geometry of the moduli space of curves*, Arithmetic and Geometry (M. Artin, J. Tate eds.) (1983), Birkhäuser, Boston, Basel-Stuttgart, 271-326.
6. D. Mumford, *Curves and their Jacobians*, U. Michigan Press, Ann Arbor (1975).
7. R. Parthasarathy, *Holomorphic forms on* $\Gamma \setminus G/K$ *and chern classes*, Topology **21** (1982), 152-175.
8. K. Petri, *Über die invariante Darstellung algebraischer Funktioner einer Veränderlichen*, Math. Ann. **88** (1922), 242-289.
9. R. Weissauer, *Vektorwertige Siegelsche Modulformen kleinen Gewichtes*, J. reine angew. Math. **343** (1983), 184-202.
10. R. Weissauer, *Divisors of the Siegel modular variety*, Number Theory, New York 1984-85, Lecture Notes **1240**, Springer-Verlag.