A PARAMETRIZED INDEX THEOREM FOR THE ALGEBRAIC K-THEORY EULER CLASS (PARTS I AND II)

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Abstract. A Riemann–Roch theorem is a theorem which asserts that some algebraically defined wrong–way map in K–theory agrees or is compatible with a topologically defined one [BFM]. Bismut and Lott [BiLo] proved a Riemann–Roch theorem for smooth fiber bundles in which the topologically defined wrong–way map is the homotopy transfer of Becker–Gottlieb and Dold. We generalize and refine their theorem. In the process, we prove a family index theorem for fiber bundles with compact topological manifold fibers, a theorem in which the relevant index equation involves algebraic K–theory. Our methods enable us to make a "universal" choice of algebraic K–theory for such bundles.

0. Introduction

The classical setting. Suppose that $p\colon E \to B$ is a bundle of smooth compact manifolds. By this we mean that p comes with an atlas of local trivializations $\phi_i\colon p^{-1}(U_i) \to U_i \times M_i$ where each M_i is a smooth compact manifold, and the changes of charts $\phi_j\phi_i^{-1}\colon (U_i \cap U_j) \times M_i \to (U_i \cap U_j) \times M_j$ are fiberwise smooth, and the induced maps $(U_i \cap U_j) \times T^r M_i \to (U_i \cap U_j) \times T^r M_j$ are continuous for each $r > 0$.

Let V be a complex vector bundle on E with discrete structure group (*flat* complex vector bundle for short), and V_i the complex vector bundle on B whose fiber over $b \in B$ is the local–coefficient homology group $H_i(p^{-1}(b); V)$. Denote by $K(\mathbb{C}_t)$ the K–theory of the topological ring $\mathbb{C}$, so that $K(\mathbb{C}_t) \simeq \mathbb{Z} \times \mathrm{BU}$ and for a space X the group of homotopy classes $[X, K(\mathbb{C}_t)]$ is the topological complex K–theory of X. The vector bundles V, V_i give classes $[V] \in [E, K(\mathbb{C}_t)]$, $[V_i] \in [B, K(\mathbb{C}_t)]$, and the Atiyah–Singer index theorem for families leads to the equation [BecSch]

$$\mathrm{tr}^*[V] = \sum (-1)^i [V_i] \quad \in [B, K(\mathbb{C}_t)] \ . \tag{0.1}$$

Here tr is the homotopy transfer of Becker–Gottlieb and Dold, which is a stable map from B_+ to E_+ determined by p, and $\mathrm{tr}^*[V]$ is the image of $[V]$ under the map on topological K–theory induced by tr.

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The Bismut–Lott theorem. Let $K(\mathbb{C})$ be the K–theory space of the discrete ring $\mathbb{C}$, in other words, the space whose homotopy groups are the algebraic K-groups of $\mathbb{C}$. Like $K(\mathbb{C}_t)$, the space $K(\mathbb{C})$ represents a cohomology theory, and the natural ring homomorphism from $\mathbb{C}$ with the discrete topology to $\mathbb{C}$ with the ordinary topology induces a map $K(\mathbb{C}) \to K(\mathbb{C}_t)$. Since the vector bundles V and V_i above are flat, the elements $[V]$ and $[V_i]$ lift back to elements in $[E, K(\mathbb{C})]$ and $[B, K(\mathbb{C})]$, respectively, which we still denote $[V]$ and $[V_i]$. It is natural to ask whether equation (0.1) holds in $[B, K(\mathbb{C})]$.

Bismut and Lott have given strong evidence that this is the case. There are certain characteristic classes for flat bundles, constructed by Kamber and Tondeur, which for odd k yield homomorphisms $c_k\colon [B, K(\mathbb{C})] \to H^k(B;\mathbb{R})$. The Riemann–Roch theorem of Bismut and Lott [BiLo] then states that equation (0.1) holds in $[B, K(\mathbb{C})]$ after the homomorphisms c_k are applied:

$$c_k\left(\mathrm{tr}^*[V]\right) = \sum (-1)^i c_k[V_i] \quad \in H^k(B;\mathbb{R}) \ . \tag{0.2}$$

To prove this Bismut and Lott use Bismut's local version of the Atiyah–Singer theorem for families. (They also assume that the fibers of p are *closed* manifolds, but we have been told that this was mostly to simplify the presentation.) See also [Lo].

An improvement. In this paper we show that the equation $\mathrm{tr}^*[V] = \sum(-1)^i[V_i]$ holds in $[B, K(\mathbb{C})]$, but we go a little beyond that, answering a question also raised by John Lott.

Suppose that $p\colon E \to B$ is a smooth fiber bundle with compact fiber F , as above. Let R be *any discrete ring.* Let V be a local coefficient system of f.g. projective left R–modules on E , in other words: a bundle of f.g. projective left R–modules on E. As above, V determines an element $[V] \in [E, K(R)]$. Each fiber $F_b = p^{-1}(b)$ of p has local–coefficient homology groups $H_i(F_b; V)$, and for fixed i these homology groups form a bundle V_i of left R–modules on B. We assume that the fibers of V_i are again projective, in which case they are also finitely generated. Each V_i determines an element $[V_i] \in [B, K(R)]$. In §8, especially 8.12, we show

$$\mathrm{tr}^*[V] = \sum (-1)^i [V_i] \qquad \in [B, K(R)] \tag{0.3}$$

Smoothness is an essential hypothesis in (0.3) and (0.2). In another paper we shall show, again in answer to a question raised by John Lott, that (0.2) and hence (0.3) can fail for a fiber bundle whose fibers are closed *topological* manifolds.

We emphasize that both sides of equation (0.3) are fiber homotopy invariants of $p\colon E \to B$. This is obvious for the right–hand side. It is true for the left–hand side because there exists a fiber homotopy invariant definition of the transfer tr. See [BeGo1] ; compare the older definition in [BeGo2]. Thus, (0.3) expresses a fiber homotopy theoretic property of bundles of smooth compact manifolds.

Like the Bismut–Lott proof of (0.2), our proof of (0.3) uses a family index theorem. It is an index theorem of Hopf–Pontryagin type, that is, an index theorem stating that Poincaré duals of certain generalized Euler classes of tangent bundles of compact manifolds agree with certain generalized Euler characteristics of those manifolds.

Generalized Euler characteristics. The notion of generalized Euler characteristic that we use relies on the work of Waldhausen, especially [Wald]. Waldhausen associates with any space Y an infinite loop space $A(Y)$. This is designed to be the universal receptacle for Euler characteristics of retractive spaces over Y subject to certain relative finiteness conditions. (A retractive space over Y is a space X together with maps $r: X \to Y$ and $i: Y \to X$ such that $ri = \mathrm{id}_Y$.) In particular, if Y itself has certain finiteness properties, then $\mathbb{S}^0 \times Y$ viewed as a retractive space over Y satisfies those relative finiteness conditions, and the relative Euler characteristic of $\mathbb{S}^0 \times Y$ is defined. We denote it by $\chi(Y) \in A(Y)$, and think of it as the "absolute" Euler characteristic of Y. Note that it lives in a space $A(Y)$ which depends on Y.

The finiteness property Y that we require of Y is that it be *homotopy finitely dominated*; this means that there exists a compact CW–space Z and maps

$$f: Z \to Y \ , \qquad g: Y \to Z \ ,$$

such that fg is homotopic to id_Y. The relative finiteness conditions that we impose on retractive spaces over Y are of a similar nature, with the result that $\pi_0 A(Y)$ is isomorphic to $K_0(\mathbb{Z}\pi_1(Y))$ in the case where Y is path connected and based. In that case

$$K_0(\mathbb{Z}\pi_1(Y)) \cong \mathbb{Z} \oplus \tilde{K}_0(\mathbb{Z}\pi_1(Y))$$

and the component of $\chi(Y)$ is the sum of the ordinary Euler characteristic of Y, in $\mathbb{Z}$, and the Wall finiteness obstruction [Wa] of Y, in the reduced K_0–group of the ring $\mathbb{Z}\pi_1(Y)$.

The characteristic $\chi(Y)$ depends "continuously" on Y in the following sense. If $p: E \to B$ is any fibration with homotopy finitely dominated fibers, then we can evaluate χ fiberwise. We obtain a section $\chi(p)$ of another fibration, $A_B(E) \to B$, whose fiber over $b \in B$ is $A(p^{-1}(b))$.

Generalized Euler classes. The generalized Euler class in our index theorem is a class $[e_n]$ in the n–th cohomology of the pair $(B\,\mathrm{TOP}(n), B\,\mathrm{TOP}(n-1))$ with twisted coefficients in the spectrum $\mathbf{A}(*)$ associated with the infinite loop space $A(*)$. Here $B\,\mathrm{TOP}(n)$ is the classifying space for euclidean n–bundles, fiber bundles with fibers homeomorphic to $\mathbb{R}^n$. We work at this level of generality because our index theorem is for topological manifolds, and tangent bundles of topological manifolds are euclidean bundles. Compare [Ki].

Assembly. The final ingredient in our index theorem is the assembly transformation [WWa]. Let $\mathbf{A}(Y)$ be the spectrum associated with the infinite loop space $A(Y)$. Assembly is essentially a natural transformation $\alpha: Y_+ \wedge \mathbf{A}(*) \to \mathbf{A}(Y)$. It agrees with the obvious identification $*_+ \wedge \mathbf{A}(*) \cong \mathbf{A}(*)$ when $Y = *$, and is essentially characterized by that property. (Every functor from spaces to spectra which respects homotopy equivalences has such an assembly transformation.) We like to think of α as a map of infinite loop spaces, from the space $A^{\%}(Y) := \Omega^\infty(Y_+ \wedge \mathbf{A}(Y))$ to $A(Y)$.

(Most index theorems involve some map from the home of the symbols to the home of the indices. The map is usually some form of assembly, or assembly following on Poincaré duality, depending on whether those symbols are viewed as representatives of generalized homology or cohomology. However, it is not always easily identified as such. See for example [BD1], [BD2] and the reformulation of the Mishchenko–Fomenko index theorem [MF] in [R1,3.1],[R2,3.3],[R3, 2.2].)

The index theorem. Assume now that $p: E \to B$ is a bundle of compact *topological* manifolds, possibly with boundary. Let τ be the vertical tangent bundle pair of p. The fiberwise Poincaré dual of $[e_n](\tau)$ turns out to be a vertical homotopy class of sections of a fibration $A^{\%}_B(E) \to B$ with fiber $A^{\%}(p^{-1}(b))$ over $b \in B$. The index theorem states that fiberwise assembly $\alpha: A^{\%}_B(E) \to A_B(E)$ takes the fiberwise Poincaré dual of $[e_n](\tau)$ to the fiberwise characteristic, up to a vertical homotopy:

$$\alpha\wp[e_n](\tau) \quad \simeq \quad \chi(p)\,. \tag{0.4}$$

A vanishing theorem. Becker [Be] has defined a *stable cohomotopy Euler class*, living in n–th cohomology of the pair $(B\,\mathrm{O}(n), B\,\mathrm{O}(n-1))$ with twisted coefficients in the sphere spectrum $\mathbf{S}^0$. Using the unit map $\mathbf{S}^0 \to \mathbf{A}(*)$ determined by the point $\chi(*) \in A(*) = \Omega^\infty \mathbf{A}(*)$, we can view $[b_n]$ as a class with twisted coefficients in the spectrum $\mathbf{A}(*)$. It turns out that

$$[b_n] = j^*[e_n] \tag{0.5}$$

where $j : (B\,\mathrm{O}(n), B\,\mathrm{O}(n-1)) \longrightarrow (B\,\mathrm{TOP}(n), B\,\mathrm{TOP}(n-1))$ is the inclusion. The unit map $\mathbf{S}^0 \to \mathbf{A}(*)$ has a canonical splitting up to homotopy, so that (0.5) identifies the stable cohomotopy component of $[e_n]$ applied to any vector bundle (pair), and implies the vanishing of the other component. The reasoning behind (0.5) is that the forgetful passage from vector bundles to euclidean bundles factors through disk bundles, bundles with fibers homeomorphic to $\mathbb{D}^n$; and we can learn something about those disk bundles by applying (0.4) with p equal to the disk bundle projection. Briefly, (0.5) is a distant corollary of (0.4).

Outline of proof of (0.3). One needs to know that the fiberwise Poincaré dual of $[b_n](\tau)$, viewed as a vertical homotopy class of sections of a fibration on B with fiber $Q(p^{-1}(b)_+)$ over $b \in B$, refines the Becker–Gottlieb–Dold transfer, viewed as a homotopy class of maps $B \to Q(E_+)$. (This refinement is still a fiber homotopy invariant of $p: E \to B$.) Combining (0.4) with (0.5) gives

$$\alpha\wp[b_n](\tau) \quad \simeq \quad \chi(p)\,. \tag{0.6}$$

Applying an appropriate map of infinite loop spaces $A_B(E) \to K(R)$ determined by the module bundle V to the two sides of equation (0.6), we obtain (0.3).

This outline of proof suggests that (0.6) should be viewed as essentially a special case of (0.3), the universal case — with suitably generalized notions of ring and flat vector bundle. We therefore sometimes refer to (0.6) as the *universal Riemann–Roch formula* for bundles of smooth compact manifolds.

Further remarks. • The two sides of equation (0.6) are defined for *any* fibration with homotopy finitely dominated fibers, $p: E \to B$. It is therefore tempting to look for a geometric characterization of those fibrations $p: E \to B$ for which the universal Riemann–Roch equation (0.6) is a *true* equation. We will show in part III of this work [DWWIII] that (0.6) holds for a fibration $p: E \to B$ with homotopy finitely dominated fibers if and only if p is fiber homotopy equivalent to a bundle of smooth compact manifolds. This converse depends very strongly on Waldhausen's work relating stabilized h–cobordism theory (alias concordance theory, alias pseudoisotopy theory) to algebraic K–theory.

• Our construction of the characteristic class $[e_n]$ is designed to give the shortest possible proof of the index theorem (0.4). There is an alternative description, given in [We], in which $[e_n]$ appears as the stable part of the obstruction to splitting off trivial line bundles from euclidean n–bundles. For motivation we recall that Becker designed his characteristic class $[b_n]$ to have the following property:

> *Let ξ be an n–dimensional vector bundle on a CW–space X of dimension $\leq 2n-4$. Then $[b_n](\xi) = 0$ if and only if ξ has a vector bundle splitting $\xi \cong \zeta \oplus \varepsilon$, where ε is a trivial line bundle.*

Becker found that the correct coefficient spectrum for a characteristic class $[b_n]$ with this property had to be a spectrum with i–th term $\Sigma(\mathrm{O}(i)/\mathrm{O}(i-1))$, in other words the sphere spectrum. By analogy, in the alternative definition of $[e_n]$ the coefficient spectrum has i–th term $\Sigma(\mathrm{TOP}(i)/\mathrm{TOP}(i-1))$. Again Waldhausen's theory is needed to identify the coefficient spectrum with $\mathbf{A}(*)$.

• After reading a preliminary version of the present paper, Waldhausen found another proof of (0.6) for bundles of smooth compact manifolds, and the "converse" mentioned above. His proof is based on his manifold approach to (his) algebraic K–theory of spaces. Compared with our proof, it has some advantages and disadvantages. The main advantage is that it is very direct, for a reader who knows his way through Waldhausen's work on h–cobordism theory and the algebraic K–theory of spaces. The main disadvantage is that it does not make the connection with index theory, and as a result does not suggest the generalizations that our proof suggests. The alternative proof will be sketched in Part III.

Section headings

Guide. In §1 we introduce the notion of a characteristic for a functor F from a small category to $\mathcal{C}$ to spaces. We use it to explain how certain generalized Euler characteristics can be evaluated on "families" of spaces, such as the fibers of a fibration. What one should have in mind here is the A–theory Euler characteristic described above.

In §§2–5 we have a much narrower framework, as follows. Let $\mathcal{E}^\bullet$ be the category of euclidean neighborhood retracts (ENR's), where a morphism from X to Y is a proper map from an open subset V of X to Y. Such a morphism is a *localization* if the underlying map $V \to Y$ is cell–like [Lac1–3]. The localization morphisms form a subcategory $\ell\mathcal{E}^\bullet$ of $\mathcal{E}^\bullet$. We now fix a functor F from $\mathcal{E}^\bullet$ to pointed spaces which is pro–excisive ; this means that the abelian group valued functor $\pi_* F$ is a locally finite generalized homology theory on $\mathcal{E}^\bullet$. We then fix a characteristic χ for $F|\ell\mathcal{E}^\bullet$.

The F that we have in mind in §§2–5 is the pro–excisive functor corresponding to $\mathbf{A}(*)$, the connective spectrum determined by the infinite loop space $A(*)$. The χ that we have in mind in §§2–5 is a refinement of the A–theory Euler characteristic outlined above — where that makes sense, i.e. for compact ENR's and cell–like maps between such. Our strategy is to prove the index theorem (0.4) by stating and proving a pre–assembly version of (0.4), a much sharper index theorem, in the abstract framework of pro–excisive functors F on $\mathcal{E}^\bullet$ and characteristics for $F|\ell\mathcal{E}^\bullet$.

The characteristic classes needed in the (sharper) index theorem are developed in §2. The index theorem itself occupies §3. In §4 we prove the vanishing theorem (0.5), in the abstract framework. §5 is about a pre–assembly version of (0.6), a consequence of §3 and §4.

In §6 and §7 we finally present those examples of characteristics that we had in mind in §1 and §§2–5, respectively. It is shown in §8 that the §7 example does indeed refine the main example of §6. This completes the proofs of (0.4) and (0.6). In §8 we show how (0.3) follows from (0.6), and discuss how (0.3) needs to be corrected to be valid for bundles of compact topological manifolds. In the last subsection of §8 we explain how our results give rise to a family Reidemeister torsion ; compare [BiLo] and [IgK1], [IgK2].

Conventions. • As in [Gd] we use the word *space* to mean a compactly generated Hausdorff space, unless otherwise stated. Products and mapping spaces are formed in the category of these spaces in the usual way. • All *cofibrations* are closed maps having the homotopy extension property. • A *homotopy cartesian square* is a commutative square of spaces in which the canonical map from the initial term to the homotopy pullback of the other three terms is a weak homotopy equivalence. • We tend to use boldface notation for spectra. Specifically, if X is an infinite loop space, then $\mathbf{X}$ is the default notation for the corresponding (-1)–connected spectrum.

Part I: An index theorem

1. Characteristics

Overview. A finitely dominated space X determines an Euler characteristic in the group $K_0(\mathbb{Z}\pi_1(X))$ whose image in the reduced K_0 group is the Wall finiteness obstruction. With a view to extracting Wall finiteness obstructions fiberwise, i.e. from a fibration with finitely dominated fibers, we propose an abstraction of the notion *Euler characteristic* which is applicable to *parametrized families* (of finitely dominated spaces, or other objects).

To keep notation simple, we work in an abstract setting which does not even mention algebraic K–theory. For illustrations with algebraic K–theory, see §6, which can be read right after §1.

Let $\mathcal{C}$ be a small category and let F be a functor from $\mathcal{C}$ to the category of spaces. For each C in $\mathcal{C}$ let $\mathcal{C}/C$ be the *over category* whose objects are the morphisms in $\mathcal{C}$ with codomain C. Later we will also need the *under category* $C/\mathcal{C}$ whose objects are the morphisms in $\mathcal{C}$ with domain C.

1.1. Definition. A *characteristic* for F is a natural transformation χ from the functor $C \mapsto |\mathcal{C}/C|$ to F.

Remarks. (i) The *space* of characteristics for F is exactly $\operatorname{holim} F$, the homotopy inverse limit of F.

(ii) For C in $\mathcal{C}$ let $\chi(C) := \chi(\mathrm{id}_C) \in F(C)$, the image under χ of the identity vertex in $|\mathcal{C}/C|$. In our favorite examples $\chi(C)$ is something like an Euler characteristic of C. It is a point in a space $F(C)$ which may depend on C.

1.2. Example. Let F be the constant functor with constant value equal to the classifying space $|\mathcal{C}|$. For each C in $\mathcal{C}$ we have the forgetful functor $\mathcal{C}/C \to \mathcal{C}$ which, on passage to classifying spaces, yields $\chi\colon |\mathcal{C}/C| \to F(C)$.

1.3. Example. Let $\mathcal{F}$ be a functor from $\mathcal{C}$ to the category of small categories. Suppose that we have a rule selecting, for each object C in $\mathcal{C}$, an object $C^!$ in $\mathcal{F}(C)$, and for each morphism $e\colon C \to D$ in $\mathcal{C}$, a morphism $e^!\colon e_*(C^!) \to D^!$ in $\mathcal{F}(D)$. We assume that $e^!$ is an identity morphism whenever e is an identity morphism and that the chain rule alias 1-cocycle condition is satisfied: $(ef)^! = e^! \cdot e_*(f^!)$ whenever e, f are composable.

In these circumstances we can define a functor F from $\mathcal{C}$ to spaces by $F(C) := |\mathcal{F}(C)|$. We can also define a characteristic for F, as follows. For fixed D in $\mathcal{C}$ the rule taking $f\colon C \to D$ to $f_*(C^!)$ is a functor from $\mathcal{C}/D$ to $\mathcal{F}(D)$, inducing $\chi\colon |\mathcal{C}/D| \to F(D)$.

Suppose again that F is a functor from $\mathcal{C}$ to spaces. The *homotopy colimit* of F, denoted $\operatorname{hocolim} F$, is the geometric realization of the simplicial space

$$[n] \mapsto \coprod_{u:[n]\to\mathcal{C}} Fu(0)\,.$$

See [BK]. When it is convenient we write $\operatorname{hocolim}_C F(C)$ or $\operatorname{hocolim} F(?)$ instead of $\operatorname{hocolim} F$. The homotopy colimit of F comes with a canonical map to $|\mathcal{C}|$, because $|\mathcal{C}|$ is the homotopy colimit of the terminal functor from $\mathcal{C}$ to spaces. A characteristic for F determines, *up to homotopy*, a section of the canonical map $\operatorname{hocolim} F \to |\mathcal{C}|$. More precisely:

1.4. Observation. *A characteristic χ for F induces a map from* $\operatorname{hocolim}|\mathcal{C}/?|$ *to* $\operatorname{hocolim} F$. *The composition* $\operatorname{hocolim}|\mathcal{C}/?| \to \operatorname{hocolim} F \to |\mathcal{C}|$ *is a homotopy equivalence.*

1.5. Remark. Suppose that χ is a *strict* characteristic, that is, an element in $\lim F \subset \operatorname{holim} F$, or equivalently, a natural transformation $* \to F(C)$ defined for C in $\mathcal{C}$. Then χ induces a map $|\mathcal{C}| \to \operatorname{hocolim} F$ which is a section of the projection $\operatorname{hocolim} F \to |\mathcal{C}|$.

If χ is not strict, then we can improve it to a strict characteristic in the following way. Let $F_1(C) = \operatorname{holim}_{C\to D} F(D)$ where the homotopy limit is taken over the category $C/\mathcal{C}$ of $\mathcal{C}$–objects under C. Since $F(C) \cong \lim_{C\to D} F(D)$, there is a natural map $v: F(C) \to F_1(C)$, the inclusion of a limit in the corresponding homotopy limit. The forgetful functor $f_C: C/\mathcal{C} \to \mathcal{C}$ given by $(C \to D) \mapsto D$ induces a map from $\operatorname{holim} F$ to $\operatorname{holim} F f_C = F_1(C)$. Let $\chi_1(C) \in F_1(C)$ be the image of $\chi \in \operatorname{holim} F$ under that map. Then χ_1 is a strict characteristic for F_1. It is not equal to $v_*\chi$, but there is a canonical path ω in $\operatorname{holim} F_1$ connecting $v_*\chi$ with χ_1.

Proof. It is enough to construct ω in the universal case where $F(C) = |\mathcal{C}/C|$ and $\chi_C: |\mathcal{C}/C| \to |\mathcal{C}|$ is the identity for each C. For this case we introduce $\operatorname{Ar}\mathcal{C}$, the category whose objects are the morphisms $f: A \to B$ in $\mathcal{C}$ and whose morphisms are commutative squares

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ A' \arrow[r, "f'"] & B' . \end{tikzcd}$$

Using remark (i) after 1.1, we can view points in $\operatorname{holim} F_1$ as collections of maps

$$|\mathcal{C}/d_f| \times |(d_f/\mathcal{C})/f| \longrightarrow |\mathcal{C}/c_f| ,$$

one for each f in $\operatorname{Ar}\mathcal{C}$ with domain d_f and codomain c_f. These collections are subject to naturality conditions which we need not spell out. Now $\chi_1 \in \operatorname{holim} F_1$ corresponds to the collection of maps which are induced by the functors p_f from $\mathcal{C}/d_f \times (d_f/\mathcal{C})/f$ to $\mathcal{C}/c_f$ taking $(g,(f_1,f_2))$ to f_2 ; here g is an object of $\mathcal{C}/d_f$ and f_1 , f_2 are composable morphisms in $\mathcal{C}$ with $f_2 f_1 = f$, so that the pair (f_1,f_2) is a typical object of $(d_f/\mathcal{C})/f$. Also $v_*\chi \in \operatorname{holim} F_1$ corresponds to the collection of maps induced by the functors q_f from $\mathcal{C}/d_f \times (d_f/\mathcal{C})/f$ to $\mathcal{C}/c_f$ which take $(g,(f_1,f_2))$ to $f_2 f_1 g$. For each f there is a natural transformation ω_f from q_f to p_f , given on an object $(g,(f_1,f_2))$ of $\mathcal{C}/d_f \times (d_f/\mathcal{C})/f$ by $f_1 g$, a morphism in $\mathcal{C}/c_f$ from $f_2 f_1 g$ to f_2. The natural transformations ω_f for $f \in \operatorname{Ar}\mathcal{C}$ determine a path ω from $v_*\chi$ to χ_1. $\square$

Suppose now that F takes all morphisms in $\mathcal{C}$ to weak homotopy equivalences. Then we like to think of points $\chi \in \operatorname{holim} F$ as sections of a (quasi–)fibration. The quasifibration is the projection $\operatorname{hocolim} F_1 \to |\mathcal{C}|$, and we obtain sections of it from characteristics for F by composing the homeomorphism

$$\operatorname{holim} F \longrightarrow \lim F_1 \quad ; \quad \chi \mapsto \chi_1$$

with the map $\lim F_1 \to \Gamma(\operatorname{hocolim} F_1 \to |\mathcal{C}|)$ just described. We can further compose with the inclusion $\Gamma(\operatorname{hocolim} F_1 \to |\mathcal{C}|) \longrightarrow \Gamma\tilde{}(\operatorname{hocolim} F_1 \to |\mathcal{C}|)$, where $\Gamma\tilde{}(\dots)$ is the section space of the associated fibration. It follows from [Dwy] that the composition

$$\operatorname{holim} F \longrightarrow \cdots \longrightarrow \Gamma\tilde{}(\operatorname{hocolim} F_1 \to |\mathcal{C}|)$$

is a weak homotopy equivalence. Summarizing, if F takes all morphism to homotopy equivalences, then $\operatorname{holim} F$ can be interpreted as a section space.

1.6. Main application. Suppose that $p: E \to B$ is a fibration, where B is the geometric realization of a simplicial set $B^\natural$. For a simplex x in $B^\natural$, let E_x be the pullback of E under the characteristic map $\Delta^{|x|} \to B$.

Suppose further that $\mathcal{C}$ is a full subcategory of the category of all spaces, with homotopy equivalences as morphisms; suppose that all the E_x are in $\mathcal{C}$. Let F be a functor from $\mathcal{C}$ to spaces taking all morphisms to homotopy equivalences.

Modulo canonical homotopy equivalences, we will create a fibration $F_B(E) \to B$ by applying F to the fibers of p. If F comes with a characteristic χ , we can evaluate it on the fibers of p to obtain a section $\chi(p)$ of $F_B(E) \to B$.

Details: Let $\operatorname{simp}(B)$ be the category of simplices of $B^\natural$; we should really call it $\operatorname{simp}(B^\natural)$. The objects are the simplices of $B^\natural$; a morphism $x \to y$ is a monotone map $f: \{0, 1, \dots, |x|\} \to \{0, 1, \dots, |y|\}$ for which $f^* y = x$. Then $x \mapsto E_x$ is a functor from $\operatorname{simp}(B)$ to $\mathcal{C}$. Let

$$\chi(p) \in \operatorname*{holim}_x F(E_x) \tag{1.7}$$

be the element determined by χ. Using the last line of 1.5, we can view $\chi(p)$ as a section of the quasifibration

$$\operatorname{hocolim} F_1(E_x) \to |\operatorname{simp}(B)| \tag{1.8}$$

(or of the associated fibration). We emphasize that (1.7) is our rigorous definition of $\chi(p)$. But we find the informal notation $F_B(E) \to B$ and $\chi(p): B \to F_B(E)$ for (1.8) and (1.7) more suggestive, and will use it when appropriate (not in proofs, as a rule).

2. Excisive characteristics and characteristic classes

Overview. In this section we introduce *excisive* characteristics $\chi(Y) \in F(Y)$ defined for euclidean neighborhood retracts Y. We assume therefore that F is defined on the category $\mathcal{E}^\bullet$ whose morphisms are pointed maps between one–point compactifications of euclidean neighborhood retracts, and that π_*F is a locally finite generalized homology theory. However, we do not assume that the characteristics $\chi(Y)$ behave naturally for *all* morphisms in $\mathcal{E}^\bullet$; they behave naturally only for what we call *localization* morphisms.

Again we work in an abstract setting to keep the notation simple. For illustrations with algebraic K–theory, see §9, which can be read right after §2.

Locally finite homology theories

A map $X \to Y$ between locally compact spaces is *proper* if it extends to a (continuous) pointed map $X^\bullet \to Y^\bullet$ of the one–point compactifications. Euclidean neighborhood retracts (ENR's) are locally compact, so we can make a category $\mathcal{E}$ whose objects are the ENR's and whose morphisms are the proper maps. There is a larger category $\mathcal{E}^\bullet$ with the same objects, where a morphism $X \to Y$ is a (continuous) pointed map $f: X^\bullet \to Y^\bullet$. We call such a morphism f a *localization* if it restricts to a cell-like map [Lac1–3] from $f^{-1}(Y) \subset X$ to Y. We denote the subcategory of $\mathcal{E}^\bullet$ consisting of all objects and the localization morphisms by $\ell\mathcal{E}^\bullet$.

Remark. A pointed map $f: X^\bullet \to Y^\bullet$ determines an open subset $V = f^{-1}(Y)$ of X, and a proper map $g = f|V$ from V to Y. Conversely, any such pair (V, g) with $V \subset X$ open and $g: V \to Y$ proper determines a pointed map $f: X^\bullet \to Y^\bullet$. As a morphism in $\mathcal{E}^\bullet$, the pointed map f is a localization if and only if g is cell–like.

2.1. Terminology (compare [WWx, 1.1]. A commutative square of locally compact spaces and maps

$$\begin{array}{ccc} X_1 & \longrightarrow & X_2 \\ \downarrow & & \downarrow \\ X_3 & \longrightarrow & X_4 \end{array}$$

is a *proper homotopy pushout square* if the resulting map from the homotopy pushout of $X_3 \leftarrow X_1 \to X_2$ to X_4 is a proper homotopy equivalence. A covariant functor F from $\mathcal{E}$ to pointed spaces is *homotopy invariant* if it takes proper homotopy equivalences to weak homotopy equivalences. A homotopy invariant F is *excisive* if it takes proper homotopy pushout squares in $\mathcal{E}$ to weak homotopy pullback squares (also called *homotopy Cartesian* squares) of spaces, and $F(X \times [0, \infty[)$ is weakly contractible (weakly homotopy equivalent to a point) for all X in $\mathcal{E}$.

Finally we call F *pro–excisive* if it is homotopy invariant, excisive, and satisfies an appropriate wedge axiom as follows. Suppose that $X = \amalg_{i\in\mathbb{N}} X_i$ in $\mathcal{E}$. For each $j \geq 0$ let $X_j^\sharp$ be the coproduct of X_j and all $X_i \times [0, \infty[$ for $i \neq j$. The obvious inclusions induce

$$\pi_*F(X) \to \pi_*F(X_j^\sharp) \leftarrow \pi_*F(X_j)$$

where the right–hand arrow is an isomorphism by excision. We use this as an identification, take the product over all j, and obtain $u: \pi_*F(X) \to \prod_{j\geq 0} \pi_*F(X_j)$. The wedge axiom we want stipulates that u be an isomorphism.

2.2. Remark. Suppose that F is pro–excisive. Then each $F(X)$ is an infinite loop space. Indeed, excision implies that the commutative diagram of pointed spaces

$$\begin{array}{ccc} F(X) & \longrightarrow & F(X \times [0,\infty[\,) \\ \Big\downarrow & & \Big\downarrow \\ F(X \times (-\infty,0]\,) & \longrightarrow & F(X \times \mathbb{R}) \end{array}$$

is a homotopy Cartesian square with contractible lower left and upper right term. Let $\mathbf{F}(X)$ be the CW–spectrum made from the spaces $F(X \times \mathbb{R}^i)$, using their singular simplicial sets and the method of [Gd, 0.1]. Then $\mathbf{F}$ is a functor from $\mathcal{E}$ to CW–spectra ; it is pro–excisive in the sense of [WWx, 1.1]. Conversely, if $\mathbf{J}$ from $\mathcal{E}$ to CW–spectra is pro–excisive as in [WWx, 1.1], then $\Omega^\infty \mathbf{J}$ is pro–excisive in the sense of 2.1. above. (Use [WWx, 2.1].) Here Ω^∞ of a spectrum can be defined as the geometric realization of the simplicial set of maps to that spectrum from the sphere spectrum.

Consequently, the classification theorem [WWx, 1.2] is also a classification theorem for pro–excisive functors as in 2.1 above. The result of the classification is that every pro–excisive F from $\mathcal{E}$ to pointed spaces is related by a chain of natural weak equivalences to the functor $X \mapsto \Omega^\infty(X^\bullet \wedge \mathbf{F}(*))$. (A natural transformation between functors F_1, F_2 from $\mathcal{E}$ to pointed spaces is a *weak equivalence* if the map $F_1(X) \to F_2(X)$ is a weak homotopy equivalence for every X in $\mathcal{E}$.)

2.3. More Terminology. A functor from $\mathcal{E}^\bullet$ to pointed spaces is *pro–excisive* if its restriction to $\mathcal{E}$ is pro–excisive. For such functors there is a similar classification theorem: every pro–excisive F from $\mathcal{E}^\bullet$ to pointed spaces is related by a chain of natural weak equivalences to the functor $X \mapsto \Omega^\infty(X^\bullet \wedge \mathbf{F}(*))$. See [WWx, 2.1]. If F from $\mathcal{E}^\bullet$ to pointed spaces is pro–excisive, then the functor $X \mapsto \pi_* F(X)$ on $\mathcal{E}^\bullet$ has all the properties that one expects from a *locally finite homology theory.*

2.4. Assumption. For the remainder of this section (and for use in §3, §4, §5, §6) we fix a pro–excisive functor F from $\mathcal{E}^\bullet$ to pointed spaces, and a characteristic χ for $F|\ell\mathcal{E}^\bullet$. We call such a characteristic an *excisive* characteristic.

Excisive characteristics for bundles

Let $p\colon E \to B$ be a fiber bundle whose fibers are compact topological manifolds. Assume that $B = |B^\natural|$ for a simplicial set $B^\natural$. Our goal is to associate with this setup a section up to homotopy $\chi(p)$ of $F_B(E) \to B$.

This is of course the informal notation of 1.6. The true home of $\chi(p)$ should be the homotopy inverse limit $\operatorname{holim}_x F(E_x)$, where x runs over the objects of $\operatorname{simp}(B)$. The space $\operatorname{holim}_x F(E_x)$ can be viewed as a section space ; see 1.5.

Unfortunately the characteristic χ does not determine a point in $\operatorname{holim}_x F(E_x)$. The difficulty stems from the fact that χ is a characteristic for $F|\ell\mathcal{E}^\bullet$, while $x \mapsto E_x$ is not a functor from $\operatorname{simp}(B)$ to $\ell\mathcal{E}^\bullet$ except in the trivial case $B = \emptyset$. Heavy guns are needed to overcome this difficulty.

Let $tB^\natural$ be the simplicial set whose k–simplices are pairs (x,θ) , where x is a k–simplex in $B^\natural$ and θ is an equivalence relation on E_x , with quotient space E_x^θ , such that the two projections make up a homeomorphism $E_x \to \Delta^{|x|} \times E_x^\theta$.

2.5. Theorem. *The forgetful map $tB^\natural \to B^\natural$ is a Kan fibration. Its fiber over any vertex x is isomorphic to $\mathrm{TOP}(E_x, \partial E_x)/\mathrm{TOP}_0(E_x, \partial E_x)$, where $\mathrm{TOP}(E_x, \partial E_x)$ is the simplicial group of automorphisms of the pair $(E_x, \partial E_x)$ and $\mathrm{TOP}_0(E_x, \partial E_x)$ is the corresponding discrete group. Hence by* [McD1] *the fibers of $tB^\natural \to B^\natural$ are acyclic.*

Remark. Suppose that B is connected, with base vertex z. Let c from B to $B\,\mathrm{TOP}(E_z, \partial E_z)$ be the classifying map for p. Then we think of $tB := |tB^\natural|$ as the homotopy pullback of

$$B \xrightarrow{c} B\,\mathrm{TOP}(E_z, \partial E_z) \hookleftarrow B\,\mathrm{TOP}_0(E_z, \partial E_z)\,.$$

Proof of 2.5. The Kan fibration statement is true by inspection. Now fix a vertex x in B and write $L = E_x = p^{-1}(x)$. There is a map from $\mathrm{TOP}(L, \partial L)$ to the fiber of $tB^\natural \to B^\natural$ over x which takes a k–simplex $f\colon L \times \Delta^k \to L \times \Delta^k$ to the equivalence relation θ on $L \times \Delta^k$ given by

$$z_1\,\theta\,z_2 \quad \Leftrightarrow \quad f_1(z_1) = f_1(z_2) \in L\ ,$$

where f_1 is the first component of f. The map factors through the space of cosets $\mathrm{TOP}(L, \partial L)/\mathrm{TOP}_0(L, \partial L)$, giving the isomorphism claimed in 2.5. The main result of [McD1], building on work of Mather [Math], Thurston [Thu] and Segal [Se1], is that $\mathrm{TOP}(L, \partial L)/\mathrm{TOP}_0(L, \partial L)$ is acyclic. $\square$

Let $tB = |tB^\natural|$, and let $\mathrm{simp}(tB)$ be the category of simplices of $tB^\natural$. The projections $E_x \to E_x^\theta$ for (x,θ) in $\mathrm{simp}(tB)$ constitute a natural transformation which induces a map of homotopy inverse limits:

$$\operatorname*{holim}_{x} F(E_x) \longrightarrow \operatorname*{holim}_{(x,\theta)} F(E_x^\theta)\,. \tag{2.6}$$

2.7. Corollary. *This map is a weak homotopy equivalence.*

Proof. Since we are comparing two functors — on two different but related categories, $\mathrm{simp}(B)$ and $\mathrm{simp}(tB)$ — taking all morphisms to homotopy equivalences, we can translate the statement into one about section spaces. That statement is a special case of the following: Let

$$\begin{array}{ccc} E_1 & \longrightarrow & E_2 \\ \Big\downarrow{\scriptstyle q_1} & & \Big\downarrow{\scriptstyle q_2} \\ B_1 & \xrightarrow{\;g\;} & B_2 \end{array}$$

be a commutative pullback square, where q_2 is a fibration whose fibers are componentwise nilpotent [HMR], and g is a map whose homotopy fiber over any point in B_2 is acyclic. Suppose also that B_1 and B_2 are homotopy equivalent to CW–spaces. Then the pullback map from $\Gamma(q_2)$ to $\Gamma(q_1)$ is a homotopy equivalence.

The proof of the general statement is as follows. We can assume that g is a fibration. Then $\Gamma(q_1)$ can be identified with a space of sections $\Gamma(q_2')$ where $q_2': E_2' \to B_2$ is the fibration whose fiber over $x \in B_2$ is the space of maps from $g^{-1}(x)$ to $q_2^{-1}(x)$. Our map $\Gamma(q_2) \to \Gamma(q_1)$ becomes a map $\Gamma(q_2) \to \Gamma(q_2')$ induced by an evident map $E_2 \to E_2'$ over B. This is given on the fibers over $x \in B$ by the inclusion of the space constant maps from $g^{-1}(x)$ to $q_2^{-1}(x)$ in the space of all maps from $g^{-1}(x)$ to $q_2^{-1}(x)$. By our hypotheses, $E_2 \to E_2'$ is a fiberwise homotopy equivalence. $\square$

2.8. Definition–Summary. Let $p: E \to B$ be a fiber bundle with compact topological manifold fibers. Up to contractible choice, p determines a section $\chi(p)$ of $F_B(E) \to B$, in the informal notation of 1.6 (but with the assumptions of 2.4).

Details. We assume $B = |B^\natural|$ as in 2.5 and observe that $(x, \theta) \mapsto E_x^\theta$ is a functor taking all morphisms to *homeomorphisms*, hence a functor from $\mathrm{simp}(B)$ to $\ell\mathcal{E}^\bullet$. Therefore the characteristic χ for F determines a point

$$\chi(p) \in \operatorname*{holim}_{(x,\theta)} F(E_x^\theta)\,. \tag{2.9}$$

The homotopy fiber C of (2.6) over $\chi(p)$ is a weakly contractible space. Each point $c \in C$ determines a point in $\mathrm{holim}_x\, F(E_x)$ which in turn determines as in 1.5 and 1.6 a section of

$$\operatorname*{hocolim}_{x} F_1(E_x) \to |\,\mathrm{simp}(B)|$$

(see remark just below). We emphasize once again that (2.9) is our rigorous definition of $\chi(p)$; but we find the informal notation $\chi(p): B \to F_B(E)$ suggestive. $\square$

Remark. Since χ is a characteristic for $F|\ell\mathcal{E}^\bullet$, not for F , the definition of F_1 given in 1.5 cannoit be used here. Instead we let $F_1(X) = \mathrm{holim}_{X\to Y}\, F(Y)$ where the homotopy limit is taken over $X/\ell\mathcal{E}^\bullet$.

Characteristic classes

We will show that χ in 2.4 determines a characteristic cohomology class $[\bar{e}_n]$ for euclidean n–bundles, with twisted coefficients in the spectrum $\mathbf{F}(*)$. For more precision, write $\mathrm{TOP}(n) = \mathrm{TOP}(\mathbb{R}^n)$ and let $\gamma(n)$ be the tautological euclidean n–bundle on $B\,\mathrm{TOP}(n)$, with fiberwise one–point compactification $\gamma(n)^\bullet$. We write

$$[\bar{e}_n] \in H^{\gamma(n)}(B\,\mathrm{TOP}(n)\ ; \mathbf{F}(*))$$

to indicate that $[\bar{e}_n]$ is a vertical homotopy class of sections of a bundle with base $B\,\mathrm{TOP}(n)$ and fiber $\Omega^\infty(\gamma(n)_x^\bullet \wedge \mathbf{F}(*))$ over $x \in B\,\mathrm{TOP}(n)$.

2.10. Remark. We take the following view of euclidean n–bundles. Let $\mathcal{B}(n)$ be the category of *balls*, i.e. the category of all spaces homeomorphic to $\mathbb{R}^n$, with *reverse* embeddings as morphisms. That is, a morphism from V to W is an embedding of W in V. Then

$$|\mathcal{B}(n)| \simeq B\,\mathrm{TOP}(n)\,. \tag{2.11}$$

by [McD1, 2.15]. A map from a paracompact space X to $|\mathcal{B}(n)|$ is therefore as good as a euclidean n–bundle on X.

2.12. Remark. We take the following view of cohomology. Let $\mathcal{C}$ be a small category and let $\mathbf{E}$ be a functor from $\mathcal{C}$ to the category of Ω–spectra taking all morphisms in $\mathcal{C}$ to weak homotopy equivalences. (The morphisms in the category of Ω–spectra are *functions* in the sense of [A, p.140].) In particular, for fixed $n \in \mathbb{Z}$ we have a functor $C \mapsto E_n(C)$ from $\mathcal{C}$ to spaces, picking out the n–th term of $\mathbf{E}(C)$. Now we regard $\mathbf{E}$ as a (possibly twisted) coefficient system on $|\mathcal{C}|$ for generalized cohomology and write

$$\pi_k \operatorname{holim} E_n = H^{n-k}(\,|\mathcal{C}|;\mathbf{E}(\text{—})) \tag{2.13}$$

for $k \geq 0$. We justify this by interpreting $\operatorname{holim} E_n$ as the section space of a suitable fibration on $|\mathcal{C}|$, with fiber $\simeq E_n(C)$ over a vertex C of $|\mathcal{C}|$. The details are as in remark 1.5. $\square$

We identify $\mathcal{B}(n)$ with a subcategory of $\ell\mathcal{E}^\bullet$, consisting of the objects $V, W, \ldots$ which are homeomorphic to $\mathbb{R}^n$, and where a morphism from V to W is a pointed map $f\colon V^\bullet \to W^\bullet$ which restricts to a homeomorphism $f^{-1}(W) \to W$. Let

$$\bar{e}_n := \chi|\mathcal{B}(n) \;\in\; \operatorname{holim}\,(F|\mathcal{B}(n))\,. \tag{2.14}$$

Now 2.13 says that $\pi_0 \operatorname{holim}\,(F|\mathcal{B}(n))$ is the 0–th cohomology of $|\mathcal{B}(n)|$ with coefficients in the spectrum–valued functor $\mathbf{F}|\mathcal{B}(n)$. Combining this observation with 2.10 and 2.3, we can say that $[\bar{e}_n] \in \pi_0 \operatorname{holim}\,(F|\mathcal{B}(n))$ is a characteristic cohomology class for euclidean n–bundles,

$$[\bar{e}_n] \in H^{\gamma(n)}(B\,\mathrm{TOP}(n)\;;\mathbf{F}(*))\,.$$

Relative characteristic classes

2.15. Remark. We take the following view of relative cohomology. Let $\mathcal{A}$ be a small category, $\partial\mathcal{A} \subset \mathcal{A}$ a subcategory. Suppose that any morphism in $\mathcal{A}$ whose codomain is in $\partial\mathcal{A}$ belongs to $\partial\mathcal{A}$. Let $\mathbf{E}$ be a functor from $\mathcal{A}$ to Ω–spectra taking all morphisms to weak homotopy equivalences. Let $\partial\mathbf{E}$ be a functor from ∂A to *weakly contractible* Ω–spectra, and let $\psi\colon \partial\mathbf{E} \to \mathbf{E}|\partial\mathcal{A}$ be a natural transformation. We make another functor $\mathbf{E}^{\mathrm{rel}}$ from $\mathcal{A}$ to Ω–spectra by

$$\mathbf{E}^{\mathrm{rel}}(a) := \begin{cases} \partial\mathbf{E}(a) & \text{if } a \text{ is in } \partial\mathcal{A} \\ \mathbf{E}(a) & \text{otherwise,} \end{cases}$$

$$\mathbf{E}^{\mathrm{rel}}(f) := \begin{cases} \partial\mathbf{E}(f) & \text{if } f \text{ is in } \partial\mathcal{A} \\ \mathbf{E}(f) & \text{if the domain of } f \text{ is not in } \partial\mathcal{A} \\ \mathbf{E}(f)\cdot\psi & \text{otherwise.} \end{cases}$$

Then we allow ourselves to write

$$\pi_k \operatorname{holim} E_n^{\mathrm{rel}} \quad = \quad H^{n-k}(|\mathcal{A}|, |\partial\mathcal{A}|; \mathbf{E})$$

for $k \geq 0$. To justify this, we refer to 2.12 and observe in addition that $\operatorname{holim} E_n^{\mathrm{rel}}$ fits into a homotopy cartesian square with contractible lower left–hand term,

$$\begin{array}{ccc} \operatorname{holim} E_n^{\mathrm{rel}} & \longrightarrow & \operatorname{holim} E_n \\ \big\downarrow & & \big\downarrow \\ \operatorname{holim} \partial E_n & \longrightarrow & \operatorname{holim}(E_n|\partial\mathcal{A}) . \end{array} \qquad \square$$

For $n \geq 0$ let $\mathcal{GB}(n)$ be the category of *generalized balls* whose objects are the spaces homeomorphic to $\mathbb{R}^n$ or $[0,\infty[\times\mathbb{R}^{n-1}$, with reverse embeddings taking boundary to boundary as morphisms. That is, a morphism from V to W is an embedding of $(W, \partial W)$ in $(V, \partial V)$. Note that the inclusion $\mathcal{B}(n) \to \mathcal{GB}(n)$ induces a homotopy equivalence of the classifying spaces — because it has a left adjoint, the *delete boundary* functor. Let $\partial\mathcal{GB}(n) \subset \mathcal{GB}(n)$ be the full subcategory consisting of the objects homeomorphic to $[0,\infty[\times\mathbb{R}^{n-1}$. Then we have

$$\bar{e}_n = \chi|\mathcal{GB}(n) \in \operatorname{holim}(F|\mathcal{GB}(n)), \tag{2.16}$$

improving on (2.14). It is shown in 2.17 below that the pair $(|\mathcal{GB}(n)|, |\partial\mathcal{GB}(n)|)$ is homotopy equivalent to $(B\,\mathrm{TOP}(n), B\,\mathrm{TOP}(n-1)^{\urcorner})$ for $n \geq 0$, where $\mathrm{TOP}(n-1)^{\urcorner}$ is the group of homeomorphisms $]-\infty,+\infty]\times\mathbb{R}^{n-1} \longrightarrow]-\infty,+\infty]\times\mathbb{R}^{n-1}$, viewed as a subgroup of $\mathrm{TOP}(n)$ by restriction. (For $n = 0$ read $B\,\mathrm{TOP}(n-1)^{\urcorner} = \emptyset$.) Therefore, using 2.15, we may write

$$[\bar{e}_n] \in H^{\gamma(n)}(B\,\mathrm{TOP}(n), B\,\mathrm{TOP}(n-1)^{\urcorner}; \mathbf{F}(*)) .$$

2.17. Lemma. $(|\mathcal{GB}(n)|, |\partial\mathcal{GB}(n)|) \quad \simeq \quad (B\,\mathrm{TOP}(n), B\,\mathrm{TOP}(n-1)^{\urcorner})$.

Proof. Let $\mathcal{B}_t(n)$ and $\partial\mathcal{GB}_t(n)$ be the topological categories corresponding to $\mathcal{B}(n)$ and $\mathcal{GB}(n)$. It is enough to show that the vertical arrows in the diagram of classifying spaces and inclusion maps

$$\begin{array}{ccc} |\partial\mathcal{G}\mathcal{B}(n)| & \longrightarrow & |\mathcal{B}(n)| \\ \big\downarrow & & \big\downarrow \\ |\partial\mathcal{G}\mathcal{B}_t(n)| & \longrightarrow & |\mathcal{B}_t(n)| \end{array}$$

are homotopy equivalences (the horizontal arrows are induced by the *delete boundary* functors). This is done in [McD1, 2.18] for the right–hand vertical arrow. An entirely analogous argument works for the left–hand arrow. $\square$

2.18. Remark. Pairs of categories $(\mathcal{A},\partial\mathcal{A})$ as in 2.15 tend to arise in the following way. Let $v\colon\mathcal{B}\to\mathcal{C}$ be any functor between small categories. The *right cylinder* of v is the category obtained from $\mathcal{B}\amalg\mathcal{C}$ by adjoining, for each object b in $\mathcal{B}$ and each object c in $\mathcal{C}$, a morphism set $\mathrm{mor}(b,c)$ equal to $\mathrm{mor}(v(b),c)$. The composition maps from $\mathrm{mor}(c_1,c_2)\times\mathrm{mor}(b,c_1)$ to $\mathrm{mor}(b,c_2)$ and from $\mathrm{mor}(b_2,c)\times\mathrm{mor}(b_1,b_2)$ to $\mathrm{mor}(b_1,c)$ are the obvious ones. See [Tho1] for a generalization of this construction.

Now let $\mathcal{A}$ be the right cylinder of v and let $\partial\mathcal{A}:=\mathcal{B}$, viewed as a subcategory of $\mathcal{A}$. Then the pair $(\mathcal{A},\partial\mathcal{A})$ has the properties listed in 2.15.

For example, the pair $(\mathcal{G}\mathcal{B}(n),\partial\mathcal{G}\mathcal{B}(n))$ can be obtained in this way: $\mathcal{G}\mathcal{B}(n)$ is isomorphic to the right cylinder of the *delete boundary* functor, a functor from $\partial\mathcal{G}\mathcal{B}(n)$ to $\mathcal{B}(n)$.

3. The index theorem

Overview. We keep the notation and hypotheses of §2, specifically 2.4. The goal is to show that, for a closed topological manifold M^n, the element $\chi(M)\in F(M)$ is Poincaré dual to e_n of the tangent bundle τ of M. (This formulation indicates that we think of Poincaré duality for M as a map between certain infinite loop spaces.) The strategy is to look for a description of the classifying map for τ adapted to the view of euclidean bundles developed in 2.10. In the process we will be led to a description of Poincaré duality which is adapted to the view of cohomology developed in 2.12.

The classifying map for the tangent bundle

Assume for now that M^n is a manifold without boundary, not necessarily compact. Let $\mathcal{O}=\mathcal{O}(M)$ be the set of open subsets of M which are homeomorphic to $\mathbb{R}^n$. This is a poset, ordered by *reverse* inclusion. We will see that

$$|\mathcal{O}|\simeq M\,.$$

For a more precise statement, we introduce an open subset $W\subset|\mathcal{O}|\times M$. Note first that $|\mathcal{O}|$ is the geometric realization of a simplicial set whose k–simplices are of the form $(U_0,U_1,\dots,U_k)$ with $U_i\in\mathcal{O}$ and $U_i\supset U_{i+1}$. We decree that $(x,y)\in|\mathcal{O}|\times M$ belongs to W if the (open) cell containing x corresponds to a nondegenerate simplex $(U_0,\dots,U_k)$ which has $y\in U_0$.

3.1. Proposition. *The projections $W \to |\mathcal{O}|$ and $W \to M$ are homotopy equivalences.*

Proof. Since W is open in the product $|\mathcal{O}| \times M$, the projections in question are *almost locally trivial* in the sense of [Se1, A.1]. By [Se1, A.2] it is enough to verify that both have contractible fibers. Each fiber of $W \to |\mathcal{O}|$ is homeomorphic to euclidean space $\mathbb{R}^n$.

Let W_y be the fiber of $W \to M$ over $y \in M$. Under the projection this embeds in $|\mathcal{O}|$ and we can describe it as the union of all open cells corresponding to nondegenerate simplices $(U_0, \dots, U_k)$ where U_k contains y. There is a subspace $V_y \subset W_y$ defined as the union of all open cells corresponding to nondegenerate simplices $(U_0, \dots, U_k)$ where $y \in U_k$. Note the following:

- V_y is a deformation retract of W_y. Namely, suppose that x in W_y belongs to a cell corresponding to a simplex $(U_0, \dots, U_k)$ with $y \in U_0$. Let $(x_0, x_1, \dots, x_k)$ be the barycentric coordinates of x in that simplex, all $x_i > 0$, and let $j \leq k$ be the largest integer such that $y \in U_j$. Define a deformation retraction by

$$h_{1-t}(x) := tx_{\text{no}} + x_{\text{yes}})^{-1}(x_0, \dots, x_j, tx_{j+1}, \dots, tx_k)$$
$$x_{\text{no}} := \sum_{i>j} x_i \qquad x_{\text{yes}} := \sum_{i \leq j} x_i$$

  for $t \in [0,1]$, using the barycentric coordinates in the same simplex.
- V_y is a CW–subspace of $|\mathcal{O}|$ which can also be described as the classifying space of a sub–poset $\mathcal{O}_y \subset \mathcal{O}$, consisting of the $U \in \mathcal{O}$ containing y.

Finally we observe that the category $\mathcal{O}_y$ is a directed poset. Therefore $|\mathcal{O}_y|$ is contractible. $\square$

3.2. Theorem. *With the identifications of* 3.1 *and* 2.10, *the inclusion of* $|\mathcal{O}(M)|$ *in* $|\mathcal{B}(n)|$ *is the classifying map for the tangent bundle* τ *of* M^n.

Proof. We cannot prove anything without explaining to some extent why 2.10 holds. Write $\mathcal{B} := \mathcal{B}(n)$ and let $|\mathcal{B}|_{\text{big}}$ be the geometric realization of the Δ–set [RS1], [RS2] alias incomplete simplicial set whose k–simplices are the functors from the poset $\{0, \dots, k\}$ to $\mathcal{B}$. (In other words, in making $|\mathcal{B}|_{\text{big}}$ we choose to ignore degeneracy operators.) It is well known that the identification map $|\mathcal{B}|_{\text{big}} \to |\mathcal{B}|$ is a homotopy equivalence. $|\mathcal{B}|_{\text{big}}$ is the codomain of a map with fibers homeomorphic to $\mathbb{R}^n$, say $V \to |\mathcal{B}_0|$. The description of V is similar to that of W in proposition 3.1. Namely, V is the coend (diagonal colimit, see [MaL]) of

$$(s, t) \mapsto s(\min) \times \text{star}(t)$$

where the variables s and t are simplices of $|\mathcal{B}|_{\text{big}}$, or equivalently, functors from one of the posets $\{0, \dots, k\}$ to $\mathcal{B}$; we write $s(\min)$ for the value of s on the minimal element, which is a space homeomorphic to $\mathbb{R}^n$. We project V to the coend of

$$(s, t) \mapsto \text{star}(t)$$

which is $|\mathcal{B}|_{\mathrm{big}}$. The projection is almost locally trivial in the sense of [Se1, A.1]. If $f\colon X \to |\mathcal{B}|_{\mathrm{big}}$ is any map where X is a CW–space, then $f^*V \to X$ is another almost locally trivial map with fibers homeomorphic to $\mathbb{R}^n$. It has no preferred section, but if we are willing to replace X by the homotopy equivalent f^*V, then we have

$$(3.3) \qquad f^*V \times_X f^*V \longrightarrow f^*V \quad ; \quad (z_1, z_2) \mapsto z_1$$

which has a preferred section (the diagonal), and qualifies therefore as a microbundle. This is how maps to $|\mathcal{B}_0|_{\mathrm{big}}$ give rise to microbundles on the domain, and it is not a very serious objection that we had to modify the domain in order to see a microbundle on it.

Returning to our business, let f be the inclusion map from $|\mathcal{O}(M)|$ to $|\mathcal{B}|_{\mathrm{big}}$. (We get an automatic factorization through $|\mathcal{B}|_{\mathrm{big}}$ because the nondegenerate simplices in the nerve of $\mathcal{O}(M)$ form a Δ–set.) Then f^*V is homeomorphic to what we previously (in the proof of 3.1) called W. We now have to find an isomorphism from the microbundle (3.3) on $f^*V = W$ to the pullback of the tangent microbundle τ under the projection $W \to M$. This is easy. $\square$

Inverse Poincaré duality by scanning

Our next order of business is to come up with a description of (inverse) Poincaré duality adapted to 2.12. The method we use is inspired by [McD2], [Böd], [Se3].

We note that $\mathcal{O}(M) \subset \mathcal{B}(n) \subset \ell\mathcal{E}^\bullet$ and that the inclusions $U \to M$ for U in $\mathcal{O}(M)$ induce wrong–way maps $M^\bullet \to U^\bullet$ and then wrong–way maps $F(M) \to F(U)$. These wrong way maps for the various U constitute a natural transformation from the constant functor on $\mathcal{O}(M)$ with value $F(M)$ to the functor $F|\mathcal{O}(M)$. The natural transformation induces

$$(3.4) \qquad F(M) \longrightarrow \operatorname{holim} F|\mathcal{O}(M)\,.$$

We have already observed in 2.12 that the homotopy groups of the codomain of (3.4) are the cohomology groups of M with twisted coefficients in $\mathbf{F}(*)$. We will show that (3.4) is a variant of (inverse) Poincaré duality.

To begin, we simplify (3.4). By the classification theorem for pro–excisive functors, we may assume that $F(X) = \Omega^\infty(X^\bullet \wedge \mathbf{J})$ for some Ω–CW–spectrum $\mathbf{J}$. Then our map takes the form

$$(3.5) \qquad \Omega^\infty(M_+ \wedge \mathbf{J}) \longrightarrow \underset{U \in \mathcal{O}(M)}{\operatorname{holim}}\, \Omega^\infty(U^\bullet \wedge \mathbf{J})$$

and is induced by the collapse maps from $M_+ = M^\bullet$ to the various $U^\bullet$. We will compare it with another map

$$(3.6) \qquad \Omega^\infty(M_+ \wedge \mathbf{J}) \longrightarrow s\Gamma(\tilde{\tau}^\bullet; \mathbf{J})\,.$$

Explanations. Here $\tilde{\tau}^\bullet$ is the bundle on M with fiber $M_z \times z$ over $z \in M$, where M_z is the mapping cone of the inclusion $M \smallsetminus z \to M$. We call it $\tilde{\tau}^\bullet$ because it is clearly fiber homotopy equivalent, rel zero section, to $\tau^\bullet$, the fiberwise one–point compactification of the tangent bundle τ.

Let q be any fibration on a space Y with a distinguished "trivial" section (which we assume is a fiberwise cofibration; compare [Jm]). We write $s\Gamma(q;\mathbf{J})$ for the space of *stable sections of* q *with coefficient spectrum* $\mathbf{J}$, that is, the space of sections of a fibration on Y with fiber $\Omega^\infty(q^{-1}(y)\wedge\mathbf{J})$ over $y\in Y$.

The map (3.6) is induced by the inclusions $M\to M_z\cong M_z\times z$.

To compare (3.5) with (3.6) we set up a commutative diagram

$$\begin{CD} \Omega^\infty(M_+\wedge\mathbf{J}) @>{(3.5)}>> \operatorname{holim}_U\Omega^\infty(U^\bullet\wedge\mathbf{J}) \\ @V{(3.6)}VV @VV{(3.6)}V \\ s\Gamma(\tilde\tau^\bullet;\mathbf{J}) @>>> \operatorname{holim}_U s\Gamma(\tilde\tau^\bullet|U;\mathbf{J})\,. \end{CD}$$

3.7. Observation. *The lower horizontal and right–hand vertical arrows in the diagram are homotopy equivalences.*

Proof. For the right–hand vertical arrow, this follows from the homotopy invariance property of homotopy limits, since $\Omega^\infty(U^\bullet\wedge\mathbf{J})\to s\Gamma(\tilde\tau^\bullet|U;\mathbf{J})$ is a homotopy equivalence for every U in $\mathcal{O}(M)$. The lower horizontal arrow can be rewritten in the form

$$s\Gamma(\tilde\tau^\bullet;\mathbf{J})\longrightarrow s\Gamma(p^*\tilde\tau^\bullet;\mathbf{J})$$

where $p\colon\operatorname{hocolim}_U U\to M$ is the obvious map. Thus we need to know that p is a homotopy equivalence. This follows from 3.1, since p factors through W in 3.1 and the factorizing map $\operatorname{hocolim}_U U\to W$ is a homotopy equivalence. $\square$

We conclude that (3.5) and (3.6) can be identified by means of the diagram preceding 3.7. It remains to understand (3.6). For this purpose we need "standard" Poincaré duality, a map

$$\wp\colon s\Gamma(\tau^\bullet;\mathbf{J})\longrightarrow\Omega^\infty(M_+\wedge\mathbf{J}) \tag{3.8}$$

which is commonly defined as the composition of a Thom homotopy equivalence (inducing the Thom isomorphism on homotopy groups) with an Atiyah–Milnor–Spanier–Whitehead duality map.

Details: Let $\nu=\nu(M)$ be a normal bundle of an embedding of M in $\mathbb{R}^w$, with collapse map $c\colon\mathbb{S}^w\to\operatorname{th}(\nu)$. The Thom homotopy equivalence

$$s\Gamma(\tau^\bullet;\mathbf{J})\quad\longrightarrow\quad\operatorname{map}(\operatorname{th}(\nu)\wedge\mathbf{S}^0,\mathbb{S}^w\wedge\mathbf{J})$$

associates to $t\in s\Gamma(\tau^\bullet;\mathbf{J})$ the map whose restriction to $\nu_x^\bullet\wedge\mathbf{S}^0$ for $x\in M$ is $\mathrm{id}\wedge t(x)$ from $\nu_x^\bullet\wedge\mathbf{S}^0$ to $\nu_x^\bullet\wedge\tau_x^\bullet\wedge\mathbf{J}\cong\mathbb{S}^w\wedge\mathbf{J}$. The duality map

$$\operatorname{map}(\operatorname{th}(\nu)\wedge\mathbf{S}^0,\mathbb{S}^w\wedge\mathbf{J})\quad\longrightarrow\quad\Omega^\infty(M_+\wedge\mathbf{J})$$

associates to $\mathbf{g}\colon \mathrm{th}(\nu)\wedge \mathbf{S}^0 \longrightarrow \mathbb{S}^w \wedge \mathbf{J}$ the composition

$$\mathbb{S}^w \wedge \mathbf{S}^0 \xrightarrow{c\wedge \mathbf{id}} \mathrm{th}(\nu)\wedge \mathbf{S}^0 \xrightarrow{(\mathrm{proj},\mathbf{g})} M_+ \wedge \mathbb{S}^w \wedge \mathbf{J}\,.$$

Strictly speaking, that composition is an element of $\Omega^w\Omega^\infty(M_+\wedge\mathbb{S}^w\wedge\mathbf{J})$. However, we have an inclusion $\Omega^\infty(M_+\wedge\mathbf{J}) \to \Omega^w\Omega^\infty(M_+\wedge\mathbb{S}^w\wedge\mathbf{J})$ which is a homotopy equivalence.

3.9. Definition. We define a map called *reflection*, $t\mapsto \bar t$, from $s\Gamma(\tau^\bullet;\mathbf{J})$ to $s\Gamma(\tau^\bullet;\mathbf{J})$. It is the map induced by the sphere bundle automorphism

$$\tau^\bullet \wedge_M \tau^\bullet \wedge_M \nu^\bullet \longrightarrow \tau^\bullet \wedge_M \tau^\bullet \wedge_M \nu^\bullet$$

which permutes the first two factors, and which we regard here as a stable fiberwise automorphism of $\tau^\bullet$ via identifications $\tau^\bullet_x \wedge(\tau^\bullet_x\wedge\nu^\bullet_x)\cong \tau^\bullet_x\wedge\mathbb{S}^w$. (Strictly speaking, reflection is an involution on $\Omega^w s\Gamma((\tau\oplus\varepsilon^w)^\bullet;\mathbf{J})$ where ε^w is a trivial euclidean bundle of fiber dimension w.)

Remark. If τ is a vector bundle, the involution $t\mapsto\bar t$ has a simpler description. Up to homotopy it is the map induced by the automorphism of τ which is scalar multiplication by -1 on each fiber. For the proof, let μ_s for $s\in\mathbb{C}$ be the real vector bundle automorphism of $\tau\oplus\tau\cong\tau\otimes_{\mathbb{R}}\mathbb{C}$ given by complex multiplication with s. Then conjugation with μ_s for s on the unit circle, between 1 and $e^{i\pi/4}$, deforms the automorphism of $\tau\oplus\tau$ given by permutation of the summands into that given by $(x,y)\mapsto(-x,y)$.

3.10. Observation–Definition–Summary. *The composition* (3.6) $\cdot$ (3.8) *is given by reflection,* $t\mapsto\bar t$. *Therefore the maps* (3.6) *and* (3.4) *can be labeled*

$$\bar\wp^{\,-1}\colon \Omega^\infty(M_+\wedge\mathbf{J})\to s\Gamma(\tilde\tau^\bullet;\mathbf{J})\,,$$
$$\bar\wp^{\,-1}\colon F(M)\to \mathrm{holim}\ F|\mathcal{O}(M)\,.$$

The index theorem

Returning to (3.5), we must ask how the image of $\chi(M)\in F(M)$ under $\bar\wp^{\,-1}$ is related to the image of $\bar e_n\in \mathrm{holim}\ F|\mathcal{B}(n)$ under the restriction map

$$\mathrm{holim}\,F|\mathcal{B}(n)\to \mathrm{holim}\ F|\mathcal{O}(M)\,.$$

3.11. Observation (the index theorem for a single manifold)**.** *The two image points are connected by a canonical path* $\omega(M)$ *in* $\mathrm{holim}\,(F|\mathcal{O}(M))$.

Proof. Enlarge $\mathcal{O}=\mathcal{O}(M)$ by adding M as an initial object. We can still write $(\mathcal{O}\cup\{M\})\subset\ell\mathcal{E}^\bullet$, so the characteristic χ gives us an element z in the homotopy limit of $F|(\mathcal{O}\cup\{M\})$, which is the same as the homotopy limit of the diagram

$$F(M)\xrightarrow{\bar\wp^{\,-1}}\mathrm{holim}\ F|\mathcal{O}\,.$$

Therefore z consists of a point z_0 in $F(M)$ and a path ω in holim $F|\mathcal{O}$ starting at $\wp^{-1}(z_0)$. Inspection shows that $z_0 = \chi(M)$ and that the endpoint of ω is the image of $\bar{e}_n$ in holim $F|\mathcal{O}$. $\square$

The case of a manifold with boundary

3.12. Remark. We indicate how 3.2 can be generalized to manifolds with boundary. Let M^n be a manifold with boundary ∂M and interior M_0, where $n > 0$. Let $\mathcal{O}(M)$ be the poset whose objects are the open subsets of M homeomorphic to either $\mathbb{R}^n$ or $]-\infty,\infty] \times \mathbb{R}^{n-1}$. The partial order on $\mathcal{O}(M)$ is reverse inclusion as usual. Let $\partial\mathcal{O}(M) \subset \mathcal{O}(M)$ consist of the open subsets of M which are homeomorphic to $]-\infty,\infty] \times \mathbb{R}^{n-1}$.

The *delete interior* functor from $\partial\mathcal{O}(M)$ to $\mathcal{O}(\partial M)$ induces a homotopy equivalence of the classifying spaces. (Namely, by Quillen's theorem A [Qui] it is enough to show that for every $U \in \mathcal{O}(\partial M)$ the poset of $U' \in \partial\mathcal{O}(M)$ with $U' \supset U$ has a contractible classifying space. This is clearly a directed poset.) Also, the inclusion $\mathcal{O}(M_0) \to \mathcal{O}(M)$ induces a homotopy equivalence of the classifying spaces, since it has a left adjoint. It follows that the pair $(|\mathcal{O}(M)|, |\partial\mathcal{O}(M)|)$ is homotopy equivalent to $(M, \partial M)$. The following inclusion can be thought of as the classifying map for the tangent bundle pair $(\tau(M), \tau(\partial M))$:

$$(|\mathcal{O}(M)|, |\partial\mathcal{O}(M)|) \longrightarrow (|\mathcal{GB}(n)|, |\partial\mathcal{GB}(n)|). \tag{3.13}$$

3.14. Remark. We indicate how 3.6 and 3.11 can be generalized to compact manifolds M with boundary. The analog of (3.5) has the form

$$\wp^{-1}\colon F(M) \to \operatorname{holim}(F|\mathcal{O}(M)). \tag{3.15}$$

It is a homotopy equivalence and can be analyzed as in 3.10. Note that $F|\mathcal{O}(M)$ takes objects in $\partial\mathcal{O}(M) \subset \mathcal{O}(M)$ to contractible spaces ; therefore the codomain of (3.15) can be interpreted as *relative cohomology* of the pair $(M, \partial M)$. Compare 2.15. The domain of (3.15) should be interpreted as absolute homology of M.

The analog of the index theorem 3.11 states that the images of $\chi(M) \in F(M)$ and $\bar{e}_n \in \operatorname{holim}(F|\mathcal{GB}(n))$ in $\operatorname{holim}(F|\mathcal{O}(M))$ are connected by a canonical path $\omega(M)$.

The index theorem for families

Let M^n be a compact manifold. As we have just observed, stating and proving the index theorem for M amounts to specifying a point $\mathcal{I}_M = (\bar{e}_n, \omega(M), \chi(M))$ in the homotopy pullback of the diagram

$$\operatorname{holim}(F|\mathcal{GB}(n)) \xrightarrow{\text{restriction}} \operatorname{holim}(F|\mathcal{O}(M)) \xleftarrow{\ \wp^{-1}\ } F(M). \tag{3.16}$$

To make a family index theorem, we must understand how $\mathcal{I}_M$ depends on M as a variable. We therefore return to the assumptions and notation of 2.8. In particular, $p\colon E \to B$ is a bundle with compact manifold fibers (fiber dimension n).

3.17. Observation. *The restriction map in* (3.16) *has a refinement, or parametrized version,*

$$\operatorname{holim}(F|\mathcal{GB}(n)) \quad\longrightarrow\quad \underset{(x,\theta)}{\operatorname{holim}}\ \operatorname{holim}(F|\mathcal{O}(E_x^\theta))\,.$$

3.18. Observation. *The point* $\mathcal{I}_M$ *in the homotopy pullback of* (3.16) *has a refinement or parametrized version which is a point in the homotopy pullback of*

$$\operatorname{holim}(F|\mathcal{GB}(n)) \longrightarrow \underset{(x,\theta)}{\operatorname{holim}}\ \operatorname{holim}(F|\mathcal{O}(E_x^\theta)) \xleftarrow{\ \bar\wp^{-1}\ } \underset{(x,\theta)}{\operatorname{holim}}\ F(E_x^\theta)\,.$$

Now let C be the homotopy fiber of $\operatorname{holim}_x F(E_x) \to \operatorname{holim}_{(x,\theta)} F(E_x^\theta)$ over the point $\chi(p)$ determined by χ. We observed in 2.7 that C is weakly contractible. We are now ready for the family index theorem:

3.19. Observation (the family index theorem)**.** *Each* $c \in C$ *determines a point* $\mathcal{I}_p = (\bar e_n, \omega(p), \chi(p))$ *in the homotopy pullback of*

$$\operatorname{holim}(F|\mathcal{GB}(n)) \longrightarrow \underset{(x,\theta)}{\operatorname{holim}}\ \operatorname{holim}(F|\mathcal{O}(E_x^\theta)) \xleftarrow{\ \simeq\ } \underset{x}{\operatorname{holim}}\ F(E_x)\,.$$

where the right–hand arrow is the composition of $\bar\wp^{-1}$ *in* 3.18 *with the forgetful weak homotopy equivalence* $\operatorname{holim}_x F(E_x) \to \operatorname{holim}_{(x,\theta)} F(E_x^\theta)$.

3.20. Remark. A portable form of 3.19 is $\bar\wp(\bar e_n(\tau)) \simeq \chi(p)$ or $\wp(e_n(\tau)) \simeq \chi(p)$, where $\bar e_n(\tau)$ and $e_n(\tau)$ have the following meaning.

- Formally, $\bar e_n(\tau)$ is the image of $\bar e_n$ under the left–hand arrow in 3.19.
- Informally, $\bar e_n(\tau)$ is the pullback of a certain section $\bar e_n$ under the classifying map $(E, \partial E) \to (B\,\mathrm{TOP}(n), B\,\mathrm{TOP}(n-1)^\urcorner)$ for the vertical tangent bundle τ of p. Furthermore (and still informally) $\bar e_n$ is a section of a certain fibration on $B\,\mathrm{TOP}(n)$ whose fibers are infinite loop spaces homotopy equivalent to $F(\mathbb{R}^n)$. It vanishes on $B\,\mathrm{TOP}(n-1)^\urcorner)$ and so *represents* a class in the n–th cohomology of $(B\,\mathrm{TOP}(n), B\,\mathrm{TOP}(n-1)^\urcorner)$ with twisted coefficients in the spectrum $\mathbf{F}(*)$. Hence $\bar e_n(\tau)$ is a section of a fibration on E with fibers $\simeq F(\mathbb{R}^n)$. It vanishes on ∂E and so *represents* a class in the cohomology of $(E, \partial E)$ with twisted coefficients in the spectrum $\mathbf{F}(*)$.
- Informally, $e_n(\tau)$ is the image of $\bar e_n(\tau)$ under reflection, 3.9, with the informal definition of $\bar e_n(\tau)$. We do not give a formal definition of $e_n(\tau)$.

4. Disk bundles and the unit transformation

The unit transformation

4.1. Definition. We make a category of pairs (F, z) where F is a pro–excisive functor from $\mathcal{E}^\bullet$ to pointed spaces, see 2.1–2.3, and z is a point in $F(*)$. A *morphism* from (F, z) to (F', z') is a natural transformation $F \to F'$ taking z to z'. Such a morphism is a *weak equivalence* if $F(X) \to F'(X)$ is a weak homotopy equivalence for all X. We say that two objects (F, z) and (F', z') are *weakly equivalent* if they can be related by a chain of weak equivalences.

4.2. Lemma. *For every* (F, z) *, there exists a morphism* $\eta\colon (C, y) \to (F, z)$ *where* (C, y) *is weakly equivalent to the pair*

$$(X \mapsto \Omega^\infty\Sigma^\infty X^\bullet \ , \ 1)\,.$$

Proof. By the classification theorem for excisive functors, mentioned in 2.2, there exists a diagram of natural transformations

$$F(X) \xrightarrow{\ g\ } F'(X) \xleftarrow{\ h\ } F''(X) \tag{4.3}$$

where $F''(X) = \Omega^\infty(X^\bullet \wedge \mathbf{F}(*))$ and g, h are homotopy equivalences for every X. (The classification theorem speaks of a *chain* of natural transformations, but it is easy to reduce to a chain of length 2.) Specializing (4.3) to the case $X = *$, choose a point $\bar{z}$ in the homotopy fiber of h over $g(z)$. An argument involving homotopy pullbacks shows that it is enough to prove 4.2 for the excisive functor F'' and the point $z'' \in F''(*)$ which is the image of $\bar{z}$ under the projection. However, that case is easy because z'' determines a map of spectra $\mathbf{S}^0 \to \mathbf{F}(*)$ and then a natural transformation from $Q(X^\bullet) := \Omega^\infty(X^\bullet \wedge \mathbf{S}^0)$ to $\mathbf{F}(*)$. This takes the unit element in $Q\mathbb{S}^0 = Q(*^\bullet)$ to z''. The functor $X \mapsto Q(X^\bullet)$ is excisive and freely generated by the unit element in $Q(*^\bullet)$. $\square$

Remark. The existence statement 4.2 implies a uniqueness statement, as follows. Suppose that $(C_1, y_1) \to (F, z)$ and $(C_2, y_2) \to (F, z)$ are two morphisms as in 4.2. Assume or arrange that $C_1(X) \to F(X)$ and $C_2(X) \to F(X)$ are Serre fibrations for every X. Make another object (P, x) where $P(X)$ is the pullback of

$$C_1(X) \to F(X) \leftarrow C_2(X)$$

and $x \in P(*)$ is the point determined by y_1 and y_2. Finally choose a morphism $(C_3, y_3) \to (P, x)$ as in 4.2. Then the following compositions are weak equivalences over (F, z):

$$(C_3, y_3) \to (P, x) \to (C_1, y_1)$$
$$(C_3, y_2) \to (P, x) \to (C_2, y_2)\,.$$

4.4. Definition. If F is any pro–excisive functor from $\mathcal{E}^\bullet$ to pointed spaces, and χ is a characteristic for $F|\ell\mathcal{E}^\bullet$, then we can take $z = \chi(*) \in F(*)$ in 4.2. The resulting natural transformation $\eta: C \to F$ constructed as in 4.2 is the *unit transformation* for F and χ. We shall also use informal notation such as $\eta: Q(X^\bullet) \to F(X)$ or even $\eta: Q^\bullet(X) \to F(X)$. Further, we write $\boldsymbol{\eta}: \mathbf{S}^0 \to \mathbf{F}(*)$ for the map of spectra corresponding to $\eta: Q(X^\bullet) \to F(X)$ via the classification theorem ; see 2.3.

The Becker–Euler Classes

4.5. Definition. Let $\mathrm{G}(n)$ be the simplicial monoid of homotopy equivalences $\mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$. The classifying space $B\,\mathrm{G}(n)$ carries a tautological spherical quasifibration $\zeta(n)$ with fibers $\simeq \mathbb{S}^{n-1}$. Let $\mathrm{G}(n-1)^\urcorner \subset \mathrm{G}(n)$ be the submonoid consisting of the homotopy equivalences of the form $1 * g: \mathbb{S}^0 * \mathbb{S}^{n-2} \to \mathbb{S}^0 * \mathbb{S}^{n-2}$; here we use an identification of the join $\mathbb{S}^0 * \mathbb{S}^{n-2}$ with $\mathbb{S}^{n-1}$. Of course, $\mathrm{G}(n-1)^\urcorner$ is isomorphic as a simplicial monoid to $\mathrm{G}(n-1)$, under this isomorphism the tautological quasifibration $\zeta(n-1)$ on $B\,\mathrm{G}(n-1)$ corresponds to a quasifibration $\zeta(n-1)^\urcorner$ on $B\,\mathrm{G}(n-1)^\urcorner$.

4.6. Definition. Let θ be a spherical quasifibration on a space X and let $\mathbf{J}$ be any spectrum. Let ε be the trivial fibration $\mathbb{S}^0 \times X \to X$ and let $\varepsilon \oplus \theta$ be the fiberwise join of ε and θ. We denote by $\Gamma^{\sim}(\varepsilon \oplus \theta\ ;\ \mathbf{J})$ the space of sections of the fibration associated with the quasifibration on X with fiber $\Omega^\infty((\varepsilon \oplus \theta)_x \wedge \mathbf{J})$ over x. Here the base point of $\mathbb{S}^0 \cong \varepsilon_x$ serves as base point for $(\varepsilon \oplus \theta)_x$, for each $x \in X$. We write

$$H^\theta(X;\mathbf{J})\ ,$$
$$H^\theta(X,Y;\mathbf{J})$$

for π_0 of $\Gamma^{\sim}(\varepsilon \oplus \theta\ ;\ \mathbf{J})$ and π_0 of the subspace consisting of the sections vanishing on Y , respectively. (A section *vanishes* on Y if its restriction to Y is the zero section.)

Suppose now that X has the homotopy type of a compact CW–space. Choose a spherical quasifibration ξ on X such that the fiberwise join $\theta \oplus \xi$ admits a trivialization, i.e. a fiber homotopy equivalence $\theta \oplus \xi \to \varepsilon^w$ where ε^w is the trivial fibration $\mathbb{S}^{w-1} \times X \to X$. We can use this to define an involution $s \mapsto \bar{s}$ called *reflection* (compare 3.9) on $\Gamma^{\sim}(\varepsilon \oplus \theta\ ;\ \mathbf{J})$, strictly speaking on the homotopy equivalent space $\Omega^w\Gamma^{\sim}(\varepsilon \oplus \theta \oplus \theta \oplus \xi;\ \mathbf{J})$. This is induced by the automorphism of $\theta \oplus \theta$ which permutes the factors. Reflection induces involutions on $H^\theta(X;\mathbf{J})$ and $H^\theta(X,Y;\mathbf{J})$.

4.7. Definition. Following [Be], we now describe characteristic classes for spherical quasifibrations: the Becker–Euler classes

$$[b_n]\ , [\bar{b}_n] \in H^{\zeta(n)}(B\,\mathrm{G}(n), B\,\mathrm{G}(n-1)^\urcorner; \mathbf{S}^0)\,. \tag{4.8}$$

Becker designed $[b_n]$ to have the following property. Let (X,Y) be a CW–pair of relative dimension $\leq 2n-4$. A map of pairs $f\colon (X,Y) \to (B\,\mathrm{G}(n), B\,\mathrm{G}(n-1)^{\urcorner})$ factors up to homotopy through $(B\,\mathrm{G}(n-1)^{\urcorner}, B\,\mathrm{G}(n-1)^{\urcorner})$ if and only if $f^*[b_n]=0$ in $H^{\zeta(n)}(X,Y\,;\mathbf{S}^0)$. We will see that $[\bar{b}_n]$ has exactly the same property. —

We begin by making a section of ε , the trivial bundle $\mathbb{S}^0 \times B\,\mathrm{G}(n) \to B\,\mathrm{G}(n)$: the section picking the non–base point of $\mathbb{S}^0$ in each fiber. Then we regard it as a section s of $\varepsilon \oplus \zeta(n)$. The section s is vertically nullhomotopic over $B\,\mathrm{G}(n-1)^{\urcorner}$. Namely, $\varepsilon \oplus \zeta(n)\,|\,B\,\mathrm{G}(n-1)^{\urcorner}$ is canonically identified with $\varepsilon \oplus \varepsilon \oplus \zeta(n-1)^{\urcorner}$, and s is already vertically nullhomotopic when viewed as a section of the sub–quasifibration $\varepsilon \oplus \varepsilon \quad \subset \quad \varepsilon \oplus \varepsilon \oplus \zeta(n-1)^{\urcorner}$. However, s has *two* canonical vertical nullhomotopies, as a section of $\varepsilon \oplus \varepsilon$. We may identify $\varepsilon \oplus \varepsilon$ with a trivial bundle with fibers $\cong \mathbb{S}^1 \subset \mathbb{C}$; then s picks out -1 in each fiber, and the nullhomotopies use canonical contractions of $\mathbb{S}^1(+)$, the intersection of $\mathbb{S}^1$ with the upper half plane, and $\mathbb{S}^1(-)$, the intersection of $\mathbb{S}^1$ with the lower half plane. These two vertical nullhomotopies of $s|B\,\mathrm{G}(n-1)^{\urcorner}$ allow us to deform s into sections trivial on $B\,\mathrm{G}(n-1)^{\urcorner}$ of the fibration on $B\,\mathrm{G}(n)$ associated with $\varepsilon \oplus \zeta(n)$. We view these sections as sections b_n and $\bar{b}_n$ of a larger fibration on $B\,\mathrm{G}(n)$ obtained from $\varepsilon \oplus \zeta(n)$ by first applying $\Omega^\infty\Sigma^\infty$ to each fiber, and then making fibrant. The relative vertical homotopy classes are denoted $[b_n]$ and $[\bar{b}_n]$, respectively.

We need to understand the homotopy theoretic relationship between b_n and $\bar{b}_n$. As the notation suggests, they are related by *reflection*, see 4.6. However, this is only meaningful when we restrict to a compact CW sub–pair (X,Y) of the CW–pair $(B\,\mathrm{G}(n), B\,\mathrm{G}(n-1)^{\urcorner})$. To show that $\bar{b}_n|X$ is indeed the reflection of $b_n|X$, up to a contractible choice of vertical homotopies relative to Y , use an argument as in the remark after 3.9, applied to $\varepsilon \oplus \varepsilon\,|\,Y$ as a subfibration of

$$\varepsilon \oplus \zeta(n-1)^{\urcorner} \oplus \varepsilon \oplus \zeta(n-1)^{\urcorner}\ |\ Y \quad \cong \quad \zeta(n) \oplus \zeta(n)\ |\ Y\,.$$

Characteristic classes associated with disk bundles

Let $W =]-\infty,\infty] \times \mathbb{R}^{n-1}$. Then $W^\bullet$ is an n–disk. There is a commutative diagram of topological groups and homomorphisms

$$\begin{array}{ccc}
\mathrm{TOP}(W,\partial W) & \xrightarrow{\ \subset\ } & \mathrm{TOP}(W^\bullet,\partial(W^\bullet)) \\
\Big\downarrow{=} & & \Big\downarrow{\text{restr.}} \\
\mathrm{TOP}(W,\partial W) & \xrightarrow{\text{restr.}} & \mathrm{TOP}(W \smallsetminus \partial W).
\end{array}$$

It leads to a map ι of pairs:

$$\tag{4.9} \begin{array}{c}
(B\,\mathrm{TOP}(W^\bullet,\partial(W^\bullet))\,, B\,\mathrm{TOP}(W,\partial W)) \\
\Big\downarrow{\iota} \\
(B\,\mathrm{TOP}(W \smallsetminus \partial W)\,, B\,\mathrm{TOP}(W,\partial W)) \\
\| \\
(B\,\mathrm{TOP}(n), B\,\mathrm{TOP}(n-1)^{\urcorner})\,.
\end{array}$$

4.10. Theorem. *The equation* $\iota^*[\bar{e}_n] = \iota^*\boldsymbol{\eta}_*[\bar{b}_n]$ *holds in*

$$H^{\gamma(n)}(B\,\mathrm{TOP}(W^\bullet, \partial(W^\bullet))\,, B\,\mathrm{TOP}(W, \partial W)\ ; \mathbf{F}(*))\,.$$

Informally, 4.10 says that $[\bar{e}_n]$ applied to a euclidean n–bundle ξ equals $\boldsymbol{\eta}_*[\bar{b}_n]$ applied to ξ , provided ξ has been obtained from an n–disk bundle by deleting the boundary sphere bundle. In particular, $[\bar{e}_n]$ applied to an n–dimensional vector bundle ξ is $\boldsymbol{\eta}_*[\bar{b}_n]$ of ξ. (Note however that the formal statement is not only more formal but also more relative, and therefore considerably stronger.)

Outline of proof. Let V_1 be a space homeomorphic to $\mathbb{R}^n$. We need to know that any embedding of V_1 into a manifold V_2 homeomorphic to the n–disk determines a “lift” of $\chi(V_1) \in F(V_1)$ to $Q^\bullet(V_1)$, across the unit map $\eta : Q^\bullet(V_1) \to F(V_1)$. By “lift” we mean a point in the homotopy fiber of $\eta : Q^\bullet(V_1) \to F(V_1)$ projecting to $\chi(V_1) \in F(V_1)$. The naturality property of χ applied to the morphism $V_2 \to V_1$ in $\ell\mathcal{E}^\bullet$ determined by the embedding $V_1 \to V_2$ shows that it is enough to specify a lift of $\chi(V_2) \in F(V_2)$ to $Q^\bullet(V_2)$. The naturality property of χ applied to the morphism $V_2 \to *$ in $\ell\mathcal{E}^\bullet$, which induces homotopy equivalences $Q^\bullet(V_2) \to Q^\bullet(*)$ and $F(V_2) \to F(*)$ since it is a proper homotopy equivalence, shows that it is enough to specify a lift of $\chi(*) \in F(*)$ to $Q^\bullet(*)$, across $\eta : Q^\bullet(*) \to F(*)$. But such a specified lift is part of the definition of η.

For the honest proof of 4.10 we replace the classifying spaces of topological groups involved by classifying spaces of appropriate discrete categories. We describe those discrete categories first.

Our model for the pair $(B\,\mathrm{TOP}(W \smallsetminus \partial W)\,, B\,\mathrm{TOP}(W, \partial W))$ is that of 2.17, the classifying space pair of the category pair $(\,\mathcal{GB}(n)\ , \partial\mathcal{GB}(n)\,)$. Our model for the pair $(B\,\mathrm{TOP}(W^\bullet, \partial(W^\bullet))\,, B\,\mathrm{TOP}(W, \partial W))$ will be a homology approximation only, again a classifying space pair determined by a category pair $(\mathcal{OD}(n), \partial\mathcal{OD}(n))$ which we now describe. It comes with a forgetful functor to $(\,\mathcal{GB}(n)\ , \partial\mathcal{GB}(n)\,)$ which we call again ι because it is our categorical model for the map ι mentioned in 4.10.

The objects of $\mathcal{OD}(n)$ are pairs (D, H) where D is homeomorphic to the n–disk, and $H \in \mathcal{O}(D)$. A morphism from (D, H) to (D', H') is a homeomorphism $f : D' \to D$ for which $f(H') \subset H$. The forgetful functor ι from $\mathcal{OD}(n)$ to $\mathcal{GB}(n)$ is given by $(D, H) \mapsto H$. Let $\partial\mathcal{OD}(n)$ be the inverse image of $\partial\mathcal{GB}(n)$ under the forgetful functor.

By construction, the category pair $(\mathcal{OD}(n), \partial\mathcal{OD}(n))$ is equivalent to a semidirect product of the discrete group $\mathrm{TOP}_0(\mathbb{D}^n, \partial\mathbb{D}^n)$ with the pair $(\mathcal{O}(\mathbb{D}^n), \partial\mathcal{O}(\mathbb{D}^n))$. So the classifying space pair determined by $(\mathcal{OD}(n), \partial\mathcal{OD}(n))$ is homotopy equivalent to the homotopy orbit pair of the canonical action of $\mathrm{TOP}_0(\mathbb{D}^n, \partial\mathbb{D}^n)$ on

$$(\,|\mathcal{O}(\mathbb{D}^n)|, |\partial\mathcal{O}(\mathbb{D}^n)|\,) \quad \simeq \quad (\mathbb{D}^n, \partial\mathbb{D}^n)\,.$$

It admits therefore a *homology* equivalence to the homotopy orbit pair of the canonical action of $\mathrm{TOP}(\mathbb{D}^n, \partial\mathbb{D}^n)$ on $(\mathbb{D}^n, \partial\mathbb{D}^n)$, which in turn is homotopy equivalent to $(B\,\mathrm{TOP}(W^\bullet,\partial(W^\bullet))\,,B\,\mathrm{TOP}(W,\partial W))$ of 4.10. More precisely, we can make a commutative square of pairs

$$
\begin{array}{ccc}
(\,|\mathcal{OD}(n)|,|\partial\mathcal{OD}(n)|\,) & \longrightarrow & (B\,\mathrm{TOP}(W^\bullet,\partial(W^\bullet))\,,B\,\mathrm{TOP}(W,\partial W)) \\
\big\downarrow{\scriptstyle |\iota|} & & \big\downarrow{\scriptstyle \iota} \\
(\,|\mathcal{GB}(n)|,|\partial\mathcal{GB}(n)|\,) & \xrightarrow{\ \simeq\ } & (B\,\mathrm{TOP}(W\smallsetminus\partial W)\,,B\,\mathrm{TOP}(W,\partial W))
\end{array}
\tag{4.11}
$$

where the upper horizontal arrow is a homology equivalence. When we apply $H^{\gamma(n)}(\dots;\mathbf{F}(*))$ or $H^{\gamma(n)}(\dots;\mathbf{S}^0)$ to (4.11), we obtain a commutative square of abelian groups and homomorphisms in which the two horizontal arrows are isomorphisms. Here $\gamma(n)$ denotes the usual bundle over $B\,\mathrm{TOP}(n)=B\,\mathrm{TOP}(W\smallsetminus\partial W)$, but also its pullback to the various other spaces appearing in (4.11).

The proof of 4.10, and also a sharper formulation of 4.10, will come out of the commutative diagram

$$
\begin{array}{ccc}
Q^\bullet(*) & \xrightarrow{\ \eta\ } & F(*) \\
\big\downarrow & & \big\downarrow \\
\operatorname*{holim}_D Q^\bullet(*) & \xrightarrow{\ \eta\ } & \operatorname*{holim}_D F(*) \\
\big\uparrow{\scriptstyle\simeq} & & \big\uparrow{\scriptstyle\simeq} \\
\operatorname*{holim}_D Q^\bullet(D) & \xrightarrow{\ \eta\ } & \operatorname*{holim}_D F(D) \\
\big\downarrow{\scriptstyle\simeq} & & \big\downarrow \\
\operatorname*{holim}_{(D,H)} Q^\bullet(H) & \xrightarrow{\ \eta\ } & \operatorname*{holim}_{(D,H)} F(H)
\end{array}
\tag{4.12}
$$

where D runs through the category of all n–disks, with homeomorphisms as morphisms, and (D,H) runs through $\mathcal{OD}(n)$. The characteristic χ for F , and the defining property of η , supply a distinguished element $\chi|$ in the homotopy inverse limit of the subdiagram

$$
\begin{array}{ccc}
Q^\bullet(*) & \xrightarrow{\ \eta\ } & F(*) \\
 & & \big\downarrow \\
 & & \operatorname*{holim}_D F(*) \\
 & & \big\uparrow{\scriptstyle\simeq} \\
 & & \operatorname*{holim}_D F(D) \\
 & & \big\downarrow \\
 & & \operatorname*{holim}_{(D,H)} F(H)\,.
\end{array}
\tag{4.13}
$$

4.14. Reformulation of 4.10 — Definition. *Let Φ_n be the fiber over $\chi|$ of the restriction map from the homotopy inverse limit of* (4.12) *to the homotopy inverse limit of* (4.13). *Then Φ_n is a contractible space. It comes with a forgetful map f to the homotopy fiber of*

$$\operatorname*{holim}_{(D,H)} Q^{\bullet}(H) \xrightarrow{\;\eta\;} \operatorname*{holim}_{(D,H)} F(H)$$

over the point $\iota^*\bar{e}_n$. The map obtained by composing f with the projection to $\operatorname{holim}_{(D,H)} Q^{\bullet}(H)$ *hits the component $\iota^*[\bar{b}_n]$; we call it $\iota^*\bar{b}_n$.*

Proof and explanation. Note first of all that π_0 of $\operatorname{holim}_{(D,H)} F(H)$ is the generalized n–th cohomology group, with twisted coefficients in $\mathbf{F}(*)$, which appears explicitly in 4.10, and π_0 of $\operatorname{holim}_{(D,H)} Q^{\bullet}(H)$ is another generalized cohomology group, with twisted coefficients in the sphere spectrum, which is implicit in 4.10.

The restriction map from holim of (4.12) to holim of (4.13) is a fibration and a homotopy equivalence. Hence Φ_n is contractible. All other statements are true by inspection. □

5. The transfer of Becker–Gottlieb and Dold

Our goal here is to combine the family index theorem 3.19 with theorem 4.10 stating equality of $[\bar{e}_n]$ and $\boldsymbol{\eta}_*[\bar{b}_n]$ for n–disk bundles. Therefore we need to understand $\bar{\wp}[\bar{b}_n]$, the reflected Poincaré dual of $[\bar{b}_n]$ applied to the vertical tangent bundle pair of a bundle $p\colon E \to B$ with compact n–manifold fibers. This is a vertical homotopy class of sections of $Q^{\bullet}_B(E) \to B$ which we identify with a well–known fiber homotopy invariant of the bundle $p\colon E \to B$ due to Becker–Gottlieb [BeGo1] and Dold [Do]. See also [Cla] and [DoP].

The Poincaré dual of the Becker–Euler class

Let M^n be compact, with tangent bundle τ. We write b_M and $\bar{b}_M$ for the Becker–Euler sections $b_n(\tau)$ and $\bar{b}_n(\tau)$, understood as sections of $\tau^{\bullet}$ which are trivial over ∂M. The goal here is to describe explicitly $\wp b_M$, the Poincaré dual of b_M. Our interest in $\wp b_M$ comes from the fact that it agrees with $\bar{\wp}\bar{b}_M$.

Embed M in $\mathbb{R}^w$ as a locally flat submanifold. Choose a normal bundle ν , increasing w if necessary. Let $\partial\nu = \nu|\partial M$. Let $c\colon \mathbb{S}^w \to \operatorname{th}(\nu)/\operatorname{th}(\partial\nu)$ be the collapse map. Now the Poincaré dual $\wp b_M$ is the composition

$$\mathbb{S}^w \xrightarrow{\;c\;} \operatorname{th}(\nu)/\operatorname{th}(\partial\nu) \xrightarrow{\;(\mathrm{id},b_M)\;} \operatorname{th}(\nu \oplus \tau) \tag{5.1}$$

where the map labelled (id, b_M) takes $z \in \nu_x$ to $(z, b_M(x))$ in $\nu_x \times \tau_x$. We think of (5.1) as an element in $\Omega^w(\mathbb{S}^w \wedge M^{\bullet}) \subset \Omega^w\Omega^{\infty}\Sigma^{\infty}(\mathbb{S}^w \wedge M^{\bullet})$, to be consistent.

The explicit description of $\wp b_M$ involves a contractible choice of embedding of M in a high–dimensional euclidean half–space, and normal bundle $\nu(M)$. Also, b_M is only determined up to contractible choice.

Becker–Euler class and Spanier–Whitehead duality

We recall the homotopy transfer of Becker–Gottlieb and Dold. This is defined for any fibration $p\colon E \to B$ where each fiber $p^{-1}(b)$ is homotopy equivalent to a homotopy finitely dominated CW–space. It is a section tr_p of a fibration

$$(Q_+)_B(E) \longrightarrow B$$

with fiber $Q_+(p^{-1}(x)) = Q(p^{-1}(x)_+)$ over $x \in B$, where the $+$ denotes an added base point. (Here we cannot write $Q^\bullet$ because the fibers $p^{-1}(x)$ might not be compact.) The composition of tr_p with the inclusion of $(Q_+)_B(E)$ in $Q_+(E)$ is a map $B \to Q_+(E)$, in other words a stable map from B_+ to E_+ ; hence the word *transfer*.

First we assume $B = *$. Let E_+^* be a Spanier–Whitehead 0–dual of E_+. In other words, E_+^* is a CW–spectrum with finitely many cells, equipped with a map

$$\kappa\colon \mathbf{S}^0 \to E_+^* \wedge E_+$$

which is nondegenerate, i.e., slant product with $[\kappa]$ is an isomorphism from the cohomology of E_+ to the homology of E_+^* (with integer coefficients). In this case tr_p is a point in $Q_+(E)$, namely, the composition

$$\begin{array}{c} \mathbf{S}^0 \\ \downarrow{\scriptstyle \kappa} \\ E_+^* \wedge E_+ \\ \downarrow{\scriptstyle \mathrm{id}\wedge\mathrm{diag}} \\ E_+^* \wedge (E_+ \wedge E_+) \\ \downarrow{\scriptstyle \cong} \\ (E_+^* \wedge E_+) \wedge E_+) \\ \downarrow{\scriptstyle \kappa^*\wedge\mathrm{id}} \\ \mathbf{S}^0 \wedge E_+\,. \end{array} \tag{5.2}$$

Here $\kappa^*\colon E_+^* \wedge E_+ \longrightarrow \mathbf{S}^0$ is the transpose of κ. It must be chosen together with a homotopy h connecting the maps of bispectra

$$\mathbf{S}^0 \wedge \mathbf{S}^0 \xrightarrow{\mathrm{id}\wedge\kappa} \mathbf{S}^0 \wedge (E_+^* \wedge E_+)\ ,$$

$$\mathbf{S}^0 \wedge \mathbf{S}^0 \xrightarrow{\sigma_{(24)}(\kappa\wedge\kappa)} (E_+^* \wedge E_+) \wedge (E_+^* \wedge E_+) \xrightarrow{\kappa^*\wedge\mathrm{id}} \mathbf{S}^0 \wedge (E_+^* \wedge E_+)$$

where $\sigma_{(24)}$ is the automorphism of $(E_+^* \wedge E_+) \wedge (E_+^* \wedge E_+)$ interchanging the two factors E_+. The choice of a pair (κ^*, h) is a contractible choice. The point is that $E_+^* \wedge E_+$ is self–dual, the self–duality being given by $\sigma_{(24)}(\kappa \wedge \kappa)$.

In the case of an arbitrary base B, the choices of dual etc. must be made fiberwise, for each fiber of $p: E \to B$. We omit the details.

5.3. Example. Returning to the case $B = *$, assume that $E = M$ is a compact n–manifold, embedded in $\mathbb{R}^w$ with normal bundle ν and so on as in (5.1). Then we have geometric choices for E_+^*, κ, and κ^*. For E_+^* we take the formal w–fold desuspension of $\mathrm{th}(\nu)/\mathrm{th}(\partial\nu)$. For κ we take the formal w–fold desuspension of the composition

$$\mathbb{S}^w \xrightarrow{\;c\;} \mathrm{th}(\nu)/\mathrm{th}(\partial\nu) \xrightarrow{\;\mathrm{diag}\;} \mathrm{th}(\nu)/\mathrm{th}(\partial\nu) \ \wedge\ M_+$$

where "diag" denotes the map taking $z \in \nu_x$ to (z,x) in $\nu_x \times M$. To make κ^* we choose an embedding $u: M \to \mathrm{int}(M)$, isotopic to id_M. Then we choose a function $\psi: [0,\infty) \to [0,\infty)$ with support equal to $[0,d]$, where d is the distance from $u(M)$ to the complement in $\mathbb{R}^w$ of the total space of ν. Let κ^* be the formal w–fold desuspension of the map

$$\begin{array}{ccc} \mathrm{th}(\nu)/\mathrm{th}(\partial\nu) \ \wedge\ M_+ & \longrightarrow & \mathbb{R}^w \cup \infty \ \cong\ \mathbb{S}^w \\ (x,y) & \mapsto & \dfrac{x - u(y)}{\psi(|x-u(y)|)}\,. \end{array}$$

With these choices, (5.2) is the composition

$$\mathbb{S}^w \xrightarrow{\;c\;} \mathrm{th}(\nu)/\mathrm{th}(\partial\nu) \xrightarrow{\;j\;} \mathbb{S}^w \wedge M_+$$

where j is induced by a certain map over M from the total space of ν to $\mathbb{R}^w \times M$. By inspection, j agrees with (id, b_M) in (5.1), more precisely, the two agree after composition with $\mathbb{S}^w \wedge M_+ \to \mathbb{S}^w/K \wedge M_+$ where $K \subset \mathbb{S}^w \cong \mathbb{R}^w \cup \infty$ is a suitable disk containing the base point ∞. Hence (5.2) and (5.1) agree in the case at hand. We spell out the fiberwise version of this insight (returning to the $Q^\bullet$ notation since the fibers are compact):

5.4. Theorem. *Let $p: E \to B$ be a bundle with compact n–manifold fibers. Then $\mathrm{tr}_p: B \to Q_B^\bullet(E)$ agrees with $\wp b_n$, the fiberwise Poincaré dual of b_n of the vertical tangent bundle pair of p.*

5.5. Remark. Let Γ be the space of sections of $Q_B^\bullet(E) \to B$. Then tr_p is, strictly speaking, not a point in Γ, but a map $C_1 \to \Gamma$ where C_1 is a contractible space (the space of choices needed in the construction). Similarly, $\wp b_n$ is a map $C_2 \to \Gamma$ with contractible C_2. Our proof of 5.4 produces another map $C_3 \to \Gamma$ with contractible C_3, and maps $C_3 \to C_1$ as well as $C_3 \to C_2$ over Γ. In this sense tr_p and $\wp b_n$ *agree.*

The index theorem for regular manifolds

5.6. Definition. A *regular* manifold M^n is a topological manifold (with boundary) together with

- an n–disk bundle $q: L \longrightarrow M$;
- an open embedding $j: U \to L$ over M , where U is an open neighborhood of the diagonal in $M \times M$ (viewed as a space over M by means of the first projection).

We require in addition that $j(U_z)$ be the interior of L_z for $z \notin \partial M$, and L_z minus a point in ∂L_z for $z \in \partial M$. Here $U_z := U \cap (z \times M)$.

Remark. The homotopy theoretic content of 5.6 is that the classifying map for the tangent bundle pair of $(M, \partial M)$ comes with a specified lift across (4.9) to a map of pairs

$$(M, \partial M) \longrightarrow (B\,\mathrm{TOP}(W^\bullet, \partial(W^\bullet)), B\,\mathrm{TOP}(W, \partial W))$$

where $W = (-\infty, +\infty] \times \mathbb{R}^{n-1}$.

By a *bundle of regular n–manifolds*, we mean a bundle $p: E \to B$ with n–manifold fibers, together with a disk bundle $D \to E$, a neighborhood U of the diagonal in $E \times_B E$, and an embedding $U \to D$ over E which has the properties listed above fiberwise ; that is, the embedding $U|b \to D|b$ has those properties for each $b \in B$.

Example: A bundle $p: E \to B$ of smooth n–manifolds has a structure of bundle of regular manifolds, canonical up to contractible choice.

5.7. Theorem. *Let $p: E \to B$ be a bundle of compact regular n–manifolds. Then $\chi(p): B \to F_B(E)$ is vertically homotopic to the composition*

$$B \xrightarrow{\mathrm{tr}_p} Q^\bullet_B(E) \xrightarrow{\boldsymbol{\eta}_*} F_B(E)\,.$$

Remark. Our proof gives a contractible choice of preferred vertical homotopies.

Proof. Let τ stand for the vertical tangent bundle pair of $p: E \to B$. We showed $\chi(p) \simeq \bar{\wp}\bar{e}_n(\tau)$ in 3.19, see also 3.20 ; and $\bar{e}_n(\tau) = \boldsymbol{\eta}_* \bar{b}_n(\tau)$ in 4.10. Therefore $\chi(p) \simeq \boldsymbol{\eta}_*(\bar{\wp}\bar{b}_n(\tau)) = \boldsymbol{\eta}_*(\wp b_n(\tau)) \simeq \boldsymbol{\eta}_* \,\mathrm{tr}_p$ by 5.4. $\square$

Part II: Applications

In Part II we present the functors F and characteristics χ that we had in mind in Part I, beginning with 1.1. Specifically, §6 builds and explores characteristics (for certain spaces) which are natural under homotopy equivalences, whereas §9 builds an excisive characteristic (see 2.4).

6. Homotopy invariant characteristics

The A-Theory Characteristic

Notation. A space Y is *homotopy finite* if it is homotopy equivalent to a compact CW–space. It is *homotopy finitely dominated* if there exists a compact CW–space Z and maps $r\colon Z \to Y$, $i\colon Y \to Z$ such that $ri \simeq \mathrm{id}_Y$.

6.1. Example: The A–theory characteristic. We want to construct a characteristic as in 1.1 for Waldhausen's A–theory functor, restricted to the category $\mathcal{C}$ of homotopy finitely dominated spaces with homotopy equivalences as morphisms. (To ensure that $\mathcal{C}$ is small, we insist that any space in $\mathcal{C}$ must have underlying set contained in S_1 , where S_1 is a fixed uncountable set.) First we need to recall the definition of $A(Y)$ for a space Y.

K-theory of Waldhausen categories.
For any category $\mathcal{D}$ with cofibrations $\mathrm{cof}\,\mathcal{D}$ and weak equivalences $w\mathcal{D}$, Waldhausen [Wald, Sec.1.3] has constructed an infinite loop space $\Omega|wS_\bullet(\mathcal{D})|$ which we shall denote by $K(\mathcal{D})$. Furthermore, Waldhausen constructs a natural transformation $|w\mathcal{D}| \to K(\mathcal{D})$ which he observes is reminiscent of Segal's "group completion" process. Following Thomason we call a category with cofibrations and weak equivalences a *Waldhausen category.*

Retractive spaces over a space Y.
Waldhausen defines $A(Y)$ as the K-theory of a certain Waldhausen category of retractive spaces over Y which we now describe ; see [Wald, Sec. 2.1]. A retractive space over Y consists of a space X and a diagram

$$X \underset{s}{\overset{r}{\rightleftarrows}} Y$$

such that $rs = \mathrm{id}_Y$ and s is a closed embedding having the homotopy extension property. The retractive spaces over Y form a category where the morphisms are maps over and relative to Y. A morphism is a *cofibration* if the underlying map of spaces is a closed embedding having the homotopy extension property. It is a *weak equivalence* if the underlying map of spaces is a homotopy equivalence. With these notions of cofibration and weak equivalence, the category of retractive spaces over Y is a Waldhausen category. This follows from [Str]. A retractive space X

over Y is *homotopy finite* if it is the codomain of a weak equivalence from another retractive space over Y which is a CW–space relative to Y, with finitely many cells. A retractive space X over Y is *homotopy finitely dominated* if it fits into a diagram $X' \to W \to X$ of retractive spaces over Y such that W is homotopy finite and the composite morphism from X' to X is a weak equivalence.

Let $\mathcal{R}^{fd}(Y)$ be the Waldhausen category of homotopy finitely dominated retractive spaces over Y, and define $A(Y)$ as the K–theory of $\mathcal{R}^{fd}(Y)$.

A continuous map $f: Y_1 \to Y_2$ induces a functor $f_*: \mathcal{R}^{fd}(Y_1) \to \mathcal{R}^{fd}(Y_2)$, pushout with f, and then a map $A(Y_1) \to A(Y_2)$. This suggests that $Y \mapsto A(Y)$ is a functor. Unfortunately, to establish that we have to say exactly what we mean by pushout, and the pushout must be strictly associative and unital.

Fix another uncountable set S_2 and a bijection $b: S_1 \to S_2$. We decree, belatedly, that $A(Y)$ is only defined when the underlying set of Y is contained in S_1. In the definition of $\mathcal{R}^{fd}(Y)$ and $A(Y)$, we allow only retractive spaces $X \rightleftarrows Y$ over Y for which the section $Y \to X$ is an inclusion, and the underlying set of $X \smallsetminus Y$ is contained in S_2. Given a map $f: Y \to Y'$, where $Y, Y' \in S_1$, and an object $X \rightleftarrows Y$ in $\mathcal{R}^{fd}(Y)$, let $f_*X = (X \smallsetminus Y) \cup Y'$, with the pushout topology and the obvious retraction to Y'. Then f_* is a functor from $\mathcal{R}^{fd}(Y)$ to $\mathcal{R}^{fd}(Y')$. Most important: the rule $f \mapsto f_*$ respects composition, so that $g_* f_* = (gf)_*$.

We now construct a characteristic χ for the A–theory functor defined on the category $\mathcal{C}$. To be precise, let the objects of $\mathcal{C}$ be the finitely dominated spaces Y with $Y \subset S_1$. The morphisms in $\mathcal{C}$ are the continuous maps which are homotopy equivalences.

Since we have a natural map $|w\mathcal{R}^{fd}(Y)| \to A(Y)$ (the one that is reminiscent of group completion) it is enough to construct a characteristic χ for the functor $Y \mapsto |w\mathcal{R}^{fd}(Y)|$. We will do this using 1.3, with $\mathcal{F} = w\mathcal{R}^{fd}$. Accordingly, we must associate with every Y in $\mathcal{C}$ an object $Y^!$ in $w\mathcal{R}^{fd}(Y)$. Informally, $Y^!$ is just $\mathbb{S}^0 \times Y$ with the obvious retraction to and section from Y. Formally,

$$Y^! = b(Y) \cup Y, \tag{6.2}$$

topologized as a disjoint union, and viewed as a retractive space over Y by means of $b^{-1} \cup \mathrm{id}$. For a morphism $e: X \longrightarrow Y$ in $\mathcal{C}$ let $e^!$ from $e_*(X^!) = b(X) \cup Y$ to $Y^! = b(Y) \cup Y$ be the map $beb^{-1} \cup \mathrm{id}_Y$. This is a morphism in $w\mathcal{R}^{fd}(Y)$. The 1–cocycle condition is satisfied: $(ef)^! = e^! \cdot e_*(f^!)$ whenever e, f are composable. By 1.3 we get the desired characteristic χ for $|w\mathcal{R}^{fd}|$.

Remark. Often we are interested in a composition $A \cdot F$ where A is the A–theory functor, and F is a functor from some category to spaces. Strictly speaking we should then assume that the values of F are spaces whose underlying sets are contained in S_1. Usually, however, we will not mention this subtlety.

6.3. Definition–Summary. Let $p: E \to B$ be a fibration with homotopy finitely dominated fibers, where B is the geometric realization of a simplicial set $B^\natural$. We apply 1.6 with $\mathcal{C}$ equal to the category of homotopy finitely dominated spaces, with homotopy equivalences as morphisms, and $F = A|\mathcal{C}$. The characteristic χ from (6.1) and (6.2) then yields a section $\chi(p): B \to A_B(E)$, in the informal notation of 1.6. We call it the *parametrized A–theory Euler characteristic* of p. We will also sometimes write $\chi_h(p): B \to A_B(E)$, to stress the homotopy invariant nature of the construction.

Remark. Let $B = B\,\mathrm{G}(Y)$, where $\mathrm{G}(Y)$ is the simplicial monoid of homotopy automorphisms of Y. Let $p: E \to B\,\mathrm{G}(Y)$ be the universal fibration with fibers homotopy equivalent to Y. We obtain $\chi(p): B \to A_B(E)$ as above. In homotopy theoretic terms, the fibration $A_B(E) \to B$ amounts to an A_∞–action of $\mathrm{G}(Y)$ on $A(Y)$, and the section $\chi(p)$ means that $\chi(Y) \in A(Y)$ has been promoted to a homotopy fixed point of the action.

In the case where Y is connected and based, this refines the rather trivial observation that the sum of the ordinary Euler characteristic of Y, in $\mathbb{Z}$, and the Wall finiteness obstruction of Y, in the reduced K_0–group of $\mathbb{Z}\pi_1(Y)$, is an element of $K_0(\mathbb{Z}\pi_1(Y))$ which is invariant under the action of $\pi_0(G(Y))$.

Linearized characteristics

Let R be a (discrete) ring, with unit. Let Y be a space, homotopy equivalent to a CW–space. The bundles of f.g. projective left R–modules on Y, in other words covering spaces of Y where the fibers are continuously equipped with a structure of f.g. projective left R–module, are the objects of an exact category in the sense of Quillen [Qui]. See also [Kar]. Let $K_0(Y,R)$ be the Grothendieck group of that exact category. There is a canonical homomorphism

$$\beta_Y: K_0(Y,R) \to [Y, K(R)]$$

where the square brackets denote a set of homotopy classes of maps. It takes the class of a module bundle V to the homotopy class of ιc_V, where c_V is a classifying map for the bundle V, and ι is the inclusion of the classifying space for such bundles into its group completion, which is $K(R)$. (The target of c_V is $\coprod B\mathrm{GL}(P)$ where P runs through a set of representatives of the isomorphism classes of f. g. projective left R–modules.)

Now let $p: E \to B$ be a fibration, where the base is homotopy equivalent to a CW–space, and the fibers are homotopy finitely dominated. The Riemann–Roch problem raised by Bismut and Lott, as we see it, is to factorize a certain homomorphism

$$\beta_{E\downarrow B}: K_0(E,R) \longrightarrow [B, K(R)]$$

through $\beta_E: K_0(E,R) \to [E, K(R)]$. In §8 we obtain such factorizations under special hypotheses on $p: E \to B$. In this subsection, we merely define $\beta_{E\downarrow B}$, relate it to the parametrized characteristics of 6.1 and make it more explicit in special cases.

Definition of $\beta_{E\downarrow B}$. Following Waldhausen, we regard $K(R)$ as the K–theory of a Waldhausen category $\mathcal{C}(R)$. The objects of $\mathcal{C}(R)$ are the chain complexes of left projective R–modules which are homotopy equivalent to finitely generated ones (bounded below and above). The morphisms are the R–module chain maps ; a morphism is a weak equivalence if it is a homotopy equivalences, and a cofibration if it is split injective in each dimension.

As for $p: E \to B$, we adopt the notation of 1.6, assuming that B is the geometric realization of a simplicial set $B^\natural$. Let V be a bundle of f.g. projective R–modules on E. For each simplex z in $B^\natural$ let $J_V(z)$ be the singular chain complex of E_z with twisted coefficients in V. Then J_V is a functor from simp(B) to $w\mathcal{C}(R)$ and so induces a map from $|\operatorname{simp}(B)| \simeq B$ to $K(R)$, via the subspace $|w\mathcal{C}(R)|$ of $K(R)$. This map depends only on the class $[V] \in K_0(E, R)$. We call it $\beta_{E\downarrow B}([V])$.

Relation with the A–theory characteristic. Fix a bundle V of f.g. left projective R–modules on E. For a homotopy finitely dominated retractive space

$$X \underset{s}{\overset{r}{\rightleftarrows}} E$$

let $\lambda_V(X \leftrightarrows E)$ be the relative singular chain complex of the pair (X, E) with (twisted) coefficients in the bundle of modules $r^*(V)$. The functor λ_V induces a map of K–theory spaces, $A(E) \to K(R)$; see the remark just below.

> *Remark.* John Klein points out that the functor λ_V is not an *exact* functor. It respects cofibrations and weak equivalences, but it does not respect cobase change. For example, suppose that $X \to Y$ is a cofibration (and also an inclusion) in the domain category of λ_V. Then there is a canonical map from $\lambda_V(X)/\lambda_V(Y)$ to $\lambda_V(X/Y)$, but the map is rarely an isomorphism. Fortunately however, it is a chain homotopy equivalence. This suggests the remedy: Instead of using Waldhausen's $S_\bullet$ construction to define the K–theory of a category with cofibrations and weak equivalences, use the variation due to Thomason, given at the end of [Wald, §1.3]. This is natural with respect to functors which are slightly less than exact. In particular, λ qualifies.

6.4. Observation. *$\beta_{E\downarrow B}$ applied to the class of a bundle V is the homotopy class of the composition (explanation follows)*

$$B \xrightarrow{\chi(p)} A_B(E) \longrightarrow A(E) \xrightarrow{\lambda_V} K(R).$$

Explanation of terms. We still assume $B = |B^\natural|$. Let $A_B(E) = \operatorname{hocolim}_z A(E_z)$ as in 1.6. $A_B(E) \to A(E)$ is induced by the canonical maps $E_z \to E$. The characteristic determines as in 1.4, 1.6 a section up to homotopy of the projection $A_B E \to |\operatorname{simp}(B)| \sim B$. $\square$

Assuming that for each $b \in B$ and $i \geq 0$ the homology module $H_i(p^{-1}(b); V)$ has a finite length resolution by f.g. projective left R–modules, we shall give a homology theoretic description of $\beta_{E\downarrow B}$ applied to the class of V. Here is some K–theory background.

Recall that an exact category $\mathcal{M}$ is determined by an embedding of an additive category as a full subcategory of an abelian category $\mathcal{A}$ where $\mathcal{M}$ is closed under extensions in $\mathcal{A}$. See [Qui, p.16] and [Tho2, App.A]. A map f in $\mathcal{M}$ is an admissible monomorphism if it is a monomorphism in $\mathcal{A}$ and the cokernel is isomorphic to an object in $\mathcal{M}$. Dually, a map f in $\mathcal{M}$ is an admissible epimorphism if it is an epimorphism in $\mathcal{A}$ and the kernel is isomorphic to an object in $\mathcal{M}$. The two main examples for us are

$$\mathcal{M} = \mathcal{P}\mathcal{A}\,,$$

the category of projective objects in the abelian category $\mathcal{A}$; and

$$\mathcal{M} = \mathcal{N}\mathcal{P}\mathcal{A}\,,$$

consisting of those objects in the abelian category $\mathcal{A}$ which have finite length resolutions by objects in $\mathcal{P}\mathcal{A}$. The letter $\mathcal{N}$ can be read as *nearly.* If $\mathcal{A}$ is the category of finitely generated left modules over a ring R, then we write $\mathcal{P}_R$ and $\mathcal{N}\mathcal{P}_R$ instead of $\mathcal{P}\mathcal{A}$ and $\mathcal{N}\mathcal{P}\mathcal{A}$.

Notice that any morphism in $\mathcal{P}\mathcal{A}$ which is epi in $\mathcal{A}$ is admissible. A morphism in $\mathcal{P}\mathcal{A}$ which is mono in $\mathcal{A}$ is admissible iff it splits. However, all morphisms in $\mathcal{N}\mathcal{P}\mathcal{A}$ which are epi/mono in $\mathcal{A}$ are admissible. See [Ba, I.6.2]. Quillen's Q–construction associates to an exact category $\mathcal{M}$ an infinite loop space $K(\mathcal{M})$. See [Qui]. Quillen's resolution theorem [Qui] implies that the inclusion of $K(\mathcal{P}\mathcal{A})$ in $K(\mathcal{N}\mathcal{P}\mathcal{A})$ is a homotopy equivalence.

If $\mathcal{M}$ is an exact category, we can make $\mathcal{M}$ into a category with cofibrations and weak equivalences, by letting $\operatorname{cof}\mathcal{M}$ be the admissible monomorphisms in $\mathcal{M}$ and by letting $w\mathcal{M}$ be the isomorphisms in $\mathcal{M}$. Then there is natural equivalence from $K(\mathcal{M})$ in the sense of Quillen to $K(\mathcal{M})$ in the sense of Waldhausen [Wald, 1.9], [Gi, 9.3].

For geometric applications we want to have a chain complex theoretic description of $K(\mathcal{M})$. Let $\mathrm{ch}(\mathcal{M})$ be the category of chain complexes in $\mathcal{M}$ which are graded over $\mathbb{Z}$ and bounded above and below. We make $\mathrm{ch}(\mathcal{M})$ into a category with cofibrations and weak equivalences by letting $\operatorname{cof}\mathrm{ch}(\mathcal{M})$ be the chain maps which are degreewise admissible monomorphisms, and by letting $w\,\mathrm{ch}(\mathcal{M})$ be the quasi-isomorphisms, i.e. chain maps which induce isomorphisms in homology. (The homology groups of a chain complex in $\mathcal{M}$ are objects of $\mathcal{A}$.) The chain complexes concentrated in degree zero form a full subcategory which we identify with $\mathcal{M}$. In many cases the inclusion of Waldhausen K–theories, $K(\mathcal{M}) \to K(\mathrm{ch}(\mathcal{M}))$, is a homotopy equivalence.

6.5. Example. *The commutative square of inclusion maps*

$$\begin{tikzcd} K(\mathcal{P}\mathcal{A}) \arrow[r] \arrow[d] & K(\mathrm{ch}(\mathcal{P}\mathcal{A})) \arrow[d] \\ K(\mathcal{N}\mathcal{P}\mathcal{A}) \arrow[r] & K(\mathrm{ch}(\mathcal{N}\mathcal{P}\mathcal{A})) \end{tikzcd}$$

consists entirely of homotopy equivalences.

Proof. The upper horizontal arrow is a homotopy equivalence by [Wald, 1.7.1] and the approximation theorem [Wald, 1.6.7]. See [Tho2]. The right–hand vertical arrow is a homotopy equivalence by the approximation theorem, and the left–hand vertical arrow is a homotopy equivalence by Quillen's resolution theorem, mentioned earlier. $\square$

Remark. Let $\mathcal{A}$ be the category of finitely generated left R–modules. Replacing $\mathrm{ch}(\mathcal{PA})$ by the larger category of chain complexes of left projective R–modules which are *homotopy equivalent* to objects in $\mathrm{ch}(\mathcal{PA})$ does not change the homotopy type of the K–theory space. This follows directly from the approximation theorem.

Returning to an arbitrary exact category $\mathcal{M} \subset \mathcal{A}$, we introduce certain full subcategories of $\mathrm{ch}(\mathcal{M})$. Let $\mathrm{tch}(\mathcal{M})$ consist of the *trivial* chain complexes (with trivial differential), and let $\mathrm{sch}(\mathcal{M})$ consist of the very *special* chain complexes C whose homology groups H_iC belong to $\mathcal{M}$ for all i. Thus

$$\mathrm{tch}(\mathcal{M}) \subset \mathrm{sch}(\mathcal{M}) \subset \mathrm{ch}(\mathcal{M})$$

and we define $w\,\mathrm{tch}(\mathcal{M}) := \mathrm{tch}(\mathcal{M}) \cap w\mathcal{M}$, $\quad w\,\mathrm{sch}(\mathcal{M}) := \mathrm{sch}(\mathcal{M}) \cap w\mathcal{M}$. Note: it is irrelevant to us whether $\mathrm{sch}(\mathcal{M})$ is an exact subcategory of $\mathrm{ch}(M)$ or not.

In the proposition just below, we regard the homology functor H_* as a functor from $\mathrm{sch}(\mathcal{M})$ to $\mathrm{tch}(\mathcal{M})$. We also use a *restricted* product Π' of pointed spaces. It consists of those points (x_i) in the honest product for which $x_i \neq *$ for only finitely many i.

6.6. Proposition. *The following diagram commutes up to homotopy:*

$$\begin{tikzcd}
{|w\,\mathrm{sch}(\mathcal{M})|} \arrow[r, "H_*"] \arrow[d, "\subset"] & {|w\,\mathrm{tch}(\mathcal{M})|} \arrow[r, "\cong"] \arrow[d, "\subset"] & \prod'_{i\in\mathbb{Z}} |w\mathcal{M}| \arrow[d, "\subset"] \\
K(\mathrm{ch}(\mathcal{M})) \arrow[r, "="] & K(\mathrm{ch}(\mathcal{M})) & \prod'_{i\in\mathbb{Z}} K(\mathrm{ch}(\mathcal{M})) . \arrow[l, "\text{altern. sum}"']
\end{tikzcd}$$

Proof. Given a chain complex C in $\mathrm{sch}(\mathcal{M})$, let P_kC be the k–the Postnikov approximation to C. Thus $(P_kC)_i = C_i$ for $i \le k$, $(P_kC)_{k+1} = \mathrm{im}(\partial\colon C_{k+1} \to C_k)$, and $(P_kC)_i = 0$ for $i > k+1$. Let Q_kC be the kernel of the canonical projection from P_kC to $P_{k-1}C$. Then

$$Q_kC \longrightarrow P_kC \longrightarrow P_{k-1}C$$

is a functorial cofibration sequence in $\mathrm{sch}(\mathcal{M})$. The projection

$$Q_kC \longrightarrow H_*(Q_kC)$$

is a weak equivalence in $\mathrm{sch}(\mathcal{M})$, and of course $H_*(Q_kC)$ is concentrated in degree k and equal to H_kC there. Using this and an observation [Wald, 1.3.3] related to the additivity theorem, one finds that the left–hand square in 6.6 commutes up to homotopy. Commutativity of the right–hand square is a consequence of [Wald, 1.6.2] □

We note that 6.6 remains valid in the case $\mathcal{M} = \mathcal{P}_R$ if the chain complexes in $\mathrm{sch}(\mathcal{P}_R)$ and $\mathrm{ch}(\mathcal{R})$ are allowed to be chain complexes of projective left R–modules which are homotopy equivalent to finitely generated ones.

We return to $\beta_{E\downarrow B}$. Using 6.5 and 6.6, we get the following result.

6.7. Proposition. *Suppose that the homology groups of the fibers of* p *with coefficients in* V *admit finite length resolutions by f.g. projective* R*–modules. Then* $\beta_{E\downarrow B}$ *applied to the class of* V *is homotopic to the alternating sum* $\Sigma(-1)^i \ldots$ *of the maps*

$$B \xrightarrow{k(i)} |\,\mathrm{iso}(\mathcal{N}\mathcal{P}_R)| \xrightarrow{\subset} K(\mathcal{N}\mathcal{P}_R) \simeq K(R) \qquad (i \geq 0)$$

where $\mathcal{N}\mathcal{P}_R$ *consists of all left* R*–modules having a finite resolution by finitely generated projective ones, and* $k(i)$ *classifies the bundle on* B *with fiber* $H_i(p^{-1}(b); V)$ *over* $b \in B$. □

7. An excisive characteristic

Controlled A–theory

7.1. Definitions. Following [ACFP], [CP], [CPV] we introduce control. For us, a *control space* is a pair (P, Q) of locally compact Hausdorff spaces, with Q closed in P. Let $P_0 := P \smallsetminus Q$. An open subset $V \subset P_0$ is *admissible* if $V \cup Q$ is open in P.

Let X_1 and X_2 be retractive spaces over P_0 (with retractions $r_i : X_i \to P_0$ for $i = 1, 2$). A retractive map $X_1 \to X_2$ is, as usual, a map over P_0 and relative to P_0. A *retractive map germ* from X_1 to X_2 is an equivalence class of pairs (U, f), where $U \subset P_0$ is admissible open, and $f : r_1^{-1}(U) \longrightarrow X_2$ is a retractive map (that is, a map relative to U and over P_0). Two such pairs (U, f) and (V, g) are equivalent if f and g agree on $r^{-1}(W)$ for some admissible open W contained in $U \cap V$.

A *controlled map* from X_1 to X_2 is a map which is relative to P_0, and satisfies the following *control condition*:

> For every $z \in Q$ and neighborhood L of z in P, there exists a smaller neighborhood $L' \subset L$ of z in P such that $r_2 f(x) \in L$ whenever $r_1(x) \in L'$, and $r_1(x) \in L$ whenever $r_2 f(x) \in L'$ (for $x \in X_1$).

A *controlled map germ* from X_1 to X_2 is an equivalence class of pairs (U, f), where $U \subset P_0$ is admissible open, and $f : r^{-1}(U) \longrightarrow X_2$ is a map relative to U which satisfies the the control condition above (for all $x \in r^{-1}(U)$). Again, two such pairs

(U,f) and (V,g) define the same controlled map germ if f and g agree on $r^{-1}(W)$ for sufficiently small admissible open $W \subset P_0$. Note that a retractive map germ is also a controlled map germ.

Let $\mathcal{R}(P,Q)$, $\mathcal{RG}(P,Q)$, $\mathcal{C}(P,Q)$, $\mathcal{CG}(P,Q)$ be the categories whose objects are the retractive spaces over P_0 and whose morphisms are the retractive maps, retractive map germs, controlled maps, and controlled map germs, respectively. (To ensure that these categories depend functorially on (P,Q), we assume, as in 6.1, that the underlying set of P_0 is contained in S_1, and allow only retractive spaces $X \rightleftarrows P_0$ where $X \smallsetminus P_0$ is contained in S_2, as a set.) Then

$$\mathcal{R}(P,Q) \subset \mathcal{C}(P,Q),$$
$$\mathcal{RG}(P,Q) \subset \mathcal{CG}(P,Q).$$

We will define (below) notions of *homotopy* in $\mathcal{C}(P,Q)$ and $\mathcal{CG}(P,Q)$. This will lead very naturally to notions of *cofibration* and *weak equivalence* and *homotopy locally finitely dominated object* in $\mathcal{C}(P,Q)$ and $\mathcal{CG}(P,Q)$. We will then use these notions in $\mathcal{R}(P,Q)$ and $\mathcal{RG}(P,Q)$, respectively. In particular, the homotopy locally finitely dominated objects form full subcategories $\mathcal{R}^{\ell d}(P,Q)$, $\mathcal{RG}^{\ell d}(P,Q)$ of $\mathcal{R}(P,Q)$ and $\mathcal{RG}(P,Q)$, respectively. These subcategories become Waldhausen categories with the notions of cofibration and weak equivalence inherited from $\mathcal{C}(P,Q)$ and $\mathcal{CG}(P,Q)$, respectively.

Homotopy. Let X be retractive over P_0. We write $X \curlywedge I$ for the colimit of

$$X \times I \hookleftarrow P_0 \times I \to P_0$$

which is again a retractive space over P_0. Two morphisms $\phi, \psi \colon X \to Y$ in $\mathcal{C}(P,Q)$ are homotopic if there exists a morphism $\gamma \colon X \curlywedge I \longrightarrow Y$ in $\mathcal{C}(P,Q)$ such that $\gamma i_0 = \phi$ and $\gamma i_1 = \psi$, where $i_0 \colon X \to X \curlywedge I$ and $i_1 \colon X \to X \curlywedge I$ are the maps (morphisms) given by $x \mapsto (x,0)$ and $x \mapsto (x,1)$, respectively. Likewise, two morphisms $\phi, \psi \colon X \to Y$ in $\mathcal{CG}(P,Q)$ are homotopic if there exists a morphism $\gamma \colon X \curlywedge I \longrightarrow Y$ in $\mathcal{CG}(P,Q)$ such that $\gamma i_0 = \phi$ and $\gamma i_1 = \psi$.

Weak equivalences. We obtain homotopy categories $\mathcal{HC}(P,Q)$ and $\mathcal{HCG}(P,Q)$ by identifying homotopic morphisms in $\mathcal{C}(P,Q)$ and in $\mathcal{CG}(P,Q)$, respectively. A morphism in $\mathcal{C}(P,Q)$ is a *weak equivalence* if it becomes an isomorphism in $\mathcal{HC}(P,Q)$. A morphism in $\mathcal{CG}(P,Q)$ is a *weak equivalence* if it becomes an isomorphism in $\mathcal{HCG}(P,Q)$.

Cofibrations. Let $\phi \colon X_1 \to X_2$ be a morphism in $\mathcal{C}(P,Q)$. Let $r_i \colon X_i \to P_0$ be the retractions. We say that ϕ is a *cofibration* if it is a closed embedding and has the homotopy extension property (for homotopies in $\mathcal{CG}(P,Q)$, as defined above).

Let $\phi \colon X_1 \to X_2$ be a morphism in $\mathcal{CG}(P,Q)$. Let $r_i \colon X_i \to P_0$ be the retractions. We say that ϕ is a *closed embedding* if, for some representative (U,f) of ϕ, there exists an open admissible $V \subset P_0$ such that $(r_2 f)^{-1}(V) \subset r_1^{-1}(U)$ and f restricts to a closed embedding from $(r_2 f)^{-1}(V)$ to $r_2^{-1}(V)$. We say that ϕ is a *cofibration* if it is a closed embedding and has the homotopy extension property (for homotopies in $\mathcal{CG}(P,Q)$, as defined above).

Homotopy locally finite domination. Let X be a retractive space over P_0 with retraction $r\colon X \to P_0$. A *controlled CW–structure* on X is a structure of relative CW–space on X (relative to P_0) with the following property.

> For every $z \in Q$ and neighborhood L of z in P, there exists a smaller neighborhood L' of z in Q such that any cell of X which has nonempty intersection with $r^{-1}(L')$ is contained in $r^{-1}(L)$.

An object Y in $\mathcal{C}(P,Q)$ or $\mathcal{CG}(P,Q)$ is *homotopy locally finitely dominated* if, in $\mathcal{HC}(P,Q)$ or in $\mathcal{HCG}(P,Q)$, as appropriate, it is a retract of some X with a controlled CW–structure which is

- locally finite (i.e., X is locally compact) ;
- finite dimensional (i.e., X equals its relative n–skeleton for some n).

Now we define $\mathcal{R}^{\ell d}(P,Q)$ and $\mathcal{RG}^{\ell d}(P,Q)$ as explained above. We leave it to the reader to verify that $\mathcal{R}^{\ell d}(P,Q)$ and $\mathcal{RG}^{\ell d}(P,Q)$ are Waldhausen categories.

7.2. Observation. *Let (P,Q) and (P',Q') be control spaces. A proper map w from P to P' with $w^{-1}(Q') = Q$ induces an exact functor from $\mathcal{R}^{\ell d}(P,Q)$ to $\mathcal{R}^{\ell d}(P',Q')$, and another from $\mathcal{RG}^{\ell d}(P,Q)$ to $\mathcal{RG}^{\ell d}(P',Q')$, by pushout along w (compare 6.1).*

Proof. We begin with the following fact. Suppose given a closed $C' \subset P'$. For any neighborhood V of $w^{-1}(C')$, there exists a neighborhood V' of C' such that $w^{-1}(V') \subset V$.

Now take $C' = Q'$ and deduce that, for any neighborhood V of Q in P, there exists a neighborhood V' of Q' in P' such that $w^{-1}(V') \subset V$. Also, try letting C' consist of a single point z' in Q', and deduce that for any neighborhood V of $w^{-1}(z')$ there exists a neighborhood V' of z' such that $w^{-1}(V') \subset V$. Together, these observations show that pushout along w takes controlled maps to controlled maps, and controlled map germs to controlled map germs. They also show that pushout along w takes objects with controlled CW–structures to objects with controlled CW–structures. It follows easily that $w_*\colon\mathcal{R}(P,Q) \to \mathcal{R}(P',Q')$ and $w_*\colon\mathcal{RG}^{\ell d}(P,Q) \to \mathcal{RG}^{\ell d}(P',Q')$ respect cofibrations (and pushouts), and weak equivalences, and homotopy locally finite dominations. $\square$

7.3. Observation. *Let (P,Q) be a control space. For open $U \subset P$, there is an exact restriction functor from $\mathcal{RG}^{\ell d}(P,Q)$ to $\mathcal{RG}^{\ell d}(U,Q\cap U)$.*

Explanation. Let $U_0 = U \cap P_0$. The restriction functor is $X \mapsto X|U_0$, where X is short for a retractive space over P_0 with retraction $r\colon X \to P_0$, and $X|U_0$ is $r^{-1}(U_0)$, a retractive space over U_0. It is clear that $X \mapsto X|U$ respects cofibrations (and pushouts). The surprise is that it also respects weak equivalences. And the biggest surprise, perhaps, is that it also takes homotopy locally finitely dominated objects to homotopy locally finitely dominated objects. All of this is true by inspection. $\square$

7.4. Corollary. *The rule* $Y \mapsto K(\mathcal{R}\mathcal{G}^{\ell d}(Y \times [0,\infty], Y\times\infty))$ *is a functor from* $\mathcal{E}^\bullet$ *to pointed spaces.*

Proof. Let $f\colon Y_1^\bullet \to Y_2^\bullet$ be a pointed map, where Y_1 and Y_2 are ENR's. Let $V = f^{-1}(Y_2)$. The restriction of f to V is a proper map $V \to Y_2$ which induces an exact functor

$$\mathcal{R}\mathcal{G}^{\ell d}(V \times [0,\infty], V\times\infty) \longrightarrow \mathcal{R}\mathcal{G}^{\ell d}(Y_2 \times [0,\infty], Y_2\times\infty)$$

as in 7.2. There is another exact functor

$$\mathcal{R}\mathcal{G}^{\ell d}(Y_1 \times [0,\infty], Y_1\times\infty) \longrightarrow \mathcal{R}\mathcal{G}^{\ell d}(V \times [0,\infty], V\times\infty)$$

given by restriction as in 7.3. The map of K–theories induced by f that we need to define is the map induced by the composite exact functor

$$\mathcal{R}\mathcal{G}^{\ell d}(Y_1\times[0,\infty], Y_1\times\infty) \to \mathcal{R}\mathcal{G}^{\ell d}(V\times[0,\infty], V\times\infty) \to \mathcal{R}\mathcal{G}^{\ell d}(Y_2\times[0,\infty], Y_2\times\infty)\,. \quad \square$$

7.5. Theorem. *The functor* $Y \mapsto K(\mathcal{R}\mathcal{G}^{\ell d}(Y \times [0,\infty], Y\times\infty))$ *on* $\mathcal{E}^\bullet$ *is pro-excisive.*

Proof. This is essentially in [CPV], but we give some guidance. A functor from $\mathcal{E}^\bullet$ to spaces is pro–excisive if and only if its restriction to $\mathcal{E}$ is pro–excisive, by definition. Our verification that $Y \mapsto K(\mathcal{R}\mathcal{G}^{\ell d}(Y \times [0,\infty], Y\times\infty))$ is pro–excisive on $\mathcal{E}$ has two steps.

(1) The first step consists in observing that a functor from $\mathcal{E}$ to pointed spaces is pro–excisive if and only if its restriction to the subcategory $\mathcal{C}\mathcal{X}$ of (countable, finite dimensional, locally finite) simplicial complexes, with simplicial embeddings as morphisms, is pro–excisive. Of course, we will have to say exactly what we mean by a pro–excisive functor on $\mathcal{C}\mathcal{X}$.

(2) In a second step we show that $K(\mathcal{R}\mathcal{G}^{\ell d}(Y \times [0,\infty], Y\times\infty))$ for Y in $\mathcal{C}\mathcal{X}$ is related by a chain of natural homotopy equivalences to $\Omega^\infty\mathbf{J}(Y^\bullet)$, where $\mathbf{J}$ is a *reduced Steenrod functor* from compact metrizable spaces to spectra. (We deduce this from [CPV, 3.1].) The axioms for a reduced Steenrod functor [CPV] imply directly that $\Omega^\infty\mathbf{J}$ is pro–excisive on $\mathcal{C}\mathcal{X}$.

Step 1. Let T be a functor from $\mathcal{C}\mathcal{X}$ to spaces. We assume that T is pro–excisive, which means (by definition):

- for any morphism $f\colon Y \to Z$ in $\mathcal{C}\mathcal{X}$ which is a proper homotopy equivalence, the induced map $T(Y) \to T(Z)$ is a weak homotopy equivalence ;
- for any Y in $\mathcal{C}\mathcal{X}$ which is a union of subobjects Y_1 and Y_2, the following commutative square is homotopy Cartesian:

$$\begin{array}{ccc} T(Y_1 \cap Y_2) & \longrightarrow & T(Y_1) \\ \big\downarrow & & \big\downarrow \\ T(Y_2) & \longrightarrow & T(Y)\ ; \end{array}$$

- $T(Y \times [0,\infty[\,)$ is weakly contractible for any Y in $\mathcal{C}\mathcal{X}$;
- for $Y = \amalg_{j\in\mathbb{N}} Y_j$ the canonical homomorphism (compare 2.1) from $\pi_*T(Y)$ to $\prod_j \pi_*T(Y_j)$ is an isomorphism.

Suppose that $T^{\sim}$ is an extension of T to all of $\mathcal{E}$. We are going to verify that $T^{\sim}$ is pro–excisive on $\mathcal{E}$.

- *Homotopy invariance.* It is enough to show that for X in $\mathcal{E}$, the inclusion $X \times 0 \hookrightarrow X \times I$ induces a weak homotopy equivalence from $T^{\sim}(X \times 0)$ to $T^{\sim}(X \times I)$. Choose a proper retraction

$$Y \underset{s}{\overset{r}{\rightleftarrows}} X$$

with Y in $\mathcal{CX}$. Then $T^{\sim}(X)$ is a retract of $T(Y)$, and similarly $T^{\sim}(X \times I)$ is a retract of $T(Y \times I)$ because $X \times I$ is a retract of $Y \times I$. Since the inclusion–induced map $T(Y \times 0) \to T(Y \times I)$ is a weak homotopy equivalence (for any simplicial complex structure on $Y \times I$ extending the given one on $Y \times 0$) and respects the retractions, $T^{\sim}(X \times 0)$ maps by a weak homotopy equivalence to $T^{\sim}(X \times I)$.
- *Excision.* Because of the homotopy invariance, it is enough to check that if X in $\mathcal{E}$ is the pushout of a diagram of proper embeddings $X_1 \hookleftarrow X_0 \hookrightarrow X_2$ with X_0, X_1, X_2 in $\mathcal{E}$, then the commutative diagram

$$\begin{array}{ccc} T^{\sim}(X_0) & \longrightarrow & T^{\sim}(X_1) \\ \downarrow & & \downarrow \\ T^{\sim}(X_2) & \longrightarrow & T^{\sim}(X) \end{array}$$

is homotopy Cartesian. Now it is possible to find Y in $\mathcal{CX}$ and a proper map $r\colon Y \to X$ with section $s\colon X \to Y$ such that $Y_1 := r^{-1}(X_1)$ and $Y_2 := r^{-1}(X_2)$, and therefore also $Y_0 := r^{-1}(X_0)$, are simplicial subcomplexes. Homotopy Cartesian–ness of the above square then follows from homotopy Cartesian–ness of the square

$$\begin{array}{ccc} T(Y_0) & \longrightarrow & T(Y_1) \\ \downarrow & & \downarrow \\ T(Y_2) & \longrightarrow & T(Y)\,. \end{array}$$

- *Contractibility axiom.* The retraction argument shows that $T^{\sim}(X \times [0, \infty[\,)$ is contractible for any X in $\mathcal{E}$.
- *Coproduct axiom.* The retraction argument shows that the canonical homomorphism $\pi_* T^{\sim}(\amalg X_j) \longrightarrow \prod_j \pi_* T^{\sim}(X_j)$ is an isomorphism.

STEP 2. Here we find it more convenient to use $I = [0,1]$ instead of $[0, \infty]$. If a base point in I is needed, let it be 0. Let $\mathbf{K}$ be the K–theory functor from Waldhausen categories to spectra. By [CPV, 3.1], there exists a reduced Steenrod functor $\mathbf{J}$ from compact metrizable spaces to spectra such that

$$\mathbf{K}(\mathcal{R}^{\ell d}(Y^\bullet \wedge I, Y^\bullet \wedge \partial I)) \quad \simeq \quad \mathbf{J}(Y^\bullet)$$

for Y in $\mathcal{CX}$, by a chain of natural homotopy equivalences. So it only remains for us to find a natural homotopy equivalence

$$\mathbf{K}(\mathcal{R}^{\ell d}(Y^\bullet \wedge I, Y^\bullet \wedge \partial I)) \longrightarrow \mathbf{K}(\mathcal{R}\mathcal{G}^{\ell d}(Y \times [0,1], Y \times 1))\,.$$

We note that the objects of $\mathcal{R}^{\ell d}(Y^\bullet \wedge I, Y^\bullet \wedge \partial I)$ are retractive spaces over $Y\times]0,1[$ subject to certain conditions, and the objects of $\mathcal{R}\mathcal{G}^{\ell d}(Y\times]0,1], Y\times 1)$ are also retractive spaces over $Y\times]0,1[$, subject to less stringent conditions. So there is an obvious exact functor

$$\mathcal{R}^{\ell d}(Y^\bullet \wedge I, Y^\bullet \wedge \partial I) \longrightarrow \mathcal{R}\mathcal{G}^{\ell d}(Y\times]0,1], Y\times 1)\,.$$

To show that this induces a homotopy equivalence of the K–theory spectra, we factorize it as

$$\mathcal{R}^{\ell d}(Y^\bullet \wedge I, Y^\bullet \wedge \partial I) \longrightarrow \mathcal{Q} \longrightarrow \mathcal{R}\mathcal{G}^{\ell d}(Y\times]0,1], Y\times 1)$$

where $\mathcal{Q}$ is $\mathcal{R}^{\ell d}(Y^\bullet \wedge I, Y^\bullet \wedge \partial I)$, but with a weaker notion of weak equivalence, namely, the one pulled back from $\mathcal{R}\mathcal{G}^{\ell d}(Y\times]0,1], Y\times 1)$. An application of the approximation theorem [Wald] shows that the second functor in the factorization induces a homotopy equivalence of K–theories. The first functor in the factorization induces a map of the K–theories which is clearly onto on π_0. The fibration theorem [Wald] identifies its homotopy fiber as the K–theory of $\mathcal{Z}$, the full subcategory of $\mathcal{R}^{\ell d}(Y^\bullet \wedge I, Y^\bullet \wedge \partial I)$ consisting of the objects which become weakly equivalent to $*$ in $\mathcal{Q}$ (which means, weakly equivalent to $*$ in $\mathcal{R}\mathcal{G}^{\ell d}(Y\times]0,1], Y\times 1)$).

It remains to show that $\mathbf{K}(\mathcal{Z}) \simeq *$. Let $\mathcal{Z}' \subset \mathcal{Z}$ be the full subcategory consisting of the objects which become *isomorphic* to $*$ in $\mathcal{R}\mathcal{G}^{\ell d}(Y\times]0,1], Y\times 1)$. Another application of the approximation theorem shows that the inclusion of $\mathbf{K}(\mathcal{Z}')$ in $\mathbf{K}(\mathcal{Z})$ is a homotopy equivalence. Finally, an Eilenberg swindle as in [Vo1] shows that $\mathbf{K}(\mathcal{Z}')$ is contractible. □

7.6. Observation. *The pro–excisive functor $Y \mapsto K(\mathcal{R}\mathcal{G}^{\ell d}(Y\times[0,\infty], Y\times\infty))$ on $\mathcal{E}^\bullet$ has coefficient spectrum homotopy equivalent to $\mathbb{S}^1 \wedge \mathbf{A}(*)$. Here $\mathbf{A}(*)$ is the spectrum determined by the infinite loop space $A(*)$.*

Proof. By step 2 in the proof of 7.5, the coefficient spectrum is

$$\{K(\mathcal{R}^{\ell d}(\mathbb{D}^{i+1}, \mathbb{S}^i)) \mid i \geq 0\} \tag{7.7}$$

up to a weak homotopy equivalence. But it is well known from [Vo1], ... , [Vo2], [Vo3] that (7.7) is a connected delooping of $\mathbf{A}(*)$. □

The excisive A–theory characteristic

The goal is to define an excisive characteristic as in 2.4. The pro–excisive functor F on $\mathcal{E}^\bullet$ which we have in mind is essentially Ω of the one in 7.5. The characteristic is only defined for $F|\ell\mathcal{E}^\bullet$, as in §2.

Outline. Given Y in $\mathcal{E}^\bullet$, we define a full subcategory $\mathcal{V}(Y)$ of $\mathcal{R}^{\ell d}(Y\times[0,\infty], Y\times\infty)$ and make it into a Waldhausen category in its own right. (The symbol $\mathcal{V}$ should suggest something in the direction of *vanishing.*) The important properties of $\mathcal{V}(Y)$ are these:

- the inclusion functor $\mathcal{V}(Y) \longrightarrow \mathcal{R}^{\ell d}(Y \times [0,\infty], Y \times \infty)$ is exact ;
- $K(\mathcal{V}(Y))$ is contractible ;
- The space $(Y \times 0) \amalg (Y \times [0,\infty[)$, as a retractive space over $Y \times [0,\infty[$, belongs to $\mathcal{V}(Y)$.

We see that the object $(Y \times 0) \amalg (Y \times [0,\infty[)$ of $\mathcal{V}(Y)$ becomes isomorphic to the zero object in $\mathcal{RG}^{\ell d}(Y \times [0,\infty], Y \times \infty)$, and so determines a point $\chi(Y)$ in

$$\begin{aligned} F(Y) &:= \text{hofiber}\,\big[\,K(\mathcal{V}(Y)) \longrightarrow K(\mathcal{RG}^{\ell d}(Y \times [0,\infty], Y \times \infty))\,\big] \\ &\simeq \Omega K(\mathcal{RG}^{\ell d}(Y \times [0,\infty], Y \times \infty)) \,. \end{aligned}$$

Remark. In the remainder of this section, we want to make sure that all Waldhausen categories in sight have a unique zero object (an object which is initial and terminal). This means, for example, that our earlier definition of $\mathcal{RG}^{\ell d}(Y \times [0,\infty], Y \times \infty)$ needs to be modified: identify all zero objects.

7.8. Details.

Definition of $\mathcal{V}(Y)$ and exactness of the inclusion functor. The objects of $\mathcal{V}(Y)$ are, briefly, the proper retractive ENR's over $Y \times [0,\infty[$. A *proper retractive* ENR over $Y \times [0,\infty[$ is a retractive space of the form

$$X \underset{s}{\overset{r}{\rightleftarrows}} Y \times [0,\infty[$$

where X is an ENR and r is a proper map. The morphisms in $\mathcal{V}(Y)$ are the retractive maps. A morphism in $\mathcal{V}(Y)$ is a *cofibration* if it is injective. A morphism $f\colon X_1 \to X_2$ is a weak equivalence in $\mathcal{V}(Y)$ if and only if, for any open cover $\mathcal{U}$ of Y, there exists a controlled map $g\colon X_2 \to X_1$ and controlled homotopies $h\colon gf \simeq \text{id}$ and $j\colon gf \simeq \text{id}$ such that each of the tracks $\{r_1 j_t(x) | t \in I\}$ and $\{r_2 h_t(x) | t \in I\}$ is contained in $U \times [0,\infty[$ for some $U \in \mathcal{U}$.

We have now made $\mathcal{V}(Y)$ into a Waldhausen category. The inclusion functor from $\mathcal{V}(Y)$ to $\mathcal{R}^{\ell d}(Y \times [0,\infty], Y \times \infty)$ is exact.

Contractibility of $K(\mathcal{V}(Y))$. This is based on an Eilenberg swindle. We will show that there exists self–maps

$$a\colon K(\mathcal{V}(Y)) \to K(\mathcal{V}(Y)), \qquad b\colon K(\mathcal{V}(Y)) \to K(\mathcal{V}(Y))$$

such that $a \simeq \text{id}$ and $b \simeq \text{id} + ab$ (where we use the infinite loop space structure on $\mathcal{V}(Y)$ to make sense of the addition). Hence the identity map of $K(\mathcal{V}(Y))$ is nullhomotopic.

For $s \geq 0$ let $\rho^s\colon Y \times [0,\infty[\to Y \times [0,\infty[$ be the map $(y,t) \mapsto (y, t+s)$. We write $\rho^1 = \rho$. The map a is induced by the exact functor ρ_* from $\mathcal{V}(Y)$ to $\mathcal{V}(Y)$. The map b is induced by the exact functor

$$X \mapsto \coprod_{i=0,1,2,\dots} \rho^i_* X$$

from $\mathcal{V}(Y)$ to $\mathcal{V}(Y)$. The additivity theorem implies immediately that $b \simeq \mathrm{id} + ab$. To show that $a \simeq \mathrm{id}$, we introduce another exact functor

$$\psi\colon \mathcal{V}(Y) \longrightarrow \mathcal{V}(Y)$$

and natural weak equivalences $X \to \psi(X) \leftarrow \rho_*(X)$. Namely, $\psi(X)$ is the quotient of $X \times I$ by the relations $(x,s) \simeq (\rho^s(x), 0)$ for $x \in Y \times [0,\infty[$. (Remember that X contains $Y \times [0,\infty[$ as a retract.) Then $\psi(X)$ maps to $Y \times [0,\infty[$ by means of $(x,s) \mapsto \rho^s(r(x))$, where r is the retraction for X, and $Y \times [0,\infty[$ maps to $\psi(X)$ by $z \mapsto (j(z), 0)$ where $j\colon Y \times [0,\infty[\to X$ is the structural section for X. Therefore $\psi(X)$ is retractive over $Y \times [0,\infty[$. The natural weak equivalences $X \to \psi(X)$ and $\rho_*(X) \to \psi(X)$ are given by $x \mapsto (x,0)$ and $\rho(x) \mapsto (x,1)$. (We have written $\rho(x)$ for the image of $x \in X$ under an obvious embedding $X \to \rho_*(X)$ which covers the map $\rho\colon Y \times [0,\infty[\longrightarrow Y \times [0,\infty[$.)

The characteristic object. $(Y \times 0) \amalg (Y \times [0,\infty[)$ is an object of $\mathcal{V}(Y)$, which we denote by $Y^!$. Moreover, $Y^!$ maps to the zero object (see remark preceding 7.8) in $\mathcal{R}\mathcal{G}^{\ell d}(Y \times [0,\infty[, Y \times \infty)$. Hence it determines a point $\chi(Y) \in F(Y)$.

Naturality. Let $f\colon Y_1^\bullet \to Y_2^\bullet$ be a pointed map, where Y_1 and Y_2 are ENR's. We decompose f as in the proof of 7.4:

$$Y_1^\bullet \to W^\bullet \to Y_2^\bullet$$

where $W = f^{-1}(Y_2)$. Using that decomposition, we found that f induces an exact functor

$$\mathcal{R}\mathcal{G}^{\ell d}(Y_1 \times [0,\infty[, Y_1 \times \infty) \longrightarrow \mathcal{R}\mathcal{G}^{\ell d}(Y_2 \times [0,\infty[, Y_2 \times \infty)\,.$$

This led to 7.4. The same recipe gives an exact functor from $\mathcal{V}(Y_1)$ to $\mathcal{V}(Y_2)$, and we conclude that $Y \mapsto K(\mathcal{V}(Y))$ is a functor on $\mathcal{E}^\bullet$. Hence F is a functor on $\mathcal{E}^\bullet$.

Moreover, it is clear that the pullback functor $\mathcal{V}(Y_1) \to \mathcal{V}(W)$ takes the characteristic object $Y_1^!$ to $W^!$. If $f|W$ from W to Y_2 is cell–like, then the retractive map

$$(W \times \varepsilon) \amalg (Y_2 \times [0,\infty[) \longrightarrow (Y_2 \times \varepsilon) \amalg (Y_2 \times [0,\infty[)$$

given by $f|W$ on $(W \times \varepsilon) \cong W$ is a weak equivalence in $\mathcal{V}(Y_2)$. Hence we obtain by the recipe of 1.3 a characteristic χ for the functor on $\ell\mathcal{E}^\bullet$ given by

$$Y \quad\mapsto\quad \mathrm{fiber}\left(|w\mathcal{V}(Y)| \to |w\mathcal{R}\mathcal{G}^{\ell d}(Y \times [0,\infty[, Y \times \infty)|\right)\,.$$

This functor is a subfunctor of $F|\ell\mathcal{E}^\bullet$, so that, finally, we have a characteristic χ for the functor $F|\ell\mathcal{E}^\bullet$.

7.9. Remark. The functor F on $\mathcal{E}^\bullet$ is pro–excisive, with coefficient spectrum

$$\mathbf{F}(*) \simeq \mathbf{A}(*)\,.$$

Proof. This follows from 7.6. □

7.10. Notation. Because of 7.9, we write $F(Y) =: {}_{\ell f}A^{\%}(Y)$ or, if Y is compact, $F(Y) = A^{\%}(Y)$. See §8 for more justification.

7.11. Definition–Summary. Let $p: E \to B$ be a fiber bundle with compact topological manifold fibers, where B is the geometric realization of a simplicial set $B^{\natural}$. Applying 2.8 with $F = {}_{\ell f}A^{\%}$ and χ as in 7.8 yields

$$\chi(p): B \to A^{\%}_B(E) ,$$

a section of $A^{\%}_B(E) \to B$. This is in the informal notation of 1.6. We call $\chi(p)$ the *parametrized $A^{\%}$–theory Euler characteristic* of p.

8. Riemann–Roch theorems

Assembly

8.1. Definitions. Let $\mathcal{C}$ be the category of all spaces which are homotopy equivalent to compact CW–spaces. A functor $\mathbf{F}$ from $\mathcal{C}$ to CW–spectra is *homotopy invariant* if it takes homotopy equivalences to homotopy equivalences. A homotopy invariant $\mathbf{F}$ is *excisive* if $\mathbf{F}(\emptyset)$ is contractible and if $\mathbf{F}$ preserves homotopy pushout squares.

8.2. Proposition–Definition. *For any homotopy invariant functor $\mathbf{F}$ from $\mathcal{C}$ to CW–spectra, there exists an excisive (and homotopy invariant) $\mathbf{F}^{\%}$ from $\mathcal{C}$ to CW–spectra and a natural transformation*

$$\alpha = \alpha_{\mathbf{F}}: \mathbf{F}^{\%} \longrightarrow \mathbf{F}$$

such that $\alpha: \mathbf{F}^{\%}(*) \to \mathbf{F}(*)$ is a homotopy equivalence. There is a construction of $\mathbf{F}^{\%}$ and $\alpha_{\mathbf{F}}$ which is natural in $\mathbf{F}$.

Proposition 8.2 is a mild variation on [WWa, Thm.1.1]. The proof goes through. We call α the *assembly.* The concept and the name originated in Frank Quinn's thesis [Qun1], [Qun2].

Remark. There is a chain of natural homotopy equivalences relating $\mathbf{F}^{\%}(X)$ to $X_+ \wedge \mathbf{F}(*)$. See [WWa, 1.2].

Remark. Natural dependence of $\alpha_{\mathbf{F}}$ on $\mathbf{F}$ implies "universality" of $\alpha_{\mathbf{F}}$. That is, $\alpha_{\mathbf{F}}: \mathbf{F}^{\%} \to \mathbf{F}$ is the best approximation (from the left) of $\mathbf{F}$ by an excisive functor. Namely, suppose that $\beta: \mathbf{E} \to \mathbf{F}$ is another natural transformation where $\mathbf{E}$ is excisive on $\mathcal{C}$. We have a commutative square

$$\begin{array}{ccc} \mathbf{E}^{\%}(X) & \xrightarrow{\alpha_{\mathbf{E}}} & \mathbf{E}(X) \\ \downarrow{\scriptstyle \beta^{\%}} & & \downarrow{\scriptstyle \beta} \\ \mathbf{F}^{\%}(X) & \xrightarrow{\alpha_{\mathbf{F}}} & \mathbf{F}(X) \end{array}$$

which is natural in X. The arrow $\alpha_{\mathbf{E}}$ is a homotopy equivalence, showing that β essentially factors through $\alpha_{\mathbf{F}}$. If $\beta: \mathbf{E}(*) \to \mathbf{F}(*)$ happens to be a homotopy equivalence, then $\beta^{\%}: \mathbf{E}^{\%}(X) \to \mathbf{F}^{\%}(X)$ is also homotopy equivalence for all X. See [WWa, §1]. For this reason, we will say that $\beta: \mathbf{E} \to \mathbf{F}$ is *an assembly transformation* if $\mathbf{E}$ is excisive on $\mathcal{C}$ and $\beta: \mathbf{E}(*) \to \mathbf{F}(*)$ is a homotopy equivalence.

8.3. Notation–Example. The functor $Y \mapsto A(Y)$ has an Ω–spectrum–valued version, $Y \mapsto \mathbf{A}(Y)$, so that the zero–th term of $\mathbf{A}(Y)$ is $A(Y)$. We abbreviate $\Omega^\infty \mathbf{A}^{\%}(Y)$ to $A^{\%}(Y)$, for Y in $\mathcal{C}$.

Comments. The space $A(Y)$ is the underlying space of a Γ–space [Se2] determined by the coproduct of retractive spaces over Y , and the abelian monoid $\pi_0 A(Y)$ is an abelian group. Therefore Segal's machine delivers an Ω–spectrum $\mathbf{A}(Y)$ with zero–th term $A(Y)$.

We already have a definition of $A^{\%}(Y)$ when Y is a compact ENR, from 7.10 ; but there is no serious clash, since both versions of $A^{\%}(Y)$ are related to $\Omega^\infty(Y_+ \wedge \mathbf{A}(*))$ by a chain of natural homotopy equivalences.

Universal Riemann–Roch

Assume as in 1.6 that $p: E \to B$ is a fibration with homotopy finitely dominated fibers. In the informal notation of 1.6, we have the fiberwise assembly map

$$A^{\%}_B(E) \to A_B(E)\,,$$

a map over B. For more precision, we have to assume that B is the geometric realization of a simplicial set $B^\natural$, and we obtain the fiberwise assembly map as a map over $|\,\mathrm{simp}(B)|$. In the theorems just below, we use the informal notation.

8.4. Theorem. *Let $p: E \to B$ be a bundle of compact topological manifolds (possibly with boundary). Then the section $\chi_h(p): B \to A_B(E)$ from 6.3 is vertically homotopic to the composition of $\chi(p): B \to A^{\%}_B(E)$ from 7.11 with fiberwise assembly, $\alpha: A^{\%}_B(E) \to A_B(E)$.*

The proof will be given in the next subsection.

8.5. Theorem. *Let $p: E \to B$ be a bundle of compact regular manifolds (possibly with boundary). Then the section $\chi_h(p): B \to A_B(E)$ from 6.3 is vertically homotopic to the composition of Becker–Gottlieb–Dold transfer $\mathrm{tr}_p: B \to Q^\bullet_B(E)$ with $\alpha\eta: Q^\bullet_B(E) \to A_B(E)$.*

This follows directly from 8.4 and 5.7. We emphasize that smooth manifolds are regular manifolds, so that 8.5 can be seen as a statement about smooth manifold bundles.

Proof of 8.4

Here we must unravel the relationship between the excisive characteristic of 7.11, and the homotopy invariant characteristic of 6.3, in the case when Y is a compact ENR. At this point we clearly need different symbols for the two characteristics, so we write $\chi(Y) \in F(Y)$ for the excisive one, 7.11, and $\chi_h(Y) \in F_h(Y)$ for the homotopy invariant one, 6.3. Thus, F is the pro–excisive functor of 7.8 and 7.9, with coefficient spectrum $\simeq \mathbf{A}(*)$, and F_h is the A–theory functor $Y \mapsto A(Y)$. We shall find that, modulo formal inversion of a natural homotopy equivalence, there is an *assembly* transformation $F(Y) \to F_h(Y)$ which is defined when Y is a compact ENR and takes $\chi(Y)$ to $\chi_h(Y)$. The *assembly* word is justified because α is a natural map between infinite loop spaces.

For a more precise statement we need the commutative diagram of Waldhausen categories and exact functors

$$\begin{array}{ccccc}
\mathcal{V}(Y) & \xrightarrow{\subset} & \mathcal{R}^{\ell d}(Y \times [0,\infty], Y \times \infty) & \xleftarrow{\supset} & \mathcal{R}^{fd}(Y) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{V}(Y) & \xrightarrow{\subset} & \mathcal{R}\mathcal{G}^{\ell d}(Y \times [0,\infty], Y \times \infty) & \xleftarrow{\supset} & *
\end{array} \tag{8.6}$$

where Y in the right–hand column has been identified with $Y \times 0$. We apply the K–theory functor and take homotopy pullbacks of the resulting rows to obtain a forgetful map

$$\begin{array}{c} F'(Y) \\ \downarrow \\ F(Y)\,. \end{array}$$

8.7. Lemma. *That map is a homotopy equivalence.*

Proof. The right–hand square in (8.6) is homotopy Cartesian. See [CPV, 2.12]. □

8.8. Observation. *There is a canonical lift $\chi'(Y) \in F'(Y)$ of $\chi(Y) \in F(Y)$. Under the projection $F'(Y) \to F_h(Y)$, this lift $\chi'(Y)$ maps to $\chi_h(Y)$.*

Explanation. Note that the object $Y^!$ in $\mathcal{V}(Y)$ also belongs to the subcategory $\mathcal{R}^{fd}(Y) \subset \mathcal{R}^{\ell d}(Y \times [0,\infty], Y \times \infty)$. □

8.9. Observation. *The projection $F' \to F_h$ is an assembly transformation.*

Explanation and proof. We view $F' \to F_h$ as a natural transformation between infinite loop space valued functors, and show that the corresponding natural transformation $\mathbf{F}' \to \mathbf{F}_h$ between spectrum–valued functors is an assembly transformation.

We know already that $\mathbf{F}'$ is excisive. It remains to show that $\mathbf{F}'(*) \to \mathbf{F}_h(*)$ is a homotopy equivalence, or equivalently, that $F'(*) \to F_h(*)$ is a homotopy equivalence. But this is obvious from the definition of F' and the contractibility of $K(\mathcal{R}^{\ell d}([0,\infty],\infty))$. $\square$

Remark. Why did we bother with χ when χ' is apparently so much more convenient ? The answer is, of course, that χ has better naturality properties: it is a characteristic on $\ell\mathcal{E}^\bullet$.

Conclusion of proof of 8.4. The best policy here is to stick with homotopy inverse limits, and to avoid mentioning section spaces explicitly. Therefore we use the rigorous definitions of (2.9) and (1.7):

$$\chi'(p) \in \operatorname*{holim}_{(x,\theta)} F'(E_x^\theta)\ ,$$
$$\chi_h(p) \in \operatorname*{holim}_{x} F_h(E_x)\,.$$

To find out how $\chi'(p)$ and $\chi_h(p)$ might be related, we set up a commutative square with forgetful vertical arrows

$$\begin{array}{lccc} & \operatorname{holim}_{(x,\theta)}\ F'(E_x^\theta) & \xleftarrow{\ \simeq\ } & \operatorname{holim}_x F'(E_x) \\ (8.10) & \downarrow & & \downarrow \\ & \operatorname{holim}_{(x,\theta)}\ F_h(E_x^\theta) & \xleftarrow{\ \simeq\ } & \operatorname{holim}_x F_h(E_x)\,. \end{array}$$

We are searching for a path connecting the images of $\chi'(p)$ and $\chi_h(p)$ in the lower left–hand term of (8.10).

Now recall the natural transformation $E_x \to E_x^\theta$ for (x,θ) in $\operatorname{simp}(tB)$, which induces the horizontal arrows in (8.10). View it as a functor T on $\operatorname{simp}(tB)\times\{0,1\}$. Using χ_h once more, we obtain a point $z \in \operatorname{holim} F_hT$. But $\operatorname{holim} F_hT$ is exactly the space of natural transformations from the diagram $I \hookleftarrow 0$ to the lower row of (8.10). In particular, z determines a path of the kind we are searching for. That path does what we want. $\square$

Linear Riemann–Roch: Generalities

In the remaining subsections we work mostly with a bundle $p\colon E \to B$ of compact topological manifolds, and a ring R, and produce a transfer homomorphism t from $[E, K(R)]$ to $[B, K(R)]$ such that

$$\beta_{E\downarrow B} = t\beta_E\colon K_0(E;R) \longrightarrow [B, K(R)]$$

(see §6, subsection on *Linearized characteristics*). The case where p is a bundle of regular compact manifolds is easier to understand, so we deal with it first. In this case t is induced by the Becker–Gottlieb–Dold transfer tr_p, and so depends only on the fiber homotopy type of p.

Linear Riemann–Roch for regular manifolds

8.11. Theorem. *Let $p: E \to B$ be a bundle of regular compact manifolds, and let R be any ring. Then $\beta_{E\downarrow B}: K_0(E;R) \longrightarrow [B, K(R)]$ equals $\mathrm{tr}_p^* \cdot \beta_E$, where $\mathrm{tr}_p^*: [E, K(R)] \longrightarrow [B, K(R)]$ is induced by the Becker–Gottlieb–Dold transfer, a stable map from B_+ to E_+.*

8.12. Corollary. *Let V be a bundle of f.g. projective left R–modules on E. Suppose that the homology groups $H_i(E_b;V)$ all admit finite length resolutions by f.g. projective left R–modules ; let V_i be the R–module bundle on B with fiber $H_i(E_b;V)$ over $b \in B$. Then*

$$\mathrm{tr}_p^*[V] = \sum_i (-1)^i [V_i] \qquad \in [B, K(R)].$$

Proof of 8.12 modulo 8.11. Use 6.7. □

Proof of 8.11. Assume first that B is the geometric realization of a simplicial set with finitely many nondegenerate simplices only. Combining 6.4 and 8.5, we find that $\beta_{E\downarrow B}$ applied to the class of some module bundle V on E is the homotopy class of the composition

$$B \xrightarrow{\mathrm{tr}_p} Q_B^\bullet(E) \xrightarrow{\alpha\eta} A_B(E) \longrightarrow A(E) \xrightarrow{\lambda_V} K(R)$$

which is also the homotopy class of the composition

$$B \xrightarrow{\mathrm{tr}_p} Q_B^\bullet(E) \to Q^\bullet(E) \xrightarrow{\alpha\eta} A(E) \xrightarrow{\lambda_V} K(R).$$

Note that $\lambda_V \alpha\eta: Q^\bullet(E) \longrightarrow K(R)$ is a map of infinite loop spaces, and is therefore determined by its restriction to E. We now examine that restriction.

To this end we observe first that $\eta: Q^\bullet(E) \to A^\%(E)$ is homotopic to the composition of $\chi(\mathrm{id}_E): E \to A_E^\%(E)$ with $A_E^\%(E) \to A^\%(E)$. This follows from 5.7, since $\mathrm{id}: E \to E$ is a bundle of regular compact 0–manifolds. Then we conclude that $\alpha\eta: Q^\bullet(E) \to A(E)$ is homotopic to the composition of $\chi_h(\mathrm{id}_E): E \to A_E(E)$ with $A_E(E) \longrightarrow A(E)$. This makes it obvious that $\lambda_V \alpha\eta \,|\, E$ is the map which "classisifies" the module bundle V ; so it is in the homotopy class $\beta_E(V)$.

This completes the proof in the case where B is the geometric realization of a finitely generated simplicial set. In the case where B is the geometric realization of a larger simplicial set, we reduce to the previous case using naturality arguments. □

Products

For now we assume only that $p: E \to B$ is a fibration with finitely dominated fibers. We shall reformulate $\beta_{E\downarrow B}: K_0(E;R) \to [B, K(R)]$ in terms of a product

$$\mu: \mathbf{K}(E,R) \wedge \mathbf{A}(E) \longrightarrow \mathbf{K}(R)$$

where $\mathbf{K}(E,R)$ is the K–theory spectrum of the Waldhausen category of bundles of f.g. projective left R–modules on E.

Ingredients. We need three Waldhausen categories. The first is $\mathcal{P}(E,R)$, the category of bundles of f.g. projective left R–modules on E. We make it into a Waldhausen category by decreeing that the split monomorphisms are the cofibrations, and the isomorphisms are the weak equivalences. The second Waldhausen category that we need is $\mathcal{R}^{fd}(E)$, defined in 8.1. The third is $\mathrm{ch}(R)$, the category of chain complexes C of projective left R–modules which, up to chain homotopy equivalence, are bounded and f.g. projective in each degree. The cofibrations in $\mathrm{ch}(R)$ are the chain maps which are split monomorphisms of R–modules in each degree, and the weak equivalences are the homotopy equivalences. The $\mathrm{ch}(R)$ notation is not entirely consistent with §6, but it is short. The K–theory spaces of these categories are then $K(E,R)$ (by definition) and $A(E)$ and $K(R)$, respectively. To be more precise, however, we decree

$$\begin{aligned} K(E,R) &:= \Omega|w\mathcal{T}_\bullet\mathcal{P}(E,R)| \\ A(E) &:= \Omega|w\mathcal{T}_\bullet\mathcal{R}^{fd}(E)| \\ K(R) &:= \Omega^2|ww\mathcal{T}_\bullet\mathcal{T}_\bullet\,\mathrm{ch}(R)| \end{aligned}$$

where $\mathcal{T}_\bullet$ is the Thomason construction, a mild variation on Waldhausen's $\mathcal{S}_\bullet$ construction which we have used earlier (just before 6.4). It is describe at the end of §1.3 in [Wald]. Our unusual definition of $K(R)$ is blessed by the digression following 1.5.3 in [Wald].

The product at the space level. Here we describe a functor from $\mathcal{P}(E,R) \times \mathcal{R}^{fd}(E)$ to $\mathrm{ch}(R)$. The functor is given by

$$(V, (X \rightleftarrows E)) \mapsto \lambda_V(X \rightleftarrows E)$$

where λ_V is the relative singular chain complex of (X,E) with coefficients in V, as in 6.4. The functor is not quite bi–exact because λ_V is not quite exact for fixed V, but it does induce (with the definitions just above) a map

$$\mu: K(E,R) \wedge A(E) \to K(R)\,.$$

The product at the spectrum level. We make the spectra we need out of Γ–spaces, using Segal's machine [Se2]. A Γ–space is a contravariant functor from a certain category Γ to spaces, subject to certain conditions. The category Γ^{op} has as objects the finite sets S, and as morphism from S to T the pointed maps $S_+ \to T_+$.

The *underlying space* of a Γ–space $S \to Z(S)$ is the space $Z(\{1\})$. In particular, $K(E,R)$ and $A(E)$ are underlying spaces of Γ–spaces given by

$$S \mapsto K(S \times E, R)$$
$$S \mapsto A(S \times E)$$

respectively, with structure maps whose definition we leave to the reader. We like to think of $K(R)$ as the underlying space of a *bi*–Γ–space (contravariant functor from $\Gamma \times \Gamma$ to spaces, subject to some conditions), as follows:

$$(S,T) \mapsto K(R^{S\times T}).$$

(Note in this connection that the category of left modules over $R^{S\times T}$ is equivalent to the product of $S \times T$ copies of the category of left R–modules.) The Segal machine makes this bi–Γ–space into a bi–spectrum, which we denote (here) by $\mathbf{K}(R)$. In conclusion, we see that it is easy to promote $\mu: K(E,R) \wedge A(E) \to K(R)$ to a bi–natural map

$$K(S \times E, R) \wedge A(T \times E) \longrightarrow K(R^{S\times T})$$

for S, T in Γ. The Segal machine translates this into a map of bispectra,

$$\mu: \mathbf{K}(E,R) \wedge \mathbf{A}(E) \longrightarrow \mathbf{K}(R). \quad \square$$

Description of $\beta_{E\downarrow B}$ in terms of μ. We remember $K_0(E,R) = \pi_0 K(E,R)$ and recover $\beta_{E\downarrow B}$ as a composition

$$K_0(E,R) \longrightarrow [A(E), K(R)] \longrightarrow [B, K(R)],$$

where the first arrow is obtained from μ by adjunction, and the second is composition with $\chi(p)$. Compare 6.4.

Assembly (second encounter) and coassembly

Here we need an explicit description of the assembly transformation $\alpha_{\mathbf{J}}: \mathbf{J}^{\%} \to \mathbf{J}$ for a *covariant* functor $\mathbf{J}$ from compact ENR's to spectra. The standard description, taken from [WWa], is

$$\mathbf{J}^{\%}(Y) = \underset{x \in Y^s}{\mathrm{hocolim}}\, \mathbf{J}(\Delta^{|x|})$$

where Y^s denotes the singular simplicial set of Y, and the simplices of Y^s are viewed as objects of a category in the usual way. The characteristic maps $\Delta^{|x|} \to Y$ for the various x in Y^s give rise to a natural transformation $\mathbf{J}(\Delta^{|x|}) \to \mathbf{J}(Y)$ which in turn induces the required map α from $\mathbf{J}^{\%}(Y)$, above definition, to $\mathbf{J}(Y)$.

Coassembly is the contravariant analog of assembly. Therefore let $\mathbf{J}$ be a *contravariant* homotopy functor from compact ENR's to spectra. Then there exists an essentially unique natural transformation $\omega: \mathbf{J} \to \mathbf{J}^{\mho}$ with the following properties:

- $\omega: \mathbf{J}(*) \to \mathbf{J}^{\mho}(*)$ is a weak equivalence ;
- $\mathbf{J}^{\mho}(*)$ respects pushout squares and $\mathbf{J}(\emptyset) \simeq *$.

One possible definition of $\mathbf{J}^{\mho}$: First make sure that the values of $\mathbf{J}$ are Ω–spectra, made up of (geometric realizations of) Kan simplicial sets. Then let

$$\mathbf{J}^{\mho}(Y) := \operatorname*{holim}_{x \in Y^s} \mathbf{J}(\Delta^{|s|})$$

where the homotopy limit is again taken over the category of singular simplices of Y. Proofs are very much like those in the covariant case.

Now let $\mathbf{J}_i$ for $i = 1, 2$ be homotopy functors from the category of compact ENR's to the category of spectra. Suppose that $\mathbf{J}_1$ is contravariant, and $\mathbf{J}_2$ is covariant. Let $\mathbf{J}_3$ be a single bispectrum. Let

$$\mu : \mathbf{J}_1(X) \wedge \mathbf{J}_2(X) \longrightarrow \mathbf{J}_3$$

be a *dinatural* transformation. (This is an informal extension of, or variation on, the definition of *dinatural* given in [MaL] ; what it means here is that for any map $f: X \to Y$ of ENR's, the following diagram of bispectra is commutative:

$$\begin{tikzcd}
\mathbf{J}_1(Y) \wedge \mathbf{J}_2(X) \arrow[r] \arrow[d] & \mathbf{J}_1(X) \wedge \mathbf{J}_2(X) \arrow[d, "\mu"] \\
\mathbf{J}_1(Y) \wedge \mathbf{J}_2(Y) \arrow[r, "\mu"] & \mathbf{J}_3 .
\end{tikzcd}$$

The example to have in mind is: $\mathbf{J}_1(X) = \mathbf{K}(X, R)$ and $\mathbf{J}_2(X) = \mathbf{A}(X)$ and $\mathbf{J}_3 = \mathbf{K}(R)$ (bispectrum version) and μ as in the previous subsection.

8.13. Observation. *In this situation there exists a dinatural transformation* $\mathbf{J}_1^{\mho}(X) \wedge \mathbf{J}_2^{\%}(X) \longrightarrow \mathbf{J}_3$ *making the following diagram of dinatural transformations commutative:*

$$\begin{tikzcd}
\mathbf{J}_1 \wedge \mathbf{J}_2^{\%} \arrow[r, "\omega \wedge \mathrm{id}"] \arrow[d, "\mathrm{id} \wedge \alpha"] & \mathbf{J}_1^{\mho} \wedge \mathbf{J}_2^{\%} \arrow[d] \\
\mathbf{J}_1 \wedge \mathbf{J}_2 \arrow[r, "\mu"] & \mathbf{J}_3 .
\end{tikzcd}$$

Proof. We use the above definitions of α and ω. Fix Y, a compact ENR. Write simp for the category of singular simplices of Y. For x in simp let $E(x)$ be the classifying space of the category of simp–objects under x, or equivalently, the classifying space of the category of $\mathrm{simp}^{\mathrm{op}}$–objects over x. Then

$$\mathbf{J}_1^{\mho}(Y) = \operatorname*{holim}_{x \in \mathrm{simp}^{\mathrm{op}}} \mathbf{J}_1(\Delta^{|x|}) = \mathrm{end}\,[\,(x, y) \mapsto \mathbf{J}_1(\Delta^{|x|})^{E(y)}\,]$$

$$\mathbf{J}_2^{\%}(Y) = \operatorname*{hocolim}_{x \in \mathrm{simp}} \mathbf{J}_2(\Delta^{|x|}) = \mathrm{coend}\,[(x, y) \mapsto E(x)_+ \wedge \mathbf{J}_2(\Delta^{|y|})\,] .$$

Now define the missing map $\mathbf{J}_1^{\mho}(Y) \wedge \mathbf{J}_2^{\%}(Y) \longrightarrow \mathbf{J}_3$ in such a way that, for each x in simp, the following commutes:

$$\begin{array}{ccc}
\mathbf{J}_1^{\mho}(Y) \wedge (E(x)_+ \wedge \mathbf{J}_2(\Delta^{|x|})) & \longrightarrow & \mathbf{J}_1^{\mho}(Y) \wedge \mathbf{J}_2^{\%}(Y) \\
\downarrow & & \downarrow \\
\mathbf{J}_1(\Delta^{|x|})^{E(x)} \wedge (E(x)_+ \wedge \mathbf{J}_2(\Delta^{|x|})) & \longrightarrow & \mathbf{J}_3
\end{array}$$

where the arrow in the lower row is the composition of an evaluation map to $\mathbf{J}_1(\Delta^{|x|}) \wedge \mathbf{J}_2(\Delta^{|x|})$ with $\mu: \mathbf{J}_1(\Delta^{|x|}) \wedge \mathbf{J}_2(\Delta^{|x|}) \to \mathbf{J}_3$. $\square$

Linear Riemann–Roch for topological manifolds

8.14. Example. With μ and $\mathbf{K}(E,R)$ as in the previous subsection, we easily get $\mathbf{K}(E,R)^{\mho} \simeq \mathbf{K}(R)^E$ and then, from 8.13, a homotopy commutative square of bispectra

$$\begin{array}{ccc}
\mathbf{K}(E,R) \wedge \mathbf{A}^{\%}(E) & \xrightarrow{\omega \wedge \mathrm{id}} & \mathbf{K}(R)^E \wedge \mathbf{A}^{\%}(E) \\
\Big\downarrow{\scriptstyle \mathrm{id} \wedge \alpha} & & \Big\downarrow \\
\mathbf{K}(E,R) \wedge \mathbf{A}(E) & \xrightarrow{\mu} & \mathbf{K}(R)
\end{array}$$

and from that, a square of spaces and maps which commutes up to homotopy,

$$\begin{array}{ccc}
K_0(E,R) \times A^{\%}(E) & \longrightarrow & [E, K(R)] \times A^{\%}(E) \\
\Big\downarrow & & \Big\downarrow \\
K_0(E,R) \times A(E) & \longrightarrow & K(R)\,.
\end{array} \tag{8.15}$$

If now $p: E \to B$ is a bundle with compact manifold fibers, then $\chi(p): B \to A_B(E)$ has a canonical lift $B \to A_B^{\%}(E)$ by 8.4, and we obtain from (8.15) another square which commutes up to homotopy,

$$\begin{array}{ccc}
K_0(E,R) \times B & \xrightarrow{\beta_E \times \mathrm{id}} & [E, K(R)] \times B \\
\Big\downarrow{\scriptstyle \mathrm{id} \times \chi(p)} & & \Big\downarrow \\
K_0(E,R) \times A(E) & \xrightarrow{\mu} & K(R)\,.
\end{array} \tag{8.16}$$

8.17. Definition. Let $t: [E, K(R)] \to [B, K(R)]$ be the adjoint of the right–hand vertical arrow in the commutative square (8.16).

8.16. Observation. $t\beta_E = \beta_{E\downarrow B}: K_0(E,R) \longrightarrow [B, K(R)]$. $\square$

Reidemeister torsion

Let $p : E \to B$ be a fibration with homotopy finitely dominated fibers E_b , for $b \in B$. Let R be a ring, and V a bundle of f.g. projective left R-modules on E. For a space X over E with reference map $r : X \to E$ let

$$\Phi^h(X) := \text{hofiber}[\lambda_V r_*\colon A(X) \to K(R)\,]$$
$$\Phi^t(X) := \text{hofiber}[\lambda_V r_*\alpha\colon A^{\%}(X) \to K(R)\,]$$
$$\Phi^d(X) := \text{hofiber}[\lambda_V r_*\alpha\eta\colon Q(X_+) \to K(R)\,]$$

with $\lambda_V : A(E) \to K(R)$ as in 6.4, and α, η the assembly and unit maps (defined under mild conditions on X which we suppress). Now suppose that $H_i(E_b; V) = 0$ for all $b \in B$; equivalently, the singular chain complexes with V–coefficients of the fibers E_b are all contractible. Then the composition

$$B \xrightarrow{\chi_h(p)} A_B(E) \longrightarrow K(R)$$

is canonically nullhomotopic. In other words, we have lifted $\chi_h(p)$ to a section of a fibration

$$\Phi^h_B(E) \longrightarrow B$$

with fiber $\Phi^h(E_b)$ over $b \in B$. We call this lift the *homotopy Reidemeister torsion* of p. If $p : E \to B$ happens to be a bundle of compact topological manifolds, then using 8.4 we have a lift of the excisive family characteristic $\chi(p) : B \to A^{\%}_B(E)$ to a section of

$$\Phi^t_B(E) \longrightarrow B\,.$$

This is the *topological* Reidemeister torsion of p. And finally, if p happens to be a bundle of compact regular manifolds, we have from 8.5 a lift of the Becker–Gottlieb–Dold section $B \to (Q_+)_B(E)$ to a section of

$$\Phi^d_B(E) \longrightarrow B$$

which we call the *regular* Reidemeister torsion of p. In particulr, if p is a bundle of compact *smooth* manifolds, its regular Reidemeister torsion is defined — in that case we also call it the smooth Reidemeister torsion.

Remark. Earlier, Igusa and Klein [IgK1], [IgK2] used parameterized generalized Morse functions to define the (parametrized) Reidemeister torsion of a bundle of smooth compact manifolds, $p : E \to B$. Their Reidemeister torsion is a map from B to the homotopy fiber of $\lambda_V\alpha\eta : Q(E_+) \to K(R)$. Also Bismut and Lott have used a differential form version of their Riemann-Roch theorem to construct an analytic version of parametrized Reidemeister torsion in the special case $R = \mathbb{C}$. It is not clear at this stage whether the parameterized Reidemeister torsions produced by [BiLo] are in agreement with ours. In the unparametrized setting, $B = *$, they are ; this is the theorem of Cheeger and Müller, [Che], [Mü]. Put differently again, our theorem 8.5 and [BiLo] together make it possible to *state* a family version of the Cheeger–Müller theorem, and we would like to know whether this is in fact true.

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