INTERSECTION OF SUBGROUPS IN FREE GROUPS AND HOMOTOPY GROUPS

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Abstract. Let K be a two-dimensional CW-complex with subcomplexes K_1, K_2, K_3 such that $K = K_1 \cup K_2 \cup K_3$ and $K_1 \cap K_2 \cap K_3$ is the 1-skeleton K^1 of K. We construct a natural homomorphism of $\pi_1(K)$-modules

$$\pi_3(K) \to \frac{R_1 \cap R_2 \cap R_3}{[R_1, R_2 \cap R_3][R_2, R_3 \cap R_1][R_3, R_1 \cap R_2]},$$

where $R_i = ker\{\pi_1(K^1) \to \pi_1(K_i)\},\ i = 1, 2, 3$ and the action of $\pi_1(K) = F/R_1R_2R_3$ on the right hand abelian group is defined via conjugation in F. In certain cases, the defined map is an isomorphism.

1. Introduction

Given a free group F and normal subgroups ($n \geq 2$)

$$R_1, \dots, R_n \subset F,$$

we consider the quotient group

$$I_n(F, R_1, \dots, R_n) = \frac{R_1 \cap \dots \cap R_n}{\prod_{I \cup J = \{1,\dots,n\}} [\bigcap_{i \in I} R_i, \bigcap_{j \in J} R_j]}.$$

Here $\bigcap$ denotes the intersection of subgroups in the free group F and $\prod$ is the product of commutator subgroups as indicated. In fact, the abelian group I_n has the natural structure of an $F/R_1 \dots R_n$-module, with the group action defined via conjugation in F.

The computation of the abelian group I_n is highly non-trivial. In fact, Wu [6] showed for the special case $F = \langle x_1, \dots, x_{n-1} \rangle, R_i = \langle x_i \rangle^F,\ i = 1, \dots, n-1, R_n = \langle x_1 \dots x_{n-1} \rangle^F$ that

$$I_n(F, R_1, \dots, R_n) = \pi_n(S^2)$$

is the n-th homotopy group of the 2-sphere.

It is one of the deep problems of algebraic topology to compute homotopy groups $\pi_n(S^2)$. In low degrees one has (see [5]):

n	2	3	4	5	6	7	8	9
$\pi_n(S^2)$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_4 \oplus \mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$

On the other hand, for $n = 2$, one has a general description of the group $I_2(F, R_1, R_2)$ in terms of homotopy groups of certain spaces. For this we consider a connected 2-dimensional CW-complex K with subcomplexes

$$K_1, \dots, K_n \subset K,$$

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for which $K_1 \cup \cdots \cup K_n = K$ and $K_1 \cap \cdots \cap K_n$ is the 1-skeleton K^1 of K, with $F = \pi_1(K^1)$ and

$$R_i = ker\{\pi_1(K^1) \to \pi_1(K_i)\},\ i = 1, \ldots, n.$$

In fact, Gutierrez-Ratcliffe [3] show that for $n = 2$ one has an exact sequence of $\pi_1(K)$-modules

$$0 \to i_1\pi_2(K_1) + i_2\pi_2(K_2) \to \pi_2(K) \to I_2(F, R_1, R_2) \to 0,$$

where i_j is the map induced by the inclusion $K_j \to K,\ j = 1, 2$. In this case,

$$I_2(F, R_1, R_2) = \frac{R_1 \cap R_2}{[R_1, R_2]}.$$

It is the purpose of this paper to combine the results of Wu and Gutierrez-Ratcliffe respectively and to study a corresponding generalization. We conjecture that each element $\alpha \in \pi_n(S^2)$ determines a natural function $(n \geq 2)$

$$\alpha_* : \pi_2(K)/(i_1\pi_2(K_1) + \cdots + i_n\pi_2(K_n)) \to I_n(F, R_1, \ldots, R_n).$$

For the example of Wu above K can be chosen to be the 2-sphere S^2 and α_* carries in this case the identity of S^2 to α showing that α_* is non-trivial. In general, α_* is not a homomorphism of abelian groups.

Proposition. *Let $n = 2$. If α is a generator of $\pi_2(S^2) = \mathbb{Z}$, then α_* exists and is given by the map $\pi_2(K) \to I_2(F, R_1, R_2)$ of Gutierrez-Ratcliffe* [3].

Moreover, as a main result of this paper we prove the following

Theorem. *Let $n = 3$. If $\alpha \in \pi_3(S^2)$ is a generator, then there is a well-defined function α_* which is a quadratic map inducing a natural homomorphism of $\pi_1(K)$-modules*

$$\alpha_\# : \pi_3(K) \to I_3(F, R_1, R_2, R_3).$$

For the example of Wu, one has $K = S^2$ and in this case $\alpha_\#$ is an isomorphism.

2. The example of Wu

Recall the description of homotopy groups of the 2-sphere due to Wu [6]. Let $F[S^1]$ be Milnor's F-construction applied to the simplicial circle S^1. This is the free simplicial group with $F[S^1]_n$ a free group of rank $n \geq 1$ with generators $x_0, \ldots, x_{n-1}$. Changing the basis of $F[S^1]_n$ in the following way: $y_i = x_i x_{i+1}^{-1},\ y_{n-1} = x_{n-1}$, we get another basis $\{y_0, \ldots, y_{n-1}\}$ in which the simplicial maps can be written easier. A combinatorial group-theoretical argument then shows that the functor I_{n+1} applied to the example of Wu in the introduction gives exactly the n-th homotopy group of the loop space $\Omega\Sigma S^1$, which is isomorphic to the homotopy group of $\pi_{n+1}(S^2)$ (see [6] for explicit computations). In fact, we have

$$\pi_{n+1}(S^2) \cong \frac{\langle y_{-1}\rangle^F \cap \langle y_0\rangle^F \cap \cdots \cap \langle y_{n-1}\rangle^F}{[[y_{-1}, y_0, \ldots, y_{n-1}]]},$$

where F is a free group with generators $y_0, \dots, y_{n-1}$, $y_{-1} = (y_0 \dots y_{n-1})^{-1}$, the group $[[y_{-1}, y_0, \dots, y_{n-1}]]$ is the normal closure in F of the set of left-ordered commutators

$$[z_1^{\varepsilon_1}, \dots, z_t^{\varepsilon_t}] \tag{1}$$

with the properties that $\varepsilon_i = \pm 1$, $z_i \in \{y_{-1}, \dots, y_{n-1}\}$ and all elements in $\{y_{-1}, \dots, y_{n-1}\}$ appear at least ones in the sequence of elements z_i in (1). A standard commutator calculus argument, given essentially in Corollary 3.5 of [6] shows that

$$[[y_{-1}, y_0, \dots, y_{n-1}]] = \prod_{I \cup J = \{1, \dots, n+1\}} [\bigcap_{i \in I} R_i, \bigcap_{j \in J} R_j],$$

where $R_{i+1} = \langle y_i \rangle^F$, $i = 0, \dots, n-1$, $R_{n+1} = \langle y_{-1} \rangle^F$. Hence we have the following isomorphism

$$I_{n+1}(F, R_1, \dots, R_{n+1}) \cong \pi_{n+1}(S^2). \tag{2}$$

Consider first the most elementary case $n = 2$. In this case we view the 2-sphere S^2 as a standard complex constructed from the group presentation

$$\langle x_1 \mid x_1, x_1^{-1} \rangle.$$

Clearly then

$$I_2(\bar{S}^2) = \frac{\langle x_1 \rangle \cap \langle x_1^{-1} \rangle}{[\langle x_1 \rangle, \langle x_1^{-1} \rangle]} \simeq \mathbb{Z}$$

with x_1 a generator of this infinite cyclic group.

3. The category $\mathcal{K}_n$

For $n \geq 2$, denote by $\mathcal{K}_n$ the category with objects $\bar{K} = (K, K_1, \dots, K_n)$. Here K is a two-dimensional CW-complex, K_i is a subcomplex of K, $i = 1, \dots, n$, such that $K = K_1 \cup \dots \cup K_n$, and $K^1 = K_1 \cap \dots \cap K_n$. A morphism in $Hom_{\mathcal{K}_n}(\bar{K}, \bar{L})$ for $\bar{K}, \bar{L} \in \mathcal{K}_n$ is a map

$$f : K^1 \to L^1$$

between 1-skeletons of K and L, such that f can be extended to a map $\bar{f} : K \to L$, with the property $\bar{f}(K_i) \subseteq L_i$, $i = 1, \dots, n$.

Denote by $\mathcal{R}_n$ $(n \geq 2)$ the category with objects $(F, R_1, \dots, R_n)$, where F is a free group and R_i is a normal subgroup in F. A morphism in $\mathcal{R}_n$ between two objects $(F, R_1, \dots, R_n)$ and $(F', R'_1, \dots, R'_n)$ is a group homomorphism $g : F \to F'$ such that $g(R_i) \subseteq R'_i$, $i = 1, \dots, n$. This category was also considered in [1].

There is a natural functor between these two categories,

$$\mathcal{F}_n : \mathcal{K}_n \to \mathcal{R}_n,$$

defined by setting

$$\mathcal{F}_n : (K, K_1, \dots, K_n) \mapsto (\pi_1(K^1), R_1, \dots, R_n),$$

where $R_i = \ker\{\pi_1(K^1) \to \pi_1(K_i)\}$.

Proposition 1. *The functor $\mathcal{F}_n$ defines an equivalence of the categories $\mathcal{K}_n$ and $\mathcal{R}_n$.*

For $n \geq 2$, define the functor

$$I_n : \mathcal{R}_n \to \mathcal{A}b,$$

where $\mathcal{A}b$ is the category of abelian groups, by setting

$$I_n : \bar{R} = (F, R_1, \dots, R_n) \mapsto I_n(\bar{R}) := \frac{R_1 \cap \dots \cap R_n}{\prod_{I \cup J = \{1,\dots,n\}} [\bigcap_{i \in I} R_i, \bigcap_{j \in J} R_j]}.$$

Clearly, for any $\bar{R} \in \mathcal{R}_n$, the abelian group $I_n(\bar{R})$ has a natural structure of $F/R_1 \dots R_n$-module, where the group action viewed via conjugation in F.

4. The surjection q and the conjecture on α_*

In this section we show the following result.

Proposition 2. *For an object $\bar{S}_n$ in $\mathcal{K}_n$ associated to Wu's example in $\mathcal{R}_n$ there is a surjection*

$$q : Hom_{\mathcal{K}_n}(\bar{S}_n, \bar{K}) \twoheadrightarrow \pi_2(K)/(i_1(K_1) + \dots i_n(K_n)),$$

which is natural in $\bar{K} \in \mathcal{K}_n$.

For $\alpha \in \pi_n(S^2) = I_n \mathcal{F}_n(\bar{S}_n)$ we thus obtain the following diagram

$$\begin{tikzcd}
Hom_{\mathcal{K}_n}(\bar{S}_n, \bar{K}) \arrow[r, two heads, "q"] \arrow[d, "\alpha^*"] & \pi_2(K)/(i_1\pi_2(K_1) + \dots + i_n\pi_2(K_n)) \arrow[dl, dashed, "\alpha_*"] \\
I_n(\mathcal{F}_n(\bar{K})) &
\end{tikzcd}$$

where $\alpha^*(f) = f_*(\alpha)$.

Conjecture 1. *For each $\alpha \in \pi_n(S^2)$ there exists a function α_* for which the diagram commutes. Hence α_* is well defined and natural provided $q(f) = q(g)$ implies $\alpha^*(f) = \alpha^*(g)$.*

Recall that for a given two-dimensional complex K, the free crossed module

$$\partial : \pi_2(K, K^1) \to \pi_1(K^1)$$

can be defined as follows. The group $\pi_2(K, K^1)$ is generated by the set

$$\{e_\alpha^w \mid \alpha \text{ is a 2-cell in } K,\ w \in \pi_1(K^1)\}$$

with the set of relations

$$\{e_\alpha^v e_\beta^w e_\alpha^{-v} e_\beta^{-u},\ u = v r_\alpha v^{-1} w\}, \tag{3}$$

where $r_\alpha \in \pi_1(K^1)$ is the attaching element representing e_α (see, for example, [2]). The homomorphism ∂ is defined by setting $\partial : e_\alpha^w \mapsto r_\alpha^w$. Hence every element from $ker(\partial) = \pi_2(K)$ can be represented by an element $e_{\alpha_1}^{\pm w_1} \dots e_{\alpha_m}^{\pm w_m}$, such that $r_{\alpha_1}^{\pm w_1} \dots r_{\alpha_m}^{\pm w_m}$ is trivial in $\pi_1(K^1)$.

Consider the two-dimensional sphere S^2 as the standard two-complex constructed from the following presentation of the trivial group:

$$\langle x_1, \dots, x_{n-1} \mid x_1, \dots, x_{n-1},\ x_{n-1}^{-1} \cdots x_1^{-1} \rangle. \tag{4}$$

This presentation defines an element $\bar{S}_n$ from $\mathcal{K}_n$:

$$\bar{S}_n = (S^2, L_1, \dots, L_n), \tag{5}$$

with $L_i = \vee_{i=1}^{n-1} S^1 \cup e_i$, where e_i is the 2-cell corresponding to the relation word x_i, $i = 1, \dots, n-1$, e_n is the 2-cell corresponding to the relation word $x_{n-1}^{-1} \cdots x_1^{-1}$.

Let $f \in Hom_{\mathcal{K}_n}(\bar{S}_n, \bar{K})$. It means that there exists a homomorphism between two free groups $f : F_{n-1} := F(x_1, \dots, x_n) \to \pi_1(K^1)$ such that

$$f(x_i) \in \ker\{\pi_1(K^1) \to \pi_i(K_i)\},\ i = 1, \dots, n-1 \tag{6}$$

and f can be extended to a homomorphism between two crossed modules:

$$\begin{array}{ccc} \pi_2(S^2, \vee_{i=1}^{n-1} S^1) & \xrightarrow{\partial_1} & F_{n-1} \\ {\scriptstyle f'}\Big\downarrow & & {\scriptstyle f}\Big\downarrow \\ \pi_2(K, K^1) & \xrightarrow{\partial_2} & \pi_1(K^1) \end{array} \tag{7}$$

For a given group homomorphism $f : F_{n-1} \to \pi_1(K^1)$ with the property (6), the necessary and sufficient condition of the existence of the extension (8) is the condition

$$f(x_1 \cdots x_n) \subseteq R_n := \ker\{\pi_1(K^1) \to \pi_1(K)\}.$$

For $\bar{K} = (K, K_1, \dots, K_n) \in \mathcal{K}_n$, we now define the canonical (forgetful) map

$$q : Hom_{\mathcal{K}_n}(\bar{S}_n, \bar{K}) \to \pi_2(K)/(i_1\pi_2(K_1) + \dots i_n\pi_2(K_n)),$$

which carries a morphism $\bar{S}^2 \to \bar{K}$ to the underlying map $S^2 \to K$. Here the natural maps $i_j : \pi_2(K_j) \to K$ are induced by inclusions $K_j \to K$. Using the language of crossed modules, we can describe the map q as follows. Denote by $\{s_1, \dots, s_n\}$ the set of 2-cells in S^2 viewed as the standard two-complex for the group presentation (4). Then the map f' defines elements $f'(s_\alpha) \in \pi_2(K, K^1)$. Observe that $\partial_1(s_1 \dots s_n) = 1$ and the element $s_1 \dots s_n$ presents the generator of $\pi_2(S^2)$. Since the diagram (8) is commutative, $\partial_2(f'(s_1) \dots f'(s_n)) = 1$ and the element $f'(s_1) \dots f'(s_n)$ represents certain element from $ker(\partial_2) = \pi_2(K)$, which is exactly $q(f)$. Let us show that this map does not depend on an extension (8). Suppose we have another extension of the homomorphism f:

$$\begin{array}{ccc} \pi_2(S^2, \vee_{i=1}^{n-1} S^1) & \xrightarrow{\partial_1} & F_{n-1} \\ {\scriptstyle f''}\Big\downarrow & & {\scriptstyle f}\Big\downarrow \\ \pi_2(K, K^1) & \xrightarrow{\partial_2} & \pi_1(K^1) \end{array} \tag{8}$$

with $f''(s_j) \neq f'(s_j)$ at least for one j $(1 \leq j \leq n)$. It follows that $\partial_2(f'(s_j)f''(s_j)^{-1}) = 1$, hence

$$f'(s_j)f''(s_j)^{-1} \in im\{i_j : \pi_2(K_j) \to \pi_2(K)\}.$$

Therefore, the images of elements $f'(s_1 \dots s_n)$ and $f''(s_1 \dots s_n)$ are equal in the quotient $\pi_2(K)/(i_1\pi_2(K_1) + \dots i_n\pi_2(K_n))$ and the map q is well-defined.

Lemma 1. *The map q is surjective.*

Proof. Consider the diagram (8). Now let $c = e_{\alpha_1}^{\pm w_1} \dots e_{\alpha_m}^{\pm w_m}$ be an arbitrary element from $ker(\partial_2)$. Lets enumerate all cells of K in the following order: $e_{1,\alpha}, \dots, e_{n,\alpha}$ with $e_{i,\alpha} \in K_i,\ i = 1, \dots, m$. Clearly, the set of relations (3) in $\pi_2(K, K^1)$ gives a possibility to present the element c in the form

$$c = \prod_* e_{1,*}^{\pm w_{1,*}} \cdots \prod_* e_{n,*}^{\pm w_{n,*}}$$

with some $w_{i,*} \in \pi_1(K^1)$. Then we define the map $f : F_{n-1} \to \pi_1(K^1)$ by setting $f(x_i) = \prod_* r_{i,*}^{\pm w_{n,*}}$. Then we can extend it to $f' : \pi_2(S^2, \vee_{i=1}^{n-1} S^1) \to \pi_2(K, K^1)$ by $f'(s_i) = \prod_* r_{i,*}^{\pm w_{n,*}}$. This is correct, since

$$\partial_1(f'(s_n)) = \partial_2(f'(s_1) \dots f'(s_{n-1}))^{-1} = f(\partial_1(s_1 \dots s_{n-1})^{-1}).$$

Then the homotopy class corresponding to the element $c \in \pi_2(K, K^1)$ coincides with $q(f)$ and the surjectivity of q is proved. □

5. Proof of the conjecture for $n = 2$ and $n = 3$

We now describe the map q in the conjecture by use of identity sequences which represent elements in $\pi_2(K)$, see [4]. Let F be a free group with basis X and $\mathcal{R}$ a certain set of words in F. Consider the group presentation

$$\mathcal{P} = \langle X \mid \mathcal{R} \rangle \tag{9}$$

$c_i,\ i = 1, \dots, m$ are words in F, which are conjugates of elements from $\mathcal{R}$, i.e. $c_i = t_i^{\pm w_i},\ t_i \in \mathcal{R},\ w_i \in F$. Then the sequence

$$c = (c_1, \dots, c_m) \tag{10}$$

is called an *identity sequence* if the product $c_1 \dots c_m$ is the identity in F. For a given identity sequence (10), define its inverse:

$$c^{-1} = (c_m^{-1}, \dots, c_1).$$

For a given element $w \in F$, the conjugate c^w is the sequence:

$$c^w = (c_1^w, \dots, c_m^w),$$

which clearly is again an identity sequence. Define the following operation in the class of identity sequences, called *Peiffer operations*:
(i) replace each w_i by any word equal to it in F;
(ii) delete two consequtive terms in the sequence if one is equal identically to the inverse of the other;
(iii) add two consequtive terms in the sequence if one is equal identically to the inverse of the other;
(iv) replace two consequtive terms c_i, c_{i+1} by terms $c_{i+1}, c_{i+1}^{-1} c_i c_{i+1}$;
(v) replace two consequitive terms c_i, c_{i+1} by terms $c_i c_{i+1} c_i^{-1}, c_i$.

Two identity sequences are called *equivalent* if one can be obtained from the other by a finite number of Peiffer operations. This defines an equivalence relation in the class of identity sequences. The set of equivelence classes of identity sequences for a given group presentation (9) denote by $E_{\mathcal{P}}$. Then $E_{\mathcal{P}}$ can be viewed as a group, with a binary operation defined as a class of justaposition of two sequences: for identity sequences c_1, c_2 and their equivalence classes $\langle c_1\rangle, \langle c_2\rangle \in E_{\mathcal{P}}$, $\langle c_1\rangle + \langle c_2\rangle = \langle c_1c_2\rangle$. The inverse element of the class $\langle c\rangle$ is $\langle c^{-1}\rangle$ and the identity in $E_{\mathcal{P}}$ is the empty sequence. It is easy to see that $E_{\mathcal{P}}$ is Abelian. For two identity sequences $c = (c_1,\dots,c_m)$ and $d = (d_1,\dots,d_k)$, we have

$$\langle cd\rangle = \langle (c_1,\dots,c_m,d_1,\dots,d_k)\rangle = \langle (d_1,\dots,d_k,c_1^{d_1\dots d_m},\dots,c_m^{d_1\dots d_m})\rangle$$

by the relation (iv). Since $d_1\dots d_m = 1$ in F, we have

$$\langle cd\rangle = \langle (d_1,\dots,d_k,c_1,\dots,c_m)\rangle = \langle dc\rangle.$$

Furthermore, $E_{\mathcal{P}}$ is a F-module, where the action is given by

$$\langle c\rangle \circ f = \langle c^f\rangle,\ f\in F.$$

It is easy to show that

$$\langle c\rangle \circ r = \langle c\rangle,\ r\in R,$$

i.e. the subgroup R acts trivially at $E_{\mathcal{P}}$. To see this, let $r = r_1^{\pm v_1}\dots r_k^{\pm v_k},\ r_i\in\mathcal{R},\ v_i\in F$. Then for any identity sequence $c = (c_1,\dots,c_m)$, by (ii), (iii), (iv),

$$\langle (c_1,\dots,c_m)\rangle = \langle (c_1,\dots,c_m,r_1^{\pm v_1},\dots,r_k^{\pm v_k},r_k^{\mp v_k},\dots,r_1^{\mp v_1})\rangle = \\ \langle (r_1^{\pm v_1},\dots,r_k^{\pm v_k},c_1^r,\dots,c_m^r,r_k^{\mp v_k},\dots,r_1^{\mp v_1})\rangle = \langle (c_1^r,\dots,c_m^r)\rangle.$$

Thus $E_{\mathcal{P}}$ can be viewed as a G-module. It is not hard to show that for a given presentation $\mathcal{P}$, the second homotopy module $\pi_2(K_{\mathcal{P}})$ is isomorphic to the identity sequence module $E_{\mathcal{P}}$ (see, for example, [4]).

For a given $\bar{K}$ choose the elements $e_{i,\alpha}\in\pi_2(K,K^1),\ i=1,\dots,n, \alpha\in A$ which represent the corresponding two-dimensional cells in $K_i,\ i=1,\dots,n$ with the natural property

$$\partial(e_{i,\alpha})\in R_i,$$

where $R_i = \ker\{\pi_1(K^1)\to\pi_1(K_i)\},\ i=1,\dots n$, and the normal closure of the set $\{\partial(e_{i,\alpha})\mid \alpha\in A\}$ in $\pi_1(K^1)$ is equal to R_i. Clearly, K is homotopically equivalent to a wedge

$$K\simeq\bigvee_{j\in J}S^2\vee K_{\mathcal{P}},$$

where $K_{\mathcal{P}}$ is the standard two-complex constructed from the group presentation

$$\langle X\mid \partial(e_{i,\alpha}),\ i=1,\dots,n,\ \alpha\in A\rangle,$$

with X being a basis of $\pi_1(K^1)$. Then we have the following natural isomorphism of $\pi_1(K)$-modules:

$$\pi_2(K)/(i_1\pi_2(K_1)+\dots+i_n\pi_2(K_n))\simeq\pi_2(K_{\mathcal{P}})/(i_1\pi_2(K_{\mathcal{P}_1})+\dots+i_n\pi_2(K_{\mathcal{P}_n})),$$

where $\mathcal{P}_i$ is the following presentation of the group $\pi_1(K_i)$:

$$\langle X\mid \partial(e_{i,\alpha}),\ \alpha\in A\rangle$$

for $i=1,\dots,n$.

Let $f,g\in Hom_{\mathcal{K}_n}(\bar{S}_n,\bar{K})$. Then we can present

$$f(x_i) = r_1^{(i)^{\pm w_{1,i}}}\dots r_{k_i}^{(i)^{\pm w_{k_i,i}}},\ i=1,\dots,n-1,$$
$$f(x_1\cdots x_n) = r_1^{(n)^{\pm w_{1,n}}}\dots r_{k_n}^{(n)^{\pm w_{k_n,n}}}$$

for some $r_j^{(i)} \in \{\partial(e_{i,\alpha}),\ \alpha \in A\}$ and $w_{j,i} \in \pi_1(K^1)$. Analogically for $g \in Hom_{\mathcal{K}_n}(\bar{S}_n, \bar{K})$:

$$g(x_i) = r_1'^{(i)\pm w_{1,i}'} \dots r_{k_i'}'^{(i)\pm w_{k_i',i}'},\ i = 1, \dots, n-1,$$
$$g(x_1 \cdots x_n) = r_1'^{(n)\pm w_{1,n}'} \dots r_{k_n'}'^{(n')\pm w_{k_n',n}'}$$

The following Lemma follows directly from the definition of the map q and the above description of the second homotopy module for the standard complex in terms of identity sequences.

Lemma 2. *Using the above notation, $q(f) = q(g)$ if and only if the identity sequence*

$$(r_1^{(1)\pm w_{1,1}}, \dots, r_{k_n}^{(n)\pm w_{k_n,n}}, r_{k_n'}'^{(n')\mp w_{k_n',n}'}, \dots, \dots, r_1'^{(1)\mp w_{1,1}'})$$

is equivalent to an identity sequence of the form

$$(s_1^{(1)}, \dots, s_{l_1}^{(1)}, \dots, s_1^{(n)}, \dots, s_{l_n}^{(n)})$$

with $s_j^{(i)} \in \{\partial(e_{i,\alpha})^{\pm w},\ w \in \pi_1(K^1)\}$ such that $s_1^{(i)} \dots s_{l_i}^{(i)}$ is trivial in $\pi_1(K^1)$ for every $i = 1, \dots, n$.

Let $(K, K_1, K_2) \in \mathcal{K}_2$. The $\pi_1(K)$-module $\pi_2(K)/(i_1\pi_2(K_1)+i_2\pi_2(K_2))$ can be identified to the module of the identity sequences of the type

$$(c_1, \dots, c_m),\ c_j \in \{\partial(c_{i,\alpha})^w,\ w \in \pi_1(K^1),\ \alpha \in A,\ i = 1, 2\} \tag{11}$$

modulo the sequences of the form $(c_1, \dots, c_{m_1}, c_{m_1+1}, \dots, c_m)$ with $c_1, \dots, c_{m_1} \in \{\partial(c_{1,\alpha})^w,\ w \in \pi_1(K^1)\}$, $c_{m_1+1}, \dots, c_m \in \{\partial(c_{2,\alpha})^w,\ w \in \pi_1(K^1)\}$ with

$$c_1 \dots c_{m_1} = c_{m_1+1} \dots c_m = 1$$

in $\pi_1(K^1)$.

Every identity sequence (11) with the help of Peiffer operations of the type (iv) can be reduced to the sequence of the form $(c_1, \dots, c_{m_1}, c_{m_1+1}, \dots, c_m)$ with $c_1, \dots, c_{m_1} \in \{\partial(c_{1,\alpha})^w,\ w \in \pi_1(K^1)\}$, $c_{m_1+1}, \dots, c_m \in \{\partial(c_{2,\alpha})^w,\ w \in \pi_1(K^1)\}$. Then for a generator $x \in \pi_2(S^2)$, the map

$$\Lambda_x : \pi_2(K)/(i_1\pi_2(K_1) + i_2\pi_2(K_2)) \to \frac{R_1 \cap R_2}{[R_1, R_2]}$$

is given in the above notation by

$$\Lambda_x : (c_1, \dots, c_{m_1}, c_{m_1+1}, \dots, c_m) \mapsto c_1 \cdots c_{m_1}.[R_1, R_2].$$

First observe that Λ_x is the homomorphism of $\pi_1(K) = \pi_1(K^1)/R_1R_2$-modules. Secondly, Λ_x clearly is an epimorphism. The fact that Λ_x is a monomorphism is not difficult (see Theorem 1.3 [4] for the complete proof). Hence we have the following exact sequence of $\pi_1(K)$-modules due to Gutierrez and Ratcliffe [3]:

$$0 \to i_1\pi_2(K_1) + i_2\pi_2(K_2) \xrightarrow{\alpha} \pi_2(K) \to \frac{R_1 \cap R_2}{[R_1, R_2]} \to 0. \tag{12}$$

Theorem 1. *Conjecture 1 is true for $n = 3$.*

Proof. In this case we view S^2 as the standard complex constructed for the group presentation

$$\langle x_1, x_2 \mid x_1, x_2, x_2^{-1}x_1^{-1}\rangle$$

with

$$I_3(\mathcal{F}_3(\bar{S}^2)) = I_3(F(x_1,x_2), \langle x_1\rangle^{F(x_1,x_2)}, \langle x_2\rangle^{F(x_1,x_2)}, \langle x_2^{-1}x_1^{-1}\rangle^{F(x_1,x_2)}) \simeq \mathbb{Z}$$

with a generator given by the commutator $[x_1, x_2]$.

Let $\bar{K} = (K, K_1, K_2, K_3) \in \mathcal{K}_3$. Denote $F = \pi_1(K^1)$. Denote the sets of words in F: $\mathcal{R}_i = \{\partial(e_{i,\alpha},\ \alpha \in A\},\ i = 1,2,3$. By $\mathcal{R}_i^F$ we mean the set $\{r^w,\ r \in \mathcal{R}_i,\ w \in F\}$. The $\pi_1(K)$-module $\pi_2(K)/(i_1\pi_2(K_1)+i_2\pi_2(K_2)+i_3\pi_2(K_3))$ can be identified with the module of the identity sequences

$$c = (c_1, \dots, c_m),\ c_j \in \mathcal{R}_1^F \cup \mathcal{R}_2^F \cup \mathcal{R}_3^F \tag{13}$$

modulo the sequences of the type

$$(c_1, \dots, c_{m_1}, c_{m_1+1}, \dots, c_{m_2}, c_{m_2+1}, \dots, c_m) \tag{14}$$

with $c_1, \dots, c_{m_1} \in \mathcal{R}_1^F$, $c_{m_1+1}, \dots, c_m \in \mathcal{R}_2^F$, $c_{m_2+1}, \dots, c_m \in \mathcal{R}_3^F$ and

$$c_1 \dots c_{m_1} = c_{m_1+1} \dots c_{m_2} = c_{m_2+1} \dots c_m = 1, \tag{15}$$

in F.

Divide the sequence (13) into the three ordered subsequences

$$(c_{r_1}, \dots, c_{r_l}),\quad (c_{s_1}, \dots, c_{s_k}),\quad (c_{t_1}, \dots, c_{t_n}), \tag{16}$$

where $c_{r_i} \in \mathcal{R}_1^F,\ i = 1, \dots, l$, $c_{s_i} \in \mathcal{R}_2^F,\ i = 1, \dots, k$, $c_{t_i} \in \mathcal{R}_3^F,\ i = 1, \dots, h$ and

$$r_1 < r_2 < \dots < r_l,\quad s_1 < s_2 < \dots < s_k,\quad t_1 < t_2 < \dots < t_h,$$

$$\{r_1, \dots, r_l\} \cup \{s_1, \dots, s_k\} \cup \{t_1, \dots, t_h\} = \{1, \dots, m\}.$$

Denote

$$\begin{aligned}
&\bar{c}_i = c_{r_i},\ i = 1, \dots, l,\\
&\bar{c}_{l+i} = c_{s_i}^{\prod_{r_j > s_i} c_{r_j}},\ i = 1, \dots, k,\\
&\bar{c}_{l+k+i} = c_{t_1}^{(\prod_{r_z > t_i} c_{r_z}) \prod_{s_j > t_1} c_{s_j}^{\prod_{r_z > s_j} c_{r_z}}},\ i = 1, \dots, h.
\end{aligned}$$

Clearly,

$$\bar{c}_1, \dots, \bar{c}_l \in R_1,\ \bar{c}_{l+1}, \dots, \bar{c}_{l+k} \in R_2,\ \bar{c}_{l+k+1}, \dots, \bar{c}_{l+k+h} \in R_3$$

and the sequence

$$(\bar{c}_1, \dots, \bar{c}_{l+k+h}) \tag{17}$$

is made of the sequence (13), applying the Peiffer operations of type (iv). At the first step we replace all terms c_{r_i} to the left side of the sequence. At the second step we replace all terms c_{s_i} between elements c_{r_i}-s and c_{t_i}-s and get the sequence (17). Denote

$$\begin{aligned}
&r_c := \bar{c}_1 \dots \bar{c}_l \in R_1,\\
&s_c := \bar{c}_{l+1} \dots \bar{c}_{l+k} \in R_2,\\
&t_c := \bar{c}_{l+k+1} \dots \bar{c}_{l+k+h} \in R_3.
\end{aligned}$$

In these notations, for the generator $x := [x_1, x_2]$ of $I_3(\mathcal{F}_3(\bar{S}^2))$ construct the map

$$\Lambda_x : \pi_2(K)/(i_1\pi_2(K_1) + i_2\pi_2(K_2) + i_3\pi_2(K_3)) \to \frac{R_1 \cap R_2 \cap R_3}{[R_1, R_2 \cap R_3][R_2, R_3 \cap R_1][R_3, R_1 \cap R_2]}, \tag{18}$$

where $F = \pi_1(K^1)$, $R_i = \ker\{F \to \pi_1(K_i)\}$, $i = 1, 2, 3$, by setting

$$\Lambda_x : (c_1, \dots, c_m) \mapsto [r_c, s_c].[R_1, R_2 \cap R_3][R_2, R_3 \cap R_1][R_3, R_1 \cap R_2].$$

Since $r_c s_c t_c = 1$ in F, we have $[r_c, s_c] \in R_1 \cap R_2 \cap R_3$.

Lets show that the above map Λ_x is well-defined. Let c' be an identity sequence equivalent to the sequence c. Defining elements $r_{c'}, s_{c'}, t_{c'}$ as above, we have to show that

$$[r_c, s_c] \equiv [r_{c'}, s_{c'}] \mod [R_1, R_2 \cap R_3][R_2, R_3 \cap R_1][R_3, R_1 \cap R_2]. \tag{19}$$

Since we above defined map Λ_x is trivial for any sequence of the type (14) with conditions (15), the equivalence (19) is necessary and sufficient for the correctness of the map Λ_x.

First observe that if the sequences c and c' differ by the Peiffer operations of the type (ii) or (iii), the equivalence 19 holds. The only nontrivial Peiffer operations needed to check are operations (iv) and (v). Since (v) is converse to (iv), it is enough to prove the equivalence (19) for the case c' is obtained from c by the single Peiffer operation of the type (iv):

$$c'_i = c_{i+1},\ c'_{i+1} = c_{i+1}^{-1} c_i c_{i+1},\ c'_j = c_j,\ j \neq i, i+1$$

for some $1 \leq i < m$.

The cases $i, i+1 \in \{r_1, \dots, r_l\}$, $i, i+1 \in \{s_1, \dots, s_k\}$, $i, i+1 \in \{t_1, \dots, t_h\}$ are trivial. In these cases $r_c = r_{c'}, s_c = s_{c'}$, hence the needed equivalence (19) follows. In the case $i+1 \in \{r_1, \dots, r_l\}$ also nothing to prove since the definition of the elements r_c, s_c involves the process of repeating of such operations. In the case $i \in \{t_1, \dots, t_h\}$ or $i+1 \in \{t_1, \dots, t_h\}$ we clearly have $[r_c, s_c] \equiv [r_{c'}, s_{c'}] \mod [R_1, R_2 \cap R_3][R_2, R_3 \cap R_1]$.

Only nontrivial case to consider is $i \in \{r_1, \dots, r_l\}$, $i+1 \in \{s_1, \dots, s_k\}$. Clearly then, $[r_{c'}, s_{c'}] = [r_{c''}, s_{c''}]$, where the sequence c'' is obtained by applying again the operation (iv) to the sequence c' :

$$c''_i = c_{i+1}^{-1} c_i c_{i+1},\ c''_{i+1} = c_{i+1}^{-1} c_i^{-1} c_{i+1} c_i c_{i+1}.$$

Let $c_i = c_{r_j}$, $c_{i+1} = c_{s_e}$. Repeating the operation (iv), we can deform the sequences c and c'' to the form

$$r_2 = r_1 + 1, \dots, r_{j-1} = r_{j-2} + 1,\ r_{j+2} = r_{j+1} + 1, \dots, r_l = r_{l-1} + 1$$

without changing $[r_c, s_c]$ and $[r_{c''}, s_{c''}]$. Now we can form the triple of words in F:

$$\mathcal{R}'_1 = \mathcal{R}_1 \cup \{c_{r_1} \dots c_{r_{j-1}}, c_{j+1} \dots c_l\},\ \mathcal{R}'_2 = \mathcal{R}_2,\ \mathcal{R}'_3 = \mathcal{R}_3.$$

Clearly, this triple preserves the triple of normal subgroups R_1, R_2, R_3 and we can consider the new identity sequences for the triple of words $\mathcal{R}'_1 \cup \mathcal{R}'_2 \cup \mathcal{R}'_3$ formed by gluing the elements $c_{r_1}, \dots, c_{r_{j-1}}$, and $c_{r_{j+1}}, \dots, c_{r_l}$:

$$c''' = (*, \dots, *, c_{r_1} \dots c_{r_{j-1}}, *, \dots, *, c_{r_{j+1}} \dots c_{r_l}, *, \dots, *).$$

It is easy to see that

$$[r_c, s_c] = [r_{c'''}, s_{c'''}]$$

in F. Hence, we can always assume that $l=3$, $c_{r_2}=c_i$ and reduce arbitrary case to this one using the described procedure. In these notations, we have sequences

$$c=(*,\dots,*,c_{r_1},*,\dots,*,c_{r_2},c_{s_e},*,\dots,*,c_{r_3},*,\dots,*),$$
$$c''=(*,\dots,*,c_{r_1},*,\dots,*,c_{r_2}^{c_{s_e}},c_{s_e}^{c_{r_2}c_{s_e}},*,\dots,*,c_{r_3},*,\dots,*).$$

Then

$$[r_c,s_c]=[c_{r_1}c_{r_2}c_{r_3},S_1],$$
$$[r_{c'''},s_{c'''}]=[c_{r_1}c_{r_2}^{c_{s_e}}c_{r_3},S_2],$$

where

$$S_1=(\prod_{s_j<r_1}c_{s_j}^{c_{r_1}c_{r_2}c_{r_3}})(\prod_{r_1<s_j<s_e}c_{s_j}^{c_{r_2}c_{r_3}})\cdot c_{s_e}^{c_{r_3}}\cdot(\prod_{s_e<s_j<r_3}c_{s_j}^{c_{r_3}})(\prod_{r_3<s_j}c_{s_j}),$$
$$S_2=(\prod_{s_j<r_1}c_{s_j}^{c_{r_1}c_{r_2}^{c_{s_e}}c_{r_3}})(\prod_{r_1<s_j<s_e}c_{s_j}^{c_{r_2}^{c_{s_e}}c_{r_3}})\cdot c_{s_e}^{c_{r_2}c_{s_e}c_{r_3}}\cdot(\prod_{s_e<s_j<r_3}c_{s_j}^{c_{r_3}})(\prod_{r_3<s_j}c_{s_j}).$$

We then have

$$\begin{aligned}[c_{r_1}c_{r_2}^{c_{s_e}}c_{r_3},S_2]&=c_{r_3}^{-1}c_{r_2}^{-1}[c_{r_2}^{-1},c_{s_e}]c_{r_1}^{-1}S_2^{-1}c_{r_1}c_{r_2}^{c_{s_e}}c_{r_3}S_2=\\&c_{r_3}^{-1}c_{r_2}^{-1}[c_{r_2}^{-1},c_{s_e}]c_{r_1}^{-1}S_2^{-1}c_{r_3}^{-1}c_{r_2}^{-c_{s_e}}c_{r_2}^{c_{s_e}}c_{r_3}c_{r_1}c_{r_2}^{c_{s_e}}c_{r_3}S_2\equiv\\&c_{r_3}^{-1}c_{r_2}^{-1}c_{r_1}^{-1}S_2^{-1}c_{r_3}^{-1}c_{r_2}^{-c_{s_e}}[c_{r_2}^{-1},c_{s_e}]c_{r_2}^{c_{s_e}}c_{r_3}c_{r_1}c_{r_2}^{c_{s_e}}c_{r_3}S_2\mod[R_3,R_1\cap R_2],\end{aligned}$$

since $S_2^{-1}c_{r_3}^{-1}c_{r_2}^{-c_{s_e}}\in R_3$, $[c_{r_2}^{-1},c_{s_e}]\in R_1\cap R_2$. Therefore,

$$[c_{r_1}c_{r_2}^{c_{s_e}}c_{r_3},S_2]\equiv c_{r_3}^{-1}c_{r_2}^{-1}c_{r_1}^{-1}S_2^{-1}c_{r_3}^{-1}c_{r_2}^{-c_{s_e}}c_{r_2}c_{r_3}c_{r_1}c_{r_2}^{c_{s_e}}c_{r_3}S_2\mod[R_3,R_1\cap R_2].$$

However,

$$c_{r_1}c_{r_2}^{c_{s_e}}c_{r_3}S_2=c_{r_1}c_{r_2}c_{r_3}S_1\in R_3,$$

therefore, $S_2=c_{r_3}^{-1}c_{r_2}^{-c_{s_e}}c_{r_2}c_{r_3}S_1$ and we have

$$[c_{r_1}c_{r_2}^{c_{s_e}}c_{r_3},S_2]\equiv[c_{r_1}c_{r_2}c_{r_3},S_1]\mod[R_3,R_1\cap R_2].$$

Hence, we always have the needed equivalence (19) and we proved that the map Λ_x is well-defined. □

For the generator $x\in\pi_3(S^2)$ denote by Λ the composite map of the natural projection $\pi_2(K)\to\pi_2(K)/i_1\pi_2(K_1)+i_2\pi_2(K_2)+i_3\pi_2(K_3)$ and the map Λ_x:

$$\Lambda:\pi_2(K)\to\frac{R_1\cap R_2\cap R_3}{[R_1,R_2\cap R_3][R_2,R_3\cap R_1][R_3,R_1\cap R_2]}.$$

Proposition 3. *Let $b\in i_{12}\pi_2(K_1\cup K_2)+i_{13}\pi_2(K_1\cup K_3)+i_{23}\pi_2(K_2\cup K_3)\subseteq\pi_2(K)$ where the maps i_{12},i_{13},i_{23} are induced by the inclusions*

$$i_{12}:K_1\cup K_2\to K,\ i_{13}:K_1\cup K_3\to K,\ i_{23}:K_2\cup K_3\to K.$$

Then $\Lambda(a+b)=\Lambda(a)$ for every $a\in\pi_2(K)$.

Proof. Let a be an element from $\pi_2(K)$ presented by identity sequence (16) and the element b be an element from $i_{12}(K_1\cup K_2)\subseteq\pi_2(K)$ presented by the identity sequence $(d_1,\dots,d_{l'},e_1,\dots,e_{k'})$ with $d_i\in\mathcal{R}_1^F$, $e_i\in\mathcal{R}_2^F$. Then the element $a+b$ can be presented by the following identity sequence

$$c(a+b)=(c_{r_1},\dots,c_{r_l},d_1,\dots,d_{l'},c_{s_1}^{d_1\dots d_{l'}},\dots,c_{s_k}^{d_1\dots d_{l'}},e_1,\dots,e_{k'},f_1,\dots,f_{h'}),$$

with $f_1, \dots, f_{h'} \in \mathcal{R}_3^F$. Denote $a_1 = c_{r_1} \dots c_{r_l}, a_2 = d_1 \dots d_{l'}, b_1 = c_{s_1} \dots c_{s_k},\ b_2 = e_1 \dots e_{k'}$. Then we have

$$[a_1a_2, b_1^{a_2}b_2] = a_2^{-1}a_1^{-1}b_2^{-1}a_2^{-1}b_1^{-1}a_2a_1b_1a_2b_2 \equiv$$
$$a_1^{-1}b_2^{-1}a_2^{-1}b_1^{-1}a_1b_1a_2b_2 \equiv a_1^{-1}b_1^{-1}a_1b_1 \mod [R_3, R_1 \cap R_2],$$

since $a_2 \in R_1 \cap R_2, a_1^{-1}b_2^{-1}a_2^{-1}b_1^{-1} \in R_3, a_2b_2 = 1$. Hence $\Lambda(a+b) = \Lambda(a)$.

In the case $b \in i_{13}\pi_2(K_1 \cup K_3) + i_{23}\pi_2(K_2 \cup K_3)$, we have obviously, that the elements which represent $\Lambda(a+b)$ and $\Lambda(a)$ are equal modulo $[R_1, R_2 \cap R_3][R_2, R_3 \cap R_1]$ hence $\Lambda(a+b) = \Lambda(a)$. □

The following example shows that the map Λ is not always surjective.

Example. Let F be a free group with generators x_1, x_2. Consider the following sets of words:

$$\mathcal{R}_1 = \{x_1\},\ \mathcal{R}_2 = \{[x_1, x_2]\},\ \mathcal{R}_3 = \{[x_1, x_2, x_1]\}.$$

Denoting R_1, R_2, R_3 the normal closures of the sets $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ respectively, we have

$$[R_1, R_2 \cap R_3], [R_2, R_3 \cap R_1], [R_3, R_1 \cap R_2] \subseteq \gamma_4(F),$$

where $\gamma_4(F)$ the 4-th lower central series term of F. However,

$$[x_1, x_2, x_1] \in (R_1 \cap R_2 \cap R_3) \setminus \gamma_4(F),$$

since $[x_1, x_2, x_1]$ is a basic commutator of length three in F. Suppose we have

$$\Lambda(x) = [x_1, x_2, x_3].[R_1, R_2 \cap R_3][R_2, R_3 \cap R_1][R_3, R_1 \cap R_2]$$

for some element x of the second homotopy module of the standard complex constructed for the group presentation

$$\langle x_1, x_2 \mid x_1, [x_1, x_2],\ [x_1, x_2, x_1]\rangle.$$

Then

$$[x_1, x_2, x_1] \equiv [r, s] \mod \gamma_4(F) \tag{20}$$

for some $r \in R_1, s \in R_2$, such that

$$rs \in R_3. \tag{21}$$

However, the condition (21) implies that $r \in \gamma_2(F)$, since $s \in \gamma_2(F)$. Therefore $[r, s] \in \gamma_4(F)$ and the equivalence (20) is not possible. Hence, the map Λ is not surjective.

Theorem 2. *The map Λ is a homogenous quadratic map, i.e.*

$$\Lambda(a, b) = \Lambda(a+b) - \Lambda(a) - \Lambda(b)$$

is bilinear and $\Lambda(x) = \Lambda(-x)$ for any $a, b, x \in \pi_2(K)$.

Proof. For $x, y \in \pi_2(K_{\langle X \mid \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3\rangle})$, consider the cross-effect

$$\Lambda(a, b) = \Lambda(a+b) - \Lambda(a) - \Lambda(b) \in \frac{R_1 \cap R_2 \cap R_3}{[R_1, R_2 \cap R_3][R_2, R_3 \cap R_1][R_3, R_1 \cap R_2]}.$$

Represent elements a, b by identity sequences:

$$c(a) = (c_1, \dots, c_m),\ c(b) = (c_1', \dots, c_{m'}').$$

Consider the corresponding divisions of the sequences $c(a)$ and $c(b)$:

$$\{c_{r_1},\dots,c_{r_l}\}\cup\{c_{s_1},\dots,c_{s_k}\}\cup\{c_{t_1},\dots,c_{t_n}\}=\{c_1,\dots,c_m\},$$
$$\{c'_{\bar r_1},\dots,c'_{\bar r_{l'}}\}\cup\{c'_{\bar s_1},\dots,c'_{\bar s_{k'}}\}\cup\{c'_{\bar t_1},\dots,c'_{\bar t_{n'}}\}=\{c'_1,\dots,c'_m\}$$

with $c_{r_i}, c'_{\bar r_i}\in\mathcal{R}_1^F, c_{s_i}, c'_{\bar s_i}\in\mathcal{R}_2^F,\ c_{t_i}, c'_{\bar t_i}\in\mathcal{R}_3^F$. Consider then the induced division of the sequence $c(a+b)=(c_1,\dots,c_m,c'_1,\dots,c'_{m'})$, which represents the element $a+b\in\pi_2(K_{\langle X\ \mid\ \mathcal{R}_1\cup\mathcal{R}_2\cup\mathcal{R}_3\rangle})$:

$$\{c_{r_1},\dots,c_{r_l},c'_{\bar r_1},\dots,c'_{\bar r_{l'}}\}\cup\{c_{s_1},\dots,c_{s_k},c'_{\bar s_1},\dots c'_{\bar s_{k'}}\}\cup\{c_{t_1},\dots,c_{t_n},c'_{\bar t_1},\dots,c'_{\bar t_{n'}}\}.$$

For the description of the functor $\Lambda(a,b)$, using the Peiffer operation (iv) to the sequences $c(a)$ and $c(b)$, we can reduce the general case to the case of $l=1,k=1,l'=1,k'=1$ with $r_1<s_1$, $\bar r_1<\bar s_1$. Denote $x_1=c_{r_1}, y_1=c_{s_1}, x_2=c'_{\bar r_1}, y_2=c'_{\bar s_1}$.

Then

$$\Lambda(a)=[x_1,y_1],\ \Lambda(b)=[x_2,y_2],$$
$$\Lambda(a+b)=[x_1x_2,y_1^{x_2}y_2].$$

We have

$$\begin{aligned}\Lambda(a+b)&=[x_1,y_2]^{x_2}[x_2,y_2][x_1,y_1^{x_2}]^{x_2y_2}[x_2,y_1^{x_2}]^{y_2}\\ &\equiv[x_1,y_2]^{x_2}[x_2,y_2][x_1,y_1^{x_2}][x_2,y_1]\mod [R_3,R_1\cap R_2]\\ &\equiv[x_1,y_2]^{x_2}[x_2,y_2]x_1^{-1}x_2^{-1}y_1^{-1}x_2x_1y_1\mod [R_3,R_1\cap R_2]\\ &\equiv[x_1,y_2]^{x_2}[x_2,y_2][x_2,y_1]^{x_1}[x_1,y_1]\mod [R_3,R_1\cap R_2].\end{aligned}$$

Since $x_1y_1, x_2y_2\in R_3$,

$$\begin{aligned}\Lambda(a,b)=\Lambda(a+b)-\Lambda(a)-\Lambda(b)&\equiv[x_1,y_2]^{x_2}[x_2,y_1]^{x_1}\mod [R_3,R_1\cap R_2]\\ &\equiv[x_1,y_2]^{y_2^{-1}}[x_2,y_1]^{y_1^{-1}}\mod [R_3,R_1\cap R_2]\\ &\equiv[y_2^{-1},x_1][y_1^{-1},x_2]\mod [R_3,R_1\cap R_2].\end{aligned}$$

Now lets show the linearity of the functor $\Lambda(*,*)$, i.e. that

$$\Lambda(a+b,d)=\Lambda(a,c)+\Lambda(b,d), \tag{22}$$
$$\Lambda(a,b+d)=\Lambda(a,b)+\Lambda(a,d) \tag{23}$$

for arbitrary elements $a,b,d\in\pi_2(K_{\langle X\ \mid\ \mathcal{R}_1\cup\mathcal{R}_2\cup\mathcal{R}_3\rangle})$. Let $c(a),c(b)$ and $c(d)$ be the identity sequences represented the elements a,b and d respectively. Again, without loss of generality we can assume that these elements are represented by identity sequences with single element from each class $\mathcal{R}_i$. Denote the correspondent pairs by $x_1,y_1\subset c(a)$ (the set-theoretical inclusion means that x_1, y_1 are elements of the sequence $c(a)$), $x_2,y_2\subset c(b)$,

$x_3, y_3 \subset c(d)$. In this notation, modulo $[R_1, R_2 \cap R_3][R_2, R_3 \cap R_1][R_3, R_1 \cap R_2]$, we have

$$\begin{aligned}
\Lambda(a+b,d) &\equiv [y_3^{-1}, x_1x_2][y_2^{-1}y_1^{-x_2}, x_3] \\
&\equiv [y_3^{-1}, x_2][y_3^{-1}, x_1]^{x_2}[y_2^{-1}, x_3]^{y_1^{-x_2}}[y_1^{-x_2}, x_3] \\
&\equiv [y_3^{-1}, x_2]x_2^{-1}y_3x_1^{-1}y_3x_1y_1x_2y_2x_3^{-1}y_2^{-1}x_2^{-1}y_1^{-1}x_2x_3 \\
&\equiv [y_3^{-1}, x_2]x_2^{-1}y_3x_1^{-1}y_3x_1y_1(x_2y_2x_3^{-1}y_2^{-1}x_2^{-1}x_3)x_3^{-1}y_1^{-1}x_2x_3 \\
&\equiv [y_3^{-1}, x_2]x_2^{-1}(x_2y_2x_3^{-1}y_2^{-1}x_2^{-1}x_3)y_3x_1^{-1}y_3x_1y_1x_3^{-1}y_1^{-1}x_2x_3 \\
&\equiv [y_3^{-1}, x_2][y_2^{-1}, x_3]x_3^{-1}x_2^{-1}x_3y_3x_1^{-1}y_3x_1y_1x_3^{-1}y_1^{-1}x_2x_3 \\
&\equiv [y_3^{-1}, x_2][y_2^{-1}, x_3]x_3^{-1}x_2^{-1}x_3[y_3^{-1}, x_1][y_1^{-1}, x_3]x_3^{-1}x_2x_3 \\
&\equiv [y_3^{-1}, x_2][y_2^{-1}, x_3][y_3^{-1}, x_1][y_1^{-1}, x_3] \\
&\equiv \Lambda(a,d) + \Lambda(b,d),
\end{aligned}$$

since $[y_3^{-1}, x_1][y_1^{-1}, x_3] \in R_2 \cap R_3$ and (22) follows. The equality (23) can be proved analogically.

Now let us prove that $\Lambda(-x) = \Lambda(x)$. Clearly, we can assume that our identity sequence representing the element $x \in \pi_2(K)$ has the form

$$(r_1, s_1, t_1)$$

with $r_1 \in \mathcal{R}_1, s_1 \in \mathcal{R}_2, t_1 \in \mathcal{R}_3$. The inverse sequence, which represents the element $-x$ has the form

$$(t_1^{-1}, s_1^{-1}, r_1^{-1}).$$

Then we have

$$\Lambda(-x) = [r_1^{-1}, s_1^{-r_1^{-1}}] = [s_1^{-1}, r_1] = [r_1, s_1]^{s_1^{-1}} \equiv [r_1, s_1] \equiv \Lambda(x) \mod [R_2, R_3 \cap R_1].$$

□

Theorem 3. *The function Λ induces the homomorphism of $F/R_1R_2R_3$-modules*

$$\bar{\Lambda} : \pi_3(K) \to \frac{R_1 \cap R_2 \cap R_3}{[R_1, R_2 \cap R_3][R_2, R_3 \cap R_1][R_3, R_1 \cap R_2]}.$$

Proof. Let $x \in \pi_2(K_{\langle X \mid \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 \rangle})$. Present x by the sequence

$$c(x) = (c_1, \dots, c_m).$$

For a given element $f \in \pi_1(K)$, present this element as a coset $f = w.R_1R_2R_3$ for some element $w \in F$. Then the element $f \circ x \in \pi_2(K_{\langle X \mid \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 \rangle})$ can be presented by sequence

$$c(x)^w = (c_1^w, \dots, c_m^w).$$

It follows directly from the definition of $\Lambda(x)$, that

$$\Lambda(f \circ x) \equiv \Lambda(x)^w \mod [R_1, R_2 \cap R_3][R_2, R_3 \cap R_1][R_3, R_1 \cap R_2].$$

Since $\pi_3(K) = \Gamma\pi_2(K)$, we have the needed homomorphism of $F/R_1R_2R_3$-modules due to Theorem 2. □

Example. For two-dimensional sphere S^2, clearly, Λ defines the isomorphism (2):

$$\bar{\Lambda} : \pi_3(S^2) \to I_3(\mathcal{F}_3(\bar{S}_3))$$

with $\bar{S}_3 \in \mathcal{K}_3$ defined in (5).

Example. Consider a group presentation

$$\mathcal{P} = \langle x_1, \dots, x_k \mid r_1, \dots, r_l \rangle$$

of a group G. Let $\mathcal{P}'$ be another presentation of G with $k + 2l$ generators and $3l$ relators given by

$$\mathcal{P}' = \langle x_1, \dots, x_k, y_1, \dots, y_l, z_1, \dots, z_l \mid y_1, \dots, y_l, z_1 y_1^{-1}, \dots, z_l y_l^{-1}, z_1^{-1} r_1, \dots, z_l^{-1} r_l \rangle$$

Then the standard complex $K_{\mathcal{P}'}$ is the union $K_1 \cup K_2 \cup K_3$, where K_1, K_2, K_3 are standard complexes of the following presentations

$$\langle x_1, \dots, x_k, y_1, \dots, y_l, z_1, \dots, z_l \mid y_1, \dots, y_l \rangle,$$
$$\langle x_1, \dots, x_k, y_1, \dots, y_l, z_1, \dots, z_l \mid z_1 y_1^{-1}, \dots, z_l y_l^{-1} \rangle,$$
$$\langle x_1, \dots, x_k, y_1, \dots, y_l, z_1, \dots, z_l \mid z_1^{-1} r_1, \dots, z_l^{-1} r_l \rangle$$

respectively. Denoting $\bar{K} = (K_{\mathcal{P}'}, K_1, K_2, K_3) \in \mathcal{K}_3$, we have the following isomorphism of G-modules:

$$\pi_3(K_{\mathcal{P}}) \simeq \pi_3(K_{\mathcal{P}'}) \simeq I_3(\mathcal{F}_3(\bar{K})).$$

This isomorphism follows directly from the description of Kan's loop construction $GK_{\mathcal{P}}$ and the fact that for a simplicial group G_* with G_2 generated by degeneracy elements, one has $\pi_2(G_*) \simeq I_3(G_2, ker(d_0), ker(d_1), ker(d_2))$.

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