ALMOST COMPLEX STRUCTURES WHICH ARE COMPATIBLE WITH KÄHLER STRUCTURE

Frank Connolly
Lê Hông Vân
Kaoru Ono

Department of Mathematics
University of Notre Dame
Notre Dame, IN 46556

USA

Max-Planck-Institut für Mathematik
Gottfried-Claren-Str. 26
D-53225 Bonn

GERMANY

Department of Mathematics
Ochanomizu University
Otsuka Tokyo 112

JAPAN

MPI 95-139

ALMOST COMPLEX STRUCTURES WHICH ARE COMPATIBLE WITH KÄHLER STRUCTURE

FRANK CONNOLLY (*), LÊ HÔNG VÂN (**), KAORU ONO(***)

December 7, 1995

(*) Department of Mathematics
University of Notre Dame
Notre Dame, IN 46556, USA,
(**)1 Max-Planck-Insitut für Mathematik
Gottfried-Claren-Straße 26
53225 Bonn, Germany,
(***)2 Department of Mathematics
Ochanomizu University
Otsuka Tokyo 112 Japan

§1. Introduction.

On a symplectic manifold (M^{2n}, ω) there is an almost complex structure J_ω compatible to ω (i.e. $\omega(J_\omega x, J_\omega y) = \omega(x, y)$ and $\omega(x, J_\omega x) > 0$). It is well-known that the homotopy class $[J_\omega]$ is a symplectic invariant of (M^{2n}, ω). The questions we are concerned in this note are

S: Given a homotopy class $[J]$ of an almost complex structure on a compact 4-manifold M^4 is there a symplectic structure ω which is compatible with $[J]$?

K: An analogous question for the existence of a compatible Kähler structure.

Remark. We would like to mention some results related to the questions S and K.

1) A recent result of Taubes [T1] states that, a necessary condition for the existence of such a compatible $[J]$ is that the Seiberg-Witten-Taubes (SWT) invariant of the canonical $spin^c$-structure associated to J must be ± 1 (see the next section for more details).

[1]supported by DFG/ Heisenberg Programm
[2]supported in part by Encouragement for Young Scientists, the Ministry of Education, Science and Culture, Japan

2) Using Yang-Mills Instanton theory Donaldson showed that there is a homotopy class of almost complex structures on $K3$ surfaces which does not contain any complex structure [D 1].

3) Hirzebruch conjectured that complex structures on $S^2 \times S^2$ and $\mathbf{C}P^2 \# \overline{\mathbf{C}P^2}$ are unique up to diffeomorphisms. This conjecture was recently proved by Friedmann and Qin [F-Q]. Thus the existence of an almost complex structure which is not compatible with Kähler structure on Hirzebruch's surfaces follows straightforward from their result combined with an argument in [D 1]. A similar classification theorem of symplectic structures on minimal rational and ruled surfaces was very recently proved by Taubes (for $\mathbf{C}P^2$) [T2] and Lalonde and McDuff [L-M] (see also [L-L], [O-O]).

In [D2] Donaldson showed that there is a free involution p on the set of homotopy classes of almost complex structures on a compact oriented closed manifold M^4. Using this we shall prove the following theorems

Theorem 1. *Let M^4 be a closed oriented manifold with $b_+^2 = 1$. Suppose that a homotopy class $[J]$ on M^4 is compatible with a Kähler structure. Then the homotopy class $p[J]$ is not compatible to any Kähler structure.*

Theorem 2. *Let M^4 be an oriented minimal rational or ruled surface. Suppose that a homotopy class $[J]$ is compatible with symplectic structure. Then the homotopy class $p[J]$ is not compatible with any symplectic structure.*

As an immediate corollary we see that for a manifold M^4 considered in Theorem 1 the action of the orientation preserving diffeomorphism group of M^4 on the set of homotopy classes of almost complex structures is not transitive.

A proof of our theorems will be given in section 3. In section 2 some facts on almost complex structures on 4-manifolds and the Seiberg-Witten equation (which is the main tool of our proof) will be collected.

Acknowledgement. The authors are grateful to A. Stipsicz who brought the author's attention to the cited Donaldson work [D1]. The first two authors thank the Max-Planck-Insitut for the hospitality during their stay there and they thank J. Morgan for his introductory lecture in the Seiberg-Witten theory in MPI and stimulating and enlightening discussions.

§2. Preliminaries.

2.1. Homotopy classes of almost complex structures on an oriented closed 4-manifold.

a) It is a classical result due to Ehresmann-Wu that two cohomology classes $c_1 \in H^2(M^4, \mathbf{Z})$ and $c_2 \in H^4(M^4, \mathbf{Z})$ are the first and second Chern classes of an almost complex structure J compatible with the given orientation on M^4 if and only if c_1 and c_2 satisfy the following conditions

$$c_2 = e(M^4), \tag{1}$$

$$c_1 = w_2(M) \, mod \, 2, \tag{2}$$

$$c_1^2 = 3\tau(M) + 2e(M), \tag{3}$$

where e denotes the Euler class, w_2 the second Whitney class, and τ the signature of M^4.

b) To answer the question: how many homotopy classes of almost complex structures on M^4 exist with a given "admissible" c_1 class, we can use the obstruction theory (see e.g. [K]). In [D 1], Donaldson detected difference of homotopy classes of almost complex structures in terms of a cohomological orientation. Namely, he considered the elliptic operator

$$\delta := d^* \oplus d^+ : \Omega^1 \to (\Omega^0 \oplus \Omega^2_+).$$

Using Hodge theory one can show that the kernel (and cokernel) of δ equals $H^1(M^4, \mathbf{R})$ (corr. $H^0(M^4, \mathbf{R}) \oplus H^2_+(M^4, \mathbf{R})$). An orientation of $det\, H^1(M^4, \mathbf{R}) \otimes det(H^0(M^4, \mathbf{R}) \oplus H^2_+(M, \mathbf{R}))$ of an oriented 4-manifold M is called a cohomological orientation. Given an almost complex structure J on M^4 we can deform operator δ to an complex linear operator $\delta_J^{1/2} = \frac{1}{2}(\delta - J\delta J)$. Thus $\delta_J^{1/2}$ gives a canonical way to define a cohomological orientation of M^4 preferred by $[J]$.

Fact 2.1.c [D1, D2]. *Given an admissible $c_1 \in H^2(M^4, \mathbf{Z})$ there are exactly two homotopy classes $[J]$ and $p([J])$ such that the cohomological orientations preferred by $[J]$ and $p([J])$ are opposite. If M^4 is Kähler, $H^1(M^4, \mathbf{R})$ has the canonical orientation defined by the complex structure and the choice of cohomological orientation preferred by $[J]$ is determined by $[\omega]$, because $H^2_+(M)$ is isomorphic to $H^{1,1}(M)_{\mathbf{R}} \oplus H^{0,2}(M)$ as real vector spaces and $H^{0,2}(M)$ is a complex vector space.*

If ω is an closed 2-form on M^4 then it induces a 2-form Q_ω on the linear space $H^1(M^4, \mathbf{R})$ as follows

$$Q_\omega(\alpha, \beta) = -\int_M \alpha \wedge \beta \wedge \omega. \tag{4}$$

If ω is a Kähler form then Q_ω is the Hodge-Riemann bilinear form (see e.g. [W]). Thus a Kähler form ω defines a symplectic form on $H^1(M^4, \mathbf{R})$ and therefore induces a natural orientation on it. We get easily the following observation

Remark 2.1.d. *Let (M^4, J, ω) be a Kähler manifold. Then the orientations on $H^1(M, \mathbf{R})$ defined by J and ω concide.*

2.2. Seiberg-Witten equation for symplectic 4-manifolds. (see [H], [K-M], [T1, T2]). Let us recall that the Seiberg-Witten equation for a *spin*^{c}-structure on a Riemannian 4-manifold M^4 is the pair of the following equations for A and a positive half spinor ϕ.

$$D_A(\phi) = 0 \qquad (SW1)$$

$$F_A^+ = q(\phi), \qquad (SW2)$$

where A is a connection on the associated line bundle of the *spin*^{c}-structure and q a quadratic form with value in $i\Omega^2_+(M^4)$. We can also perturb the Seiberg-Witten (SW) equation by adding a term $\mu \in i\Omega^2_+(M^4)$ in the second equation SW 2. If $b_2^+(M^4) \geq 2$ the "number"(or cobordism type of moduli space) of the solutions to (SW 1-2) (actually to any its perturbed equation) does not depend on metric g and therefore defines, roughly speaking, the Seiberg-Witten invariant of the *spin*^{c}-structure on M. If $b^2_+(M^4) = 1$ for each *spin*^{c}-structure there are exactly two chambers in the space of pairs (g, μ) of a metric and a perturbation such that the "number " of the solutions of SW-equation with respect to the metric g and perturbation μ depends only on the chamber to which the pair (g, μ) belongs. The wall dividing these two chambers is defined by the equation

$$\int (c_1(L) - \frac{i\mu}{2\pi})\omega_g = 0$$

where ω_g is the unique (up to scalar) self dual harmonic form on M^4 and L is the associated line bundle of the *spin*^{c}-structure. If $b_1(M) = 0$ then one has a (relatively simple) wall-crossing formula which relates the difference of the Seiberg-Witten invariant in two chambers [K-M]. In short it says that the difference is ± 1. A general formula in the case $b_1 \neq 0$ is well-known to specialists and can be found, for instance, in [O-O].

For a symplectic manifold (M^4, ω) (or more generally, for an almost complex manifold M^{2n}) we always have a choice of the canonical *spin*^{c}-structure S_{can} because there is a natural inclusion $U_n \to Spin^c_{2n}$. Taubes proved [T 1] that the Seiberg-Witten invariant of the canonical *spin*^{c}-structure with respect to the perturbation $\mu = ir\omega$, when r is big enough and ω is a symplectic form, (we will call it SWT-invariant), is always ± 1. It is easy to see that once we fix a *spin*^{c}-structure S on (M^4, ω) the pairs $(g, r\omega)$ and $(g', r\omega)$ are always in the same chamber if r is big enough and g and g' are metrics compatible to ω. Thus the SWT invariant is well-defined for any *spin*^{c}-structure on a symplectic manifold (M^4, ω).

§3 Proof of the Theorems.

Lemma 3.1. *Let J be an almost complex structure on M^4. Then the canonical $spin^c$-structures defined by $[J]$ and $p[J]$ are equivalent.*

Proof. Without lost of generality we can assume that two almost complex structures J and $p(J)$ coincide outside a ball B_1 of a point and inside B_1 the complex structure J is standard. Then we have a natural identification of two $spin^c$-structures outside of the ball. Let a complex line bundle L be the difference of these two $spin^c$-structures. Take a bit bigger open ball B and make a reduction along the boundary $\partial\bar{B}$. So we get a complex projective plane $\mathbf{C}P^2(B)$ which is a compactification of the open ball B. Clearly J and $p(J)$ descend to two almost complex structures on $\mathbf{C}P^2(B)$, which coincide near the complex line $\mathbf{C}P^1$ obtained by the reduction of the boundary $\partial\bar{B}$. In the same way the complex line bundle L descends to a complex line bundle L' which represents the difference of the two $spin^c$-structures on $\mathbf{C}P^2(B)$. By the construction L' has a section near $\mathbf{C}P^1$. Since the second cohomology of the complex projective plane is detected by the complex line, we have that L' is trivial. This implies the triviality of L. So the two $spin^c$-structures on M^4 are equivalent. □

Lemma 3.2. *Let M^4 be a symplectic manifold with $b_+^2 = 1$. Let ω_1 and ω_2 be symplectic forms on M^4 such that $c_1(J_{\omega_1}) = c_1(J_{\omega_2})$. Suppose that $b_1(M^4) = 0$. Then ω_1 and ω_2 are in the same connected component of the positive cone in $H^2(M;\mathbf{R})$. In particular $SWT(S,\omega_1) = \pm SWT(S,\omega_2)$ for all $spin^c$-structure S on M^4.*

Proof. Let Met be the space of metrices on M and E be the line bundle on it whose fibre $\pi^{-1}(g)$ consists of self-dual harmonic forms. Since Met is contractible there are exactly two section $s^\pm$ of E with $|s^\pm(g)|_g = \sqrt{2}$, here $|.|_g$ denotes the norm naturally induced from g. If g is a metric compatible with ω then $s^\pm(g)$ equals ω up to sign. We claim that ω and ω' (up to a positive scalar) are in the same image $s^+(Met)$ or $s^-(Met)$. Suppose the contrary. By lemma 3.1 the two canonical $spin^c$-structures defined by ω_1 and ω_2 coincide and we denote it by S_{can}. The wall crossing formula [K-M] tells us that $SWT(S_{can},\omega_1)$ and $SWT(S_{can},\omega_2)$ have different parity. On the other hand Taubes theorem says that $SWT(S_{can},\omega) = \pm 1$. Thus we get a contradiction. Hence follows the first claim of Lemma 3.2. The second one follows from the fact that $(\omega_1, r\omega_1)$ and $(\omega_2, r\omega_2)$, if r is big enough, are in the same chamber for all $spin^c$-structure S. □

Proof of Theorem 1. By Noether's theorem if M^4 is a minimal surface of general type with $p_g = 0$ then $q(M) = 0$. Hence if M^4 is Kähler with $b_+^2(M^4) = 1$, $b_1 \neq 0$ then M^4 must be an irrational ruled surface or an elliptic surface. Thus by the Enriques-Kodaira classification of complex surfaces (see e.g. [BPV, p. 188]) it suffices to prove Theorem 1 in the following cases.

CASE A: $b_1 = 0$.

CASE B. M^4 is an irrational ruled surfaces.

CASE C. M^4 is a hyperelliptic surface.

CASE A. In this case Theorem 1 follows directly from Lemma 3.2 and the fact that the choice of the preferred cohomological orientation of a Kähler manifold with $b_+^2 = 1$ is the choice of the connected component of H_+^2 containing ω (see Fact 2.1.c).

CASE B. If we imitate the argument in case A here then there are two problems arising from the condition $b_1(M^4) \neq 0$. As for the wall-crossing formula, we note that the ruled surfaces admits a positive scalar curvature metric g_0 therefore the two chambers for the canonical $spin^c$-structure on M have representatives $(g_0, \mu = 0)$ for one chamber and a pair of a metric compatible to ω and Taubes perturbation $\mu = r\omega$ for the other. Thus ω and ω' should be in the same connected component of the positive cone. The second problem is related to the preferred orientation of $H^1(M, \mathbf{R})$.

Lemma 3.3. *Suppose that $b_2^+(M) = 1$ and ω and ω' are two Kähler forms in the same connected component of the positive cone. Then the orientations defined by ω and ω' on $H^1(M^4, \mathbf{R})$ are the same.*

Proof. Our argument is similar to that in [O]. Note that for α, $\beta \in H^1(M; \mathbf{R})$, $\alpha \wedge \beta$ lies in the null-cone of $H^2(M, \mathbf{R})$. Consider a path $\{\omega_t\}$ in the positive cone from ω to ω'. Then we have a one-parameter family of bilinear forms Q_{ω_t}. If these bilinear forms are all non-degenerate, then the orientations determined by Q_{ω_t} are constant. Thus Lemma 3.2 is a consequence of the following fact.

Suppose that A and B are in the closure of a connected component of positive cone. Then $A \cdot B \geq 0$. Moreover if $A^2 > 0$ then the equality $A.B = 0$ holds if and only if $B = 0$. This fact can be easily proved by considering an orthogonal decomposition of A and B as follows: $A = a_0 x_0 + \sum_{i \geq 1} a_i x_i$, $B = b_0 x_0 + \sum_{i \geq 1} b_i x_i$. Here x_0 is a unit vector in $H_+^2(M, \mathbf{R})$ and $\{x_i, | i \geq 1\}$ is an orthonormal basis in $H_-^2(M, \mathbf{R})$. The desired fact follows by applying the Cauchy inequality to the RHS of the following inequality:
$a_0 b_0 \geq \sqrt{\sum_{i>0} a_i^2} \sqrt{\sum_{i>0} b_i^2}$. □

Clearly Lemma 3.3 and the fact that ω and ω' are in the same connected component of $H_+^2(M^4)$ contradict to the Donaldson theorem on the preferred cohomological orientation for Kähler 4-manifolds (see Remark 2.1.c). Thus case B is done.

CASE C. M^4 is a hyperelliptic surface. In this case $b_1(M^4) = 2$. We consider two subcases.

1) Suppose that ω and ω' are in the same connected component of the positive cone. The same argument as before tells us that the cohomological orientations defined by J and $p[J]$ are the same, which contradicts to a theorem of Donaldson.

2) Suppose that ω and ω' are in different connected components of the positive cone. Since $b_1(M^4) = 2$, Lemma 3.3 tells us that the orientations on $H^1(M^4, \mathbf{R})$ induced by ω and ω' are opposite. Thus the cohomological orientations defined by J and $p[J]$ are the same, which is a contradiction. This completes the proof.

□

To prove Theorem 2 we need a classification of symplectic structure up to deformation equivalence on rational and ruled surfaces ([L-L], [O-O], [T2]). In fact, a stronger statement is known, which is due to Lalonde and McDuff.

Classification Theorem. *Suppose that M is diffeomorphic to a minimal rational or ruled surface. Then any symplectic form on M is diffeomorphic to a Kähler form.*

Proof of Theorem 2. Theorem 2 follows from Classification Theorem and Theorem 1 but we would like to present here alternative argument, which is independent from Theorem 1. We consider two cases depending on the first Betti number of M^4.

Case 1: $b_1(M^4) = 0$. Suppose that J is an almost complex structure which is compatible with a symplectic structure ω and $p([J])$ contains an almost complex structure which is compatible with a symplectic structure ω'. By the classification theorem, there is diffeomorphism $g \in Diff(M^4)$ such that $g_*([J]) = p([J])$. Note that J and $p(J)$ determines the same orientation on M. (Orientation here is not a cohomological orientation.) So g is an orientation preserving diffeomorphism of M. By Donaldson's theorem, $[J]$ and $p[J]$ gives different cohomological orientation. Hence g must reserve the cohomological orientation. Since g acts trivially on $H^0(M; \mathbf{R})$ and $H^1(M; \mathbf{R}) = 0$ by assumption of Case 1,g induces an orientation reversing automorphism of H^2_+. But $c_1(M) \neq 0$ and g preserves c_1 by the assumption, which implies that g must preserve the orientation of $H^2_+(M)$. We arrive at a contradiction.

Case 2: $b_1(M^4) \neq 0$, in this case M^4 is an irrational ruled surface. By the classification theorem it suffices to show there is no diffeomorphism g such that $g([J]) = p[J]$. To imitate the argument in case 1 it suffices to show that there is an natural orientation σ on $H^1(M, \mathbf{R})$ such that if g preserves $c_1 := c_1(J) = c_1(p[J])$ then g also preserves the orientation σ. Consider the skew-symmetric bilinear form Q_{c_1}, defined in (4) replacing

ω by c_1. It is easy to see that Q_{c_1} is actually a symplectic form on the vector space $H^1(M^4, \mathbf{R})$ with the desired property. Thus we choose σ the orientation induced by Q_{c_1}. □

We end up this note with following question.

Question. Suppose that M^4 has $b_1 = 0$ and $b_2^+ = 1$. Is there a homotopy class $[J]$ such that both the two homotopy classes $[J]$ and $p[J]$ are compatible with symplectic structure.

REFERENCES

[BPV] W. Barth, C. Perters, A. Van de Ven, *Compact complex surfaces*, Springer-Verlag 1984.

[D 1] S. Donaldson, *Polynomial invariant for smooth four-manifolds*, Topology 1990, 257-315.

[D2] S. Donaldson , *The orientation of Yang-Mills moduli spaces and 4-manifold topology*, J.D.G 26 (1987), 397-428

[F-Q] R. Friedmann and Z. Qin, *On complex surfaces diffeomorphic to rational surfaces*, Inv. Math. 120 (1995), 81-117.

[H] N. Hitchin, *Harmonic spinors*, Adv. in Math., 14 (1974), 1-55.

[K] M. A. Kervaire, *A note on obstructions and characteristic classes*, A.J.M. 81, (1959), 733-784.

[K-M] P. Kronheimer and T. Mrowka, *The genus of embedded surfaces in the projective plane*, Math. Res. Letters 1 (1994), 797-808.

[L-M] F. Lalonde and D. McDuff, *The classification of ruled symplectic 4-manifolds*, preprint 9/1995.

[L-L] Li and Liu, (Math. Res. Letters ?).

[O-O] H. Ohta and K. Ono, *Note on symplectic manifolds with* $b_2^+ = 1$, *II*, preprint 9/1995.

[O] K. Ono, *Note on Ruan's example in symplectic topology*, preprint 1992.

[T 1] C. Taubes, *The Seiberg-Witten invariant and symplectic forms*, Math. Res. Letters 1 (1994), 809-822.

[T 2] C. Taubes *The Seiberg-Witten and Gromov invariants*, Math. Res. Letters 2 (1995), 221-238.

[W] R. O. Wells, *Analysis on complex manifolds*, Springer-Verlag 1980.