Cycles on algebraic models of smooth manifolds

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1 Introduction

Let X be a compact nonsingular real algebraic set (in $\mathbb{R}^n$ for some n). A cohomology class in $H^k(X, \mathbb{Z}/2)$ is said to be *algebraic* if the homology class Poincaré dual to it can be represented by an algebraic subset of X. The set $H^k_{\mathrm{alg}}(X, \mathbb{Z}/2)$ of all algebraic cohomology classes in $H^k(X, \mathbb{Z}/2)$ is a subgroup, while the direct sum $H^*_{\mathrm{alg}}(X, \mathbb{Z}/2)$ of the $H^k_{\mathrm{alg}}(X, \mathbb{Z}/2)$, for $k \geq 0$, forms a subring of the cohomology ring $H^*(X, \mathbb{Z}/2)$. Early papers dealing with algebraic cohomology (or homology) classes provided examples of X with $H^*_{\mathrm{alg}}(X, \mathbb{Z}/2) \neq H^*(X, \mathbb{Z}/2)$, cf. [1, 5, 6, 14, 19, 20]. The reader can find a survey of properties and applications of $H^*_{\mathrm{alg}}(-, \mathbb{Z}/2)$ in [11].

Every compact smooth (of class $\mathcal{C}^\infty$) manifold M is diffeomorphic to a nonsingular real algebraic set, called an *algebraic model* of M, cf. [23] (see also [7, Theorem 14.1.10] and, for a weaker but influential result, [18]). The following question is a challenging problem: *How the ring $H^*_{\mathrm{alg}}(X, \mathbb{Z}/2)$ varies as X runs through the class of algebraic models of M?* This paper provides partial answers. Due to technical difficulties it is easier to describe how the group $H^k_{\mathrm{alg}}(X, \mathbb{Z}/2)$ varies for a fixed k. Results of this type are in [8] for $k = 1$, in [10] for $k = 2$, and in [16] for $k \geq 3$. If $k \geq 2$ and especially if $k \geq 3$ they are far from complete.

We say that a subring A of $H^*(M, \mathbb{Z}/2)$ is *algebraically realizable* if there exist an algebraic model X of M and a smooth diffeomorphism $\varphi : X \to M$ with $\varphi^*(A) \subseteq H^*_{\mathrm{alg}}(X, \mathbb{Z}/2)$.

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The original goal of several researchers was to show that the whole ring $H^*(M,\mathbb{Z}/2)$ is algebraically realizable, that is, M has an algebraic model X with $H^*_{\mathrm{alg}}(X,\mathbb{Z}/2) = H^*(X,\mathbb{Z}/2)$ (such a conjecture, motivated by far-reaching potential applications, was explicitly stated in [1]). However, since the publication of [3] it has been known that for some manifolds M this is impossible. An important algebraically realizable subring of $H^*(M,\mathbb{Z}/2)$ is identified in [4, Theorem 4, Remark 8]. It is the subring $A(M)$ generated by the Stiefel-Whitney classes of all real vector bundles on M together with the cohomology classes Poincaré dual to the homology classes represented by all smooth submanifolds of M. A conjecture proposed in [3], and still open at the present time, suggests that every algebraically realizable subring of $H^*(M,\mathbb{Z}/2)$ is contained in $A(M)$. For us certain subrings of $A(M)$ will play a crucial role. We say that a subring A of $H^*(M,\mathbb{Z}/2)$ is *admissible* if it is generated by the Stiefel-Whitney classes of some real vector bundles on M and the cohomology classes Poincaré dual to the homology classes represented by some smooth submanifolds of M. Thus $A(M)$ is the largest admissible subring of $H^*(M,\mathbb{Z}/2)$. However, in general, not every subring of $A(M)$ is admissible. Given any subring A of $H^*(M,\mathbb{Z}/2)$, we set $A^k = A \cap H^k(M,\mathbb{Z}/2)$. As usual, we denote by $w_i(M)$ the ith Stiefel-Whitney class of M. Recall that M is called a spin manifold if $w_1(M) = 0$ and $w_2(M) = 0$.

Theorem 1.1 *Let M be a compact connected spin manifold. Assume that* $\dim M \geq 7$ *and the group $H_i(M,\mathbb{Z})$ has no 2-torsion for $i = 1,2$. Then for any admissible subring A of $H^*(M,\mathbb{Z}/2)$, there exist an algebraic model X of M and a smooth diffeomorphism $\varphi : X \to M$ satisfying*

$$\varphi^*(A) \subseteq H^*_{\mathrm{alg}}(X,\mathbb{Z}/2)$$

and

$$\varphi^*(A^k) = H^k_{\mathrm{alg}}(X,\mathbb{Z}/2) \textit{ for } k = 0,1,2,3.$$

Of course, of main interest here is the last assertion in Theorem 1.1. This result is particularly nice in dimension 7, 8 or 9.

Corollary 1.2 *Let M be a compact connected spin manifold of dimension m, where $m = 7, 8,$ or 9. Assume that the group $H_i(M, \mathbb{Z})$ has no 2-torsion for $i = 1, \dots, m-5$. Then for any subring A of $H^*(M, \mathbb{Z}/2)$, there exist an algebraic model X of M and a smooth diffeomorphism $\varphi : X \to M$ satisfying*

$$\varphi^*(A) \subseteq H^*_{\mathrm{alg}}(X, \mathbb{Z}/2)$$

and

$$\varphi^*(A^k) = H^k_{\mathrm{alg}}(X, \mathbb{Z}/2) \textit{ for } k = 0, 1, 2, 3.$$

It suffices to prove that under the assumptions of Corollary 1.2, every subring of $H^*(M, \mathbb{Z}/2)$ is admissible. The latter fact easily follows from known results, see the next section. One can also drop the assumption about the dimension of M in Corollary 1.2, provided that the topology of M is not too complicated, cf. Example 2.6.

For manifolds which are not necessarily spin, we have the following result.

Theorem 1.3 *Let M be a compact connected smooth manifold. Assume that* $\dim M = m \geq 5$ *and the group $H_{m-2}(M, \mathbb{Z})$ has no 2-torsion. Then for any admissible subring A of $H^*(M, \mathbb{Z}/2)$, the following conditions are equivalent:*

(a) *There exist an algebraic model X of M and a smooth diffeomorphism $\varphi : X \to M$ satisfying*

$$\varphi^*(A) \subseteq H^*_{\mathrm{alg}}(X, \mathbb{Z}/2)$$

and

$$\varphi^*(A^k) = H^k_{\mathrm{alg}}(X, \mathbb{Z}/2) \text{ for } k = 0, 1, 2.$$

(b) *$w_i(M)$ is in A^i for $i = 1, 2$.*

If $\dim M = 5$, then every homology class in $H_d(M, \mathbb{Z}/2)$, $d \geq 0$, can be represented by a smooth submanifold [22, Théorème II.26], and hence every subring of $H^*(M, \mathbb{Z}/2)$ is admissible.

In order to compare the assumptions in Theorems 1.1 and 1.3, let us note that for any orientable compact smooth manifold M of dimension m, the groups $H_1(M, \mathbb{Z})$ and $H_{m-2}(M, \mathbb{Z})$ have isomorphic torsion subgroups. Indeed, this follows from the Poincaré duality and the universal coefficient theorem for cohomology.

Theorems 1.1, 1.3 and Corollary 1.2 are proved in Section 2.

2 Proofs and further results

We will need some constructions from real algebraic geometry. Throughout this paper the term *real algebraic variety* designates a locally ringed space isomorphic to an algebraic subset of $\mathbb{R}^n$, for some n, endowed with the Zariski topology and the sheaf of $\mathbb{R}$-valued regular functions. Morphisms between real algebraic varieties will be called *regular maps*. Background material on real algebraic varieties and regular maps can be found in [7]. Every real algebraic variety carries also the Euclidean topology, which is determined by the usual metric topology on $\mathbb{R}$. Unless explicitly stated otherwise, all topological notions related to real algebraic varieties will refer to the Euclidean topology.

The Grassmannian $\mathbb{G}_{n,r}$ of r-dimensional vector subspaces of $\mathbb{R}^n$ is endowed with a canonical structure sheaf which makes it into a real algebraic variety in the sense of this paper [7, Theorem 3.4.4] (an affine real algebraic variety according to the terminology used in [7]). Moreover, $\mathbb{G}_{n,r}$ is nonsingular and

$$H^*_{\mathrm{alg}}(\mathbb{G}_{n,r}, \mathbb{Z}/2) = H^*(\mathbb{G}_{n,r}, \mathbb{Z}/2),$$

cf. [7, Proposition 3.4.3, Proposition 11.3.3]. The universal vector bundle $\gamma_{n,r}$ on $\mathbb{G}_{n,r}$ is algebraic. If ξ is an algebraic vector bundle of rank r on a real algebraic variety X and if n is a sufficiently large integer, then there is a regular map $f : X \to \mathbb{G}_{n,r}$ with $f^*\gamma_{n,r}$ algebraically isomorphic to ξ, cf. [7, Theorem 12.1.7]. Here referring to algebraic vector bundles we follow [7], while in [4, 5, 6, 8, 9, 10] such bundles are called strongly algebraic.

Given a compact nonsingular real algebraic variety X, we define

$$\mathrm{Alg}^k(X)$$

to be the set of all elements u of $H^k(X, \mathbb{Z}/2)$ for which there exist a compact nonsingular irreducible real algebraic variety T (depending on u), two points t_0 and t_1 in T and a cohomology class z in $H^k_{\mathrm{alg}}(X \times T, \mathbb{Z}/2)$ such that

$$u = i^*_{t_1}(z) - i^*_{t_0}(z),$$

where for any t in T, we let $i_t : X \to X \times T$ denote the map $i_t(x) = (x, t)$ for all x in X. An equivalent description of $\mathrm{Alg}^k(X)$, which immediately implies that $\mathrm{Alg}^k(X)$ is a subgroup of $H^k_{\mathrm{alg}}(X, \mathbb{Z}/2)$, is given in [15, 16]. The groups $H^k_{\mathrm{alg}}(-, \mathbb{Z}/2)$ and $\mathrm{Alg}^k(-)$ have the expected functorial properties. If $f : X \to Y$ is a regular map between compact nonsingular real algebraic varieties, then the induced homomorphism $f^* : H^*(Y, \mathbb{Z}/2) \to H^*(X, \mathbb{Z}/2)$ satisfies

$$f^*(H^k_{\mathrm{alg}}(Y, \mathbb{Z}/2)) \subseteq H^k_{\mathrm{alg}}(X, \mathbb{Z}/2) \text{ and } f^*(\mathrm{Alg}^k(Y)) \subseteq \mathrm{Alg}^k(X),$$

cf. [12, Section 5] or [6] for the former inclusion and [16] for the latter one.

The following fact will be very useful.

Theorem 2.1 *Let X be a compact nonsingular real algebraic variety. Then $\langle u \cup v, [X] \rangle = 0$ for all u in $\mathrm{Alg}^k(X)$ and v in $H^\ell_{\mathrm{alg}}(X, \mathbb{Z}/2)$, where $k + \ell = \dim X$.*

Reference for the proof. [15, Theorem 2.1]. □

As usual $\cup$ and $\langle\ ,\ \rangle$ denote the cup product and scalar (Kronecker) product, while $[X]$ stands for the fundamental class of X in $H_d(X, \mathbb{Z}/2)$, $d = \dim X$.

We will also need some properties of $\mathrm{Alg}^k(-)$ for very specific real algebraic varieties. Let B^n be a nonsingular irreducible real algebraic variety with precisely two connected components B^n_0 and B^n_1, each diffeomorphic to the unit n-sphere, $n \geq 1$. For example, one can take

$$B^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_0^4 - 4x_0^2 + 1 + x_1^2 + \cdots + x_n^2 = 0\}.$$

Let $B^n(d) = B^n \times \cdots \times B^n$ and $B_0^n(d) = B_0^n \times \cdots \times B_0^n$ be the d-fold products, and let $\delta : B_0^n(d) \hookrightarrow B^n(d)$ be the inclusion map. Then according to [16, Example 4.5],

$$H^q(B_0^n(d), \mathbb{Z}/2) = \delta^*(H^q(B^n(d), \mathbb{Z}/2)) = \delta^*(\mathrm{Alg}^q(B^n(d))) \tag{2.2}$$

for all $q \geq 0$.

We now recall an important result from differential topology. All manifolds that appear here are without boundary.

Theorem 2.3 *Let P be a smooth manifold. Two smooth maps $f : M \to P$ and $g : N \to P$, where M and N are compact smooth manifolds of dimension m, represent the same bordism class in the unoriented bordims group $\mathcal{N}_*(P)$ if and only if for every nonnegative integer q and every cohomology class v in $H^q(P, \mathbb{Z}/2)$, one has*

$$\langle w_{i_1}(M) \cup \ldots \cup w_{i_r}(M) \cup f^*(v), [M] \rangle = \langle w_{i_1}(N) \cup \ldots \cup w_{i_r}(N) \cup g^*(v), [N] \rangle$$

for all nonnegative integers $i_1, \ldots, i_r$ with $i_1 + \cdots + i_r = m - q$.

Reference for the proof. [13, (17.3)]. $\square$

Let M be a compact smooth manifold. For any positive integer k, we define

$$G^k(M)$$

to be the subgroup of $H^k(M, \mathbb{Z}/2)$ consisting of the cohomology classes u satisfying

$$\langle w_{i_1}(M) \cup \ldots \cup w_{i_r}(M) \cup u, [M] \rangle = 0$$

for all nonnegative integers $i_1, \ldots, i_r$ with $i_1 + \cdots + i_r = m - k$.

A cohomology class v in $H^k(M, \mathbb{Z}/2)$, $k \geq 1$, is said to be *spherical*, provided $v = f^*(c)$, where $f : M \to S^k$ is a continuous (or equivalently smooth) map from M into the unit k-sphere S^k and c is the unique generator of the group $H^k(S^k, \mathbb{Z}/2) \cong \mathbb{Z}/2$. It is well known that v is spherical if and only if the homology class Poincaré dual to v can be represented by a smooth submanifold of M with trivial normal vector bundle, cf. [22, Théorème II.2]. Denote by

$$S^k(M)$$

the set of all spherical cohomology classes in $H^k(M,\mathbb{Z}/2)$. It readily follows from the characterization of spherical cohomology classes recalled above that $S^k(M)$ is a subgroup of $H^k(M,\mathbb{Z}/2)$ if $2k \geq M+1$.

For any smooth submanifold N of M of codimension k, we denote by $[N]^M$ the cohomology class in $H^k(M,\mathbb{Z}/2)$ Poincaré dual to the homology class represented by N. As usual, if ξ is a real vector bundle on M, then $w(\xi)$ and $w_k(\xi)$ will stand for, respectively, its total and kth Stiefel-Whitney class. The total Stiefel-Whitney class of M will be denoted by $w(M)$.

Given a collection $\mathcal{F}$ of real vector bundles on M and a collection $\mathcal{G}$ of smooth submanifolds of M, we denote by

$$A(\mathcal{F},\mathcal{G})$$

the subring of $H^*(M,\mathbb{Z}/2)$ generated by $w_k(\xi)$ and $[N]^M$ for all ξ in $\mathcal{F}$, $k \geq 0$, and N in $\mathcal{G}$. Since $H^*(M,\mathbb{Z}/2)$ is a finite set, we may assume without loss of generality that the collections $\mathcal{F}$ and $\mathcal{G}$ are finite. By definition, any admissible subring of $H^*(M,\mathbb{Z}/2)$ is of the form $A(\mathcal{F},\mathcal{G})$.

Theorem 2.4 *Let M be a compact connected smooth manifold of dimension m. Let $\mathcal{F}$ be a collection of real vector bundles on M and let $\mathcal{G}$ be a collection of smooth submanifolds of M. Assume that there is an integer $k_0 \geq 2$ such that $2k_0+1 \leq m$ and $\operatorname{codim}_M N \geq k_0$ for all N in $\mathcal{G}$. Then for the subring $A = A(\mathcal{F},\mathcal{G})$ of $H^*(M,\mathbb{Z}/2)$, the following conditions are equivalent:*

(a) *There exist an algebraic model X of M and a smooth diffeomorphism $\varphi : X \to M$ satisfying*

$$\varphi^*(A) \subseteq H^*_{\mathrm{alg}}(X,\mathbb{Z}/2)$$

and

$$\varphi^*(A^k) = H^k_{\mathrm{alg}}(X,\mathbb{Z}/2)$$

for all k with $k \leq k_0$ and $G^{m-k}(M) \subseteq S^{m-k}(M)$.

(b) *$w(M)$ is in A.*

Proof. If Y is a compact nonsingular real algebraic variety, then $w(Y)$ is in $H^*_{\mathrm{alg}}(Y, \mathbb{Z}/2)$, cf. [6, 11, 12], and hence (a) implies (b).

Assume that (b) holds. Let $\mathcal{F} = \{\xi_1, \ldots, \xi_a\}$ and $\mathcal{G} = \{N_1, \ldots, N_b\}$. For the use in a latter part of the proof, we modify each submanifold N_j, without affecting the cohomology class $[N_j]^M$, so that we obtain a new N_j connected and nonorientable. This is possible since M is connected and $\mathrm{codim}_M N_j \geq 2$. Indeed, the last inequality implies that if U is an open subset of M diffeomorphic to $\mathbb{R}^m$, then there is a smooth connected nonorientable submanifold P_j of M contained in U and with $\dim P_j = \dim N_j$. Joining P_j and the connected components of N_j with tubes, we get the required modification of N_j.

By transversality, the submanifolds $N_1, \ldots, N_b$ can be chosen in general position. Hence in view of [4, Theorem 4, Remark 8], we may assume that M is a nonsingular real algebraic variety, $N_1, \ldots, N_b$ are nonsingular Zariski closed subvarieties of M, and every topological real vector bundle on M is isomorphic to an algebraic vector bundle. In particular, we may assume that $\xi_1, \ldots, \xi_a$ are algebraic vector bundles. Setting $r_i = \mathrm{rank}\xi_i$ and choosing a sufficiently large integer n, we can find a regular map $f_i : M \to \mathbb{G}_{n,r_i}$ such that ξ_i is isomorphic to $f_i^*\gamma_{n,r_i}$, and hence $w(\xi_i) = f_i^*(w(\gamma_{n,r_i}))$. Therefore

(1) $\qquad A$ is generated by $f_i^*(w_k(\gamma_{n,r_i}))$ and $[N_j]^M$, $1 \leq i \leq a$, $1 \leq j \leq b$, $k \geq 0$.

Setting

$$G = \mathbb{G}_{n,r_1} \times \cdots \times \mathbb{G}_{n,r_a} \text{ and } f = (f_1, \ldots, f_a) : M \to G,$$

and making use of Künneth's theorem, we obtain

$$f^*(H^*(G, \mathbb{Z}/2)) \subseteq A. \tag{2}$$

Let $k_1, \ldots, k_s$ be all the integers such that $k_0 \geq k_1 > k_2 > \cdots > k_s \geq 1$ and $G^{m-k_\ell}(M) \subseteq S^{m-k_\ell}(M)$ for $\ell = 1, \ldots, s$. Clearly,

$$\Gamma_\ell := \{v \in H^{m-k_\ell}(M, \mathbb{Z}/2) \mid \langle u \cup v, [M] \rangle = 0 \text{ for all } u \in A^{k_\ell}\} \tag{3}$$

is a subgroup of $G^{m-k_\ell}(M)$. Choose an integer d with $\dim_{\mathbb{Z}/2}\Gamma_\ell \le d$ for $\ell = 1,\dots,s$. Let

$$B^{m-k_\ell}(d) = B^{m-k_\ell} \times \dots \times B^{m-k_\ell} \text{ and } B_0^{m-k_\ell} = B_0^{m-k_\ell} \times \dots \times B_0^{m-k_\ell}$$

be as in (2.2) (with $n = m - k_\ell$). Since every cohomology class in Γ_ℓ is spherical, there exists a smooth map

$$g_\ell = (g_{\ell 1}, \dots, g_{\ell d}) : M \to B^{m-k_\ell}(d)$$

satisfying

$$g_\ell(M) \subseteq B_0^{m-k_\ell}(d) \text{ and } \Gamma_\ell = g_\ell^*(H^{m-k_\ell}(B^{m-k_\ell}(d), \mathbb{Z}/2)). \tag{4}$$

Set

$$B = B^{m-k_1}(d) \times \dots \times B^{m-k_s}(d),\ B_0 = B_0^{m-k_1}(d) \times \dots \times B_0^{m-k_s}(d),$$
$$g = (g_1, \dots, g_s) : M \to B.$$

Making use of Künneth's theorem and the inequalities $2(m-k_\ell) \ge 2(m-k_0) \ge m+1$ for $\ell = 1,\dots,s$, we get

$$H^q(B, \mathbb{Z}/2) = 0 \text{ for } 0 < q \le m,\ q \notin \{m-k_1, \dots, m-k_s\}. \tag{5}$$

Künneth's theorem also implies

$$\Gamma_\ell = g^*(H^{m-k_\ell}(B, \mathbb{Z}/2)) \text{ for } 1 \le \ell \le s. \tag{6}$$

Assertion 1. The restriction map $g|N : N \to B$, where $N := N_1 \cup \dots \cup N_b$, is null homotopic.

Clearly, it suffices to prove that for each pair of integers (ℓ, e), with $1 \le \ell \le s$ and $1 \le e \le d$, the map $h_{\ell e}|N : N \to B_0^{m-k_\ell}$ is null homotopic, where $h_{\ell e} : M \to B^{m-k_\ell}$ is defined by $h_{\ell e}(x) = g_{\ell e}(x)$ for all x in M. Recall that $B_0^{m-k_\ell}$ is diffeomorphic to S^{m-k_ℓ}. Let σ be a generator of $H^{m-k_\ell}(B_0^{m-k_\ell}, \mathbb{Z}) \cong \mathbb{Z}$. Since $\dim N_j \le m - k_\ell$ for $j = 1,\dots,b$, it follows from Hopf's classification theorem that $h_{\ell e}|N$ is null homotopic if and only if $(h_{\ell e}|N)^*(\sigma) = 0$ in $H^{m-k_\ell}(N, \mathbb{Z})$. By the Mayer-Vietoris exact sequence, the last condition is equivalent to $(h_{\ell e}|N_j)^*(\sigma) = 0$ in $H^{m-k_\ell}(N_j, \mathbb{Z})$ for all $j = 1,\dots,b$.

If $\dim N_j < m - k_\ell$, then trivially $(h_{\ell e}|N_j)^*(\sigma) = 0$.

Suppose that $\dim N_j = m - k_\ell$. In that case necessarily $\ell = 1$ and $k_1 = k_0$. In order to ease notation, set $h = h_{1e}$. Since N_j is connected and nonorientable, $(h|N_j)^*(\sigma) = 0$ in $H^{m-k_1}(N_j, \mathbb{Z})$ if and only if $(h|N_j)^*(\bar{\sigma}) = 0$ in $H^{m-k_1}(N_j, \mathbb{Z}/2)$ where $\bar{\sigma}$ in $H^{m-k_1}(B_0^{m-k_1}, \mathbb{Z}/2)$ is the reduction modulo 2 of σ. It follows from (4) that $h^*(\bar{\sigma})$ is in Γ_1, and hence (3) implies

$$\langle h^*(\bar{\sigma}) \cup [N_j]^M, [M] \rangle = 0,$$

Therefore denoting by $\epsilon : N_j \hookrightarrow M$ the inclusion map, we have

$$\begin{aligned}
\langle (h|N_j)^*(\bar{\sigma}), [N_j] \rangle &= \langle \epsilon^*(h^*(\bar{\sigma})), [N_j] \rangle \\
&= \langle h^*(\bar{\sigma}), \epsilon_*([N_j]) \rangle \\
&= \langle h^*(\bar{\sigma}), [N_j]^M \cap [M] \rangle \\
&= \langle h^*(\bar{\sigma}) \cup [N_j]^M, [M] \rangle \\
&= 0.
\end{aligned}$$

Since N_j is connected, we get $(h|N_j)^*(\bar{\sigma}) = 0$, as required. Assertion 1 is proved.

Choose a compact subset K of M such that N is contained in the interior of K and N is a deformation retract of K, while (M, K) is a polyhedral pair. Then $g|K : K \to B$ is null homotopic and, by the homotopy extension theorem [21, p. 118, Corollary 5], there exists a continuous map $g' : M \to B$ which is homotopic to g and $g'|K$ is a constant map. Thus there is a smooth map $g'' : M \to B$ homotopic to g' and equal to g' on N. Replacing, if necessary, g by g'', we may assume that

(7) $$g : M \to B \text{ is constant on } N = N_1 \cup \ldots \cup N_b,$$

while (4) and (6) still hold.

Let $c : M \to B$ be a constant map sending M to a point in B_0.

Assertion 2. The maps $(f, g) : M \to G \times B$ and $(f, c) : M \to G \times B$ represent the same bordism class in the unoriented bordism group $\mathcal{N}_*(G \times B)$.

In view of Theorem 2.3 and Künneth's theorem, it suffices to prove that for every pair (p,q) of nonnegative integers and all cohomology classes α in $H^p(G,\mathbb{Z}/2)$ and β in $H^q(B,\mathbb{Z}/2)$, we have

$$\begin{aligned}(8)\qquad &\langle w_{i_1}(M)\cup\ldots\cup w_{i_r}(M)\cup(f,g)^*(\alpha\times\beta),[M]\rangle\\ &\qquad=\langle w_{i_1}(M)\cup\ldots\cup w_{i_r}(M)\cup(f,c)^*(\alpha\times\beta),[M]\rangle\end{aligned}$$

for all nonnegative integers $i_1,\ldots,i_r$ with $i_1+\cdots+i_r=m-(p+q)$. Note that $(f,g)^*(\alpha\times\beta)=f^*(\alpha)\cup g^*(\beta)$ and $(f,c)^*(\alpha\times\beta)=f^*(\alpha)\cup c^*(\beta)$.

If $q=0$, then $g^*(\beta)=c^*(\beta)$, and hence (8) holds.

Suppose now $0<q\le m$. Then $c^*(\beta)=0$ and (8) is equivalent to

$$(9)\qquad \langle w_{i_1}(M)\cup\ldots\cup w_{i_r}(M)\cup f^*(\alpha)\cup g^*(\beta),[M]\rangle=0.$$

If $q\notin\{m-k_1,\ldots,m-k_s\}$, then $\beta=0$ according to (4), and hence (9) holds. If $q=m-k_\ell$ for some ℓ, then $g^*(\beta)$ is in Γ_ℓ in view of (5). Since (b) is satisfied, (2) implies that $w_{i_1}(M)\cup\ldots\cup w_{i_r}(M)\cup f^*(\alpha)$ is in A^{k_ℓ}. Thus (9) holds in view of (3). Assertion 2 is proved.

The proof can be completed as follows. We may assume that M is a Zariski closed nonsingular subvariety of $\mathbb{R}^\mu$ for some μ. Then N, being a union of finitely many Zariski closed nonsingular subvarieties of $\mathbb{R}^\mu$, is a nice set, equivalently, a quasi-regular subvariety in the terminology used in [2] and [24], respectively, cf. [24, p. 75]. Since (f,c) is a regular map, and by (7), the restriction $(f,g)|N$ is also regular, it follows from Assertion 2 that [2, Theorem 2.8.4] is applicable. Hence there exist a nonnegative integer ν, a Zariski closed nonsingular subvariety X of $\mathbb{R}^{\mu+\nu}$, a smooth diffeomorphism $\varphi: X\to M$, and a regular map $(\bar f,\bar g): X\to G\times B$ such that identifying $\mathbb{R}^\mu$ with $\mathbb{R}^\mu\times\{0\}\subseteq\mathbb{R}^{\mu+\nu}$, we have $N\subseteq X$, $\varphi(x)=x$ for all x in N, and $(\bar f,\bar g)$ is homotopic to $(f,g)\circ\varphi=(f\circ\varphi,g\circ\varphi)$. In particular, setting

$$\bar f=(\bar f_1,\ldots,\bar f_a): X\to G=\mathbb{G}_{n,r_1}\times\cdots\times\mathbb{G}_{n,r_a},$$
$$\bar g=(\bar g_1,\ldots,\bar g_s): X\to B=B^{m-k_1}(d)\times\cdots\times B^{m-k_s}(d),$$

we obtain $\bar f_i^*=\varphi^*\circ f_i^*$ and $\bar g_\ell^*=\varphi^*\circ g_\ell^*$ in cohomology for $1\le i\le a$ and $1\le\ell\le s$.

The cohomology class

$$\varphi^*(f_i^*(w(\gamma_{n,r_i})))=\bar f_i^*(w(\gamma_{n,r_i}))$$

is in $H^*_{\mathrm{alg}}(X,\mathbb{Z}/2)$, the map $\bar{f}_i$ being regular. Clearly,

$$\varphi^*([N_j]^M) = [N_j]^X$$

is also in $H^*_{\mathrm{alg}}(X,\mathbb{Z}/2)$. Hence (1) implies

$$\varphi^*(A) \subseteq H^*_{\mathrm{alg}}(X,\mathbb{Z}/2).$$

In particular,

$$\varphi^*(A^{k_\ell}) \subseteq H^{k_\ell}_{\mathrm{alg}}(X,\mathbb{Z}/2) \text{ for } \ell = 1,\dots,s. \tag{10}$$

It remains to prove that the inclusion in (10) is actually an equality. By (2.2) and (4),

$$\Gamma_\ell = g^*_\ell(\mathrm{Alg}^{m-k_\ell}(B^{m-k_\ell}(d))),$$

and hence

$$\varphi(\Gamma_\ell) = \varphi^*(g^*_\ell(\mathrm{Alg}^{m-k_\ell}(B^{m-k_\ell}(d)))) = \bar{g}^*_\ell(\mathrm{Alg}^{m-k_\ell}(B^{m-k_\ell}(d))).$$

Consequently,

$$\varphi^*(\Gamma_\ell) \subseteq \mathrm{Alg}^{m-k_\ell}(X), \tag{11}$$

the map $\bar{g}_\ell : X \to B^{m-k_\ell}(d)$ being regular. By the Poincaré duality,

$$H^{k_\ell}(M,\mathbb{Z}/2) \times H^{m-k_\ell}(M,\mathbb{Z}/2) \to \mathbb{Z}/2,\ (u,v) \mapsto \langle u \cup v, [M]\rangle$$

is a dual pairing, and therefore (3), (10), (11) and Theorem 2.1 taken together imply

$$\varphi^*(A^{k_\ell}) = H^{k_\ell}_{\mathrm{alg}}(X,\mathbb{Z}/2) \text{ for } \ell = 1,\dots,s,$$

as required. The proof is complete. □

We will need the following, purely technical, observation.

Lemma 2.5 *Let M be a compact connect smooth manifold of dimension m. Then:*

(i) $G^{m-1}(M) \subseteq S^{m-1}(M)$, *provided* $m \geq 2$.

(ii) $G^{m-2}(M) \subseteq S^{m-2}(M)$, *provided* $m \geq 5$ *and* $H_{m-2}(M, \mathbb{Z})$ *has no* 2*-torsion.*

(iii) $G^{m-2}(M) \subseteq S^{m-2}(M)$, *provided* $m \geq 5$, M *is orientable, and* $H_1(M, \mathbb{Z})$ *has no* 2*-torsion.*

(iv) $H^{m-3}(M, \mathbb{Z}/2) = S^{m-3}(M)$, *provided* $m \geq 7$, M *is a spin manifold, and* $H_2(M, \mathbb{Z})$ *has no* 2*-torsion.*

Proof. Given a smooth manifold P, we denote by τ_P its tangent bundle. The normal bundle of a smooth submanifold N of M will be denoted by ν_N. Recall that ν_N is a trivial vector bundle if and only if $[N]^M$ is in $S^k(M)$, $k = \mathrm{codim}_M N$.

(i) Let u be in $G^{m-1}(M)$, that is, $\langle w_1(M) \cup u, [M] \rangle = 0$. Since M is connected, we have

$$w_1(M) \cup u = 0.$$

Choose a smooth connected curve C in M with $u = [C]^M$. It suffices to prove that the normal bundle ν_C is trivial or, equivalently, $w_1(\nu_C) = 0$. Since $\tau_C \oplus \nu_C = \tau_M|C$ and τ_C is trivial, we have

$$w_1(\nu_C) = w_1(\tau_M|C) = e^*(w_1(M)),$$

where $e : C \hookrightarrow M$ is the inclusion map. A simple computation yields

$$\begin{aligned} e_*(e^*(w_1(M)) \cap [C]) &= w_1(M) \cap e_*([C]) \\ &= w_1(M) \cap ([C]^M \cap [M]) \\ &= (w_1(M) \cup [C]^M) \cap [M] \\ &= (w_1(M) \cup u) \cap [M] \\ &= 0. \end{aligned}$$

Since C is connected, we get $e^*(w_1(M)) \cap [C] = 0$, and hence $e^*(w_1(M)) = 0$. Thus $w_1(\nu_C) = 0$, as required.

(ii) By the universal coefficient theorem, the torsion subgroups of $H_{m-2}(M, \mathbb{Z})$ and $H^{m-1}(M, \mathbb{Z})$ are isomorphic, and hence $H^{m-1}(M, \mathbb{Z})$ has no 2-torsion. It follows from another version of the universal coefficient theorem that the reduction modulo 2 homomorphism $\rho : H^{m-2}(M, \mathbb{Z}) \to H^{m-2}(M, \mathbb{Z}/2)$ is surjective.

By Wu's theorem [17, Theorem 11.14], the second Wu class of M is equal to $w_1(M) \cup w_1(M) + w_2(M)$, and consequently the Steenrod square

$$\mathrm{Sq}^2 : H^{m-2}(M, \mathbb{Z}/2) \to H^m(M, \mathbb{Z}/2)$$

is given by $\mathrm{Sq}^2(u) = (w_1(M) \cup w_1(M) + w_2(M)) \cup u$. Therefore for u in $G^{m-2}(M)$, we have $\langle \mathrm{Sq}^2(u), [M] \rangle = 0$, which implies $\mathrm{Sq}^2(u) = 0$, the manifold M being connected. Since ρ is surjective, Steenrod's classification theorem [21, p. 460, Theorem 15] implies that the cohomology class u is spherical. Thus u is in $S^{m-2}(M)$, and the proof of (ii) is complete.

(iii) By the universal coefficient theorem, the torsion subgroups of $H^2(M, \mathbb{Z})$ and $H_1(M, \mathbb{Z})$ are isomorphic. The Poincaré duality implies $H^2(M, \mathbb{Z}) \cong H_{m-2}(M, \mathbb{Z})$, and hence (iii) follows from (ii).

(iv) Since $H_2(M, \mathbb{Z})$ has no 2-torsion, the reduction modulo 2 homomorphism $H_3(M, \mathbb{Z}) \to H_3(M, \mathbb{Z}/2)$ is surjective. Hence by Thom's theorem [22, Théorème II.27] each homology class in $H_3(M, \mathbb{Z}/2)$ can be represented by an orientable smooth submanifold of M. It remains to prove that if N is an orientable smooth submanifold of M of dimension 3, then the normal bundle ν_N is trivial. The orientability of N implies $w_i(N) = 0$ for $i = 1, 2$. Since $\tau_N \oplus \nu_N = \tau_M | N$ and M is a spin manifold, we get $w_i(\nu_N) = 0$ for $i = 1, 2$. It follows from the last equality that ν_N is stably trivial (cf. for example [9, Lemma 1.2]). Finally, ν_N is trivial, since $\operatorname{rank} \nu_N \geq 4 > 3 = \dim N$.

We are now ready to prove the results announced in Section 1.

Proof of Theorem 1.1. Every element of $H^1(M, \mathbb{Z}/2)$ is of the form $w_1(\lambda)$ for some real line bundle λ on M. Clearly

$$(*) \qquad w(\lambda) = 1 + w_1(\lambda).$$

We claim that every element of $H^2(M, \mathbb{Z}/2)$ is of the form $w_2(\xi)$ for some rank 2 real vector bundle ξ on M with $w_1(\xi) = 0$. Indeed, by the universal coefficient theorem, the torsion subgroups of $H_2(M, \mathbb{Z})$ and $H^3(M, \mathbb{Z})$ are isomorphic. Hence $H^3(M, \mathbb{Z})$ has no 2-torsion, which implies that the reduction modulo 2 homomorphism $\rho : H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{Z}/2)$ is surjective. Every element of $H^2(M,\mathbb{Z})$ is the first Chern class $c_1(\xi)$ of some complex line bundle ξ on M. Regarding ξ as a rank 2 real vector bundle, we get $w_2(\xi) = \rho(c_1(\xi))$ and $w_1(\xi) = 0$, which proves the claim. Note that

$$(**) \qquad w(\xi) = 1 + w_2(\xi).$$

Since M is a spin manifold, we have $w_i(M) = 0$ for $i = 1, 2, 3$, cf. [17, Problem 8-B]. Let B be the subring of $H^*(M, \mathbb{Z}/2)$ generated by A and $w_j(M)$ for $j \geq 0$. Then B is an admissible subring with

$$A \subseteq B \text{ and } A^k = B^k \text{ for } k = 0, 1, 2, 3.$$

In view of $(*)$ and $(**)$, one can find a collection $\mathcal{F}$ of real vector bundles on M and a collection $\mathcal{G}$ of smooth submanifolds of M such that $B = A(\mathcal{F}, \mathcal{G})$ and $\operatorname{codim}_M N \geq 3$ for all N in $\mathcal{G}$. By Theorem 2.4 and Lemma 2.5 (i), (iii), (iv), there exist an algebraic model X of M and a smooth diffeomorphism $\varphi : X \to M$ satisfying

$$\varphi^*(B) \subseteq H^*_{\mathrm{alg}}(X, \mathbb{Z}/2)$$

and

$$\varphi^*(B^k) = H^k_{\mathrm{alg}}(X, \mathbb{Z}/2) \text{ for } k = 0, 1, 2, 3.$$

The proof is complete. □

Proof of Corollary 1.2. We first recall some results due to Thom [22]. Let N be a compact n-dimensional manifold. By [22, Théorème II.26], every homology class in $H_k(N, \mathbb{Z}/2)$ can be represented by a smooth submanifold, provided $2k \leq n$ or $k = n - 1$ or $(n, k) = (7, 4)$. If N is orientable and $n \leq 9$, then according to [22, Corollaire II.28], every homology class in $H_\ell(N, \mathbb{Z})$, $\ell \geq 0$, can be represented by an oriented smooth submanifold.

We can now easily complete the proof. By the Poincaré duality and the universal coefficient theorem, the reduction modulo 2 homomorphism $H_p(M, \mathbb{Z}) \to H_p(M, \mathbb{Z}/2)$ is surjective in either of the following two cases:

(i) $m = 7$ and $p = 5$,

(ii) $m = 8$ or 9 and $m/2 < p \leq m - 2$.

Hence Thom's results recalled above imply that every homology class in $H_k(M, \mathbb{Z}/2)$, $k \geq 0$, can be represented by a smooth submanifold. In particular, every subring of $H^*(M, \mathbb{Z}/2)$ is admissible. The proof is complete in view of Theorem 1.1. $\square$

Proof of Theorem 1.3. We already recalled in the proof of Theorem 2.4 that $w(Y)$ is in $H^*(Y, \mathbb{Z}/2)$ for every compact nonsingular real algebraic variety Y. Hence (a) implies (b).

Assume that (b) holds. By Lemma 2.5, $G^{m-k}(M) \subseteq S^{m-k}(M)$ for $k = 1, 2$. Since every element of $H^1(M, \mathbb{Z}/2)$ is of the form $w_1(\lambda)$ for some real line bundle λ on M and since $w(\lambda) = 1 + w_1(\lambda)$, we have $A = A(\mathcal{F}, \mathcal{G})$, where $\mathcal{F}$ is a collection of real vector bundles on M and $\mathcal{G}$ is a collection of smooth submanifolds of M with $\mathrm{codim}_M N \geq 2$ for all N in $\mathcal{G}$. It follows from Theorem 2.4 that (a) is satisfied. $\square$

We conclude this paper by examining consequences of Theorems 1.1 and 2.4 for the n-fold product $T^n = S^1 \times \cdots \times S^1$. The interested reader will notice that there are other examples of a similar type.

Example 2.6. Every homology class in $H_p(T^n, \mathbb{Z}/2)$, $p \geq 0$, can be represented by a smooth submanifold, and hence every subring A of $H^*(T^n, \mathbb{Z}/2)$ is admissible. By Theorem 1.1, if $n \geq 7$, then there exist an algebraic model X of T^n and a smooth diffeomorphism $\varphi : X \to T^n$ satisfying

$$\varphi^*(A) \subseteq H^*_{\mathrm{alg}}(X, \mathbb{Z}/2)$$

and

$$\varphi^*(A^k) = H^k_{\mathrm{alg}}(X, \mathbb{Z}/2) \text{ for } k = 0, 1, 2, 3.$$

Furthermore, for any $n \geq 1$, if A is generated by 1 and some cohomology classes in $H^i(T^n, \mathbb{Z}/2)$, $i = 1, 2$, then X and φ can be chosen in such a way that

$$\varphi^*(A) \subseteq H^*_{\mathrm{alg}}(X, \mathbb{Z}/2)$$

and

$$\varphi^*(A^k) = H^k_{\mathrm{alg}}(X, \mathbb{Z}/2) \text{ for } 2k+1 \leq n.$$

Indeed, one readily checks that $A = A(\mathcal{F})$, where $\mathcal{F}$ is a collection of real vector bundles on T^n. Since $H^{m-k}(T^n, \mathbb{Z}/2) = S^{m-k}(T^n)$ for all k with $2k+1 \leq n$, the existence of X and φ satisfying the required properties is guaranteed by Theorem 2.4.

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