EXPLICIT HRS-TILTING

BEHRANG NOOHI

Abstract. For an abelian category A equipped with a torsion pair, we give an explicit description for the abelian category B introduced in [HaReSm], and also for the categories $\mathsf{Ch}(\mathsf{B})$ and $\mathcal{D}(\mathsf{B})$. We also describe the DG structure on $\mathsf{Ch}(\mathsf{B})$. As a consequence, we find new proofs of certain results of [ibid.]. The main ingredient is the category of *decorated* complexes.

1. Introduction

In [HaReSm] the authors associate to an abelian category A equipped with a torsion pair $(\mathcal{T},\mathcal{F})$ a new abelian category B, which is in turn equipped with its own torsion pair $(\mathcal{T}',\mathcal{F}')$. The approach in [ibid.] is somewhat indirect and uses t-structures for triangulated categories. In these notes we give an alternative construction for B that is more explicit and reveals more of the structure of B.

The main input in this work is the alternative description of morphisms in the derived category $\mathcal{D}(\mathsf{A})$ between complexes concentrated in degrees $[-1,0]$; see ([No], Section 9) and §3.1. We exploit this to give an explicit description of the category $\mathsf{Ch}(\mathsf{B})$ of chain complexes in B, its DG structure (Sections 9 and 10, especially, Theorem 10.9), and its derived category (Theorem 7.3). This is achieved via what we call a *decorated complex*; see Section 5. The correspondence between the homological algebra of B and that of decorated complexes in A is established via a functor Tot which should be thought of as a "twisted" total complexes functor.

Our approach also leads to news proofs for certain results of Happel-Reiten-Smalø; see Theorem 7.6 and Theorem 8.2.

Contents

1. Introduction 1
2. A quick review of torsion theories 2
3. The category B 3
3.1. Definition of B 3
3.2. Composition of morphisms 4
3.3. Addition of morphisms 5
3.4. Kernels, cokernels 5
3.5. The epi-mono factorization 6
3.6. A side note: abelian bicategory structure on B 7
4. Complexes in B and strict morphisms between them 7
5. Decorated complexes in A 10
5.1. Zero and full decorations 10
5.2. Various cohomologies associated to decorated complexes 11
5.3. The cohomological functor $\mathbb{H}\colon \mathsf{Ch}(\mathsf{A}) \to \mathsf{B}$ associated to a torsion pair 12
5.4. Decorations in the presence of a torsion pair 12

5.5. Homological algebra in $\mathsf{Dec}(\mathsf{A})$ 13
6. Complexes in B and the derived category $\mathcal{D}(\mathsf{B})$ 14
6.1. Description of $\mathsf{Ch}(\mathsf{B})$ via decorated complexes 14
6.2. Complexes of length 2 in B 15
6.3. An alternative way of looking at complexes in B 16
6.4. Definition of the functor Tot 17
7. Effect of Tot on derived categories 19
7.1. Effect of Tot on cohomologies 19
7.2. Effect of Tot on derived categories 20
7.3. The forgetful functor $\Phi\colon \mathcal{D}(\mathsf{B}) \to \mathcal{D}(\mathsf{A})$ 20
7.4. The equivalence $\mathcal{D}(\mathsf{B}) \xrightarrow{\sim} \mathcal{D}(\mathsf{A})$ 20
8. Iterating the construction for B 21
8.1. Objects of C 21
8.2. Morphisms of C 22
9. The DG structure 23
9.1. The symmetric monoidal category $\mathsf{Dec}(K)$ 23
9.2. The $\mathsf{Dec}(K)$ enrichment of $\mathsf{Dec}(\mathsf{A})$ 23
9.3. The $\mathsf{Dec}(K)$ enrichment of $\mathsf{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F})$ 24
9.4. DG structures on $\mathsf{Ch}(\mathsf{B})$ and $\mathsf{Ch}^{st}(\mathsf{B})$ 24
9.5. Remark: decorations of length l 25
10. The derived DG equivalence between $\mathfrak{Ch}^b(\mathsf{B})$ and $\mathfrak{Ch}^b(\mathsf{A}, \mathcal{T}, \mathcal{F})$ 25
10.1. Semi-injective and semi-projective objects in B 27
10.2. Derived equivalence of $\mathfrak{Ch}^b(\mathsf{B})$ and $\mathfrak{Ch}^b(\mathsf{A}, \mathcal{T}, \mathcal{F})$ 27
References 28

2. A quick review of torsion theories

Let A be an abelian category. A **torsion theory** in A is a pair $(\mathcal{T}, \mathcal{F})$ of full additive subcategories of A such that:

- ▶ For every $T \in \mathcal{T}$ and $F \in \mathcal{F}$, we have $\mathrm{Hom}(T, F) = 0$.
- ▶ For every $A \in \mathsf{A}$, there is a (necessarily unique) exact sequence
$$0 \to T \to A \to F \to 0, \qquad T \in \mathcal{T},\ F \in \mathcal{F}.$$

The following facts are well-known and easy to prove.

Lemma 2.1. *For a torsion theory $(\mathcal{T}, \mathcal{F})$ we have $\mathcal{T}^{\perp} = \mathcal{F}$ and ${}^{\perp}\mathcal{F} = \mathcal{T}$, that is*
$$\mathcal{F} = \{F \in \mathsf{A} \mid \forall T \in \mathcal{T},\ \mathrm{Hom}(T, F) = 0\},$$
$$\mathcal{T} = \{T \in \mathsf{A} \mid \forall F \in \mathcal{F},\ \mathrm{Hom}(T, F) = 0\}.$$

Lemma 2.2. *If $X \to Y$ is a monomorphism and Y is in $\mathcal{F}$, then X is in $\mathcal{F}$. If $Y \to Z$ is an epimorphism and Y is in $\mathcal{T}$, then Z is in $\mathcal{T}$.*

Remark that it is not true in general that a subobject of an object $Y \in \mathcal{T}$ is in $\mathcal{T}$. Similarly, it is not true in general that a quotient of an object $Y \in \mathcal{F}$ is in $\mathcal{F}$.

Lemma 2.3. *Consider the exact sequence*
$$0 \to X \to Y \to Z \to 0$$

in A. *If* X *and* Z *are both in* $\mathcal{T}$ *(respectively, in* $\mathcal{F}$*), then so is* Y.

3. The category B

Let A be an abelian category and $(\mathcal{T}, \mathcal{F})$ a torsion pair in A. To this data we associate a new abelian category B and a torsion pair $(\mathcal{T}', \mathcal{F}')$. By results of ([No], Section 9) this category is naturally equivalent to the one defined in [HaReSm].

3.1. **Definition of B.** The category B is defined as follows:

- ◇ $\mathrm{Ob}(\mathsf{B}) = \{\mathbb{X} = [X^{-1} \xrightarrow{d} X^0] \mid \ker d \in \mathcal{F}, \operatorname{coker} d \in \mathcal{T}\}$. We will usually drop d from the notation.
- ◇ $\mathrm{Hom}_{\mathsf{B}}(\mathbb{X}, \mathbb{Y})$ = isomorphism classes of commutative diagrams

$$\begin{tikzcd}
X^{-1} \arrow[dd, "d"'] \arrow[dr, "\kappa"] & & Y^{-1} \arrow[dl, "\iota"'] \arrow[dd, "d"] \\
& E \arrow[dl, "\sigma"] \arrow[dr, "\rho"'] & \\
X^0 & & Y^0
\end{tikzcd}$$

  such that the diagonal maps compose to zero and NE-SW sequences is short exact.

A morphism that comes from an actual morphism of complexes $f\colon \mathbb{X} \to \mathbb{Y}$ in $\mathsf{Ch}(\mathsf{A})$ corresponds to the diagram

$$\begin{tikzcd}
X^{-1} \arrow[dd, "d"'] \arrow[dr, "{(d,-f^{-1})}"] & & Y^{-1} \arrow[dl] \arrow[dd, "d"] \\
& X^0 \oplus Y^{-1} \arrow[dl] \arrow[dr, "f^0+d"'] & \\
X^0 & & Y^0
\end{tikzcd}$$

For simplicity, we denote such morphisms in the usual way

$$\begin{tikzcd}
X^{-1} \arrow[r, "f^{-1}"] \arrow[d, "d"'] & Y^{-1} \arrow[d, "d"] \\
X^0 \arrow[r, "f^0"'] & Y^0
\end{tikzcd}$$

and call them **strict** morphisms. Equivalently, a strict morphism in B is one for which the NE-SW sequence splits.

Lemma 3.1. *If* $\mathbb{X}$ *is such that* X^0 *is projective, then every morphism coming out of* $\mathbb{X}$ *is strict. If* $\mathbb{Y}$ *is such that* Y^{-1} *is injective, then every morphism to* $\mathbb{Y}$ *is strict.*

Proof. Trivial. □

3.2. **Composition of morphisms.** Given two morphisms

$$\begin{tikzcd}[column sep=small]
X^{-1} \arrow[dd] \arrow[dr] & & Y^{-1} \arrow[dl, "\iota"'] \arrow[dd] \\
& E \arrow[dl] \arrow[dr, "\rho"'] & \\
X^{0} & & Y^{0}
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
Y^{-1} \arrow[dd] \arrow[dr, "\kappa'"] & & Z^{-1} \arrow[dl] \arrow[dd] \\
& F \arrow[dl, "\sigma'"] \arrow[dr] & \\
Y^{0} & & Z^{0}
\end{tikzcd}$$

in B, we define their composition to be

$$\begin{tikzcd}[column sep=small]
X^{-1} \arrow[dd] \arrow[dr] & & Z^{-1} \arrow[dl] \arrow[dd] \\
& E \overset{Y^{-1}}{\underset{Y^{0}}{\oplus}} F \arrow[dl] \arrow[dr] & \\
X^{0} & & Z^{0}
\end{tikzcd}$$

Here, $E \overset{Y^{-1}}{\underset{Y^{0}}{\oplus}} F$ is the quotient of the object L consisting of pairs $(x,y) \in E \times F$ such that $\rho(x) = \sigma'(y) \in Y^0$, modulo the subobject $I = \{\big(\iota(\beta), \kappa'(\beta)\big) \in E \times F \mid \beta \in Y^{-1}\}$. More precisely, let $E \underset{Y^0}{\oplus} F$ be the fiber product of E and F over Y^0. Then

$$E \overset{Y^{-1}}{\underset{Y^{0}}{\oplus}} F := \operatorname{coker}(Y^{-1} \xrightarrow{(\iota,\kappa')} E \underset{Y^0}{\oplus} F).$$

In the case where one of the morphisms is strict, the composition takes a simpler form. When the first morphisms is strict, say

$$\begin{tikzcd}
X^{-1} \arrow[r, "f^{-1}"] \arrow[d] & Y^{-1} \arrow[d] \\
X^{0} \arrow[r, "f^{0}"'] & Y^{0}
\end{tikzcd}$$

then the composition is

$$\begin{tikzcd}[column sep=small]
X^{-1} \arrow[dd] \arrow[dr] & & Z^{-1} \arrow[dl] \arrow[dd] \\
& f^{0,*}(F) \arrow[dl, "f^{0,*}(\sigma')"] \arrow[dr] & \\
X^{0} & & Z^{0}
\end{tikzcd}$$

Here, $f^{0,*}(F)$ stands for the pull back of the extension F along $f^0 \colon X^0 \to Y^0$. More precisely, $f^{0,*}(F) = X^0 \oplus_{Y^0} F$ is the fiber product.

When the second morphisms is strict, say

$$\begin{tikzcd}
Y^{-1} \arrow[r, "g^{-1}"] \arrow[d] & Z^{-1} \arrow[d] \\
Y^{0} \arrow[r, "g^{0}"'] & Z^{0}
\end{tikzcd}$$

then the composition is

$$\begin{tikzcd} X^{-1} \arrow[dr] \arrow[dd] & & Z^{-1} \arrow[dl, "g_*^{-1}(\iota)"'] \arrow[dd] \\ & g_*^{-1}(E) \arrow[dl] \arrow[dr] & \\ X^0 & & Z^0 \end{tikzcd}$$

Here, $g_*^{-1}(E)$ stands for the push forward of the extension E along $g^{-1}\colon Y^{-1} \to Z^{-1}$. More precisely, $g_*^{-1}(E) = E \oplus^{Y^{-1}} Z^{-1}$ is the push-out.

3.3. **Addition of morphisms.** Given two elements $P, P' \in \mathrm{Hom}(\mathbb{X}, \mathbb{Y})$,

$$\begin{tikzcd} X^{-1} \arrow[dr, "\kappa"] \arrow[dd] & & Y^{-1} \arrow[dl, "\iota"'] \arrow[dd] \\ & E \arrow[dl, "\sigma"] \arrow[dr, "\rho"'] & \\ X^0 & & Y^0 \end{tikzcd} \qquad \begin{tikzcd} X^{-1} \arrow[dr, "\kappa'"] \arrow[dd] & & Y^{-1} \arrow[dl, "\iota'"'] \arrow[dd] \\ & E' \arrow[dl, "\sigma'"] \arrow[dr, "\rho'"'] & \\ X^0 & & Y^0 \end{tikzcd}$$

we define $P + P' \in \mathrm{Hom}(\mathbb{X}, \mathbb{Y})$ to be

$$\begin{tikzcd} X^{-1} \arrow[dr, "{(\kappa,\kappa')}"] \arrow[dd] & & Y^{-1} \arrow[dl, "{(0,\iota)}"'] \arrow[dd] \\ & E \overset{Y^{-1}}{\underset{X^0}{\oplus}} E' \arrow[dl, "\sigma=\sigma'"] \arrow[dr, "\rho+\rho'"'] & \\ X^0 & & Y^0 \end{tikzcd}$$

where $E \overset{Y^{-1}}{\underset{X^0}{\oplus}} E'$ is defined as in §3.2. We define $-P$ by

$$\begin{tikzcd} X^{-1} \arrow[dr, "\kappa"] \arrow[dd, "d"'] & & Y^{-1} \arrow[dl, "-\iota"'] \arrow[dd, "d"] \\ & E \arrow[dl, "\sigma"] \arrow[dr, "-\rho"'] & \\ X^0 & & Y^0 \end{tikzcd}$$

When A is R-linear for some commutative ring R, then B is also naturally R-linear. For $r \in R$ and P as above, rP is equal to the composition of P and the strict morphism $r \cdot - : \mathbb{Y} \to \mathbb{Y}$; see the end of §3.2 to see what this exactly is. In the case where r is a unit, rP is represented by $(E, \kappa, r^{-1}\iota, \sigma, r\rho)$.

3.4. **Kernels, cokernels.** Consider $P \in \mathrm{Hom}(\mathbb{X}, \mathbb{Y})$ given by

$$\begin{tikzcd} X^{-1} \arrow[dr, "\kappa"] \arrow[dd] & & Y^{-1} \arrow[dl, "\iota"'] \arrow[dd] \\ & E \arrow[dl, "\sigma"] \arrow[dr, "\rho"'] & \\ X^0 & & Y^0 \end{tikzcd}$$

The cone of P, where now we consider P as a morphism in the derived category $\mathcal{D}(\mathsf{A})$, has a natural model, namely, the NW-SE complex

$$C(P) := X^{-1} \xrightarrow{\kappa} E \xrightarrow{\rho} Y^0,$$

in which Y^0 is sitting in degree 0. The corresponding triangle

$$\mathbb{X} \to \mathbb{Y} \to C(P) \to \mathbb{X}[1]$$

is defined in the obvious way.

From this we get the following descriptions of the kernel and cokernel of P in B.

Kernel. Let $A = q^{-1}(T)$, where $T \in \mathcal{T}$ is the torsion part of $H^{-1}(C(P))$ and $q\colon \ker\rho \to H^{-1}(C(P))$ is the quotient map. Then, the kernel of P is

$$\ker P := [X^{-1} \xrightarrow{\kappa} A].$$

The map $\ker P \to \mathbb{X}$ is given by $(\mathrm{id}_{X^{-1}}, \sigma|_A)$. We have $H^{-1}(\ker(P)) = H^{-2}(C(P))$ and $H^0(\ker(P)) = T$.

Cokernel. The cokernel of P is

$$\operatorname{coker} P := [E/A \xrightarrow{\rho} Y^0].$$

The map $\mathbb{Y} \to \operatorname{coker} P$ is given by $(\iota, \mathrm{id}_{Y^0})$. We have $H^{-1}(\operatorname{coker}(P)) = F$, the free part of $H^{-1}(C(P))$, and $H^0(\operatorname{coker}(P)) = H^0(C(P))$

Corollary 3.2. *A morphism P as above is a monomorphism if and only if κ is a monomorphism and $H^{-1}(C(P)) \in \mathcal{F}$. The morphism P is an epimorphism if and only if ρ is an epimorphism and $H^{-1}(C(P)) \in \mathcal{T}$.*

Corollary 3.3. *A morphism P as above is an isomorphism if and only if the NW-SE sequence $C(P)$ is short exact. In this case, the inverse of P is obtained by flipping the diagram with respect to the vertical axis.*

The short exact sequence

$$0 \to \ker P[1] \to C(p) \to \operatorname{coker}(P) \to 0$$

of complexes gives rise to the exact sequence

$$0 \to H^0(\ker P) \to H^{-1}(C(P)) \to H^{-1}(\operatorname{coker} P) \to 0$$

of cohomologies.

We also have the following.

Proposition 3.4. *There is a long exact sequence*

$$0 \to H^{-2}(C(P)) \to H^{-1}(\mathbb{X}) \to H^{-1}(\mathbb{Y}) \to H^{-1}(C(P)) \to H^0(\mathbb{X}) \to H^0(\mathbb{Y}) \to H^0(C(P)) \to 0.$$

Proof. This is the exact sequence for the exact triangle $\mathbb{X} \to \mathbb{Y} \to C(P) \to \mathbb{X}[1]$. □

3.5. **The epi-mono factorization.** Notation being as in §3.4, it is easy to see that the cokernel of the map $\ker P \to \mathbb{X}$ is the complex

$$\operatorname{coim} P := [A \xrightarrow{\sigma} X^0],$$

and the kernel of $\mathbb{Y} \to \operatorname{coker} P$ is the complex

$$\operatorname{im} P := [Y^{-1} \xrightarrow{\mathrm{pr}\circ\iota} E/A].$$

There is a canonical isomorphism between these two given by

$$\begin{tikzcd}
A \arrow[dd] \arrow[dr] & & Y^{-1} \arrow[dl, "\iota"'] \arrow[dd] \\
& E \arrow[dl, "\sigma"] \arrow[dr] & \\
X^0 & & E/A
\end{tikzcd}$$

(See Corollary 3.3.) So the epi-mono factorization of P looks like

$$\begin{tikzcd}
X^{-1} \arrow[r, "\kappa"] \arrow[dd] & A \arrow[dd] \arrow[dr] & & Y^{-1} \arrow[dl, "\iota"'] \arrow[dd] \arrow[r, "\mathrm{id}"] & Y^{-1} \arrow[dd] \\
& & E \arrow[dl, "\sigma"] \arrow[dr] & & \\
X^0 \arrow[r, "\mathrm{id}"'] & X^0 & & E/A \arrow[r, "\rho"'] & Y^0
\end{tikzcd}$$

3.6. **A side note: abelian bicategory structure on B.** Instead of defining a morphism $P\colon \mathbb{X} \to \mathbb{Y}$ to be an *equivalence class* of a diagram as in §3.1, we can consider all such diagrams and the isomorphisms between them. That is, we define $\mathcal{H}om(\mathbb{X}, \mathbb{Y})$ to be the *groupoid* of all diagrams P. With the addition operation defined in §3.3, $\mathcal{H}om(\mathbb{X}, \mathbb{Y})$ becomes a symmetric monoidal category, and the composition defined in §3.2 becomes a bilinear functor

$$\mathcal{H}om(\mathbb{X}, \mathbb{Y}) \times \mathcal{H}om(\mathbb{Y}, \mathbb{Z}) \to \mathcal{H}om(\mathbb{X}, \mathbb{Z}).$$

One easily sees that there is a natural associator for triple compositions, and that there are (weak) identity morphisms. In this way, we find an "enrichment" of the abelian category B to something that may very well be called an *abelian bicategory.*[1]

A way of thinking about this abelian bicategory is as follows. Consider the full DG subcategory of the DG category of chain complexes in A whose objects are complexes $\mathbb{X} = [X^{-1} \to X^0]$ as above. In this category, the hom complexes do not have the right homotopy type (because "not every object is cofibrant"). In our abelian bicategory, however, the hom symmetric monoidal categories have the right homotopy type (i.e., they are invariant under quasi-isomorphism). So, our category should be thought of as a "cofibrant replacement" of the DG category above. The price to pay here is that we have to deal with weak associators, weak identities, and so on. We will not pursue the bicategory structure on B in these notes.

4. Complexes in B and strict morphisms between them

In this section we prepare ourselves for the first main result of these notes that will appear in Section 6; see §6.1. One of the major players here is the category $\mathsf{Ch}^{st}(\mathsf{B})$ defined below.

Let $\mathsf{Ch}^{st}(\mathsf{B}) \subset \mathsf{Ch}(\mathsf{B})$ be the category whose objects are complexes

$$\mathbf{X} = \cdots \to {}^{n-1}\mathbb{X} \to {}^{n}\mathbb{X} \to {}^{n+1}\mathbb{X} \to \cdots$$

of objects in B, and whose morphisms are **strict** morphisms of complexes, that is, morphisms $\mathbf{X} \to \mathbf{Y}$ such that for every n the morphism ${}^{n}\mathbb{X} \to {}^{n}\mathbb{Y}$ is strict; see §3.1.

[1]Presumably such objects have already been looked studied but I am not aware of that.

We define two classes of morphisms $\mathcal{S}_{sis} \subset \mathcal{S}_{qis}$ in $\mathsf{Ch}^{st}(\mathsf{B})$. The class $\mathcal{S}_{sis}$ consists of morphisms $s\colon \mathbf{X} \to \mathbf{Y}$ that become isomorphisms in $\mathsf{Ch}(\mathsf{B})$; *note that s^{-1} may no longer be strict, so s is not necessarily an isomorphism in* $\mathsf{Ch}^{st}(\mathsf{B})$. The class $\mathcal{S}_{qis}$ consists of all quasi-isomorphisms in $\mathsf{Ch}^{st}(\mathsf{B})$.

The class $\mathcal{S}_{sis}$ is indeed a multiplicative system. This follows from Lemma 4.2 below.

Lemma 4.1. *Let $P\colon \mathbb{X} \to \mathbb{Y}$ be a morphism in* B. *Then, there is a functorial commutative diagram*

$$\begin{tikzcd} & \mathbb{E} \arrow[dl, "s"'] \arrow[dr, "g"] & \\ \mathbb{X} \arrow[rr, "P"'] \arrow[dr, "h"'] & & \mathbb{Y} \arrow[dl, "t"] \\ & \mathbb{F} & \end{tikzcd}$$

in B *such that s, t, g and h are strict (§3.1) and s and t are isomorphisms.*

Proof. First we prove the existence of s and g. Consider the diagram for P

$$\begin{tikzcd} X^{-1} \arrow[dd] \arrow[dr, "\kappa"] & & Y^{-1} \arrow[dd] \arrow[dl, "\iota"'] \\ & E \arrow[dl, "\sigma"] \arrow[dr, "\rho"'] & \\ X^0 & & Y^0 \end{tikzcd}$$

We define

$$\mathbb{E} := [X^{-1} \underset{X^0}{\oplus} E \xrightarrow{\mathrm{pr}_2} E].$$

The strict map $s\colon \mathbb{E} \to \mathbb{X}$ given by (pr_1, σ) is easily seen to be an isomorphism. The map $g\colon \mathbb{E} \to \mathbb{Y}$ is defined by (α, ρ), where $\alpha\colon X^{-1} \underset{X^0}{\oplus} E \to Y^{-1}$ is the map that takes (x, a) to $a - \kappa(x)$; here we have identified Y^{-1} with $\iota(Y^{-1})$.

The construction of t and h is similar. We take

$$\mathbb{F} := [E \xrightarrow{\mathrm{in}_1} E \overset{Y^{-1}}{\oplus} Y^0].$$

The strict map $t\colon \mathbb{Y} \to \mathbb{F}$ given by (ι, in_2) is easily seen to be an isomorphism. The map $h\colon \mathbb{X} \to \mathbb{F}$ is defined by (κ, β), where $\beta\colon X^0 \to E \overset{Y^{-1}}{\oplus} Y^0$ takes x to $(\tilde{x}, \rho(\tilde{x}))$; here $\tilde{x}$ is an arbitrary lift of x to E.

Let us prove the functoriality of $\mathbb{E}$. Consider the commutative diagram

$$\begin{tikzcd} \mathbb{X}' \arrow[r, "P'"] \arrow[d, "u"'] & \mathbb{Y}' \arrow[d, "v"] \\ \mathbb{X} \arrow[r, "P"'] & \mathbb{Y} \end{tikzcd}$$

and let $\mathbb{E}$ and $\mathbb{E}'$ be constructed as above. Let $w\colon \mathbb{E} \to \mathbb{E}'$ be $s^{-1} \circ u \circ s'$. It is easy to see that w commutes with both s maps and the g maps; in fact w is uniquely determined by this property. (Observe that we did not require w to be strict.)

The functoriality of $\mathbb{F}$ is proved in a similar way. $\square$

Lemma 4.2. *Let $f\colon \mathbf{X} \to \mathbf{Y}$ be a morphism in $\mathsf{Ch}(\mathsf{B})$. Then, there is a commutative diagram*

$$\begin{tikzcd}
& \mathbf{E} \arrow[dl, "s"'] \arrow[dr, "g"] & \\
\mathbf{X} \arrow[rr, "f"'] \arrow[dr, "h"'] & & \mathbf{Y} \arrow[dl, "t"] \\
& \mathbf{F} &
\end{tikzcd}$$

in $\mathsf{Ch}(\mathsf{B})$ such that s, t, g and h are in $\mathsf{Ch}^{st}(\mathsf{B})$ and s and t are in $\mathcal{S}_{sis}$.

Proof. This follows immediately from Lemma 4.1. □

Proposition 4.3. *The inclusion $\mathsf{Ch}^{st}(\mathsf{B}) \hookrightarrow \mathsf{Ch}(\mathsf{B})$ induces the following equivalences of categories:*

$$\mathcal{S}_{sis}^{-1}\mathsf{Ch}^{st}(\mathsf{B}) \xrightarrow{\sim} \mathsf{Ch}(\mathsf{B}),$$

$$\mathcal{S}_{qis}^{-1}\mathsf{Ch}^{st}(\mathsf{B}) \xrightarrow{\sim} \mathcal{D}(\mathsf{B}).$$

Proof. The first equivalence follows immediately from Lemma 4.2. The second equivalence follows from the first equivalence. □

We will need the following lemma in Section 8.

Lemma 4.4. *Let $a < b$ be two integers. Then the full subcategory ${}^{[a,b]}\mathsf{Ch}^{st}(\mathsf{B})$ of $\mathsf{Ch}^{st}(\mathsf{B})$ consisting of complexes concentrated in the interval $[a,b]$ is localizing with respect to both $\mathcal{S}_{sis}$ and $\mathcal{S}_{qis}$. That is, we have fully faithful functors:*

$$P\colon \mathcal{S}_{sis}^{-1}\,{}^{[a,b]}\mathsf{Ch}^{st}(\mathsf{B}) \to \mathcal{S}_{sis}^{-1}\mathsf{Ch}^{st}(\mathsf{B}),$$

$$Q\colon \mathcal{S}_{qis}^{-1}\,{}^{[a,b]}\mathsf{Ch}^{st}(\mathsf{B}) \to \mathcal{S}_{qis}^{-1}\mathsf{Ch}^{st}(\mathsf{B}).$$

(Here, by abuse of notation, we have denoted $\mathcal{S}_{sis}^{-1} \cap {}^{[a,b]}\mathsf{Ch}^{st}(\mathsf{B})$ and $\mathcal{S}_{qis}^{-1} \cap {}^{[a,b]}\mathsf{Ch}^{st}(\mathsf{B})$ also by $\mathcal{S}_{sis}$ and $\mathcal{S}_{qis}$.) The same thing is true if we take the full subcategory of complexes $\mathbf{X}$ in ${}^{[a,b]}\mathsf{Ch}^{st}\mathsf{B}$ such that $H^a(\mathbf{X}) \in \mathcal{F}$ and $H^b(\mathbf{X}) \in \mathcal{T}$.

Proof. The case of $\mathcal{S}_{sis}$ is obvious. Let us prove the case of $\mathcal{S}_{qis}$. Let $\mathsf{K}(\mathsf{B})$ and ${}^{[a,b]}\mathsf{K}(\mathsf{B})$ be the homotopy categories of $\mathsf{Ch}^{st}(\mathsf{B})$ and ${}^{[a,b]}\mathsf{Ch}^{st}(\mathsf{B})$. It is enough to prove the statement for the full subcategory ${}^{[a,b]}\mathsf{K}(\mathsf{B}) \subseteq \mathsf{K}(\mathsf{B})$. We will denote the class of quasi-isomorphisms in $\mathsf{K}(\mathsf{B})$ by $\mathcal{S}$. Note that this is a multiplicative system. By abuse of notation, we denote $\mathcal{S} \cap {}^{[a,b]}\mathsf{K}(\mathsf{B})$ also by $\mathcal{S}$.

Let $\tau_{\geq a}, \tau_{\leq b}\colon \mathsf{Ch}^{st}(\mathsf{B}) \to \mathsf{Ch}^{st}(\mathsf{B})$ be the usual truncation functors that we know from the theory of t-structures, and let $\tau_{[a,b]} = \tau_{\geq a} \circ \tau_{\leq b}\colon \mathsf{Ch}^{st}(\mathsf{B}) \to {}^{[a,b]}\mathsf{Ch}^{st}(\mathsf{B})$. We use the same notation $\tau_{[a,b]}$ for the induced functor on the homotopy categories as well as the localized categories. Let $\mathbf{X}, \mathbf{Y} \in {}^{[a,b]}\mathsf{K}(\mathsf{B})$. We have to show that the map

$$\alpha\colon \operatorname{Hom}_{\mathcal{S}^{-1}({}^{[a,b]}\mathsf{K}(\mathsf{B}))}(\mathbf{X}, \mathbf{Y}) \to \operatorname{Hom}_{\mathcal{S}^{-1}\mathsf{K}(\mathsf{B})}(Q\mathbf{X}, Q\mathbf{Y}).$$

induced by Q is an isomorphism. Let

$$\beta\colon \operatorname{Hom}_{\mathcal{S}^{-1}\mathsf{K}(\mathsf{B})}(Q\mathbf{X}, Q\mathbf{Y}) \to \operatorname{Hom}_{\mathcal{S}^{-1}({}^{[a,b]}\mathsf{K}(\mathsf{B}))}(\mathbf{X}, \mathbf{Y})$$

be the map induced by $\tau_{[a,b]}$. We show that α and β are inverse to each other. It is clear that $\beta \circ \alpha = 1$. To prove $\alpha \circ \beta = 1$, let $\tilde{f} \in \mathrm{Hom}_{\mathcal{S}^{-1}\mathsf{K}(\mathsf{B})}(Q\mathbf{X}, Q\mathbf{Y})$ be given by the roof

$$\begin{tikzcd} & \mathbf{Z} \arrow[dl, "s"'] \arrow[dr, "f"] & \\ \mathbf{X} & & \mathbf{Y} \end{tikzcd}$$

where $s \in \mathcal{S}$. Then $\alpha \circ \beta(\tilde{f})$ is given by the roof

$$\begin{tikzcd} & \tau_{[a,b]}\mathbf{Z} \arrow[dl, "\tau_{[a,b]}(s)"'] \arrow[dr, "\tau_{[a,b]}(f)"] & \\ \mathbf{X} & & \mathbf{Y} \end{tikzcd}$$

To show that this is equal to $\tilde{f}$ we construct a commutative diagram

$$\begin{tikzcd} & & \mathbf{V} \arrow[dl, "r"'] \arrow[dr, "h"] & & \\ & \mathbf{Z} \arrow[dl, "s"'] \arrow[drrr, "f" near end] & & \tau_{[a,b]}\mathbf{Z} \arrow[dlll, "\tau_{[a,b]}(s)" near end] \arrow[dr, "\tau_{[a,b]}(f)"] & \\ \mathbf{X} & & & & \mathbf{Y} \end{tikzcd}$$

in which $s \circ r \in \mathcal{S}$. This diagram is easy to construct. Simply take $\mathbf{V} = \tau_{\leq b}\mathbf{Z}$ and let r and h be the unit and the counit of the adjunction for the functors $\tau_{\leq b}$ and $\tau_{\geq a}$, respectively.

The proof in the case of a torsion pair is exactly the same, once we take $\tau_{\geq a}$ and $\tau_{\leq b}$ to be the truncation functors of the t-structure corresponding to the torsion pair. □

5. Decorated complexes in A

Let A be an abelian category. We will not fix a torsion pair on A yet.

A **decorated complex** in A consists of the following data:

- ◇ A chain complex in A
$$E^\bullet: \quad \cdots \xrightarrow{\delta} E^{n-1} \xrightarrow{\delta} E^n \xrightarrow{\delta} E^{n+1} \xrightarrow{\delta} \cdots;$$
- ◇ A graded subobject $M^\bullet$ of the underlying graded object of $E^\bullet$. That is, a sequence $M^n \subseteq E^n$, $n \in \mathbb{Z}$, of subobjects, not necessarily respected by δ.

A morphism $(E^\bullet, M^\bullet) \to (F^\bullet, N^\bullet)$ of decorated complexes is a map $f\colon E^\bullet \to F^\bullet$ of complexes such that for every n, $f(M^n) \subseteq N^n$.

We denote the category of decorated complexes by $\mathsf{Dec}(\mathsf{A})$.

5.1. **Zero and full decorations.** There are two natural decorations on every complex $E^\bullet$: the *zero* decoration, in which all M^n are zero, and the *full* decoration, in which $M^n = E^n$, for all n. We have the corresponding fully faithful embeddings

$$i\colon \mathsf{Ch}(\mathsf{A}) \hookrightarrow \mathsf{Dec}(\mathsf{A}), \ \text{zero decoration},$$

$$j\colon \mathsf{Ch}(\mathsf{A}) \hookrightarrow \mathsf{Dec}(\mathsf{A}), \ \text{full decoration}.$$

Both of these functors have both left and right adjoints. The left adjoint to j is the forgetful functor (forgetting the decoration). The right adjoint is what is denoted by $\mathcal{H}^{-1}$ in §5.2. The left adjoint to i is what is called $\mathcal{H}^0$ in §5.2. The right adjoint to i is the forgetful functor.

5.2. **Various cohomologies associated to decorated complexes.** Other than the usual cohomology of the complex $E^\bullet$, to a decorated complex $(E^\bullet, M^\bullet)$ we can associate the following two families of cohomologies:

$$H^{-1,n}(E^\bullet, M^\bullet) := \ker(M^n \xrightarrow{\delta} E^{n+1}/M^{n+1}),$$

$$H^{0,n}(E^\bullet, M^\bullet) := \operatorname{coker}(M^n \xrightarrow{\delta} E^{n+1}/M^{n+1}).$$

These cohomologies themselves fit into two complexes:

$$\mathcal{H}^{-1}(E^\bullet, M^\bullet): \quad \cdots \to H^{-1,n-1}(E^\bullet, M^\bullet) \to H^{-1,n}(E^\bullet, M^\bullet) \to H^{-1,n+1}(E^\bullet, M^\bullet) \cdots,$$

$$\mathcal{H}^{0}(E^\bullet, M^\bullet): \quad \cdots \to H^{0,n-1}(E^\bullet, M^\bullet) \to H^{0,n}(E^\bullet, M^\bullet) \to H^{0,n+1}(E^\bullet, M^\bullet) \to \cdots.$$

Let us denote the cohomologies of these complexes by $H^n(\mathcal{H}^0(E^\bullet, M^\bullet))$ and $H^n(\mathcal{H}^{-1}(E^\bullet, M^\bullet))$, or $H^n(\mathcal{H}^{-1})$ and $H^n(\mathcal{H}^0)$, if $(E^\bullet, M^\bullet)$ is understood from the context.

Proposition 5.1. *There are natural morphisms $H^{n-1}(\mathcal{H}^0) \xrightarrow{\partial} H^{n+1}(\mathcal{H}^{-1})$ fitting in a long exact sequence*

$$\cdots \to H^{n-2}(\mathcal{H}^0) \xrightarrow{\partial} H^n(\mathcal{H}^{-1}) \to H^n(E) \to H^{n-1}(\mathcal{H}^0) \xrightarrow{\partial}$$
$$H^{n+1}(\mathcal{H}^{-1}) \to H^{n+1}(E) \to H^n(\mathcal{H}^0) \xrightarrow{\partial} H^{n+2}(\mathcal{H}^{-1}) \to \cdots.$$

Proof. For simplicity, we denote $E^\bullet$ by E. We put a filtration on the complex E by setting

$$F^nE = M^n \oplus \bigoplus_{n<i} E^i.$$

This gives rise to a spectral sequence whose second page is exactly the union of the complexes $\mathcal{H}^{-1}$ and $\mathcal{H}^0$. The differentials of the third page are the morphisms $H^{n-1}(\mathcal{H}^0) \xrightarrow{\partial} H^{n+1}(\mathcal{H}^{-1})$, and after the third page all the differential become zero for degree reasons. The fact that this spectral sequence converges to $H^n(E)$ is equivalent to the existence of the above long exact sequence.

An alternative proof can be obtained by showing that the natural map of complexes $E/\mathcal{H}^{-1} \to \mathcal{H}^0[-1]$ is a quasi-isomorphism. □

We define two classes of morphisms $\mathcal{S}_{sis}$ and $\mathcal{S}_{qis}$ in $\mathsf{Dec}(\mathsf{A})$. The former is the class of all morphisms in $\mathsf{Dec}(\mathsf{A})$ which induce isomorphisms on all $H^{i,n}$, $i = -1, 0$, $n \in \mathbb{Z}$. The latter is the class of all quasi-isomorphisms, that is, all morphisms in $\mathsf{Dec}(\mathsf{A})$ which induce isomorphisms on the usual cohomologies H^n.

Proposition 5.2. *We have $\mathcal{S}_{sis} \subset \mathcal{S}_{qis}$.*

Proof. Follows from Proposition 5.1. □

In Section 6 we will see the relation between the classes $\mathcal{S}_{sis}$ and $\mathcal{S}_{qis}$ in $\mathsf{Ch}^{st}(\mathsf{B})$, as defined in Section 4, and the classes $\mathcal{S}_{sis}$ and $\mathcal{S}_{qis}$ defined above. Presumably $\mathcal{S}_{sis}$ is not a multiplicative system in $\mathsf{Dec}(\mathsf{A})$. However, its restriction to the full subcategory $\mathsf{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F})$ defined in §5.4 is a multiplicative system.

5.3. **The cohomological functor $\mathbb{H}\colon \mathsf{Ch}(\mathsf{A}) \to \mathsf{B}$ associated to a torsion pair.** Assume now that A is equipped with a torsion pair $(\mathcal{T}, \mathcal{F})$. Let $E^\bullet$ be a complex in $\mathsf{Ch}(\mathsf{A})$. We define A^n, $n \in \mathbb{Z}$, to be the subobject of E^n such that

$$\mathrm{im}(E^{n-1} \xrightarrow{\delta} E^n) \subseteq A^n \subseteq \ker(E^n \xrightarrow{\delta} E^{n+1}),$$

and

$$A^n/\,\mathrm{im}(E^{n-1} \xrightarrow{\delta} E^n) \in \mathcal{T}, \quad \text{and } \ker(E^n \xrightarrow{\delta} E^{n+1})/A^n \in \mathcal{F}.$$

These properties uniquely determine A^n.

For a chain complex $E^\bullet$ in $\mathsf{Ch}(\mathsf{A})$ we define

$$\mathbb{H}^n(E^\bullet) := [E^n/A^n \xrightarrow{\delta} A^{n+1}] \in \mathsf{B}.$$

The cohomology $\mathbb{H}^n(E^\bullet) \in \mathsf{B}$ is not to be confused with the usual cohomology $H^n(E^\bullet) \in \mathsf{A}$, nor with hypercohomology.

Proposition 5.3. *Let $E^\bullet$ be a chain complex in A. Then, for every n, there is a short exact sequence*

$$0 \to H^0(\mathbb{H}^{n-1}(E^\bullet)) \to H^n(E^\bullet) \to H^{-1}(\mathbb{H}^n(E^\bullet)) \to 0.$$

Note that $H^0(\mathbb{H}^{n-1}(E^\bullet)) \in \mathcal{T}$ and $H^{-1}(\mathbb{H}^n(E^\bullet)) \in \mathcal{F}$.

Proof. This follows immediately from the definition of $\mathbb{H}^n(E^\bullet)$. □

Corollary 5.4. *A morphism $E^\bullet \to F^\bullet$ in $\mathsf{Ch}(\mathsf{A})$ is a quasi-isomorphism if and only if it induces isomorphisms on all $\mathbb{H}^n$.*

We will not prove that $\mathbb{H}$ is a cohomological functor as it will not be needed. It can be derived from Corollary 6.5.

Remark 5.5. The above discussion can be summarized as saying that, for every torsion pair $(\mathcal{T}, \mathcal{F})$ in A we have a "tilted" cohomological functor $\mathbb{H}\colon \mathsf{Ch}(\mathsf{A}) \to \mathsf{B}$ whose corresponding notion of quasi-isomorphism is independent of the torsion pair. Observe that when either $\mathcal{T} = \{0\}$ or $\mathcal{F} = \{0\}$, we have a natural identification $\mathsf{B} = \mathsf{A}$. In the first case, $\mathbb{H}^n = H^n$, and in the second case $\mathbb{H}^n = H^{n+1}$. For a general torsion pair, $\mathbb{H}^n$ has a little bit of H^n and a little bit of H^{n+1}, as we saw in Proposition 5.3.

5.4. **Decorations in the presence of a torsion pair.** Let $(E^\bullet, M^\bullet)$ be a decorated complex. We say that the decoration is **compatible** with the torsion pair $(\mathcal{T}, \mathcal{F})$ if the following condition is satisfied:

- ▶ For every n, $H^{-1,n}(E^\bullet, M^\bullet) \in \mathcal{F}$ and $H^{0,n}(E^\bullet, M^\bullet) \in \mathcal{T}$.

The decorated complexes whose decoration is compatible with the torsion pair form a full subcategory of $\mathsf{Dec}(\mathsf{A})$ which we denote by $\mathsf{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F})$. For an object $(E^\bullet, M^\bullet)$ in $\mathsf{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F})$ we define $\mathbb{H}^n(E^\bullet, M^\bullet) = \mathbb{H}^n(E^\bullet)$.

If we intersect the classes $\mathcal{S}_{sis}, \mathcal{S}_{qis} \subset \mathsf{Dec}(\mathsf{A})$ defined in §5.2 with the subcategory $\mathsf{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F})$, we obtain two classes of morphisms in $\mathsf{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F})$, which we denote by the same notation. The class $\mathcal{S}_{sis}$ is a multiplicative system. This follows, for example, from Theorem 6.1 below.

We will see in Section 6 that there exists a natural equivalence of categories $\mathsf{Ch}^{st}(\mathsf{B}) \xrightarrow{\sim} \mathsf{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F})$ under which the classes $\mathcal{S}_{sis}, \mathcal{S}_{qis} \subset \mathsf{Ch}^{st}(\mathsf{B})$ defined in Section 4 exactly correspond to the classes $\mathcal{S}_{sis}, \mathcal{S}_{qis} \subset \mathsf{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F})$ defined above.

5.5. **Homological algebra in** $\mathsf{Dec}(\mathsf{A})$**.** The category $\mathsf{Dec}(\mathsf{A})$ should be thought of as a generalization of the category $\mathsf{Ch}(\mathsf{A})$ in the sense that we can do homological algebra in $\mathsf{Dec}(\mathsf{A})$. This means, the usual notions of homological algebra (such as, *mapping cylinder* of a morphism, *mapping cone* on a morphism, *chain homotopy* between morphisms, and so on) can be defined in $\mathsf{Dec}(\mathsf{A})$.

Let us show how mapping cylinders are defined in $\mathsf{Dec}(\mathsf{A})$. (Essentially, all other definitions can be formally reduced to this one.) Let $f\colon (E^\bullet, M^\bullet) \to (F^\bullet, N^\bullet)$ be a morphism in $\mathsf{Dec}(\mathsf{A})$ and denote $E^\bullet \to F^\bullet$ by $\overline{f}$. We define $\mathrm{Cyl}(f)$ to be the usual mapping cylinder $\mathrm{Cyl}(\overline{f}) = E^\bullet \oplus E^{\bullet+1} \oplus F^\bullet$, endowed with the direct sum of the decorations of its components. We have natural morphisms $(E^\bullet, M^\bullet) \hookrightarrow \mathrm{Cyl}(f)$ and $(F^\bullet, N^\bullet) \hookrightarrow \mathrm{Cyl}(f)$ in $\mathsf{Dec}(\mathsf{A})$. In the case where f is the identity, $\mathrm{Cyl}(f)$ is the cylinder of $(E^\bullet, M^\bullet)$, which can be used to define decorated chain homotopies. The quotient $\mathrm{Cone}(f) := \mathrm{Cyl}(f)/(F^\bullet, N^\bullet) \in \mathsf{Dec}(\mathsf{A})$ is the mapping cone of f, and so on.

Lemma 5.6. *Let* $f\colon (E^\bullet, M^\bullet) \to (F^\bullet, N^\bullet)$ *be a morphism in* $\mathsf{Dec}(\mathsf{A})$*, and let* $\mathrm{Cone}(f)$ *be its decorated cone as defined above. Then, we have the isomorphisms*

$$H^{i,n}(\mathrm{Cone}(f)) \cong H^{i,n+1}(E^\bullet, M^\bullet) \oplus H^{i,n}(F^\bullet, N^\bullet), \quad i = -1, 0.$$

Proof. Straightforward. □

Remark 5.7. Decorated homotopies do not necessarily induce isomorphisms on $H^{i,n}$, but they induce chain homotopies on the level of complexes $\mathcal{H}^{-1}$ and $\mathcal{H}^0$.

Passing to decorated chain homotopy classes of morphisms in $\mathsf{Dec}(\mathsf{A})$ we obtain the **homotopy category** $\mathsf{KDec}(\mathsf{A})$ of decorated chain complexes. This is a triangulated category with the usual shift functor. Doing the same with $\mathsf{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F})$ we obtain a full triangulated subcategory of $\mathsf{KDec}(\mathsf{A})$ which we denote by $\mathsf{K}(\mathsf{A}, \mathcal{T}, \mathcal{F})$. (Here, we have used the fact that the cone of a morphism in $\mathsf{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F})$ is again in $\mathsf{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F})$; see Lemma 5.6 above.)

The class $\mathcal{S}_{qis} \subseteq \mathsf{KDec}(\mathsf{A})$ is a *multiplicative system*, and so is its intersection with $\mathsf{K}(\mathsf{A}, \mathcal{T}, \mathcal{F})$, which we denote by the same notation $\mathcal{S}_{qis}$. This is because both are defined by a cohomological functor. More precisely, we are using the following.

Lemma 5.8 ([We], Proposition 10.4.1)**.** *Let* K *be a triangulated category, and let* $\mathcal{S}$ *be the class of quasi-isomorphisms with respect to a certain cohomological functor. Then* $\mathcal{S}$ *is a multiplicative system.*

As in classical homological algebra, the fact that $\mathcal{S}_{qis}$ becomes a multiplicative system is very useful. Note that

$$\mathcal{S}_{qis}^{-1}\mathsf{KDec}(\mathsf{A}) \cong \mathcal{S}_{qis}^{-1}\mathsf{Dec}(\mathsf{A}) \quad \text{and} \quad \mathcal{S}_{qis}^{-1}\mathsf{K}(\mathsf{A}, \mathcal{T}, \mathcal{F}) \cong \mathcal{S}_{qis}^{-1}\mathsf{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F}).$$

There are two other triangulated subcategories of $\mathsf{K}(\mathsf{A}, \mathcal{T}, \mathcal{F})$ that are less important for us (they will only be used in the proof of Theorem 7.6). We will end this section by giving their definitions. The first category, denoted $\mathsf{F}(\mathsf{A}, \mathcal{T}, \mathcal{F})$, is the full subcategory of $\mathsf{K}(\mathsf{A}, \mathcal{T}, \mathcal{F})$ consisting of decorated complexes $(E^\bullet, M^\bullet)$ such that $E^\bullet$ is a complex of free objects, i.e., $E^n \in \mathcal{F}$ for all n. This is a triangulated category because the cone of morphism between two complexes of free objects is again a complex of free objects.

The category $\mathsf{F}(\mathsf{A}, \mathcal{T}, \mathcal{F})$ contains a subcategory $\mathsf{FF}(\mathsf{A}, \mathcal{T}, \mathcal{F})$ consisting of the complexes with the full decoration. We will need the following lemmas for the proof of Theorem 7.6.

Lemma 5.9. *Let* K *be a triangulated category, and let* $\mathcal{S}$ *a multiplicative system in* K *that is defined by a cohomological functor. Let* F *be a full triangulated subcategory of* K*, and set* $\mathcal{S}_{\mathsf{F}} = \mathcal{S} \cap \mathsf{F}$*. Assume either of the following holds:*

- ▷ *For every* $X \in \mathsf{K}$*, there exists a quasi isomorphism* $F \to X$ *with* $F \in \mathsf{F}$*;*
- ▷ *For every* $X \in \mathsf{K}$*, there exists a quasi isomorphism* $X \to F$ *with* $F \in \mathsf{F}$*.*

Then the functor $\mathcal{S}_{\mathsf{F}}^{-1}\mathsf{F} \to \mathcal{S}^{-1}\mathsf{K}$ *is an equivalence of triangulated categories.*

Proof. Essential surjectivity is obvious, so it is enough to show that F is a localizing subcategory of K. This follows from [GeMa], Proposition III.2.10 (page 151). (Note that, by Lemma 5.8, $\mathcal{S}_{\mathsf{F}}$ is automatically a multiplicative system.) □

Lemma 5.10. *The inclusion* $\mathsf{FF}(\mathsf{A}, \mathcal{T}, \mathcal{F}) \subset \mathsf{F}(\mathsf{A}, \mathcal{T}, \mathcal{F})$ *induces an equivalence of triangulated categories*

$$\mathcal{S}_{qis}^{-1}\mathsf{FF}(\mathsf{A}, \mathcal{T}, \mathcal{F}) \cong \mathcal{S}_{qis}^{-1}\mathsf{F}(\mathsf{A}, \mathcal{T}, \mathcal{F}).$$

Proof. For every $(E^\bullet, M^\bullet)$ in $\mathsf{F}(\mathsf{A}, \mathcal{T}, \mathcal{F})$ we have a quasi-isomorphism $(E^\bullet, M^\bullet) \to (E^\bullet, E^\bullet)$, where $(E^\bullet, E^\bullet) \in \mathsf{FF}(\mathsf{A}, \mathcal{T}, \mathcal{F})$. The result follows from the second case of Lemma 5.9. □

Dually, there are triangulated subcategories

$$\mathsf{ZT}(\mathsf{A}, \mathcal{T}, \mathcal{F}) \subset \mathsf{T}(\mathsf{A}, \mathcal{T}, \mathcal{F}) \subset \mathsf{K}(\mathsf{A}, \mathcal{T}, \mathcal{F}),$$

where $\mathsf{T}(\mathsf{A}, \mathcal{T}, \mathcal{F})$ consists of decorated complexes of torsion objects and $\mathsf{ZT}(\mathsf{A}, \mathcal{T}, \mathcal{F})$ is its full subcategory consisting of complexes with zero decoration. The above discussion applies to these categories as well.

Remark 5.11. The discussion of this subsection applies to the case where the complexes are bounded (above, below, or both).

6. Complexes in B and the derived category $\mathcal{D}(\mathsf{B})$

In this section we give an alternative description of $\mathsf{Ch}^{st}(\mathsf{B})$; see Theorem 6.1. This, together with Proposition 4.3 enables us to give a simple description for the derived category $\mathcal{D}(\mathsf{B})$.

6.1. Description of $\mathsf{Ch}(\mathsf{B})$ via decorated complexes.

Theorem 6.1. *There is an equivalence of categories*

$$\mathrm{Tot}\colon \mathsf{Ch}^{st}(\mathsf{B}) \xrightarrow{\sim} \mathsf{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F}).$$

Under this equivalence, the images of $\mathcal{S}_{sis}, \mathcal{S}_{qis} \subset \mathsf{Ch}^{st}(\mathsf{B})$ *are exactly* $\mathcal{S}_{sis}, \mathcal{S}_{qis} \subset \mathsf{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F})$

Recall that $\mathcal{S}_{qis} \subset \mathsf{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F})$ is the class of all quasi-isomorphisms, that is, all morphisms $(E^\bullet, M^\bullet) \to (F^\bullet, N^\bullet)$ such that $E^\bullet \to F^\bullet$ is a quasi-isomorphism in $\mathsf{Ch}(\mathsf{A})$. The class $\mathcal{S}_{sis} \subset \mathcal{S}_{qis}$ consists of those morphisms $(E^\bullet, M^\bullet) \to (F^\bullet, N^\bullet)$ that induce isomorphisms on $\mathcal{H}^{-1}$ and $\mathcal{H}^0$; see §5.2.

The following corollary is immediate from Theorem 6.1.

Corollary 6.2. *The functor* Tot *induces an equivalence of categories:*

$$\mathsf{Ch}(\mathsf{B}) \xrightarrow{\sim} \mathcal{S}_{sis}^{-1}\mathsf{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F}),$$

$$\mathcal{D}(\mathsf{B}) \xrightarrow{\sim} \mathcal{S}_{qis}^{-1}\mathsf{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F}).$$

Proof. Follows from Proposition 4.3. □

We prove Theorem 6.1 by giving a step by step of simplification of what goes into the definition of a chain complex in B. We begin by complexes of length two.

6.2. **Complexes of length 2 in B.** Consider the morphisms $P\colon \mathbb{X} \to \mathbb{Y}$ and $Q\colon \mathbb{Y} \to \mathbb{Z}$ as in the following diagram:

$$\begin{tikzcd}
X^{-1} \arrow[dd] \arrow[dr] & & Y^{-1} \arrow[dl, "\iota"'] \arrow[dd] \arrow[dr, "\kappa'"] & & Z^{-1} \arrow[dl] \arrow[dd] \\
& E \arrow[dl] \arrow[dr, "\rho"'] & & F \arrow[dl, "\sigma'"] \arrow[dr] & \\
X^0 & & Y^0 & & Z^0
\end{tikzcd}$$

Let $A \subseteq E$ be defined as in §3.4, and let $B \subseteq F$ be the corresponding object for the morphism Q.

Lemma 6.3. *The composition $Q \circ P$ is zero if and only if there is a morphism $f\colon E/A \to B$ in A making the following diagram commute:*

$$\begin{tikzcd}
& & Y^{-1} \arrow[dll, "\iota"'] \arrow[drr, "\kappa'"] & & \\
E \arrow[r] \arrow[drr, "\rho"'] & E/A \arrow[rr, dotted, "\exists f"] & & B \arrow[r] & F \arrow[dll, "\sigma'"] \\
& & Y^0 & &
\end{tikzcd}$$

In this case, the monomorphism $\operatorname{im} P \to \ker Q$ is realized by the strict morphism

$$\begin{tikzcd}
Y^{-1} \arrow[r, "\mathrm{id}"] \arrow[d, "\iota"'] & Y^{-1} \arrow[d, "\kappa'"] \\
E/A \arrow[r, "f"'] & B
\end{tikzcd}$$

Proof. Recall from §3.4 and §3.5 that $\operatorname{im} P = [Y^{-1} \to E/A]$ and $\ker Q = [Y^{-1} \to B]$. The composition $Q \circ P$ being zero is equivalent to the existence of a commutative triangle

$$\begin{tikzcd}
\operatorname{im} P \arrow[rr, "\varphi"] \arrow[dr] & & \ker P \arrow[dl] \\
& \mathbb{Y} &
\end{tikzcd}$$

If we unravel this triangle, we see that it is equivalent to the diagram required in the proposition. □

Corollary 6.4. *In the sequence $\mathbb{X} \xrightarrow{P} \mathbb{Y} \xrightarrow{Q} \mathbb{Z}$ above the cohomology at $\mathbb{Y}$ is given by*

$$\mathbb{H} := [E/A \xrightarrow{f} B].$$

Corollary 6.5. *A sequence $\mathbb{X} \xrightarrow{P} \mathbb{Y} \xrightarrow{Q} \mathbb{Z}$ as above is exact at $\mathbb{Y}$ if and only if there is an isomorphism $E/A \xrightarrow{\sim} B$ respecting the morphisms ι, ρ, κ', and σ'.*

6.3. **An alternative way of looking at complexes in B.** Let P and Q be as in §6.2. By a *link* from P to Q we mean a morphism $\delta\colon E \to F$ such that:

L) The following diagram commutes and the horizontal sequence is a complex:

$$\begin{tikzcd}
 & & & Y^{-1} \arrow[dl, "\iota"'] \arrow[dr, "\kappa'"] & & & \\
0 \arrow[r] & X^{-1} \arrow[r] & E \arrow[rr, "\delta"] \arrow[dr, "\rho"'] & & F \arrow[r] \arrow[dl, "\sigma'"] & Z^0 \arrow[r] & 0 \\
 & & & Y^0 & & &
\end{tikzcd}$$

Proposition 6.6. *Consider the sequence $\mathbb{X} \xrightarrow{P} \mathbb{Y} \xrightarrow{Q} \mathbb{Z}$. Then $Q \circ P = 0$ if and only if there exists a link from P to Q. If such a link exists then it is unique.*

Proof. One implication is trivial from Lemma 6.3. To prove the reverse implication, let $A \subseteq E$ and $B \subseteq F$ be as in Lemma 6.3. We need to show that every $\delta\colon E \to F$ as in (**L**) necessarily vanishes on A and factors through B; the result will then follow from Lemma 6.3.

Let us prove that δ vanishes on A. Since $\delta(X^{-1}) = 0$, we have an induced map $\overline{\delta}\colon H^{-1}(C(P)) \to F$. By definition, the image T of A in $H^{-1}(C(P))$ is the torsion part of $H^{-1}(C(P))$. Observe that $\overline{\delta}\colon H^{-1}(C(P)) \to F$ factors through the kernels of both $\sigma'\colon F \to Y^0$ and $\rho'\colon F \to Z^0$, and that $\ker\sigma' \cap \ker\rho' = \ker(d\colon Z^{-1} \to Z^0)$ belongs to $\mathcal{F}$. So $\overline{\delta}(T) = 0$. That is $\delta(A) = 0$.

The proof that δ factors through B is similar.

We now prove the uniqueness. Let δ and δ' be two links, and set $\epsilon = \delta - \delta'$. By the commutativity condition of (**L**), ϵ vanishes on Y^{-1}. Since the horizontal sequence is a complex, ϵ vanishes on X^{-1} as well. Hence, ϵ factors through $E/(\kappa(X^{-1}) + \iota(Y^{-1}) \cong H^0(\mathbb{X})$.

Similarly, by the commutativity condition of (**L**), $\sigma' \circ \epsilon = 0$, and since the horizontal sequence is a complex, $\rho' \circ \epsilon = 0$, where ρ' is the morphism $\rho'\colon F \to Z^0$. This implies that ϵ factors through $\ker\rho' \cap \ker\sigma' \cong H^{-1}(\mathbb{Z})$.

Putting these together, we see that ϵ factors through a map $H^0(\mathbb{X}) \to H^{-1}(\mathbb{Z})$. But such a map is necessarily zero because $H^0(\mathbb{X})$ is in $\mathcal{T}$ and $H^{-1}(\mathbb{Z})$ is in $\mathcal{F}$. Therefore, ϵ is the zero map. That is, $\delta = \delta'$. $\square$

Proposition 6.7. *The complex*

$$0 \longrightarrow \mathbb{X} \xrightarrow{P} \mathbb{Y} \xrightarrow{Q} \mathbb{Z} \longrightarrow 0$$

is short exact if and only of the middle sequence in (**L**) *is exact, $H^{-1}(C(P)) \in \mathcal{F}$, and $H^{-1}(C(Q)) \in \mathcal{T}$.*

Proof. Follows from Corollary 3.2 and Corollary 6.4. $\square$

Using the idea of link, we see that a chain complex

$$\mathbf{X} = \quad \cdots \to {}^{n-2}\mathbb{X} \to {}^{n-1}\mathbb{X} \to {}^{n}\mathbb{X} \to {}^{n+1}\mathbb{X} \to {}^{n+2}\mathbb{X} \to \cdots,$$

with ${}^n\mathbb{X} = [d : {}^nX^{-1} \to {}^nX^0]$, is equivalently described by the diagram

$$\begin{tikzcd}
& {}^{n-2}X^{-1} \arrow[dl, "\iota"'] & & {}^{n-1}X^{-1} \arrow[dl, "\iota"'] & & {}^{n}X^{-1} \arrow[dl, "\iota"'] & & {}^{n+1}X^{-1} \arrow[dl, "\iota"'] & & {}^{n+2}X^{-1} \arrow[dl, "\iota"'] & \\
\cdots \arrow[rr, "\delta"] & & {}^{n-1}E \arrow[rr, "\delta"] \arrow[dl, "\sigma"] & & {}^{n}E \arrow[rr, "\delta"] \arrow[dl, "\sigma"] & & {}^{n+1}E \arrow[rr, "\delta"] \arrow[dl, "\sigma"] & & {}^{n+2}E \arrow[rr, "\delta"] \arrow[dl, "\sigma"] & & \cdots . \arrow[dl, "\sigma"] \\
& {}^{n-2}X^{0} & & {}^{n-1}X^{0} & & {}^{n}X^{0} & & {}^{n+1}X^{0} & & {}^{n+2}X^{0} &
\end{tikzcd}$$

Literally translating all the commutativity and exactness conditions that are needed to be satisfied, we arrive at the following list of requirements:

1) Every NW-SE sequence is short exact;
2) $\sigma \circ \delta \circ \iota = d$;
3) $\delta^2 \circ \iota = 0$ and $\sigma \circ \delta^2 = 0$;

The third axiom can be improved though.

Lemma 6.8. *We have $\delta^2 = 0$.*

Proof. Let ${}^nC := \ker(\delta \circ \iota)/\operatorname{im}(\sigma \circ \delta)$, i.e., the middle cohomology of the sequence

$${}^{n-1}X^{-1} \xrightarrow{\delta\circ\iota} {}^nE \xrightarrow{\sigma\circ\delta} {}^nX^0,$$

that is a complex by (**3**). Let ${}^nA := q^{-1}(T) \subseteq {}^nE$, where $T \subseteq {}^nC$ is the torsion part of ${}^nC$. As we saw in the proof of Proposition 6.6, $\delta\colon {}^{n-1}E \to {}^nE$ factors through ${}^nA$, and $\delta\colon {}^nE \to {}^{n+1}E$ vanishes on ${}^nA$. This means

$$\delta({}^{n-1}E) \ \subseteq\ {}^nA \ \subseteq\ \ker({}^nE \xrightarrow{\delta} {}^{n+1}E).$$

Therefore, $\delta^2 = 0$. □

Summarizing the above discussion, to give a complex $\mathbf{X}$ in B is equivalent to giving a diagram as above such that (**1**) and (**2**) are satisfied. Conditions (**1**) and (**2**) are saying that the objects ${}^nX^0$ are redundant and can be deduced from the rest of the data. So we can scrape off all the ${}^nX^0$, hence also the axioms (**1**) and (**2**), without losing any information about $\mathbf{X}$. This leads to the definition of the functor $\mathrm{Tot}\colon \mathsf{Ch}^{st}(\mathsf{B}) \to \mathsf{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F})$ that is discussed in the nest subsection.

6.4. **Definition of the functor** Tot**.** We define the functor

$$\mathrm{Tot}\colon \mathsf{Ch}^{st}(\mathsf{B}) \to \mathsf{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F})$$

as follows. Let $\mathbf{X}$ be a complex in B as in §6.3. We set $\mathrm{Tot}(\mathbf{X}) = (E^\bullet, M^\bullet)$, where $E^n := {}^nE$ and $M^n := \iota({}^nX^{-1})$.

To define the effect of Tot on morphisms, let $\mathbf{X}$ and $\mathbf{Y}$ be complexes in B, and let $\mathrm{Tot}(\mathbf{X}) = (E^\bullet, M^\bullet)$ and $\mathrm{Tot}(\mathbf{Y}) = (F^\bullet, N^\bullet)$. If we use Tot and literally translate what goes into the definition of a morphism of complexes $\mathbf{X} \to \mathbf{Y}$ in $\mathsf{Ch}^{st}(\mathsf{B})$, we see that such a morphism is given by a collection of morphisms $f_n\colon E^n \to F^n$ in A satisfying the following conditions:

i) For every n, $f_n(M^n) \subseteq N^n$;
ii) For every n, the "commutator"

$$\epsilon_n := f_{n+1} \circ \delta - \delta \circ f_n \colon E^n \to F^{n+1}$$

vanishes on M^n and factors through N^{n+1}.

There are two problems here, however. The first problem is that, a priori, this is not quite the same thing as a morphism in $\mathsf{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F})$; we need that ϵ_n be actually equal to zero. This is shown to be the case in Lemma 6.9 below. The second problem is that, a morphism $\mathbf{X} \to \mathbf{Y}$ does not, a priori, uniquely determine the collection $\{f_n\}$. It only uniquely determines the effect of f_n on M^n and E^n/M^n, but not on E^n. This is taken care of in Lemma 6.10 below.

The idea of the proof of both lemmas is very similar to the proof of Proposition 6.6.

Lemma 6.9. *For every n, we have $\epsilon_n = 0$. That is, the diagrams*

$$\begin{tikzcd} E^n \arrow[r, "\delta"] \arrow[d, "f_n"'] & E^{n+1} \arrow[d, "f_{n+1}"] \\ F^n \arrow[r, "\delta"'] & F^{n+1} \end{tikzcd}$$

commute.

Proof. Note that, by definition, $E^n := {}^nE$ and $M^n := \iota({}^nX^{-1})$, $F^n := {}^nF$ and $N^n := \iota({}^nX^{-1})$.

Condition (**ii**) implies that the diagram commutes when restricted to $M^n \subseteq E^n$. It also implies that the diagram commutes when composed with the quotient map $q_{n+1} \colon F^{n+1} \to F^{n+1}/N^{n+1}$.

If we use (**i**) and (**ii**) with $n-1$, it follows that the diagram commutes when restricted to $\delta(M^{n-1})$ as well. That is, ϵ_n vanishes on $\delta(M^{n-1})$. Therefore, ϵ_n induces a morphism

$$\varphi \colon \operatorname{coker}(M^{n-1} \xrightarrow{p_n \circ \delta} E^n/M^n) \to N^{n+1},$$

where $p_n \colon E^n \to E^n/M^n$ is the quotient map.

Similarly, if we use (**i**) and (**ii**) with $n+1$, it follows that the diagram commutes when composed with $q_{n+2} \circ \delta \colon F^{n+1} \to F^{n+2}/N^{n+2}$. This implies that ϵ_n, hence also φ, factors through the kernel of $N^{n+1} \to F^{n+2}/N^{n+2}$. In other words, we obtain a morphism

$$\varphi \colon \operatorname{coker}(M^{n-1} \xrightarrow{p_n \circ \delta} E^n/M^n) \to \ker(N^{n+1} \xrightarrow{q_{n+2} \circ \delta} F^{n+2}/N^{n+2}).$$

Now observe that the left hand side is equal to $H^0({}^{n-1}\mathbf{X}) \in \mathcal{T}$, and the right hand side is equal to $H^{-1}({}^{n+1}\mathbf{X}) \in \mathcal{F}$. By the definition of a torsion pair, φ has to be the zero map. This implies that $\epsilon_n = 0$. $\square$

Lemma 6.10. *Let $\{f_n\}$ and $\{f'_n\}$ be two families of morphisms as above such that:*

- ▷ *for every n, $f_n = f'_n \colon M^n \to N^n$;*
- ▷ *for every n, $f_n = f'_n \colon E^n/M^n \to F^n/N^n$.*

Then, $f_n = f'_n$ for all n.

Proof. The proof is similar to the proof of the previous lemma. We set $h_n := f_n - f'_n$. By the first condition, f_n and f'_n coincide on M^n. They also coincide on $d(M^{n-1})$ by the previous lemma. So, h_n factors through $E^n/(M^n + dM^{n-1}) = H^0({}^{n-1}\mathbb{X})$.

By the second condition, h_n factors through N^n. It follows from the previous lemma that $d \circ h_n$ factors through N^{n+1}. That is, h^n factors through $N^n \cap d^{-1}(N^{n+1}) = H^{-1}({}^n\mathbb{X})$.

Putting the above information together, we see that h_n factors through a map $H^0({}^{n-1}\mathbb{X}) \to H^{-1}({}^{n}\mathbb{X})$. By the properties of a torsion pair, h_n is necessarily the zero map. □

We finally come to the proof of Theorem 6.1.

Proof of Theorem 6.1. We define the inverse $G\colon \mathsf{Ch}(\mathsf{A},\mathcal{T},\mathcal{F}) \to \mathsf{Ch}^{st}(\mathsf{B})$ of Tot as follows. Given $(E^\bullet, M^\bullet) \in \mathsf{Ch}(\mathsf{A},\mathcal{T},\mathcal{F})$, we define $G(E^\bullet, M^\bullet) \in \mathsf{Ch}^{st}(\mathsf{B})$ to be the complex whose n^{th} term is

$$ {}^{n}\mathbb{X} := [M^n \xrightarrow{p_{n+1}\circ\delta} E^{n+1}/M^{n+1}]. $$

Here, $p_{n+1}\colon E^{n+1} \to E^{n+1}/M^{n+1}$ stands for the projection map. The differential $d\colon {}^{n-1}\mathbb{X} \to {}^{n}\mathbb{X}$ is the morphism defined by E^n. More precisely, it is given by the diagram

$$\begin{tikzcd}
M^{n-1} \arrow[dd] \arrow[dr, "\delta"] & & M^n \arrow[dl] \arrow[dd] \\
& E^n \arrow[dl, "p_n"'] \arrow[dr, "p_{n+1}\circ\delta"] & \\
E^n/M^n & & E^{n+1}/M^{n+1}
\end{tikzcd}$$

The arguments of this and the previous subsection can be reversed, in a trivial manner, to show that G is an inverse to Tot. □

We are not quite done yet with the proof of Theorem 6.1, because we have to show that the functors Tot and G respect $\mathcal{S}_{sis}$ and $\mathcal{S}_{qis}$. We do this in the next section.

7. Effect of Tot on derived categories

In this section we study the effect of Tot on various cohomology groups and complete the proof of Theorem 6.1 by showing that Tot respects $\mathcal{S}_{sis}$ and $\mathcal{S}_{qis}$. As a corollary of this, we reprove a theorem of Happel-Reiten-Smalø asserting that in the case where the torsion pair is tilting or cotilting there is an equivalence of derived categories $\mathcal{D}(\mathsf{B}) \xrightarrow{\sim} \mathcal{D}(\mathsf{A})$; see Theorem 7.6.

7.1. **Effect of** Tot **on cohomologies.** Let $\mathbf{X}$ be a complex in B. We denote the n^{th} cohomology of $\mathbf{X}$ by

$$ \mathbb{H}^n(\mathbf{X}) := \mathrm{im}\ {}^{n-1}d/\ker\ {}^{n}d. $$

(Not to be confused with hypercohomology.) The rest of the notation appearing in the next proposition have been introduced in §5.2

Proposition 7.1. *For every* $\mathbf{X}$ *in* $\mathsf{Ch}(\mathsf{B})$*, we have natural isomorphisms*

$$ H^{-1,n}(\mathrm{Tot}(\mathbf{X})) \cong H^{-1}({}^{n}\mathbb{X}), $$

$$ H^{0,n}(\mathrm{Tot}(\mathbf{X})) \cong H^{0}({}^{n}\mathbb{X}), $$

$$ \mathbb{H}^n(\mathrm{Tot}(\mathbf{X})) \cong \mathbb{H}^n(\mathbf{X}). $$

Proof. The first two isomorphisms follow from the definition of Tot. The last one is simply a rephrasing of Corollary 6.4. □

Corollary 7.2. *The functor* $\mathrm{Tot}\colon \mathsf{Ch}^{st}(\mathsf{B}) \to \mathsf{Ch}(\mathsf{A},\mathcal{T},\mathcal{F})$ *maps* $\mathcal{S}_{sis}, \mathcal{S}_{qis} \subset \mathsf{Ch}^{st}(\mathsf{B})$ *isomorphically to* $\mathcal{S}_{sis}, \mathcal{S}_{qis} \subset \mathsf{Ch}(\mathsf{A},\mathcal{T},\mathcal{F})$. *In particular, the functor* Tot *preserves, and reflects, quasi-isomorphisms.*

Proof. Immediate. □

7.2. **Effect of** Tot **on derived categories.** The shift functor on $\mathsf{Ch}(\mathsf{A},\mathcal{T},\mathcal{F})$, $(E^\bullet, M^\bullet) \mapsto (E^\bullet[1], M^\bullet[1])$, makes the localized category $\mathcal{S}_{qis}^{-1}\mathsf{Ch}(\mathsf{A},\mathcal{T},\mathcal{F})$ into a triangulated category. We have the following.

Theorem 7.3. *The functor* $\mathrm{Tot}\colon \mathsf{Ch}^{st}(\mathsf{B}) \to \mathsf{Ch}(\mathsf{A},\mathcal{T},\mathcal{F})$ *induces a triangle equivalence*

$$\mathrm{Tot}\colon \mathcal{D}(\mathsf{B}) \xrightarrow{\sim} \mathcal{S}_{qis}^{-1}\mathsf{Ch}(\mathsf{A},\mathcal{T},\mathcal{F}).$$

In particular, we have equivalences of bounded derived categories

$$\mathrm{Tot}\colon \mathcal{D}^*(\mathsf{B}) \xrightarrow{\sim} \mathcal{S}_{qis}^{-1}\mathsf{Ch}^*(\mathsf{A},\mathcal{T},\mathcal{F}),$$

where $* = -, +, b$.

Proof. Follows from Proposition 4.3 and Corollary 7.2. Note that Tot respects any kind of boundedness. □

7.3. **The forgetful functor** $\Phi\colon \mathcal{D}(\mathsf{B}) \to \mathcal{D}(\mathsf{A})$**.** We have a forgetful triangle functor

$$\Phi\colon \mathcal{S}_{qis}^{-1}\mathsf{Ch}(\mathsf{A},\mathcal{T},\mathcal{F}) \to \mathcal{D}(\mathsf{A}),$$
$$(E^\bullet, M^\bullet) \mapsto E^\bullet.$$

By Theorem 7.3, this induces a triangle functor

$$\Phi \circ \mathrm{Tot}\colon \mathcal{D}(\mathsf{B}) \to \mathcal{D}(\mathsf{A}).$$

Furthermore, by Proposition 7.1, the following diagram commutes:

$$\begin{tikzcd} \mathcal{D}(\mathsf{B}) \arrow[dr, "\mathbb{H}"] \arrow[dd, "\Phi\circ\mathrm{Tot}"'] & \\ & \mathsf{B}. \\ \mathcal{D}(\mathsf{A}) \arrow[ur, "\mathbb{H}"'] & \end{tikzcd}$$

Here, the upper $\mathbb{H}$ stands for the usual cohomology of chain complexes, and the lower $\mathbb{H}$ is the cohomological functor defined in §5.3. The following is immediate.

Proposition 7.4. *The functor* $\Phi \circ \mathrm{Tot}\colon \mathcal{D}(\mathsf{B}) \to \mathcal{D}(\mathsf{A})$ *reflects isomorphisms.*

7.4. **The equivalence** $\mathcal{D}(\mathsf{B}) \xrightarrow{\sim} \mathcal{D}(\mathsf{A})$**.** It is a theorem of Happel-Reiten-Smalø that, in the case where $(\mathcal{T},\mathcal{F})$ is either tilting or cotilting, there is an equivalence of bounded derived categories $\mathcal{D}^b(\mathsf{B}) \to \mathcal{D}^b(\mathsf{A})$.[2]

We reprove this result using our approach. Recall that a torsion theory $(\mathcal{T},\mathcal{F})$ is called *cotilting* if for every $A \in \mathsf{A}$ there exists an epimorphism $F \to A$ with $F \in \mathcal{F}$. The torsion theory is called *tilting* if for every $A \in \mathsf{A}$ there exists a monomorphism $A \to T$ with $T \in \mathcal{T}$.

Before proving the theorem, we prove a lemma.

[2]In [HaReSm] they actually assume existence of enough injectives/projectives, but this is not necessary.

Lemma 7.5. *Let $(E^\bullet, M^\bullet) \in \mathsf{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F})$, and let $f\colon F^\bullet \to E^\bullet$ be a morphism of complexes. Assume that for every n, $f^n\colon F^n \to E^n$ is an epimorphism with kernel in $\mathcal{F}$. Then the induced decoration $f^{-1}(M^\bullet)$ on $F^\bullet$ is compatible with $(\mathcal{T}, \mathcal{F})$.*

Proof. Since $H^{-1,n}(F^\bullet, f^{-1}(M^\bullet))$ is an extension of $H^{-1,n}(E^\bullet, M^\bullet)$ with $\ker f^n$, it belongs to $\mathcal{F}$ by Lemma 2.3. It is easy to see that $H^{0,n}(F^\bullet, f^{-1}(M^\bullet)) \cong H^{0,n}(E^\bullet, M^\bullet)$, so it is in $\mathcal{T}$. □

Theorem 7.6 (Happel-Reiten-Smalø)**.** *Assume $(\mathcal{T}, \mathcal{F})$ is cotilting (respectively, tilting). Then the functor*

$$\Phi \circ \mathrm{Tot}\colon \mathcal{D}^*(\mathsf{B}) \to \mathcal{D}^*(\mathsf{A}), \quad * = -, b \quad \textit{(respectively, } * = +, b\textit{)}$$

defined in §7.3 is an equivalence of triangulated categories.

Proof. We prove the claim in the cotilting case, so let $* = -$ or $* = b$. By Theorem 7.3, it is enough to show that the forgetful functor $\Phi\colon \mathcal{S}_{qis}^{-1}\mathsf{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F}) \to \mathcal{D}(\mathsf{A})$ is an equivalence.

Let $\mathsf{FF}^*(\mathsf{A}, \mathcal{T}, \mathcal{F}) \subset \mathsf{F}^*(\mathsf{A}, \mathcal{T}, \mathcal{F})$ be as in §5.5. Let $\mathsf{F}^*(\mathsf{A})$ be the full subcategory of the homotopy category $\mathsf{K}^*(\mathsf{A})$ of chain complexes in A consisting of complexes whose terms are in $\mathcal{F}$. Clearly Φ induces an equivalence $\mathsf{FF}^*(\mathsf{A}, \mathcal{T}, \mathcal{F}) \xrightarrow{\sim} \mathsf{F}^*(\mathsf{A})$ of triangulated categories, so, in particular, it induces an equivalence of the localized categories $\mathcal{S}_{qis}^{-1}\mathsf{FF}^*(\mathsf{A}, \mathcal{T}, \mathcal{F}) \xrightarrow{\sim} \mathcal{S}_{qis}^{-1}\mathsf{F}^*(\mathsf{A})$. To prove the result, we show that the functors

$$\mathcal{S}_{qis}^{-1}\mathsf{FF}^*(\mathsf{A}, \mathcal{T}, \mathcal{F}) \to \mathcal{S}_{qis}^{-1}\mathsf{Ch}^*(\mathsf{A}, \mathcal{T}, \mathcal{F}) \quad \text{and} \quad \mathcal{S}_{qis}^{-1}\mathsf{F}^*(\mathsf{A}) \to \mathcal{D}^*(\mathsf{A}),$$

induced by the corresponding inclusion maps, are equivalences of triangulated categories.

Let us prove the equivalence on the left. By Lemma 5.10, it is enough to show that $\Xi\colon \mathcal{S}_{qis}^{-1}\mathsf{F}^*(\mathsf{A}, \mathcal{T}, \mathcal{F}) \to \mathcal{S}_{qis}^{-1}\mathsf{Ch}^*(\mathsf{A}, \mathcal{T}, \mathcal{F})$ is an equivalence. We will prove this using the first case of Lemma 5.9.

Let $(E^\bullet, M^\bullet) \in \mathsf{Ch}^*(\mathsf{A}, \mathcal{T}, \mathcal{F})$. Since the torsion theory is cotilting, we can find a complex $F^\bullet$ with terms all in $\mathcal{F}$, together with an epimorphic quasi-isomorphism $f\colon F^\bullet \to E^\bullet$. It follows from Lemma 7.5 that $(F^\bullet, f^{-1}(M^\bullet))$ is in $\mathsf{F}^*(\mathsf{A}, \mathcal{T}, \mathcal{F})$. The quasi-isomorphism $(F^\bullet, f^{-1}(M^\bullet)) \to (E^\bullet, M^\bullet)$ guarantees that Lemma 5.9 applies. This proves that Ξ is an equivalence.

The proof that $\mathcal{S}_{qis}^{-1}\mathsf{F}^*(\mathsf{A}) \to \mathcal{D}^*(\mathsf{A})$ is an equivalence is entirely similar. □

8. Iterating the construction for B

The abelian category B comes with a torsion pair $(\mathcal{T}', \mathcal{F}')$, where $\mathcal{T}' = \mathcal{F}[1]$ and $\mathcal{T}' = \mathcal{F}$. We describe the abelian category C obtained by applying the tilting procedure to this torsion pair.

8.1. **Objects of C.** It follows from the description of kernels and cokernels in §3.4 that an object in C is a diagram

$$\begin{tikzcd}
X^{-1} \arrow[dd] \arrow[dr, hook, "\kappa"] & & Y^{-1} \arrow[dd] \arrow[dl, "\iota"'] \\
& E \arrow[dl, "\sigma"] \arrow[dr, two heads, "\rho"'] & \\
X^0 & & Y^0
\end{tikzcd}$$

where κ is a monomorphism and ρ is an epimorphism. Equivalently, an object in C is a 4-tuple $[K_1 \subseteq K_2 \subseteq E \supseteq M]$ satisfying the following conditions:

- $K_2 \cap M \in \mathcal{F}$,
- $E/(K_1 + M) \in \mathcal{T}$.

(Take $K_1 = \kappa(X^{-1})$, $K_2 = \ker \rho$, and $M = \iota(Y^{-1})$.)

8.2. **Morphisms of C.** A *strict* morphism $[K_1 \subseteq K_2 \subseteq E \supseteq M] \to [K'_1 \subseteq K'_2 \subseteq E' \supseteq M']$ is a morphism $f\colon E \to E'$ in A respecting the three subobjects. Let us denote the category of such 4-tuples and strict morphisms between them by C^{st}. Note that C^{st} is naturally identified with a full subcategory of $\mathsf{Ch}^{st}(\mathsf{B})$. We denote $\mathcal{S}_{qis} \cap \mathsf{C}^{st}$ by the same notation $\mathcal{S}_{qis}$; see Section 4 for notation. In other words, a morphism in C^{st} is in $\mathcal{S}_{qis}$ if it induces an isomorphism $K_1/K_2 \xrightarrow{\sim} K'_1/K'_2$.

Proposition 8.1. *The inclusion $i\colon \mathsf{C}^{st} \to \mathsf{C}$ induces an additive equivalence of abelian categories $\mathcal{S}_{qis}^{-1}\mathsf{C}^{st} \xrightarrow{\sim} \mathsf{C}$. Furthermore, the functor*

$$H\colon \mathsf{C}^{st} \to \mathsf{A},$$

$$[K_1 \subseteq K_2 \subseteq E \supseteq M] \mapsto K_2/K_1$$

fits into the following commutative (up to a canonical natural transformation)

$$\begin{tikzcd} \mathsf{C}^{st} \arrow[r, "H"] \arrow[d, "i"'] & \mathsf{A} \\ \mathsf{C} \cong \mathcal{S}_{qis}^{-1}\mathsf{C}^{st} \arrow[ur] & \end{tikzcd}$$

(Also, note that $\mathcal{S}_{qis}^{-1} \subset \mathsf{C}^{st}$ is exactly the class of morphisms C^{st} that map to isomorphisms in A under H.)

Proof. That $i\colon \mathsf{C}^{st} \to \mathsf{C}$ induces an equivalence of additive categories $\mathcal{S}_{qis}^{-1}\mathsf{C}^{st} \xrightarrow{\sim} \mathsf{C}$ follows from Lemma 4.4. The existence of the commutative triangle is trivial. $\square$

The functor $\mathsf{C} \to \mathsf{A}$, which we will, by abuse of notation, denote by H, is known to be an equivalence of abelian categories whenever $(\mathcal{T}, \mathcal{F})$ is either tilting or cotilting. Let us give a quick proof of this.

Theorem 8.2 (Happel-Reiten-Smalø)**.** *Assume $(\mathcal{T}, \mathcal{F})$ is either tilting or cotilting. Then, the functor $H\colon \mathsf{C} \to \mathsf{A}$ defined above is an additive equivalence of categories.*

Proof. Assume $(\mathcal{T}, \mathcal{F})$ is tilting. We define the inverse functor $Q\colon \mathsf{A} \to \mathsf{C}$ as follows. For every $A \in \mathsf{A}$ choose a monomorphism $i_A\colon A \hookrightarrow T$ with $T \in \mathcal{T}$. The effect of Q on objects is $Q(A) := [0 \subseteq A \subseteq T \supseteq 0]$. To define the effect of Q on morphisms, let $f\colon A \to A'$ be a morphism in A, and let $Q(A') = [0 \subseteq A' \subseteq T' \supseteq 0]$. Set $X = [0 \subseteq A \subseteq T \oplus T' \supseteq 0]$, where $A \hookrightarrow T \oplus T'$ is the map $(i_A, i_{A'} \circ f)$. We define $Q(f)$ to be the roof

$$\begin{tikzcd} & X \arrow[dl, "(\mathrm{id},\mathrm{pr}_1)"', "\sim"] \arrow[dr, "(f,\mathrm{pr}_2)"] & \\ Q(A) & & Q(A'). \end{tikzcd}$$

It is easy to check that Q is a well-defined functor and that it is an inverse equivalence to H.

Proof in the cotilting case is similar. The functor $Q'\colon \mathsf{A} \to \mathsf{C}$ sends an object $A \in \mathsf{A}$ to $[\ker p \subseteq F \subseteq F \supseteq F]$, where $p\colon F \to A$ is a choice of an epimorphism with $F \in \mathcal{F}$. $\square$

9. The DG structure

The category $\mathsf{Ch}(\mathsf{B})$ is naturally a DG category, and so is $\mathsf{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F})$, as we will see shortly. So one would like to strengthen Theorem 7.3 to a statement about DG categories. We will do that in this and the next section.

We will assume that A is a K-linear abelian category, where K is a commutative ring.

9.1. **The symmetric monoidal category** $\mathsf{Dec}(K)$**.** Let $\mathsf{Dec}(K)$ denote the category of decorated complexes of K-modules. This is a symmetric monoidal category. Given two decorated complexes $(E^\bullet, M^\bullet)$ and $(F^\bullet, N^\bullet)$, their tensor product is the complex $E^\bullet \otimes F^\bullet$, decorated with the image of $(E^\bullet \otimes N^\bullet) \oplus (M^\bullet \otimes F^\bullet)$.

We discuss the inner homs in §9.2.

9.2. **The** $\mathsf{Dec}(K)$ **enrichment of** $\mathsf{Dec}(\mathsf{A})$**.** We will introduce a $\mathsf{Dec}(K)$ enrichment of $\mathsf{Dec}(\mathsf{A})$, which we denote by $\mathfrak{Dec}(\mathsf{A})$. This is stronger than a DG structure.

Given two objects $(E^\bullet, M^\bullet)$ and $(F^\bullet, N^\bullet)$ in $\mathsf{Dec}(\mathsf{A})$, we define the K-complex

$$\mathfrak{H}om\big((E^\bullet, M^\bullet), (F^\bullet, N^\bullet)\big)$$

to be the subcomplex of the usual mapping complex $\mathfrak{H}om\big(E^\bullet, F^\bullet\big)$ consisting of maps satisfying certain compatibility with respect to $M^\bullet$ and $N^\bullet$. More precisely, an element in the k^{th} term of the complex $\mathfrak{H}om\big((E^\bullet, M^\bullet), (F^\bullet, N^\bullet)\big)$ is a collection of morphisms $g_n\colon E^n \to F^{n+k}$, $n \in \mathbb{Z}$, such that the following conditions are satisfied:

1) For every n, $g_n(M^n) \subseteq N^{n+k}$.

2) For every n, $\delta_F \circ g_n - (-1)^k g_{n+1} \circ \delta_E$ maps M^n to N^{n+k+1}. Equivalently, the following diagram commutes:

$$\begin{tikzcd} M^n \arrow[r, "\delta_E"] \arrow[d, "g_n"'] & E^{n+1} \arrow[d, "(-1)^k g_{n+1}"] \\ F_{n+k} \arrow[r, "\delta_F"'] & F^{n+k+1}/N^{n+k+1} \end{tikzcd}$$

The differential d on $\mathfrak{H}om\big((E^\bullet, M^\bullet), (F^\bullet, N^\bullet)\big)$ is defined as usual:

$$d\colon \mathfrak{H}om\big((E^\bullet, M^\bullet), (F^\bullet, N^\bullet)\big)^k \to \mathfrak{H}om\big((E^\bullet, M^\bullet), (F^\bullet, N^\bullet)\big)^{k+1},$$

$$\{g_n\}_{n\in\mathbb{Z}} \mapsto \{\delta_F \circ g_n - (-1)^k g_{n+1} \circ \delta_E\}_{n\in\mathbb{Z}}.$$

It is not hard to verify that the sequence $\{\delta_F \circ g_n - (-1)^k g_{n+1} \circ \delta_E\}_{n\in\mathbb{Z}}$ also satisfies (**1**) and (**2**).

There is a natural decoration

$$\mathcal{M}\big((E^\bullet, M^\bullet), (F^\bullet, N^\bullet)\big) \subseteq \mathfrak{H}om\big((E^\bullet, M^\bullet), (F^\bullet, N^\bullet)\big)$$

on $\mathfrak{H}om\big((E^\bullet, M^\bullet), (F^\bullet, N^\bullet)\big)$ whose k^{th} term is, by definition, the K-submodule of $\mathfrak{H}om\big((E^\bullet, M^\bullet), (F^\bullet, N^\bullet)\big)^k$ consisting of those sequences $\{g_n\}_{n\in\mathbb{Z}}$ satisfying the following axiom:

- ▶ For every n, $g_n(M^n) = 0$ and $g_n(E^n) \subseteq N^n$. Note that this condition implies (**1**) and (**2**).

9.3. **The** $\mathsf{Dec}(K)$ **enrichment of** $\mathsf{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F})$**.** The category $\mathsf{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F})$, being a full subcategory of $\mathsf{Dec}(\mathsf{A})$, inherits a $\mathsf{Dec}(K)$ enrichment, which we denote by $\mathfrak{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F})$. This $\mathsf{Dec}(K)$ enrichment has a special feature that is described in the next proposition. We simplify the notation by denoting $\mathcal{M}\big((E^\bullet, M^\bullet), (F^\bullet, N^\bullet)\big)$ by $\mathcal{M}$ and $\mathfrak{Hom}\big((E^\bullet, M^\bullet), (F^\bullet, N^\bullet)\big)$ by $\mathfrak{Hom}$.

Proposition 9.1. *For every k, $\mathcal{M}^k \cap d^{-1}(\mathcal{M}^{k+1}) = 0$. That is $H^{-1,k}(\mathfrak{Hom}, \mathcal{M}) = 0$, for all k; see §5.2 for notation.*

Proof. Let $\{g_n\}_{n\in\mathbb{Z}}$ be in $\mathcal{M}^k \cap d^{-1}(\mathcal{M}^{k+1})$. Then the following are true for all n:

i) $g_n(M^n) = 0$,
ii) $g_n(E^n) \subseteq N^{n+k}$,
iii) $\delta_F \circ g_n - (-1)^k g_{n+1} \circ \delta_E$ vanishes on M^n, and
iv) $\delta_F \circ g_n - (-1)^k g_{n+1} \circ \delta_E$ maps E^n to N^{n+k+1}.

It follows from (i) and (iii), both with $n-1$, that g_n vanishes on $\delta_E(M^{n-1})$. This implies that g_n factors through

$$H^{0,n-1}(E^\bullet, M^\bullet) = E^n/(M^n + \delta_E(M^{n-1})).$$

Also, it follows from (iv), with n, and (ii), with $n+1$, that $g_n(E^n) \subseteq \delta_F^{-1}(N^{n+k+1})$. Therefore, g_n factors through

$$H^{-1,n+k}(F^\bullet, N^\bullet) = N^{n+k} \cap \delta_F^{-1}(N^{n+k+1}).$$

Put together, we see that g_n factors through a map

$$H^{0,n-1}(E^\bullet, M^\bullet) \to H^{-1,n+k}(F^\bullet, N^\bullet).$$

Since the left hand side belongs to $\mathcal{T}$ and the right hand side belongs to $\mathcal{F}$, this map has to be zero. Hence, $g_n = 0$. □

Corollary 9.2. *The decorated complex $(\mathfrak{Hom}, \mathcal{M})$ is naturally quasi-isomorphic to the complex $\mathcal{H}^0(\mathfrak{Hom}, \mathcal{M})$ endowed with the zero decoration; see §5.2 for notation.*

This corollary means that $\mathfrak{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F})$ should simply be thought of as a DG category, with the hom complexes being $\mathcal{H}^0(\mathfrak{Hom}, \mathcal{M})$.

Abuse of notation. We denote the DG category whose objects are the ones of $\mathsf{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F})$ and whose hom complexes are $\mathcal{H}^0(\mathfrak{Hom}, \mathcal{M})$ also by $\mathfrak{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F})$.

9.4. **DG structures on** $\mathsf{Ch}(\mathsf{B})$ **and** $\mathsf{Ch}^{st}(\mathsf{B})$**.** Being the category of chain complexes in a K-linear abelian category, $\mathsf{Ch}(\mathsf{B})$ carries a natural K-linear DG structure, which we denote by $\mathfrak{Ch}(\mathsf{B})$. The DG structure of $\mathfrak{Ch}(\mathsf{B})$ induces a DG structure on $\mathsf{Ch}^{st}(\mathsf{B})$ as well, which we denote by $\mathfrak{Ch}^{st}(\mathsf{B})$. By definition, an element in $\mathfrak{Hom}_{\mathfrak{Ch}^{st}(\mathsf{B})}(\mathbf{X}, \mathbf{Y})^k \subseteq \mathfrak{Hom}_{\mathfrak{Ch}(\mathsf{B})}(\mathbf{X}, \mathbf{Y})^k$ is a sequence of morphisms $h_n \colon {}^n\mathbb{X} \to {}^{n+k}\mathbb{Y}$, $n \in \mathbb{Z}$, satisfying the following conditions:

- ▶ Every h_n is strict;

▶ The dotted arrow in the following diagram can be filled:

$$\begin{tikzcd} {}^{n-1}X^0 \arrow[d, "h^0_{n-1}"'] & E^n \arrow[l] \arrow[d, dotted] & {}^{n}X^{-1} \arrow[l] \arrow[d, "h^{-1}_n"] \\ {}^{n-1}Y^0 & F^n \arrow[l] & {}^{n}Y^{-1} \arrow[l] \end{tikzcd}$$

Here, ${}^n\mathbb{X} = [\,{}^nX^{-1} \to {}^nX^0]$, ${}^n\mathbb{Y} = [\,{}^nY^{-1} \to {}^nY^0]$, and E^n and F^n are the extensions that define the morphisms $d\colon {}^{n-1}\mathbb{X} \to {}^n\mathbb{X}$ and $d\colon {}^{n-1}\mathbb{Y} \to {}^n\mathbb{Y}$.

That $\mathfrak{Hom}_{\mathfrak{Ch}^{st}(\mathsf{B})}(\mathbf{X}, \mathbf{Y})$ is indeed a subcomplex of $\mathfrak{Hom}_{\mathfrak{Ch}(\mathsf{B})}(\mathbf{X}, \mathbf{Y})$ is an easy exercise (it also follows from the proof of Proposition 10.1 in the next section).

9.5. **Remark: decorations of length l.** A decorated complex can be thought of as a complex each of whose terms is equipped with a length one filtration (but the differentials do not necessarily respect the filtrations). We can drop the requirement on the length of the filtration and consider complexes with *length l decorations*, $0 \le l \le \infty$. It turns out that the category $\mathsf{Dec}^l(K)$ of chain complexes of K-modules with length l decorations has a natural closed monoidal structure, and for every K-linear category A, the category $\mathsf{Dec}^l(\mathsf{A})$ of chain complexes in A with length l decorations is naturally enriched over $\mathsf{Dec}^l(K)$. The homological algebra of decorated complexes, as described in §5.5, carries over to $\mathsf{Dec}^l(K)$.

10. The derived DG equivalence between $\mathfrak{Ch}^b(\mathsf{B})$ and $\mathfrak{Ch}^b(\mathsf{A}, \mathcal{T}, \mathcal{F})$

The functor $G\colon \mathsf{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F}) \to \mathsf{Ch}(\mathsf{B})$ defined in the proof of Theorem 6.1 can be enriched to a DG functor $\mathfrak{G}\colon \mathfrak{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F}) \to \mathfrak{Ch}^{st}(\mathsf{B})$ in the obvious way; see the proof of Proposition 10.1 below. We will use this to construct a DG equivalence between $\mathfrak{Ch}^b(\mathsf{B})$ and $\mathfrak{Ch}^b(\mathsf{A}, \mathcal{T}, \mathcal{F})$ strengthening the derived equivalence of Theorem 7.3; see Theorem 10.9 below.

Proposition 10.1. *The functor $\mathfrak{G}$ induces a DG equivalence*

$$\mathfrak{G}\colon \mathfrak{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F}) \to \mathfrak{Ch}^{st}(\mathsf{B}).$$

Before proving the proposition, we prove a lemma.

Lemma 10.2. *A strict morphism $(f^{-1}, f^0)\colon [X^{-1} \xrightarrow{d_X} X^0] \to [Y^{-1} \xrightarrow{d_Y} Y^0]$ in B is the zero morphism if and only if there exists $s\colon X^0 \to Y^{-1}$ such that $f^0 = d_Y \circ s$ and $f^{-1} = s \circ d_X$, that is, (f^{-1}, f^0) is null-homotopic in $\mathsf{Ch}(\mathsf{A})$.*

Proof. Use the description of strict morphisms in §3.1. □

Proof of Proposition 10.1. Let $(E^\bullet, M^\bullet)$ and $(F^\bullet, N^\bullet)$ be objects in $\mathfrak{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F})$, and set $\mathbf{X} = G(E^\bullet, M^\bullet)$ and $\mathbf{Y} = G(F^\bullet, N^\bullet)$, as in the proof of Theorem 6.1 (page 19).

Consider an element γ in $\mathfrak{Hom}\big((E^\bullet, M^\bullet), (F^\bullet, N^\bullet)\big)^k$ given by the sequence $g_n\colon E^n \to F^{n+k}$, $n \in \mathbb{Z}$. By (**1**) of page 23, each g_n induces maps $\underline{g}_n\colon M^n \to N^{n+k}$ and $\overline{g}_n E^n/M^n \to F^{n+k}/N^{n+k}$. Condition (**2**) guarantees that the map

$$\mathfrak{G}(\gamma)_n := \big((-1)^k \underline{g}_n, \overline{g}_{n+1}\big)\colon [M^n \to E^{n+1}/M^{n+1}] \to [N^{n+k} \to F^{n+k+1}/N^{n+k+1}]$$

is a morphism in B. (Recall that, by definition, ${}^n\mathbb{X} = [M^n \to E^{n+1}/M^{n+1}]$ and ${}^{n+k}\mathbb{Y} = [N^{n+k} \to F^{n+k+1}/N^{n+k+1}]$.) We define $\mathfrak{G}(\gamma)$ to be the sequence $\mathfrak{G}(\gamma)_n$, $n \in \mathbb{Z}$. It is easy to see that the map of K-modules

$$\mathfrak{Hom}\big((E^\bullet, M^\bullet), (F^\bullet, N^\bullet)\big)^k \to \mathfrak{Hom}_{\mathfrak{Ch}^{st}(\mathsf{B})}(\mathbf{X}, \mathbf{Y})^k$$

$$\gamma \mapsto \mathfrak{G}(\gamma)$$

is surjective. We claim that its kernel is equal to

$$\mathcal{M}\big((E^\bullet, M^\bullet), (F^\bullet, N^\bullet)\big)^k + d\mathcal{M}\big((E^\bullet, M^\bullet), (F^\bullet, N^\bullet)\big)^{k-1}.$$

By Lemma 10.2, applied to $\mathfrak{G}(\gamma)_n$, a sequence $g_n \colon E^n \to F^{n+k}$, $n \in \mathbb{Z}$, is in the kernel of the above map if and only if there is a sequence $s_n \colon E^n \to F^{n+k-1}$ such that for all n

i) $s_n(M^n) = 0$,
ii) $s_n(E^n) \subseteq N^{n+k-1}$,
iii) $g_n - (-1)^k s_{n+1} \circ \delta_E$ vanishes on M^n, and
iv) $g_n - \delta_F \circ s_n$ maps E^n to N^{n+k}.

We clearly have $\sigma := \{s_n\}_{n\in\mathbb{Z}} \in \mathcal{M}\big((E^\bullet, M^\bullet), (F^\bullet, N^\bullet)\big)^{k-1}$. It is also straightforward from conditions (i)-(iv) above that

$$\gamma - d(\sigma) = \{g_n - \delta_F \circ s_n + (-1)^{k-1} s_{n+1} \circ \delta_E\}_{n\in\mathbb{Z}} \in \mathcal{M}\big((E^\bullet, M^\bullet), (F^\bullet, N^\bullet)\big)^k.$$

This proves our claim about the kernel.

Abbreviating $\mathfrak{Hom}\big((E^\bullet, M^\bullet), (F^\bullet, N^\bullet)\big)$ and $\mathcal{M}\big((E^\bullet, M^\bullet), (F^\bullet, N^\bullet)\big)$ to $\mathfrak{Hom}$ and $\mathcal{M}$, respectively, we summarize what we have proved by saying that $\mathfrak{Hom}^k \to \mathfrak{Hom}_{\mathfrak{Ch}^{st}(\mathsf{B})}(\mathbf{X}, \mathbf{Y})^k$ is a surjective map of K-modules whose kernel is $\mathcal{M}^k + d\mathcal{M}^{k-1}$. Therefore, we have an induced isomorphism

$$\mathcal{H}^0(\mathfrak{Hom}, \mathcal{M})^k = \mathfrak{Hom}^k/(\mathcal{M}^k + d\mathcal{M}^{k-1}) \xrightarrow{\sim} \mathfrak{Hom}_{\mathfrak{Ch}^{st}(\mathsf{B})}(\mathbf{X}, \mathbf{Y})^k.$$

This is exactly what we wanted to prove. ☐

Remark 10.3. The DG equivalence of the previous lemma is a DG equivalence in a strong sense: it induces *isomorphisms* on hom complexes.

It is easy to see that the functor Tot can also be enriched to a DG equivalence $\mathfrak{Tot} \colon \mathfrak{Ch}^{st}(\mathsf{B}) \to \mathfrak{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F})$, and that $\mathfrak{Tot}$ and $\mathfrak{G}$ are inverse to each other.

Of course, we expect that applying Z_0 to the above enrichments give us back the old categories.

Proposition 10.4. *We have equivalences*

$$\begin{aligned} Z_0\mathfrak{Ch}(\mathsf{B}) &\cong \mathsf{Ch}(\mathsf{B}), \\ Z_0\mathfrak{Ch}^{st}(\mathsf{B}) &\cong \mathsf{Ch}^{st}(\mathsf{B}), \\ Z_0\mathfrak{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F}) &\cong \mathsf{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F}), \\ H_0\mathfrak{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F}) &\cong \mathsf{K}(\mathsf{A}, \mathcal{T}, \mathcal{F}). \end{aligned}$$

(See §5.5 for notation.)

10.1. **Semi-injective and semi-projective objects in B.** Before we prove the derived equivalence of $\mathfrak{Ch}(\mathsf{B})$ and $\mathfrak{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F})$ we need some definitions.

We say that an object $\mathbb{X} = [X^{-1} \to X^0]$ in B is *semi-injective* (respectively, *semi-projective*), if X^{-1} an injective object in A (respectively, if X^0 a projective object in A). Note that these notions are *not* invariant under isomorphism.

Proposition 10.5. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be complexes in* $\mathsf{Ch}(\mathsf{B})$. *Assume either* $\mathbf{X}$ *is a complex of semi-projective objects, or* $\mathbf{Y}$ *is a complex of semi-injective objects. Then*

$$\mathfrak{Hom}_{\mathfrak{Ch}^{st}}(\mathbf{X}, \mathbf{Y}) \hookrightarrow \mathfrak{Hom}_{\mathfrak{Ch}}(\mathbf{X}, \mathbf{Y})$$

is an isomorphism. (We do not need any boundedness conditions on $\mathbf{X}$ *or* $\mathbf{Y}$*).*

Proof. This follows from Lemma 3.1. $\square$

Lemma 10.6. *If* A *has enough injectives, then for every object* $\mathbb{X}$ *in* B *there exists a semi-injective object* $\mathbb{I}$ *and a strict isomorphism* $\mathbb{X} \to \mathbb{I}$. *If* A *has enough projectives, then for every object* $\mathbb{X}$ *in* B *there exists a semi-projective object* $\mathbb{P}$ *and a strict isomorphism* $\mathbb{P} \to \mathbb{X}$.

Proof. Assume A has enough injectives. Take a monomorphism $X^{-1} \hookrightarrow I$ into an injective I, an set $\mathbb{I} = [I \to X^0 \overset{X^{-1}}{\oplus} I]$. Similarly, if A has enough projectives, take an epimorphism $P \twoheadrightarrow X^0$ from a projective P, and set $\mathbb{P} = [P \underset{X^0}{\oplus} X^{-1} \to P]$. $\square$

Corollary 10.7. *Let* $\mathbf{X}$ *be a complex in* B. *If* A *has enough injectives, then there is a complex* $\mathbf{I}$ *of semi-injective objects in* B *and a strict isomorphism* $\mathbf{X} \to \mathbf{I}$. *If* A *has enough projectives, then there is a complex* $\mathbf{P}$ *of semi-projective objects in* B *and a strict isomorphism* $\mathbf{P} \to \mathbf{X}$. *(No boundedness conditions needed.)*

10.2. **Derived equivalence of $\mathfrak{Ch}^b(\mathsf{B})$ and $\mathfrak{Ch}^b(\mathsf{A}, \mathcal{T}, \mathcal{F})$.** In this subsection we will need A to have either enough injectives or enough projectives. Let us say that A has enough projectives. For $\mathbf{X}, \mathbf{Y} \in \mathfrak{Ch}^{st}(\mathsf{B})$ we define

$$\mathfrak{RHom}_{\mathfrak{Ch}^{st}(\mathsf{B})}(\mathbf{X}, \mathbf{Y}) := \mathfrak{Hom}_{\mathfrak{Ch}^{st}(\mathsf{B})}(\mathbf{P}, \mathbf{Y})$$

where $\mathbf{P} \to \mathbf{X}$ is a semi-projective resolution as in Corollary 10.7. If $\mathbf{P}'$ is another semi-projective resolution for $\mathbf{X}$, it follows from Lemma 3.1 that there is a canonical strict isomorphism $\mathbf{P}' \xrightarrow{\sim} \mathbf{P}$ over $\mathbf{X}$, with a strict inverse $\mathbf{P} \xrightarrow{\sim} \mathbf{P}'$. Therefore, $\mathfrak{RHom}_{\mathfrak{Ch}^{st}(\mathsf{B})}(\mathbf{X}, \mathbf{Y})$ is well-defined up to a canonical *isomorphism*.

Proposition 10.5 implies that there is an isomorphism of K-complexes

$$\mathfrak{RHom}_{\mathfrak{Ch}^{st}(\mathsf{B})}(\mathbf{X}, \mathbf{Y}) \cong \mathfrak{Hom}_{\mathfrak{Ch}(\mathsf{B})}(\mathbf{X}, \mathbf{Y}).$$

In other words, the inclusion $\mathfrak{Ch}^{st}(\mathsf{B}) \hookrightarrow \mathfrak{Ch}(\mathsf{B})$ is a "derived" equivalence; here "derived" refers to the multiplicative system $\mathcal{S}_{sis}$ and not $\mathcal{S}_{qis}$. By Proposition 10.1, we find that $\mathfrak{G}\colon \mathfrak{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F}) \to \mathfrak{Ch}(\mathsf{B})$ is a "derived" equivalence. Similar discussion is valid in the case where A has enough injectives. We summarize this in the following theorem.

Theorem 10.8. *Assume that* A *has either enough injectives or enough projectives. Then, we have a natural "derived" equivalence*

$$\mathfrak{Ch}(\mathsf{B}) \cong \mathfrak{Ch}(\mathsf{A}, \mathcal{T}, \mathcal{F})$$

of DG categories. (Here, "derived" refers to the multiplicative system $\mathcal{S}_{sis}$*.)*

From this we deduce the DG version of Theorem 7.3.

Theorem 10.9. *Assume that* A *has either enough injectives or enough projectives. Assume further that* B *has enough injectives (respectively, enough projectives). Let* $* = +, b$ *(respectively,* $* = -, b$*.) Then, we have a derived equivalence*

$$\mathfrak{Ch}^*(\mathsf{B}) \cong \mathfrak{Ch}^*(\mathsf{A}, \mathcal{T}, \mathcal{F})$$

of DG categories. (Here, "derived" refers to either of the two multiplicative systems $\mathcal{S}_{sis}$ *or* $\mathcal{S}_{qis}$*.)*

References

[GeMa] S. I. Gelfand, Y. I. Manin, *Methods of homological algebra*, Springer-Verlag, Berlin, 1996.

[HaReSm] D. Happel, I. Reiten, S. O. Smalø, *Tilting in abelian categories and quasitilted algebras*, Mem. Amer. Math. Soc. 575.

[No] B. Noohi, *On weak maps between 2-groups*, preprint, arXiv:math/0506313v2 [math.CT].

[We] C. A. Weibel, *Introduction to homological algebra*, Cambridge Studies in Advanced Mathematics 38, Cambridge University Press, 1994.

E-mail address: behrang@alum.mit.edu

Mathematics Department, Florida State University, 208 Love Building, Tallahassee, FL 32306-4510, U.S.A.