Separation of spectra in analysis of Berezin kernels

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Let G be a classical real group, let K be a maximal compact subgroup in G. There exists a canonical embedding of a symmetric space G/K to some non-compact hermitian symmetric space G°/K° such that $\dim_{\mathbb{R}} G/K = \dim_{\mathbb{C}} G^\circ/K^\circ$, (see Addendum A). Consider a problem about restriction of a highest weight representation of the group G° to the subgroup G. We use the term "kernel-representations" (by analogy with Bergman kernel-function) for restrictions of this type.

A studying of kernel-representations was initiated by F.A.Berezin [2] shortly before his death (proofs were published only in 1994 by Upmeier and Unterberger [31]). In last years an interest to this subject appeared again. Partially it is related to continuation of Berezin work, partially this interest has independent reasons ([31], [23], [24], [28],[3], [19], [4], [33], [21]). It appeared that spectra of kernel representations are very intricate. For instance (see [20]) they include all representations which can appear in spectra of Howe dual pairs; a decomposition of Howe dual pairs is known as extremely rich spectral problem. G.I.Olshanskii and myself used kernel-representations for construction of exotic unitary representations of the groups $\mathrm{O}(p,q), \mathrm{U}(p,q), \mathrm{Sp}(p,q)$, see [23].

Emphasis that in the case of spaces of scalar-valued functions (see §1) a limit of kernel representations as a parameter α tends $+\infty$ is the usual representation of the group G in the space $L^2(G/K)$, i.e. the kernel-representations can be considered as natural "deformation" of the spaces $L^2(G/K)$[2].

The main purpose of this paper is a decomposition of kernel-representations to "blocks" having more or less uniform spectra (in fact we consider only the case $G = \mathrm{O}(p,q)$; the cases $G = \mathrm{U}(p,q), \mathrm{Sp}(p,q)$ are similar and all information necessary for repetition of our construction for $\mathrm{U}(p,q)$, $\mathrm{Sp}(p,q)$ is contained in the paper). The idea to construct decompositions of this type in various problems of harmonic analysis is old (see Gelfand–Gindikin [8]). G.I.Olshanskii [26] proposed a way (noncommutative Hardy spaces) for separation of highest weight increments (see survey [6] on this subject). There are also papers of Gindikin [9] and Molchanov [16] containing solution of the problem for hyperboloids. Our method is based on theorems about restrictions of discontinuous functions to submanifolds. In [17], [27], [23], [3],[19] this method was used for separation of the discrete part of spectrum).

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[2]In a vector-valued case, a kernel-representation can be obtained by a deformation of a space of L^2-sections of a vector bundle over G/K).

In §1 we define kernel representations for groups $\mathrm{O}(p,q)$ (in the spaces of scalar-valued functions). In §2 we evaluate some integrals over orthogonal groups, which seems pleasant itself. In §3 we prove existence of restriction operators. In §4 we prove similar statements for general kernel representations (in spaces of vector-valued functions).

Addendum A contains brief discussion of definition of kernel representations

Addendum B is devoted to analytical continuation of Plancherel formula for kernel representations.

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§1. Kernel-representations

1.0. Positive definite kernels. Let $\mathcal{X}$ be a set. Recall that a function $K(x,y)$ on $\mathcal{X}\times\mathcal{X}$ is called by *positive definite kernel* if for all $x_1,\dots,x_n\in\mathcal{X}$ the matrix

$$\begin{pmatrix} K(x_1,x_1) & \dots & K(x_1,x_n)\\ \vdots & \ddots & \vdots \\ K(x_n,x_1) & \dots & K(x_n,x_n)\end{pmatrix}$$

is positive definite. A positive definite kernel defines an unique (up to a natural equivalence) Hilbert space H and a system of vectors $\theta_x\in H$ (*supercomplete basis* or *system of coherent states*), enumerated by points $x\in\mathcal{X}$ such that

1. $\langle\theta_x,\theta_y\rangle_H = K(x,y)$ for all $y,x\in\mathcal{X}$
2. The linear span of vectors θ_x is dense in H.

For each vector $v\in H$ we define a function $f_v(x)$ on $\mathcal{X}$ by the rule

$$f_v(x)=\langle v,\theta_x\rangle_H$$

We denote by H° the space of all function obtained in this way.

1.1. Matrix balls. Let $p\leqslant q$. By $\mathrm{B}_{p,q}$ we denote the space of $p\times q$ matrices with norm <1 over $\mathbb{R}$ (a norm of a matrix is the norm of operator in Euclidean space). By $\overline{\mathrm{B}}_{p,q}$ we denote the closure of $\mathrm{B}_{p,q}$ in $\mathbb{R}^{pq}$, i.e. the set of matrices with norm $\leqslant 1$.

By $\mathrm{O}(p,q)$ we denote a group of real $(p+q)\times(p+q)$ matrices $g=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ preserving bilinear form with the matrix $\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}$. The group $\mathrm{O}(p,q)$ acts on the matrix ball $\mathrm{B}_{p,q}$ (and on $\overline{\mathrm{B}}_{p,q}$) by fractional linear transformations

$$z\mapsto z^{[g]}:=(a+zc)^{-1}(b+zd)$$

It is easy to check that the action of $\mathrm{O}(p,q)$ on $\mathrm{B}_{p,q}$ is transitive, and the stabilizer of the point $z=0$ consists of matrices $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ where $a \in \mathrm{O}(p), d \in \mathrm{O}(q)$. Hence

$$\mathrm{B}_{p,q} \simeq \mathrm{O}(p,q)/\mathrm{O}(p) \times \mathrm{O}(q)$$

.

By M_h we denote the space of matrices $z \in \overline{\mathrm{B}}_{p,q}$ such that rank of $(1-zz^t)$ is h. Obviously, the sets M_h are orbits of the group $\mathrm{O}(p,q)$ on the boundary of the matrix ball $\overline{\mathrm{B}}_{p,q}$. By $\overline{\mathrm{M}}_h$ we denote the closure of the orbit M_h:

$$\overline{\mathrm{M}}_h = \cup_{r \leqslant h} \mathrm{M}_r$$

1.2. Berezin spaces. Let

$$\alpha = 0, 1, 2, \ldots, p-1 \quad \text{or} \quad \alpha > p-1$$

The *Berezin kernel* $K_\alpha = K_\alpha(z,u)$ is the function on $\mathrm{B}_{p,q} \times \mathrm{B}_{p,q}$ given by the formula

$$K_\alpha(z,u) = \det(1-zu^t)^{-\alpha} = \det(1-uz^t)^{-\alpha}$$

By well-known Berezin theorem [1] for α indicated above the Berezin kernel is positive definite[3]. Hence we obtain the space of functions on the matrix ball. Let us describe this space.

By $\mathcal{D}'(\mathrm{B}_{p,q})$ we denote the space of distributions supported in the open matrix ball $\mathrm{B}_{p,q}$. Let us define the scalar product in $\mathcal{D}'(\mathrm{B}_{p,q})$ by the formula

$$\langle \phi, \psi \rangle_\alpha := \{\phi(z)\psi(u), \det(1-zu^t)^{-\alpha}\}$$

The Berezin space $H_\alpha = H_\alpha(\mathrm{B}_{p,q})$ is the completion of pre-Hilbert space $\mathcal{D}'(\mathrm{B}_{p,q})$ (in our case, a super-complete basis θ_x consists of δ-functions).

1.3. Dual realization of Berezin spaces and kernel-representations. By δ_a we denote the delta-function supported at the point $a \in \mathrm{B}_{p,q}$. To each element $v \in H_\alpha(\mathrm{B}_{p,q})$ we assign the function f_v on $\mathrm{B}_{p,q}$ defined by the formula

$$f_v(z) = \langle v, \delta_z \rangle_{H_\alpha}$$

Hence we obtain some space $H_\alpha^\circ(\mathrm{B}_{p,q}) \simeq H_\alpha(\mathrm{B}_{p,q})$ consisting of functions on $\mathrm{B}_{p,q}$

It is readily seen that elements of the space $H_\alpha^\circ(\mathrm{B}_{p,q})$ are real analytical functions on $\mathrm{B}_{p,q}$. Then we define the unitary representation of the group $\mathrm{O}(p,q)$ in $H_\alpha^\circ(\mathrm{B}_{p,q})$ by the formula

$$T_\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(z) = (a+zc)^{-\alpha} f\left((a+zc)^{-1}(b+zd)\right) \tag{1.1}$$

[3]There were also papers of Gindikin(1975), Vergne–Rossi(1976) and Wallach(1979) on this subject, see references in [23] or [5].

We name the representations T_α by term *kernel-representations*

REMARK. Berezin [2] in the case of hermitian symmetric spaces used the term *canonical representation.* He gave this term from the paper iz [32]. This term can have many different senses ; in fact A.M.Vershik, I.M.Gelfand, M.I.Graev themself in paper [32] and subsequent works use the term "canonical representation" for multiplicative integrals and and infinitely divisible representations. In fact the class of infinitely divisible representations have very small intersection with the class of "kernel-representations", this small intersection is a chance.

1.4. Holomorphic continuation. By $\mathrm{B}_{p,q}(\mathbb{C})$ we denote the set of all complex $p \times q$ matrices with norm < 1. The kernel K_α extends to the function $\det(1 - zu^*)^{-\alpha}$ on $\mathrm{B}_{p,q}(\mathbb{C}) \times \mathrm{B}_{p,q}(\mathbb{C})$, which is holomorphic in z and antiholomorphic in u; this function is positive definite on $\mathrm{B}_{p,q}(\mathbb{C})$ ([1]). Then we deduce that functions $f \in H^\circ_\alpha(\mathrm{B}_{p,q})$ extend to holomorphic functions on $\mathrm{B}_{p,q}(\mathbb{C})$.

§2. Some integrals over orthogonal groups

Consider a matrix A. By $[A]_p$ we denote its left upper corner having a size $p \times p$. We denote the Haar measure on $\mathrm{SO}(n)$ by $d\sigma_n(g)$.

2.1. The maps Υ^m. Let us wright a matrix $g \in \mathrm{SO}(n)$ as a $(m + (n - m)) \times (m + (n - m))$ block matrix $\begin{pmatrix} P & Q \\ R & T \end{pmatrix}$. Consider the map

$$\Upsilon^m : \begin{pmatrix} P & Q \\ R & T \end{pmatrix} \mapsto T - R(1 + P)^{-1} Q$$

defined almost everywhere

PROPOSITION 2.1.*a)* Υ^m *is a map* $\mathrm{SO}(n)$ *to* $\mathrm{SO}(n - m)$.

b) $\qquad \Upsilon^k \circ \Upsilon^m = \Upsilon^{k+m}.$

c) $\quad S = \frac{g-1}{g+1}$ *implies* $\{S\}_p = \frac{\Upsilon^{n-p}(g)-1}{\Upsilon^{n-p}(g)+1}$

where $\{S\}_p$ *denotes the right lower* $p \times p$ *corner of the matrix* S.

REMARK. The map Υ is a value of Livshic characteristic function $\chi(z) = T + zR(1 - zP)^{-1} Q$ at the point $z = -1$ (see [14]).

PROOF. Statement c) can be easily checked by direct calculation using Frobenius formula for inversion of block matrix (see [7], § II.4) ,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}$$

Statements a),b) are consequences of c).

PROPOSITION 2.2. *Let* $A, B, \in \mathrm{SO}(n - m)$. *Then*

$$\Upsilon^m \left[\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} g \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \right] = A \Upsilon^m(g) B$$

PROOF is obvious.

2.2. Another proof of Proposition 2.1. Consider the space $V_n := \mathbb{R}^n \oplus \mathbb{R}^n$ equipped with indefinite bilinear form

$$L_n((x,y),(x',y')) := \sum_{k=1}^n x_k x'_k - \sum_{k=1}^n y_k y'_k$$

A graph of an unitary operator $\mathbb{R}^n \to \mathbb{R}^n$ is a L_n-isotropic subspace in $V_n = \mathbb{R}^n \oplus \mathbb{R}^n$.

For each linear relation $M : V_n \rightrightarrows V_{n-m}$ (recall that a linear relation is a subspace $M \subset V_n \oplus V_{n-m}$, for details see [18]) and each subspace $H \subset V_n$ it is possible to define a subspace $MH \subset V_{n-m}$ in the following way. A vector $y \in V_{n-m}$ is an element of MH iff there exists $x \in H$ such that $(x,y) \in M$.

Consider the linear relation $\mathcal{Z}_n^{n-m} : V_n \rightrightarrows V_{n-m}$ consisting of all vectors having the form

$$\bigl(\{(v,h),(w,-h)\},\{v,w\}\bigr) \in \{\mathbb{R}^n \oplus \mathbb{R}^n\} \oplus \{\mathbb{R}^{n-m} \oplus \mathbb{R}^{n-m}\}$$

The condition $(u,v),(u',v') \in \mathcal{Z}_n^{n-m}$ implies $L_n(u,u') = L_{n-m}(v,v')$. Hence the transformation $H \mapsto \mathcal{Z}_n^{n-m} H$ maps isotropic subspaces to isotropic subspaces.

It is readily seen that the map Υ^m coincides with the transformation $H \mapsto \mathcal{Z}_n^{n-m} H$ of isotropic grassmannians. Now Proposition 2.1 becomes obvious.

2.3. A projection of the Haar measure. Denote by I the segment $[-1,1]$. Consider the map

$$\Xi : \mathrm{SO}(n) \to \mathrm{SO}(n-1) \times I$$

given by the formula (see a notation in the beginning of the section)

$$\Xi(g) = \bigl(\Upsilon^1(g), [g]_1\bigr)$$

PROPOSITION 2.3. *The image of the Haar measure $d\sigma_n$ under the map Ξ is*

$$\mathrm{const} \cdot (1-x^2)^{(n-3)/2} dx \, d\sigma_{n-1}$$

PROOF. By $\mathrm{SO}(n-1)$- equivariance of the map Υ^1 (Proposition 2.2), the image of the Haar measure has the form $\phi(x) dx \, d\sigma_{n-1}$. On another side the projection of the Haar measure to I is $\mathrm{const} \cdot (1-x^2)^{(n-3)/2}$ and we obtain required form of the function $\phi(x)$.

2.4. The map of the orthogonal group to the cub. Consider the maps

$$\mathrm{SO}(n) \to \mathrm{SO}(n-1) \times [-1,1] \to \mathrm{SO}(n-2) \times [-1,1] \times [-1,1] \to \dots$$

As result we obtain the map of the group $\mathrm{SO}(n)$ to the cube $[-1,1]^{n-1}$, given by the formula

$$(x_1, x_2, \dots, x_{n-1}) = \big([\Upsilon^{n-2}(g)]_1, \dots, [\Upsilon^1(g)]_1, [g]_1\big)$$

By Proposition 2.3, the image of the Haar measure under this map is

$$d\mu(x_1, x_2, \dots, x_{n-1}) = \mathrm{const} \cdot \prod_{j=1}^{n-1} (1 - x_j^2)^{(j-2)/2} d\nu(x_1) \prod_{j=1}^{n-1} dx_j$$

Hence for each function f depending on $n-1$ variables $x_1, \dots, x_{n-1} \in [-1,1]$ we have

$$\int_{\mathrm{SO}(n)} f([\Upsilon^{n-1}(g)]_1, \dots, [\Upsilon^1(g)]_1, [g]_1)\, d\sigma_n(g) = \\ = \int_{[-1,1]^{n-1}} f(x_1, x_2, \dots, x_{n-1}) d\mu(x_1, x_2, \dots, x_{n-1}) \tag{2.1}$$

2.5. Multiplicativity. PROPOSITION 2.4.*Let $g \in \mathrm{O}(n)$. Let $m < p \leqslant n$. Then*

$$\det(1 + [g]_p) = \det(1 + [g]_m) \det(1 + [\Upsilon^m(g)]_{p-m})$$

PROOF. Let us wright g as block $(m+(p-m)+(n-p))\times(m+(p-m)+(n-p))$ matrix:

$$g = \begin{pmatrix} P & Q_1 & Q_2 \\ R_1 & T_{11} & T_{12} \\ R_2 & T_{21} & T_{22} \end{pmatrix}$$

By the usual formula for determinant of a block matrix, we obtain

$$\det\left[1 + \begin{pmatrix} P & Q_1 \\ R_1 & T_{11} \end{pmatrix}\right] = \det(1 + P) \det(1 + T_{11} - R_1(1+P)^{-1}Q_1)$$

On another side

$$\Upsilon^m(g) = \begin{pmatrix} T_{11} - R_1(1+P)^{-1}Q_1 & T_{12} - R_1(1+P)^{-1}Q_2 \\ T_{21} - R_2(1+P)^{-1}Q_1 & T_{22} - R_2(1+P)^{-1}Q_2 \end{pmatrix}$$

and now the statement becomes obvious.

2.6. Some integrals. THEOREM 2.5. *Assume that the Haar measure of the whole group $\mathrm{SO}(n)$ is 1. Let $\lambda_1, \dots, \lambda_n \in \mathbb{C}, \lambda_{n+1} = 0$, and $\mathrm{Re}\,\lambda_k > -(n-k)/2$ for all $k = 0, 1, \dots, n$. Then*

$$\int_{\mathrm{SO}(n)} \prod_{k=1}^{n-1} \det(1 + [g]_k)^{\lambda_k - \lambda_{k+1}}\, d\sigma_n(g) = \prod_{k=1}^{n-1} \frac{\Gamma(n-k)\Gamma(\lambda_k + (n-k)/2)}{\Gamma((n-k)/2)\Gamma(\lambda_k + n - k)} \tag{2.2}$$

PROOF. Let us apply formula (2.1) and Proposition 2.4. Then our integral splits to the product of integrals having the form

$$\int_{-1}^{1}(1-x^2)^{(n-k-2)/2}(1-x)^{\lambda_k}dx = \tag{2.3}$$
$$= 2^{\lambda_k+n-3}B\left(\lambda_k+\tfrac{n-k}{2},\tfrac{n-k}{2}\right) = 2^{\lambda_k+n-k-3}\tfrac{\Gamma(\lambda_k+(n-k)/2)\Gamma((n-k)/2)}{\Gamma(\lambda_k+n-k)}$$

2.7. Some integrals over unitary and symplectic groups. The following integrals can be evaluated in the same way, the only the obvious integral (2.3) is replaced by more complicated integrals over ball $|x|<1$ in $\mathbb{C}$ or in $\mathcal{H}$.

$$\int_{\mathrm{U}(n)}\prod_{k=1}^{n}\det(1+[g]_k)^{\lambda_k-\lambda_{k+1}}\overline{\det(1+[g]_k)}^{\mu_k-\mu_{k+1}}\,d\sigma_n(g) =$$
$$= \prod_{k=1}^{n}\frac{\Gamma(n-k+1)\Gamma(n-k+1+\lambda_k+\mu_k)}{\Gamma(n-k+1+\lambda_k)\Gamma(n-k+1+\mu_k)} \tag{2.4}$$
$$\int_{\mathrm{Sp}(n)}\prod_{k=1}^{n}|\det(1+[g]_k)|^{\lambda_k-\lambda_{k+1}}\,d\sigma_n(g) =$$
$$= \prod_{k=1}^{n}\frac{\Gamma(2(n-k+1))\Gamma(2(n-k+1)+\lambda_k+1)}{\Gamma(2(n-k+1)+\lambda_k/2)\Gamma(2(n-k+1)+\lambda_k/2+1)} \tag{2.5}$$

(we define the compact symplectic group $\mathrm{Sp}(n)$ as the group of unitary operators in n-dimensional quaternionic space $\mathbb{H}^n$; a quaternionic operator A can be considered as an operator $A_{\mathbb{R}}:\mathbb{R}^{4n}\to\mathbb{R}^{4n}$; we define a determinant of the quaternionic operator A as $\sqrt[4]{\det A_{\mathbb{R}}}$).

2.8. Other ways of evaluation of integrals (2.2), (2.4), (2.5). We obtain a verbal evaluation of these integrals, nevertheless our way is not quite usual. There exist another way of evaluation of the same integrals. The Cayley transform reduces (2.2) to the form

$$\int\prod_{k=2}^{n}\det(1-[T]_k)^{\mu_k}\,dT \tag{2.6}$$

where integration is given over the space of all skew-symmetric matrices. Last integral can be easily evaluated by the method described in [21] (in fact (2.6) is simpler than integrals from [21]). Similar way is valid for (2.4),(2.5).

Integral (2.4) reduces to the form

$$\int\prod_{k=1}^{n}\det(1+[T]_k)^{\mu_k}\det(1-[T]_k)^{\nu_k}\,dT$$

where integration is given over the space of all anti-hermitian matrices ($T=-T^*$). In this case where are two additional possibilities: we can use Laplace transform or we can use Gindikin method of separation of variables (see [5], chapter 7).

Remark. If $\lambda_1 = \lambda_2 = \dots = \lambda_n$, then integral (2.2) is equivalent to one of Hua Loo Keng integrals [12], chapter 2

§3. Operators of restriction to the boundary

In this section we assume $p < q$.

3.1. Operators of restriction to M_r**.** Denote by $\mathcal{H}^\circ_\alpha$ the subspace in $H^\circ_\alpha(\mathrm{B}_{p,q})$ consisting of functions analytical in some neighborhood of closed matrix ball $\overline{\mathrm{B}}_{p,q}$.

LEMMA 3.1. *The subspace $\mathcal{H}^\circ_\alpha$ is dense in H°_α.*

PROOF. There exists a basis in H°_α consisting of orthogonal homogeneous polynomials (see for instance[23], §1). These functions are eigenfunctions of the operator $A_\varepsilon f(z) = f((1-\varepsilon)z)$. The eigenvalues are $(1-\varepsilon)^k$. Hence for each $\varepsilon > 0$ the operator A_ε is a continuous operator $H^\circ_\alpha \to H^\circ_\alpha$. Obviously the image of the operator A_ε is dense in H°_α and also it is contained in $\mathcal{H}^\circ_\alpha$. ■

By $L^1_{\mathrm{loc}}(\mathrm{M}_r)$ we denote the space of locally integrable functions on manifold M_r (see Subsection 1.1; if $r \neq 0$ then M_r is not compact).

The group $\mathrm{O}(p,q)$ acts in the space $L^1_{\mathrm{loc}}(\mathrm{M}_r)$ by same formula (1.1), as in $H^\circ_\alpha(\mathrm{B}_{p,q})$ (we only have to think that a matrix z is an element of M_r).

By J_r we denote the operator of restriction of a function $f \in \mathcal{H}^\circ_\alpha$ to the submanifold M_r.

THEOREM 3.2. *Let $\alpha < (q-p+2r)/2$. Then the operator J_r extends to a continuous operator $H^\circ_\alpha \to L^1_{\mathrm{loc}}(\mathrm{M}_r)$.*

REMARK. Let $\alpha \neq 0$ (if $\alpha = 0$ then the space H°_α is one-dimensional). In general, functions $f \in H^\circ_\alpha$, are discontinuous in point of boundary of the matrix ball. In other words, for points $a \in \overline{\mathrm{B}}_{p,q} \setminus \mathrm{B}_{p,q}$ the lineal functional $f \mapsto f(a)$ on H°_α is not well-defined. Hence existence of the restriction operator is not obvious.

Evidently, the operator $J_r : H^\circ_\alpha \to L^1_{\mathrm{loc}}(\mathrm{M}_r)$ is $\mathrm{O}(p,q)$-intertwining. Hence its kernel $\ker J_r$ is a subrepresentation. Hence the orthogonal complement $(\ker J_r)^\perp$ also is a subrepresentation. The operator J_r is a bijection from $(\ker J_r)^\perp$ to the image $\mathrm{im}\, J_r$ of the operator J_r. Hence we constructed a subrepresentation in H°_α, and this subrepresentation has a natural realization in a space of functions on M_r.

3.2. Proof of the theorem 3.2. In [23] (theorems 2.2–2.2) there was obtained one general statement on boundary values of holomorphic functions. In our case, it implies the following statement

Theorem 3.3. *Let M be a compact subset in the boundary of the matrix ball $\mathrm{B}_{p,q}$. Let μ be a measure supported in M. Assume that* *i) For almost all (with respect to the measure $\mu \times \mu$) points $(z, u) \in M \times M$ there exists a limit as ε tends to 0 from the right*

$$L(z,u) = \lim_{\varepsilon \to +0} \det(1 - (1-\varepsilon) z u^t)^{-\alpha}$$

ii) This limit is dominated, i.e. there exists a function $h \in L^1(\mu \times \mu)$ such that

$$\det(1 - (1-\varepsilon) z u^t)^{-\alpha} \leqslant h(z,u)$$

Then the restriction operator of a function $f \in H^\circ_\alpha$ to the subset M is a continuous operator $H^\circ_\alpha \to L^1(M)$.

First we will explain briefly a way of application of the theorem in our case. We want to apply it to the orbit M_r. Obviously the limit exists. In fact it is sufficient to check the convergence of integral of the function $L(z,u) = \det(1 - zu^*)^{-\alpha}$. Nevertheless I don't see a natural measure on M_r. Hence there is a small hope to evaluate integral if we don't know natural integrand.

An estimation of convergence also seems heavy since the singularity of the integral is very unpleasant. In fact we construct non-directly a calculable integral with given singularities. w

To apply Theorem 3.3, we consider arbitrary compact subset M on the manifold M_r, we also assume that M is a closure of its interior. Let μ be the surface Lebesgue measure on M. It is sufficient to check that the integral

$$\int_M \int_M \det(1 - zu^t)^{-\alpha} d\mu(z)\, d\mu(u) \tag{3.1}$$

is convergent. The condition i) is a consequence of the convergence. Condition ii) follows from the obvious inequality ([23],3.8):

$$\det(1 - c^2 z u^t)^{-\alpha} \leqslant 2^{p\alpha} \det(1 - zu^t)^{-\alpha}.$$

It is sufficient to check that the integral

$$\int_M \det(1 - zu^t)^{-\alpha} d\mu(z) \tag{3.2}$$

is convergent for any u and is dominated by a constant that don't depend on u. As it will be shown below, it is sufficient to prove convergence of the integral for some u.

First we consider $u = u^{(0)}$ given by block $((p-r)+r) \times ((p-r)+(q-p+r))$ matrix $\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$. Then integral (3.2) transforms to the form

$$\int_M \det[1 + z]^{-\alpha}_{p-r} d\mu(z) \tag{3.3}$$

LEMMA 3.4. *Integral* (3.3) *converges.*

LEMMA 3.5. *Let us represent* $g \in \mathrm{SO}(q+r)$ *as block matrix* $g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$ *having the size* $(p + (q + r - p)) \times (q + r)$. *Then* $g_{11} \in \overline{\mathrm{M}}_r$. *Conversely each matrices* $z \in \overline{\mathrm{M}}_r$ *is a left upper corner of some matrix* $g \in \mathrm{SO}(q + r)$.

PROOF OF LEMMA 3.5 is obvious.

PROOF OF LEMMA 3.4. By Lemma 3.5 the map $\tau : g \mapsto g_{11}$ takes $\mathrm{SO}(q+r)$ to $\overline{\mathrm{M}}_r$. Obviously, the image of the Haar measure on $\mathrm{SO}(q + r)$ under the map τ is a measure equivalent to a surface Lebesgue measure on M_r. Hence it is sufficient to check convergence of the integral

$$\int_{\mathrm{SO}(q+r)} \det[1 + g]_{p-r}^{-\alpha} d\sigma_{q+r}(g)$$

But this integral was evaluated in §2 (we substitute $\lambda_1 = \cdots = \lambda_{p-r-1} = -\alpha; \lambda_{p-r} = \cdots = 0$ to formula (2.2)). ■

LEMMA 3.6. *The kernel* $\det(1 - zu^t)^{-\alpha}$ *satisfies to the equality*

$$\det(1 - z^{[g]}(u^{[g]})^t)^{-\alpha} = \det(1 - zu^t)^{-\alpha} \det(a + zc)^{\alpha} \det(a + zu)^{\alpha} \tag{3.4}$$

PROOF is obvious.

Consider a compact subset $L \subset \mathrm{O}(p, q)$ such that for each $v \in \mathrm{M}$ there exists $g_v \in L$ satisfying the condition $v^{[g_v]} = u^{(0)}$. Denote by K the set of all points in M_r having the form $u^{[g]}$ where $u \in M$ and $g \in L$. Obviously, K is compact.

Substitute $w = z^{[g]}$ to the integral (3.2). By formula (3.4), we obtain

$$\int\limits_{M^{[g_u]}} \det(1 - u^{(0)}w^t)^{-\alpha} H(w, u)\, d\mu(w), \tag{3.5}$$

where $M^{[g_u]}$ denote the image of the set M under the map g_u, and $H(w, u)$ is some function (a product of Jacobian and additional factor which appears from (3.4)). Obviously, the function $H(w, u)$ is bounded by some constant C. Hence integral (3.5) is dominated by quantity

$$C \int_K \det(1 - u^{[0]}w^t)^{-\alpha} d\mu(w)$$

This quantity doesn't depend on u and it is finite by Lemma 3.4. Theorem is proved.

3.3. Restriction of partial derivatives to the boundary. In Subsection 3.2 we restricted functions to M_r. Also it is possible to restrict partial derivatives of the function $f \in H_\alpha^\circ$. The existence of restriction operator for partial derivative $\frac{\partial}{\partial z_{ij}}$ depends on convergence of integral

$$\int_M \int_M \left| \frac{\partial}{\partial z_{ij}} \frac{\partial}{\partial u_{ij}} \det(1 - zu^t)^{-\alpha} \right| dz\, du,$$

where $M \subset \mathrm{M}_r$ is a compact subset. The integrand has the form $|P(z,u)| \det(1-zu^t)^{-\alpha-2}$ where $P(z,u)$ is a polynomial. Hence we can apply the same arguments. As a result, we obtain the following theorem.

THEOREM 3.7. *Let $\alpha + 2k < (q-p+2r)/2$. Let ∂^ν be a partial derivative of order k. Then the operator of restriction of a function $\partial^\nu f(z)$ to M_r is a continuous operator from H°_α to $L^1_{\mathrm{loc}}(\mathrm{M}_r)$.*

Now we will slightly formalize the picture.

Denote by $\mathcal{N}(\mathrm{M}_r)$ the conormal bundle to M_r in $\mathbb{R}^{pq}$. By $\mathrm{S}^k\mathcal{N}(\mathrm{M}_r)$ we denote k-th symmetric power of the conormal bundle. By $\mathcal{F}^k_r$ we denote the set of all $f \in \mathcal{H}^\circ_\alpha$ whose partial derivatives of order $\leqslant k$ vanish on M_r. Then elements of quotient space $\mathcal{F}^{k-1}_r/\mathcal{F}^k_r$ are identified with sections of the bundle $\mathrm{S}^k\mathcal{N}(\mathrm{M}_r)$.

Theorem 3.7 implies the following statement:

Let $\alpha + 2k < (q-p+2r)/2$. Then the operator of restriction of partial derivatives of order k to M_r is a well-defined operator from the space H°_α to the space of L^1_{loc}-sections of the bundle $\mathrm{S}^k\mathcal{N}(\mathrm{M}_r)$.[4]

3.4. Some comments. Thus we obtain the intertwining operator $J_{r,k}$ from H°_α to the space of sections of bundle $\mathrm{S}^k\mathcal{N}(\mathrm{M}_r)$. Its image $\mathrm{im} J_{r,k}$ is a quotient space of a Hilbert space. Hence $\mathrm{im} J_{r,k}$ has a natural structure of a Hilbert space. (we emphasis that the Hilbert topology in $\mathrm{im} J_{r,k}$ is stronger than topology induced from L^1_{loc}). We denote the representation of the group $\mathrm{O}(p,q)$ in $\mathrm{im} J_{r,k}$ by

$$\Xi^\alpha_{r,k} = \Xi^\alpha_{r,k}(p,q) \tag{3.6}$$

The quotient space of a Hilbert space H by arbitrary closed subspace L can be canonically identified with the orthogonal complement $L^\perp$ to L. Hence the representation $\Xi^\alpha_{r,k}$ is a subrepresentation in the kernel-representation T_α.

Hence we obtained a canonical family of subspaces (3.6) in $H^\circ_\alpha(\mathrm{B}_{p,q})$, we name these subspaces by the term *blocks*.

If α is sufficiently large (i.e. $\alpha > (p+q)/2-1$), then the operators of restriction to submanifolds M_r are not defined. In this case blocks $\Xi^\alpha_{r,k}(p,q)$ don't exist. It is known that for $\alpha > (p+q)/2-1$ the representation of $\mathrm{O}(p,q)$ in $H^\circ_\alpha(\mathrm{B}_{p,q})$ is equivalent to the representation of $\mathrm{O}(p,q)$ v in L^2 on symmetric space $\mathrm{B}_{p,q}$[5].

For all $\alpha > p-1$ the summand ("block"), that equivalent to $L^2(B_{p,q})$, is present in decomposition of the kernel-representation.

For $\alpha < (p+q)/2-2$ the block $\Xi^\alpha_{p-1,0}$ appears. This summand corresponds to the operator of restriction of function to the maximal boundary orbit M_{p-1}.

[4] the algebraic version of this construction arises to [13]

[5] this statement for arbitrary scalar-valued kernel-representations is proved in cite [24]; earlier similar statement for kernel representations of some groups were obtained (in different degree of generality) in [2], [30], [10]).

For $\alpha < (p+q)/2-3$ block $\Xi^\alpha_{p-2,0}$ appears. After point $\alpha = (p+q)/2-4$ two additional blocks $\Xi^\alpha_{p-1,1}$ and $\Xi^\alpha_{p-3,0}$ appears etc.

At the point $\alpha = p-1$ (it is the end of continuous series, see 1.2) the largest block $L^2(\mathrm{B}_{p,q})$ disappears from decomposition. Also at this point all blocks $\Xi^{p-1}_{p-1,1}$, $\Xi^{p-1}_{p-1,2}$, $\Xi^{p-1}_{p-1,3}$ etc. disappear. But the block $\Xi^{p-1}_{p-1,0}$ at this point still alive.

If we continue decreasing of α (after point $\alpha = p-1$ it is a discrete process, see Subsection 1.2) "new" blocks continue their appearance. Nevertheless "old" blocks quickly disappear (this process can be checked using Plancherel formula, see Addendum B).

At the point $\alpha = 0$ only the block $\Xi^0_{0,0}$ is still alive, and this block is one-dimensional representation.

Let us briefly discuss spectra of representations $\Xi^\alpha_{r,k}$.

PROPOSITION 3.8. *Each subrepresentation in the kernel-representation T_α contains an $\mathrm{O}(p)\times\mathrm{O}(q)$-invariant vector.*

PROOF. The function $f(z) = 1$ is an $\mathrm{O}(p,q)$-cyclic vector in H°_α (since its orbit is exactly super-complete basis). Hence the projection of the function $f(z) = 1$ to any invariant subspace is a cyclic vector in this subspace. Hence the projection is non zero.

PROPOSITION 3.9. a) *The representation of the group $\mathrm{O}(p,q)$ in $\Xi^\alpha_{0,0}$ is irreducible*

b) *The representation of $\mathrm{O}(p,q)$ in $\Xi^\alpha_{0,k}$ is a finite sum of irreducible representations.*

PROOF. a) Indeed the constant is the unique $\mathrm{O}(p)\times\mathrm{O}(q)$-invariant function on M_0.

b) We have to check that the space of $\mathrm{O}(p)\times\mathrm{O}(q)$-invariant sections of the bundle $\mathrm{S}^k\mathcal{N}(\mathrm{M}_0)$ is finite-dimensional. But dimension of this space can't exceed the dimension of fiber.

REMARK. "Block" $\Xi^\alpha_{0,0}$ which corresponds to operator of restriction to the compact orbit M_0 was discovered in the paper of G.I.Olshanskii and myself [23], §3 (under more rigid conditions to parameters of representations) $\alpha < (q-2p+1)/2$.

REMARK. Spectrum in blocks $\Xi^\alpha_{r,k}$ (where $0<r<p$) is purely continuous. It consist of representations induced from unitary representations of groups $\mathrm{O}(p-r,q-r)$. The last representations are contained in spectrum of $\Xi^{\alpha+2k}_{0,k}(p-r,q-r)$.

REMARK. "Additional" block with continuous spectrum in kernel-representations was discovered earlier only for the case $G=\mathrm{U}(2,2)$ in the paper [28]. In [20] I conjectured that the spectrum of kernel-representations for small values of α contains many various series. The proof of this conjecture is the subject of this paper.

§4. General kernel-representations.

4.1. Definition. By ρ_Λ we denote finite dimensional holomorphic representation of the group $\mathrm{GL}(n,\mathbb{C})$ defined by a signature $\Lambda = (\lambda_1, \dots, \lambda_n)$, where $\lambda_1 \geqslant \dots \geqslant \lambda_n$. By V_Λ we denote the representation ρ_Λ. Recall that there exists a canonical $\mathrm{U}(n)$-invariant scalar product in V_Λ.

Consider the representations $\rho_{\Lambda'}$, $\rho_{\Lambda''}$ of the groups $\mathrm{GL}(p,\mathbb{C})$, $\mathrm{GL}(q,\mathbb{C})$ defined by signatures $\Lambda' = (\lambda'_1, \dots, \lambda'_p)$, $\Lambda'' = (\lambda''_1, \dots, \lambda''_q)$ respectively. We will use notations $\widetilde{\Lambda} = (\lambda'_1, \dots, \lambda'_p; \lambda''_1, \dots, \lambda''_q)$ and also $V_{\widetilde{\Lambda}} = V_{\Lambda'} \otimes V_{\Lambda''}$.

Consider the function

$$K_{\widetilde{\Lambda}}(z,u) = \rho_{\Lambda'}(1 - zu^t) \otimes \rho_{\Lambda''}(1 - uz^t); \qquad z, u \in \mathrm{B}_{p,q}$$

taking values in the space $\mathrm{End}(V_{\widetilde{\Lambda}})$ of linear operators in $V_{\widetilde{\Lambda}}$.

Assume that $K_{\widetilde{\Lambda}}(z,u)$ is a matrix-valued positive defined kernel. Recall that this means positive definiteness of the kernel

$$L\left((z,\xi);(u,\eta)\right) = \langle K_{\widetilde{\Lambda}}(z,u)\xi, \eta\rangle_{V_{\widetilde{\Lambda}}}; \qquad x, y \in \mathrm{B}_{p,q}; \quad \xi, \eta \in V_{\widetilde{\Lambda}}$$

on the space $\mathcal{X} = \mathrm{B}_{p,q} \times V_{\widetilde{\Lambda}}$ (for the condition of positive definiteness of the kernel $\widetilde{\Lambda}(z,u)$ we obtained in [25], see also any text on classification of highest weight unitary representations, see also [23].

The kernel $L\left((z,\xi);(u,\eta)\right)$ defines a Hilbert space consisting of functions on the space $\mathcal{X} = \mathrm{B}_{p,q} \times V_{\widetilde{\Lambda}}$. But these functions are linear in the variable $\xi \in V_{\widetilde{\Lambda}}$. Hence we can consider them as $V_{\widetilde{\Lambda}}$-valued functions on $\mathrm{B}_{p,q}$.

As a result we obtain the space $H^\circ_{\widetilde{\Lambda}}$, which consists of $V_{\widetilde{\Lambda}}$-valued functions on $\mathrm{B}_{p,q}$. The group $\mathrm{O}(p,q)$ acts on $H^\circ_{\widetilde{\Lambda}}$ by unitary operators given by the formula

$$T_{\widetilde{\Lambda}}(g) f(z) = \left[\rho_{\Lambda'}(a + zc) \otimes \rho_{\Lambda''}(-d + cz^{[g]})\right] f(z^{[g]})$$

4.2. Operators of restriction to the boundary. THEOREM 4.1.*a) Assume $\lambda''_1 - \lambda'_p < (q - p + 2r)/2$. Then the operator of restriction to an orbit M_r is a well-defined operator in $H^\circ_{\widetilde{\Lambda}}$.*

b) Assume $2k + \lambda''_1 - \lambda'_p < (q - p + 2r)/2$. Then the operator of restriction of partial derivatives of order k to the orbit M_r is a well-defined operator in $H^\circ_{\widetilde{\Lambda}}$.

PROOF is similar to the proof of Theorem 3.2, it is necessary also use simple considerations from [23],4.5.

4.3. Comments. As above, we obtain decomposition of kernel-representation to "blocks" $\Xi^\Lambda_{r,k}$, where r is a parameter enumerating orbits, and k is an order of partial derivatives.

Existence of operator of restriction to M_0 was discovered in [23],§4 (under more rigid restriction to the signature Λ). In particular this construction gives a simple way for obtaining of "exotic" unitary representations of the groups

$\mathrm{O}(p,q)$ (see [23],§§4-5). Emphasis also that this representations admit a limit as $q\to\infty$ ([23], §6).

The representations of $\mathrm{O}(p,q)$ in $\Xi^\Lambda_{0,0}$ are reducible; their decomposition is discussed in [23] , §5. As above, the representation of $\mathrm{O}(p,q)$ in all spaces $\Xi^\Lambda_{0,k}$ has a purely discrete spectrum. Nevertheless the spectrum of blocks $\Xi^\Lambda_{r,k}$ for $r>0$ is still sophisticated and contains representations of different types. Here I don't have likely conjectures.

Addendum A. Hermitization of symmetric spaces.

Consider the following list.

1. $G=\mathrm{GL}(n,\mathbb{R})$	$G^\circ=\mathrm{Sp}(2n,\mathbb{R})$
2. $G=\mathrm{O}(p,q)$	$G^\circ=\mathrm{U}(p,q)$
3. $G=\mathrm{Sp}(2n,\mathbb{R})$	$G^\circ=\mathrm{Sp}(2n,\mathbb{R})\times\mathrm{Sp}(2n,\mathbb{R})$
4. $G=\mathrm{GL}(n,\mathbb{C})$	$G^\circ=\mathrm{U}(n,n)$
5. $G=\mathrm{SO}(n,\mathbb{C})$	$G^\circ=\mathrm{SO}^*(2n)$
6. $G=\mathrm{Sp}(2n,\mathbb{C})$	$G^\circ=\mathrm{Sp}(4n,\mathbb{R})$
7. $G=\mathrm{U}(p,q)$	$G^\circ=\mathrm{U}(p,q)\times\mathrm{U}(p,q)$
8. $G=\mathrm{GL}(n,\mathbb{H})$	$G^\circ=\mathrm{SO}^*(2n)$
9. $G=\mathrm{Sp}(p,q)$	$G^\circ=\mathrm{U}(2p,2q)$
10.$G=\mathrm{SO}^*(2n)$	$G^\circ=\mathrm{SO}^*(2n)\times\mathrm{SO}^*(2n)$
11.$G=\mathrm{SO}(2,n)$	$G^\circ=\mathrm{SO}(2,n)\times\mathrm{SO}(2,n)$
12.$G=\mathrm{SO}(1,p)\times\mathrm{SO}(1,q)$	$G^\circ=\mathrm{SO}(2,p+q)$

Denote by K, K° the maximal compact subgroup in G, G° respectively. Then G°/K° is a hermitian symmetric space and $G/K\subset G^\circ/K^\circ$ is a totally real subspace, and

$$\dim_{\mathbb{R}} G/K=\dim_{\mathbb{C}} G^\circ/K^\circ$$

We say that G°/K° is *hermitization* of G/K. If G/K is hermitian then its hermitization is $G/K\times G/K$. We see that all classical riemann symmetric spaces have hermitization[6]

We define kernel-representation as restriction of an unitary highest weight representation[7] ρ of the group G° to the subgroup G[8]

[6]Unfortunately I don't know the author of this observation. First time I heard this 20 years ago.

[7]A highest weight representation of the group G° is a representation in a space of holomorphic vector-valued functions on G°/K°

[8]references: [2] (the cases $G=\mathrm{U}(p,q),\mathrm{Sp}(2n,\mathbb{R}),\mathrm{SO}^*(2n),\mathrm{SO}(2,n)$), [23] ($G=\mathrm{O}(p,q),\mathrm{U}(p,q),\mathrm{Sp}(p,q)$), [24], [20] (some discussion of other cases); there exists an idea to extend a definition of kernel representation to pseudoriemann symmetric spaces, see [4].

Addendum B. Analytic continuation of the Plancherel formula

B.1. Plancherel formula. Plancherel formula for all classical series 1-10 of kernel-representations for large values of the parameter α there was obtained in [21] (this method is valid also for future tubes).[9] In all cases the Plancherel measure has the form

$$V(\alpha)\prod_{k=1}^{r}\left|\Gamma(\frac{1}{2}(\alpha-h+is_k))\right|^2\frac{ds}{c(is)c(-is)} \quad \text{where} \quad \frac{1}{i}s_k\in\mathbb{R} \tag{B.1}$$

Here r is the rank of the group; is_1, is_2,...,is_r are the standard parameters of spherical functions, spherical functions of unitary representations of non degenerate principal series corresponds to real s_k; $\frac{ds}{c(is)c(-is)}$ is Gindikin–Karpelevich measure (see [11]), and $V(\alpha)$ is a holomorphic factor. The parameter h is the least value of α for which the kernel-representation is square integrable.

In this formula, the Plancherel measure is supported on unitary principal series. We have seen that for small α spectrum of kernel-representation is complicated and hence Plancherel formula (B.1) is not correct for small α.

B.2. Analytic continuation of the Plancherel formula. Plancherel formula for $\alpha < h$ can be obtained by the analytic continuation of formula (B.1). We will give the final formula for the cases $G = \mathrm{O}(p,q)$, $\mathrm{U}(p,q)$ (for other cases see [22]).

Denote by g_t an element of $\mathrm{O}(p,q)$ having eigenvalues $e^{\pm t_k}$. Let θ_0 be the same as in Subsection 1.2. Then

$$< T_\alpha(g_t)\theta_0,\theta_0 >= \prod_{k=1}^{p}\cosh^{-\alpha}t_k$$

Plancherel formula is equivalent to expansion of matrix element $< T_\alpha\theta_0,\theta_0 >$ by spherical functions of $\mathrm{O}(p,q)$. Let us denote spherical functions of $\mathrm{O}(p,q)$ by $\Phi_{is_1,\dots,is_r}$. By $\Phi_{is_1,\dots,is_r}(t_1,t_2,\dots t_p)$ we denote a value of spherical function on g_t.

[9] For the groups $\mathrm{U}(p,q)$, $\mathrm{Sp}(2n,\mathbb{R})$, $\mathrm{SO}^*(2n)$, $\mathrm{SO}(n,2)$ and large values of parameter α the Plancherel formula was obtained by Berezin in 1978 ([2]), Shortly after Berezin died and proof was not published until [31]. For the groups $\mathrm{O}(p,1)$, $\mathrm{Sp}(p,1)$ the Plancherel formula was obtained in [3]

In these notations the Plancherel formula for $\alpha > (p+q)/2-1$ has the form

$$\prod_{k=1}^{p} \cosh^{-\alpha} t_k =$$
$$= A\cdot \prod_{k=1}^{p} \frac{1}{\Gamma(\alpha-k+1)} \int_{s\in\mathbb{R}^n} \prod_{k=1}^{p} \Gamma((\alpha-\frac{1}{2}(p+q)+1+is_k)/2)\Gamma((\alpha-\frac{1}{2}(p+q)+1-is_k)/2)\times$$
$$\times \prod_{k=1}^{p} \frac{\Gamma((q-p)/2+is_k)\Gamma((q-p)/2-is_k)}{\Gamma(is_k)\Gamma(-is_k)} \times$$
$$\prod_{1\leqslant k<l\leqslant p} (s_k^2-s_l^2)\tanh\frac{\pi}{2}(s_k-s_l)\tanh\frac{\pi}{2}(s_k+s_l)\times$$
$$\times \Phi_{is_1,\dots,is_r}(t_1,t_2,\dots,t_r)\,ds_1\dots ds_p \qquad \text{(B.2)}$$

where A is a constant.

Now we will wright the analytic continuation of formula (B.2). Let $r = 0,1,\dots,p$. Let $u_1,u_2,\dots,u_s$ be nonnegative integers. Let

$$w_j = u_1+\dots+u_j+j/2$$

Then

$$\prod_{m=1}^{p} \cosh^{-\alpha}(t_m) =$$
$$= \sum_{r=0}^{p} \sum_{u_1,\dots,u_r: w_r<h-\alpha} C_u\cdot V_u(\alpha)\times$$
$$\times \int Q_u(\alpha|s_{r+1},\dots,s_p)\Phi_{\alpha-(p+q)/2+2w_1,\dots,\alpha-(p+q)/2+2w_r,is_{r+1},\dots,is_p}(t_1,\dots,t_p)\times$$
$$ds_{r+1}\dots ds_p \qquad \text{(B.3)}$$

where

$$C = A\frac{2^{p-r}p!(2\pi)^r}{(p-r)!}\frac{(-1)^{u_1+\dots+u_s}}{\prod u_j!}\cdot \prod_{k<m\leqslant s} \frac{\Gamma(\frac12+w_m-w_k)}{\Gamma(w_m-w_k)(\frac12+w_{k-1}-w_m)_{u_k}}$$

and A is a constant,

$$V_u(\alpha) = \frac{1}{\prod_{m=1}^{p}\Gamma(\alpha-k+1)}\times$$
$$\prod_{k=1}^{r} \frac{\Gamma(\alpha-p+1++2w_k)\Gamma(-\alpha+q-1-2w_k)}{\Gamma(-\alpha+\frac12(p+q)-2w_k)(\alpha-\frac12(p+q)+w_k+w_{k-1}+\frac12)_{u_k}}\times$$
$$\prod_{k<m\leqslant r} \frac{\Gamma(\frac12-\alpha+\frac12(p+q)-w_k-w_m)}{\Gamma(-\alpha+\frac12(p+q)-w_k-w_m)(\alpha-\frac12(p+q)+w_m+w_{k-1}+\frac12)_{u_k}}$$

and

$$Q_u(\alpha|s_{r+1},\ldots,s_p)=$$

$$=\prod_{n=r+1}^{p}|\Gamma(\frac{1}{2}(\alpha-(p+q)/2+1+is_n)+w_{n-1})|^2\times$$

$$\prod_{k\leqslant r,l>r}\left|\frac{\Gamma(\frac{1}{2}(1-\alpha+\frac{1}{2}(p+q)-2w_k+is_l)}{\Gamma(\frac{1}{2}(-\alpha+\frac{1}{2}(p+q)-2w_k+is_l))(\frac{1}{2}(\alpha-\frac{1}{2}(p+q)+w_k+is_l)_{u_k}}\right|^2\times$$

$$\prod_{l>n>r}(s_l^2-s_n^2)\tanh(\frac{\pi}{2}(s_l-s_n))\tanh(\frac{\pi}{2}(s_l+s_n))$$

In the case $G=\mathrm{U}(p,q)$ we obtain

$$\prod_{m=1}^{p}\cosh^{-\alpha}(t_m)=$$

$$=\sum_{r=0}^{p}\sum_{w_1,\ldots,w_r}C_w\cdot V_w(\alpha)\times$$

$$\times\int Q_w(\alpha|s_{r+1},\ldots,s_p)\Phi_{\alpha-(p+q)+1+2w_1,\ldots,\alpha-(p+q)+1+2w_r,is_{r+1},\ldots,is_p}(t_1,\ldots,t_p)\times$$

$$ds_{r+1}\ldots ds_p \qquad \text{(B.4)}$$

where

$$C_w=A\cdot\frac{2^{p-r}p!(2\pi)^r}{(p-r)!}\cdot\prod_{k=1}^{r}\frac{(-1)^{w_k}}{w_k!}\prod_{k<l\leqslant r}(w_k-w_l)^2$$

A is a constant,

$$V_w(\alpha)=$$

$$=\prod_{m=1}^{p}\frac{1}{\Gamma(\alpha/2-m+1)^2}\prod_{k=1}^{r}\frac{\Gamma(\frac{1}{2}\alpha-p+1+w_k)^2\Gamma(-\frac{1}{2}\alpha+q-w_k)^2}{\Gamma(-\alpha+(p+q)-1+2w_k)(\alpha-(p+q)+1+w_k)_{w_k}}\times$$

$$\prod_{k<l\leqslant r}(\alpha-(p+q)+1+w_k+w_l)^2$$

and

$$Q_w(\alpha|s_{r+1},\dots,s_p) =$$
$$= \prod_{k\leqslant r, n>r} (\frac{1}{4}(\alpha-(p+q)+1+2w_k)^2+s_n^2)^2 \times$$
$$\prod_{n=r+1}^{p} \frac{|\Gamma(\frac{1}{2}((q-p+1+is_n))|^4|\Gamma(\frac{1}{2}(\alpha-p-q+1+is_n)|^2}{|\Gamma(is_n)|^2} \times$$
$$\prod_{n>m>r}(s_n^2-s_m^2)^2$$

B.3. Negative integer α. Analytic formula (B.3),(B.4) are valid for all values of the parameter $\alpha \in \mathbb{R}$. Nevertheless it has visible sense only for

$$\alpha = 0,1,2,\dots,p-1 \quad \text{or} \quad \alpha > p-1$$

when representation T_α is unitary, and for

$$\alpha = 0,-1,-2,\dots$$

when the representation T_α is finite-dimensional. In the last case the right parts of identities contain only finite number of summands with $r = p$ (Γ-factors in denominator annihilate all summands with $r < p$). In the case $G = \mathrm{U}(p,p)$ we obtain Pickrell decomposition , [29]. In this place our work also has an intersection with a recent preprint of G.Zhang [33].

For over values of parameter α Plancherel formula (B.3), (B.4) are elements of mysterious nonunitary harmonic analysis (see, for instance [15],[4]).

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