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Abstract

A sequence of solutions to the equation of symmetry for the continuous Toda chain in $1+2$ dimensions is represented in explicit form. This fact leads to the supposition that this equation is completely integrable.

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1 Introduction

The problem of the continuous Toda chain arose as a rediscovery of the earlier results of Darboux [1] and Fermi, Pasta and Ulam [2]. In a wider context, this problem illustrates features common to all other infinite dimensional integrable chains. The problem consists of taking the limit of the interparticle distance in the discrete Toda chain in two dimensions to zero, giving the continuous version described by the equation[3][4][5];

$$\frac{\partial^2}{\partial x \partial y} \log U = \frac{\partial^2}{\partial z^2} U. \tag{1}$$

The discrete Toda chain in two dimensions takes the form of the infinite set of equations

$$\frac{\partial^2}{\partial x \partial y} \log U_n = U_{n+1} - 2U_n + U_{n-1}. \tag{2}$$

With a suitable rescaling, the limit of equation (2) is (1). Under appropriate boundary conditions the equations of the infinite dimensional Toda chain (2) reduce to a finite-dimensional (classical dynamical system). Such a Toda chain with fixed ends ($U_0 = 0,\ U_n = 0$) is an exactly integrable system. Its general solution was found in [10]. In the case of periodic boundary conditions ($U_i = U_{i+N}$) the resulting system coincides with affine Toda chain connected with Kac-Moody algebras A_N. In this case it is completely integrable and it is possible to find in explicit form only c-number parametrical soliton-like solutions [11]. In the one dimensional case, where U depends upon ($x,\ y,\ z$) only through the combinations ($x + y,\ z$) the general implicit solution has been found by R.S. Ward [6].

The obvious question arises; to what class of equations does equation (1) belong? To obtain a solution of this equation by means of a limiting procedure on a solution of (2) does not seem feasible as solutions of this set of equations appear in the form of ratios of determinants. The limiting procedure is equivalent to finding the limits of infinite determinants, which is always rather difficult. For the investigation of equation (1), we shall use classical group theoretical methods [7][8], as a technique, by constructing the solutions of the corresponding symmetry equation , which takes the form

$$\frac{\partial T}{\partial x} = U \int \frac{\partial^2 T}{\partial z^2} dy. \tag{3}$$

(The term 'symmetry equation' is the one most often used in the literature to designate the equation describing the variation of the dependent variables with respect to a parameter of the solution.) In spite of the known example of the solutions of the symmetry equation for the analogous Sine Gordon case [12],[13] which are expressed in terms of multiple derivatives, the solutions of (3) which we shall obtain here are represented as linear combinations of repeated integrals, which are constructed using precise algorithmic rules.

2 Solution of the symmetry equation

The symmetry equation for our problem can be derived either indirectly, by taking it as the limit of the symmetry equation for the discrete chain, or else directly from (1), by differentiating (1) with respect to an arbitrary parameter τ and setting $S = \dfrac{\partial U}{\partial \tau}$. Upon integration over y, this yields the equation

$$\frac{\partial}{\partial x}\left(\frac{S}{U}\right) = \int \frac{\partial^2 S}{\partial z^2} dy. \tag{4}$$

In this equation change variables to T defined by

$$S \to \frac{\partial T}{\partial x} \tag{5}$$

Then the equation becomes (3) , after integration by x and differentiation once by y. (Another formulation of the symmetry equation consists in making the substitution $S \to U\dfrac{\partial W}{\partial z}$. This yields the equation $W_x = \int dy (UW_z)_z$ which may be solved by similar methods)

The derivation of the symmetry equation for the discrete chain will be discussed in the next section. The method of approach is based upon the solution of a discrete version of equation (3) derived in [9]. First of all we may remark that $T = T^0 = U$ is a solution of (3). Let us seek a solution of (3) in the form

$$T = T^0 \alpha^0. \tag{6}$$

The equation obeyed by α^0 following from (3) (taking into account (1)) takes the form

$$\alpha^0_x + \alpha^0 \int T^0_{zz} dy = \int (T^0 \alpha^0)_{zz} dy, \tag{7}$$

Let us represent α^0 as an integral $\alpha^0 \mapsto \int \alpha^0 dy$, retaining the same symbol for the integrand and differentiate (7) with respect to the argument y. We obtain

$$\alpha^0_x + \alpha^0 \int T^0_{zz} dy = 2T^0_z \int \alpha^0_z dy + T^0 \int \alpha^0_{zz} dy. \tag{8}$$

Let us attempt to find a solution of (8) with the aid of the ansatz

$$\alpha^0 = T^0_z \alpha^1_z + T^0 \beta^1. \tag{9}$$

This representation is suggested by the fact that the right hand side of (8) contains terms with factors T^0_z, T^0. After substitution of this ansatz and setting the coefficients of T^0_z and T^0 to zero in the resulting expression (this is an addditional assumption), the pair of equations which the functions α^1 and β^1 satisfy will be

$$\begin{aligned} \alpha^1_x + 2\alpha^1 \int T^0_{zz} dy &= 2\int \alpha^0_z dy, \\ \beta^1_x + 2\beta^1 \int T^0_{zz} dy + \alpha^1 \int T^0_{zzz} dy &= \int \alpha^0_{zz} dy. \end{aligned} \tag{10}$$

If the derivative of the first equation with respect to the variable z is subtracted from twice the second we obtain as a consequence the equation

$$(2\beta^1 - \alpha^1_z)_x + 2(2\beta^1 - \alpha^1_z) \int T^0_{zz} dy = 0. \tag{11}$$

From the last equation we conclude that among the solutions of (11) are those for which

$$\beta^1 = \frac{1}{2}\alpha^1_z. \tag{12}$$

Resubstitution of this solution for β^1 into the first of (10) gives the following equation to determine α^1;

$$\alpha^1_x + 2\alpha^1 \int T^0_{zz} dy = \int (2\alpha^1 T^0_{zz} + 3\alpha^1_z T^0_z + \alpha^1_{zz} T^0) dy. \tag{13}$$

If α^1 is represented in integral form as $\alpha^1 \mapsto \int \alpha^1 dy$, the form of (13) after differentiation with respect to y becomes

$$\alpha^1_x + 2\alpha^1 \int T^0_{zz} dy = 3T^0_z \int \alpha^1_z dy + T^0 \int \alpha^1_{zz} dy. \tag{14}$$

This equation differs from the equation (8) for the determination of α^0 only by numerical factors in the various terms. Repeating the above trick for α^1, α^2 etc. we finally arrive at the sequence

$$\alpha^{k-1} = T^0_z \alpha^k_z + T^0 \beta^k, \tag{15}$$

where $\beta^k = \frac{1}{k}\alpha^k_z$ and α^k has the form

$$\alpha^k_x + (k+1)\alpha^k \int T^0_{zz} dy = \int ((k+1)\alpha^k T^0_{zz} + (k+2)\alpha^k_z T^0_z + \alpha^k_{zz} T^0) dy. \tag{16}$$

This last equation possesses the obvious solution

$$\alpha^k = 1. \tag{17}$$

The final result may be represented in two forms; one of an algorithmic nature, the second in the form of repeated integrals and derivatives. In the first, the structure of a particular solution to the symmetry equation takes the form

$$T^n = T^0 \prod_{j=1}^{n} ((j+1)D_j + \sum_{k=j+1}^{n} D_k) \overbrace{\int T^0 dy_1 \int T^0 dy_2 \cdots \int T^0 dy_n}^{n}, \tag{18}$$

where D_j denotes the operation of differentiation with respect to z of the integrand T^0 situated at the j th place in the n repeated integrals in (18). An alternative expression for this solution is given by the formula (setting $T^0 = t$);

$$T^n = t \int \frac{dy_1}{t} \frac{\partial}{\partial z} (\frac{t^2}{2} \int \frac{dy_2}{t^2} \cdots \frac{\partial}{\partial z} (\frac{t^j}{j} \int \frac{dy_j}{t^j} \cdots \int \frac{\partial t}{\partial z} dy_n)) \cdots). \tag{19}$$

(note that t^j denotes the j th power of T^0, not T^j and there are n iterated integrals in the above expression).

3 Symmetry equation for the discrete Toda chain

In this section we give the corresponding analysis for the case of the discrete Toda chain. The symmetry equation for the discrete Toda equation is obtainable from a generalisation of the Darboux transformation which takes the form, for two independent functions

$$\begin{aligned}(\log U_n)_{xy} &= V_n(U_{n+1}-U_n)-V_{n-1}(U-U_{n-1}),\\ (\log V_n)_{xy} &= U_{n+1}(V_{n+1}-V)-U(V-V_{n-1}),\end{aligned} \tag{20}$$

where (20) is viewed as a set of recurrence relations connecting functions evaluated at the nodes of a chain. More generally U_{n+k} can be viewed as either the value of the function U_n at a point k places to the right, or else the k th iterate of the function U_n.

If $V = 1$, this reproduces the standard discrete Toda chain (2). Also $f = U_n V_{n+1}$ and $g = U_n V_n$ also satisfy the same discrete equation and in the continuum limit $f \to g$ the continuous Toda chain (1). The symmetry equation

$$(T_n)_x = U\int dy[V_n(T_{n+1}-T_n)-V_{n-1}(T_n-T_{n-1})] \tag{21}$$

may be solved by analogous methods to those in the previous section. (This equation is equivalent to equation (2.16) of [9])

First of all, it is clear that $T_n = U_n$ is a solution. Then set $T_n = U_n \int \alpha_n^0 dy$. The equation then becomes

$$\begin{aligned}&(\alpha_n^0)_x+\alpha_n^0\int[V(U_{n+1}-U_n)-V_{n-1}(U_n-U_{n-1})]dy = \\ &V_nU_{n+1}\int(\alpha^0{}_{n+1}-\alpha_n^0)dy - V_{n-1}U_{n-1}\int(\alpha_n^0-\alpha^0{}_{n-1})dy\end{aligned} \tag{22}$$

This equation may be rewritten in the form

$$\begin{aligned}&\alpha_x^0+\alpha^0\int[f_{n+1}-f_n+g_{n-2}-g_n)]dy\\ &= f_{n+1}\int(\alpha^0{}_{n+1}-\alpha_n^0)dy - gn-1\int(\alpha_n^0-\alpha^0{}_{n-1})dy\end{aligned} \tag{23}$$

which is equation (3.1) of [9]. The discussion proceeds as in that article; α_n^0 is represented in terms of the coefficients of the integrals of f, g and two new unknown functions α_n^1, β_n^1 as

$$\alpha_n^0 = \alpha_n^1\int f_{n+1}dy + \beta_n^1\int g_{n-1}dy \tag{24}$$

and a pair of equations similar to (23) for α_n^1, β_n^1 obtained. It turns out that $\beta_n^1 = -\alpha^1{}_{n-1}$ is a sufficient condition for the consistency of this pair, and the remaining equation for α_1^1 takes a similar form to (23). The iteration proceeds as before, and in this manner a class of solutions can be obtained in the form

$$\begin{aligned} T_n^N &= T_n^0 \prod_{i=1}^{N} (1 - L_i \exp[-(i+1)d_i - \sum_{k=i+1}^{N} d_k]) \\ &\times \int dy_1 f_{n+1} \int dy_2 f_{n+2} \ldots\ldots \int dy_N f_{n+N} \end{aligned} \tag{25}$$

where the symbol $\exp d_s$ means that the argument of the s-th term of repeated integral $(\ldots. \int dy f_{n+h} \ldots \to \ldots \int dy f_{h+n+1} \ldots)$ in (25) should be shifted by unity and the symbol L_p means the exchange of f_{n+r} and g_{n+r} in the corresponding p-th term $\ldots \int dy f_{n+r} \ldots \to \ldots \int dy g_{n+r} \ldots$. The solution of the previous section can be recovered by setting $f_n = g_n = U$.

4 General solution in the $1+1$ dimensional case

In this section we demonstrate the connection between the exact integrability, in the sense of the existence of a general implicit solution of the $(1+1)$ dimensional continuous Toda chain and the complete solution of its symmetry equation It is well known that in the 'one dimensional' case, $(\frac{\partial}{\partial x} = \frac{\partial}{\partial y})$ equation (1) is exactly integrable. Its solution has been obtained in [6]. We now want to show that in this case the general solution of the symmetry equation can be obtained. The result will be quoted leaving the reader to check its validity by direct computation.

The continuous Toda system in this case may be written in terms of two functions $u,\ v$ which obey the equations

$$u_t = u v_x, \quad v_t = v u_x. \tag{26}$$

The corresponding symmetry equations for the functions $U,\ V$ are

$$U_t = U v_x + u V_x, \quad V_t = V u_x + v U_x. \tag{27}$$

Let T, X be an arbitrary solution of the system

$$u\frac{\partial T}{\partial u} = \frac{\partial X}{\partial v}, \quad v\frac{\partial T}{\partial v} = \frac{\partial X}{\partial u} \tag{28}$$

The complete solution of this equation has been recorded in [14], where T and X are the conserved charge and current densities respectively for the system (26) Then the solution of the symmetry equation may be represented as

$$U = Tu_t + Xu_x, \ V = Tv_t + Vv_x \tag{29}$$

The general solution of (26)[14] depends on two arbitrary functions of one argument and the solution of the symmetry equation also depends upon two arbitrary functions so (29) is indeed the general solution. Thus existence of the exact integrability of the symmetry equation and that of the continuous Toda chain in the one dimensional case is demonstrated.

5 Conclusion

The analysis of a sequence of solutions to the symmetry equations of the discrete Toda chain found in [9] has been successfully adapted to the case of the continuous Toda chain. In this case , however we have found that the solution can be encapsulated into a single iterative formula. The solutions obtained suggest the possibility that the continuous Toda chain is a completely integrable system. If this is so, then it will possess multisoliton solutions, which may be found in explicit form. This is in agreement with the general ideas put forward in [15]. We have also demonstrated that in the case of the one dimensional chain, where exact integrability holds, the existence of a complete solution to the symmetry equation.

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