On Compact Shimura Surfaces

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Abstract

We study surfaces constructed from groups of units in quaternion orders Λ over the integers in real quadratic fields k. A short presentation of some general theory of such surfaces is given, in particular, we construct certain "modular curves" on the surfaces and examine their properties. We apply our results to some surfaces related to the case $k = \mathbb{Q}(\sqrt{13})$ and $d(\Lambda) = (3)$, and find their place in the Kodaira classification of algebraic surfaces.

Key words: Shimura surface, quaternion order, Kodaira classification
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The purpose of this paper is to describe some tools to study particular examples of quaternionic Shimura surfaces from the point of view of algebraic geometry. To do that one needs of course standard numerical invariants, but also more explicit knowledge of the geometry of the curves on these surfaces. In section 2, we construct a family of "modular curves", and state some results regarding these. Proofs of all claims can be found in [5]. In section 3 we consider an explicit example over the base field $\mathbb{Q}(\sqrt{13})$. For more details about the underlying computations, we refer to [5].

1 Introduction

Let $k = \mathbb{Q}(\sqrt{d})$ be a real quadratic field with $R = \mathcal{O}_k$ and $D = d(k)$. Let A be a totally indefinite quaternion algebra over k which allows an involution of type 2, i.e. a map $A \to A$, denoted $x \mapsto \overline{x}$, such that $\overline{\overline{x}} = x$, $\overline{x+y} = \overline{x} + \overline{y}$ and $\overline{xy} = \overline{x}\,\overline{y}$ for all $x, y \in A$, and such that the restriction of the involution to k is the non-trivial automorphism of k. If Λ is a maximal R-order in A, then we let $\Lambda_0 = \{\lambda \in \Lambda \mid \overline{\lambda} = \lambda\}$. It can be shown that, replacing the involution of type 2 with another one if necessary, one can assume that D and $d(\Lambda_0)$ are coprime. In that case, we say that the involution is special with respect to Λ.

The canonical involution of a quaternion algebra A is denoted $x \mapsto x^*$. The reduced norm is $\mathrm{nr}(x) = xx^*$ and the reduced trace is $\mathrm{tr}(x) = x + x^*$. The reduced discriminant of an order Λ is denoted $d(\Lambda)$, and we let $d_0(\Lambda)$ be the positive rational integer that generates $d(\Lambda)$ if such exists.

Choose a maximal order Λ and let $\Lambda^1 = \{\lambda \in \Lambda \mid \mathrm{nr}(\lambda) = 1\}$. Choose a real representation $A \to M_2(\mathbb{R})$, and consider the induced action of Λ^1 on $\mathcal{H}$ given by Möbius transformations. We get an action on $\mathcal{H}\times\mathcal{H}$ by $\lambda(z_1, z_2) = (\lambda z_1, \overline{\lambda} z_2)$. We let Γ denote the image of Λ^1 in $\mathrm{Aut}(\mathcal{H}\times\mathcal{H})$ and $X = \mathcal{H}\times\mathcal{H}/\Gamma$.

X is a so called Hilbert modular surface if $A \cong M_2(k)$. These are not compact, but can be compactified by adding a finite number of points, the so called cusps. These points give complicated singularities which were first resolved by Hirzebruch, see [9]. Studying numerical invariants of the surfaces and configurations of certain curves on them, curves coming from the resolution of the singularities and so called modular curves, one managed to identify many of these surfaces in terms of the Kodaira classification of algebraic surface. For example, if $\Lambda = M_2(R)$, then X is rational if and only of $D = 5$, 8, 12, 13, 17, 21, 24, 28, 33, or 60 (see [8]). See also [4].

From now on we will consider non-split algebras, i.e. we assume that $A \not\cong M_2(k)$. In this case the quotient $X = \mathcal{H}\times\mathcal{H}/\Lambda^1$ is compact. However, in general X has quotient singularities coming from elements of finite order in Γ, so called elliptic elements. We let Y denote the canonical minimal resolution of X. We now summarise some known results about the numerical invariants of X and Y in this case. Let ω be the Gauss-Bonnet form on $\mathcal{H}\times\mathcal{H}$:

$$\omega = \frac{1}{(2\pi)^2}\frac{dx_1 \wedge dy_1}{y_1^2} \wedge \frac{dx_2 \wedge dy_2}{y_2^2},$$

where $z_k = x_k + iy_k$, $k = 1, 2$ are the standard coordinates on the two factors of $\mathcal{H}\times\mathcal{H}$. The Euler characteristic $e(X)$ is given by

$$e(X) = \int_{\mathcal{F}} \omega + \sum_{r>1} e_r \frac{r-1}{r}, \tag{1}$$

where $\mathcal{F}$ is a fundamental domain of the group action and e_r is the number of classes of elliptic points of order r (see [6], p. 197). The formula for the volume of a fundamental domain in $\mathcal{H}\times\mathcal{H}$ under the action of Γ can be found for example in [12], p. 193. In our case, when Λ is a maximal order and the quaternion algebra A admits an involution of type 2, the formula may be written as

$$\int_{\mathcal{F}} \omega = 2\zeta_k(-1) \prod_{p \mid d_0(\Lambda)} (p-1)^2. \tag{2}$$

Here ζ_k is the zeta-function of the field k. The value $\zeta_k(-1)$ can be easily computed using a formula of Siegel (see e.g. [6], p. 192). By Satz 8 in [3], we have that the irregularity q of the surface Y vanishes: $q(Y) = 0$. We also have, since A is a skew field, that $\chi(X) = \frac{1}{4}e(X)$ (see [12], p. 206–207).

Apart from the general facts above, not very much is known about the geometry of specific examples. In [10] all cases were determined when X is non-singular and has geometric genus $p_g = 0$. There are 3 such cases, one with $k = \mathbb{Q}(\sqrt{2})$ and two with $k = \mathbb{Q}(\sqrt{3})$. One can also find there results about the corresponding question for certain extensions of the group Γ. This gives many more cases with $p_g = 0$. Note however, that in none of these cases the algebra allows an involution of type 2.

Notation. For p a prime, we let v_p denote the valuation on $\mathbb{Q}_p$ and $(a,b)_p$ the Hilbert symbol, $a, b \in \mathbb{Q}_p^*$. For integers a, b, c, we let $\varphi = [a, b, c]$ denote the quadratic form $ax^2 + bxy + cy^2$. The discriminant of φ is $d(\varphi) = b^2 - 4ac$, and the content $m(\varphi)$ of φ is the ideal generated by a, b and c.

2 The modular curves

Consider the following 4-dimensional $\mathbb{Q}$-vector space:

$$W = \{\beta \in A \mid \overline{\beta}^* = \beta\}.$$

Clearly, if $\beta \in W$, then $\mathrm{nr}(\beta) \in \mathbb{Q}$. If $\beta \in W$ and $\mathrm{nr}(\beta) > 0$, then we define the following curve on $\mathcal{H} \times \mathcal{H}$:

$$C_\beta = \{(z, \beta z) \mid z \in \mathcal{H}\}.$$

This curve maps to a subvariety $F_\beta \subset X$.

We let $\Gamma_\beta = \{\lambda \in \Lambda^1 \mid \lambda \cdot C_\beta = C_\beta\} = \{\lambda \in \lambda^1 \mid \beta\lambda = \pm\overline{\lambda}\beta\}$, the stabiliser group of C_β. We define $A_\beta = \{a \in A \mid \beta a = \overline{a}\beta\}$, which is a quaternion algebra over $\mathbb{Q}$, and let $\Lambda_\beta = A_\beta \cap \Lambda$. We have that Λ_β^1 is a subgroup of Γ_β of index 1 or 2.

If we consider A as a right A-module, then we say that a map $\Phi : A \times A \to A$ is an A-hermitian form if $\Phi(x + y, z) = \Phi(x, z) + \Phi(y, z)$, $\Phi(xa, y) = \Phi(x, y)a$ and $\Phi(x, y) = \overline{\Phi(y, x)}^*$ for all $a, x, y, z \in A$. If Λ is a maximal order in A and the type 2 involution on A is special with respect to Λ, then we say that the A-hermitian form Φ is integral with respect to the order Λ if

i) $\Phi(x, y) \in \Lambda$ for all $x, y \in \Lambda$,

ii) $\Phi(x, x) \in R + \sqrt{D}\Lambda$ for all $x \in \Lambda$.

For any β in W, we define a hermitian form Φ_β by $\Phi_\beta(x, y) = \overline{y}^*\beta x$. We define a $\mathbb{Z}$-lattice L of rank 4 by

$$L = \{\beta \in W \mid \Phi_\beta \text{ is integral}\}. \tag{3}$$

Consider a quadratic form q on L by $q(\beta) = \dfrac{d_0(\Lambda)}{d_0(\Lambda)}\,\mathrm{nr}(\beta)$. We define a lattice $L^{\#}$ dual to L by

$$L^{\#} = \{l \in W \mid \mathrm{tr}(l^* L) \subseteq \mathbb{Z}\}, \tag{4}$$

and a quadratic form $q^{\#}$ on $L^{\#}$ given by $q^{\#}(l) = Dd_0(\Lambda_0)\,\mathrm{nr}(l)$. One can show that (L, q) and $(L^{\#}, q^{\#})$ are primitive integral quadratic forms, and that their genera are independent of the choice of involution.

Consider now the quaternary quadratic lattice $(L^{\#}, q^{\#})$. Recall the definition of the even Clifford algebra $C_0(L^{\#}, q^{\#})$ as the quotient of the tensor algebra $\mathcal{T}_0(L^{\#}) = \bigoplus_{k=0}^{\infty} (L^{\#})^{\otimes 2k}$ by the ideal generated by all elements $x \otimes x - q^{\#}(x)$, where $x \in L^{\#}$. Define a function $\phi : L^{\#} \otimes_{\mathbb{Z}} L^{\#} \to A$ by

$$\phi(l_1 \otimes l_2) = Dd_0(\Lambda_0) l_1^* l_2.$$

We can extend ϕ in a natural way to the even tensor algebra, so we get a map $\phi : \mathcal{T}_0(L^{\#}) \to A$. This map clearly vanishes on elements of the form $x \otimes x - q^{\#}(x)$, where $x \in L^{\#}$, giving an embedding of the ring $C_0(L^{\#}, q^{\#})$ into the algebra A. In fact, the image is a subring (but not a suborder) of Λ:

$$\phi : C_0(L^{\#}, q^{\#}) \to \Lambda. \tag{5}$$

The map (5) is one of the main ingredients in the proofs of theorem 2 and theorem 7 below.

Combining the following two results, the genus of the orders Λ_β is determined.

Proposition 1. *(Cf. prop. 1.12 in [2]) Let F be a local field, K a separable maximal commutative subalgebra of A and S a maximal order of K. Let $\Lambda_1, \Lambda_2 \subset A$ be two S-primitive orders (i.e. they allow an embedding of S). If $d(\Lambda_1) = d(\Lambda_2)$, then $\Lambda_1 \cong \Lambda_2$.*

Theorem 2. *(See [5], thm. 7.16 and prop. 7.23) The discriminant of Λ_β equals the least common multiple of $q(\beta)$ and $d_0(\Lambda)$. Furthermore, if p is a prime, then the following holds:*

- *i) If p is split in k, then $(A_\beta)_p$ is split if and only if $p \nmid d(\Lambda)$. If p is unramified in k, then $(A_\beta)_p$ is split if and only if $v_p(q(\beta))$ is even. If p is ramified in k, then $(A_\beta)_p$ is split if and only if $\mathrm{nr}(\beta) \in \mathrm{nr}_{k_p/\mathbb{Q}_p}(k_p^*)$.*
- *ii) The order $(\Lambda_\beta)_p$ is R_p-primitive unless p is split in k and $p \mid d(\Lambda)$.*

An element $\beta \in L$ is called primitive if $\{x \in \mathbb{Q} \mid x\beta \in L\} = \mathbb{Z}$. For N a positive integer, let $L(N)$ denote the set of primitive elements of L with $q(\beta) = N$.

Proposition 3. *(See [5], prop. 8.8) If $\beta \in L(N)$, then $[\Gamma_\beta : \Lambda_\beta^1] = 2$ if and only if the following conditions hold for every prime p such that $p \mid D$:*

i) if $(-1,D)_p = 1$*, then* $p \mid N$*,*

ii) if $(-1,D)_p = -1$*, then* $v_p(d) \le v_p(N) < 2v_p(D)$*.*

By proposition 3, there exists a function κ defined on the set of all positive integers primitively represented by (L,q), such that $[\Gamma_\beta : \Lambda^1_\beta] = \kappa(N)$ for all $\beta \in L(N)$. For every positive integer N, we define

$$F_N = \bigcup_{\beta \in L(N)} F_\beta.$$

Let f_N denote the number of irreducible components of F_N. For every prime p dividing D, we define a character $\chi_{D,p} : \mathbb{Z}_p \to \{\pm 1, 0\}$, by $\chi_{D,p}(N) = (D,N)_p$ if $p \nmid N$ and $\chi_{D,p}(N) = 0$ if $p \mid N$. Furthermore, if $N \in \mathbb{Z}_p$ and $N \neq 0$, then we define $a_{D,p}(N) = 2$ if $v_p(N) \ge 2v_p(D)$ and $a_{D,p}(N) = 1$ otherwise.

Theorem 4. *(See [5], thm. 8.9) If there exists a prime p such that* $p \mid d(\Lambda)$ *and* $p^2 \mid N$*, then* $f_N = 0$*. Otherwise*

$$f_N = \frac{\kappa(N)}{2} \prod_{p \mid D} (\chi_{D,p}(NB) + 1) a_{D,p}(N),$$

where $B = d_0(\Lambda_0)/d_0(\Lambda)$*.*

Now we have the tools to determine the number of components of F_N and their genus. We turn to the problem of determining how these curves intersect. If $z = (z_1, z_2)$ is a point in $\mathcal{H} \times \mathcal{H}$, then we let

$$L_z = \{\beta \in L \mid q(\beta) > 0 \text{ and } \beta z_1 = z_2\} \cup \{0\}.$$

We have that q restricted to L_z is positive definite, so the rank of L_z is at most 2. The point z is called special if the lattice L_z has rank 2. It is possible to choose an orientation on the 2-dimensional $\mathbb{R}$-vector bundle $\mathcal{L} = \{(z,\beta) \in (\mathcal{H} \times \mathcal{H}) \times M_2(\mathbb{R}) \mid \det(\beta) > 0 \text{ and } \beta z_1 = z_2\} \cup \mathcal{H} \times \mathcal{H} \times \{0\}$ on $\mathcal{H} \times \mathcal{H}$. We fix such a choice, and get induced orientations on the lattices L_z. Let $q_z = q|_{L_z}$ be the oriented binary form associated with the special point z.

We introduce (as in [7]) a function s counting the number of special points. Let φ be an oriented binary form. Then we put

$$s(\varphi) = \sum_{\zeta \in X(\varphi)} \frac{w_\varphi}{v_\zeta}, \tag{6}$$

where w_φ is the number of elements in the group of oriented automorphisms of φ, and v_ζ is the order of the isotropy group Λ^1_z in Λ^1 of a point $z \in \mathcal{H} \times \mathcal{H}$ representing ζ (so w_φ and v_ζ both take values in the set $\{2,4,6\}$).

From now on, we will assume that $(D,6) = 1$. In this case, every elliptic point is of order 2, 3 or 5 (the latter case is only possible if $k = \mathbb{Q}(\sqrt{5})$), and one can show that every elliptic point of order 2 or 3 is a special point.

Proposition 5. *(See [5], cor. 9.9) Let q_z be the quadratic form of an elliptic point z of order $n = 2$ or $n = 3$. There are two cases:*

i) $q_z \cong mD\varphi$, where $\varphi = [1, 0, 1]$ if $n = 2$ or $\varphi = [1, -1, 1]$ if $n = 3$, and where m is a positive integer with $m \mid d(\Lambda)$. If $n = 2$, then we have $\Lambda_z^1 \cap \Lambda_\beta^1 = \{\pm 1\}$ and $\Lambda_z^1 \subseteq \Gamma_\beta$ for all $\beta \in L_z \setminus \{0\}$, and if $n = 3$, then we have $\Lambda_z^1 \cap \Gamma_\beta = \{\pm 1\}$ for all $\beta \in L_z \setminus \{0\}$.

ii) $q_z \cong m\varphi$, where $d_0(\varphi) = -4D$ if $n = 2$ or $d_0(\varphi) = -3D$ if $n = 3$, and m is a positive integer with $m \mid d(\Lambda)$. We have $\Lambda_z^1 \subseteq \Lambda_\beta^1$ for all $\beta \in L_z \setminus \{0\}$.

Lemma 6. *(See [5], lem. 9.5) Let p be a prime and φ_p a non-degenerate binary quadratic form over $\mathbb{Z}_p$. Then (L_p, q_p) represents φ_p primitively if and only if the following conditions are satisfied:*

i) if p is unramified in k, then $p \nmid m(\varphi_p)$,

ii) if $p \mid d(\Lambda)$, then φ_p is anisotropic, and it is either a modular form with $p^2 \nmid m(\varphi_p)$ or a non-modular form such that $d(\varphi_p) = (p)$ if $p \neq 2$ and such that $d(\varphi_p) \mid 8$ and φ_p do not primitively represent 4 if $p = 2$,

iii) if $p \mid D$, then $p \mid d(\varphi_p)$ and $p^2 \nmid m(\varphi_p)$. Furhermore, if $m(\varphi_p) = (1)$, then φ_p represents $B = d_0(\Lambda_\mathbb{Z})/d_0(\Lambda)$, and if $m(\varphi_p) = (p)$, then φ_p represents $-BD$.

If φ is a positive definite oriented binary form such that $\varphi \cong q_z$ for some point $z \in \mathcal{H} \times \mathcal{H}$, then we let $\Delta = d(\varphi)$, and $m > 0$ be a generator of $m(\varphi)$. We write $m = m_1 m_2 m_3$, where m_1 contains those prime factors which divide D, m_2 those prime factors which divide $d(\Lambda)$ and m_3 contains all other prime factors. By lemma 6, we have that $m_1 \mid D$, that $m_2 \mid d(\Lambda)$ and that every prime dividing m_3 is split in k. If φ is a binary form over $\mathbb{Z}$ and p a prime, then we let $\alpha_p(\varphi) = 1$ if φ_p is modular, and $\alpha_p(\varphi) = 2$ otherwise. We also introduce the modified class number h' (compare e.g. [7]), which counts primitive binary forms φ with a given discriminant $N < 0$ with multiplicity $2/w_\varphi$. Hence $h'(-3) = 1/3$, $h'(-4) = 1/2$ and $h'(N) = h(N)$ otherwise.

Theorem 7. *(See [5], thm. 9.16) A positive definite oriented binary form φ is represented by q if and only if φ_p satisfies the conditions of lemma 6 for every prime p. For such a form, we have*

$$s(\varphi) = 2^{a-1} h'\left(\frac{\Delta}{m_2^2 D}\right) \prod_{p \mid D} \alpha_p(\varphi) \prod_{p \mid d(\Lambda)} \frac{2}{\alpha_p(\varphi)}, \tag{7}$$

where a is the number of different primes dividing m_3.

Theorem 7 is our tool to determine intersection points of the modular curves. It remains to determine the self-intersections of the curves F_β. For that one can use the theory of Chern divisors (see [6], p. 252).

3 Some surfaces over $\mathbb{Q}(\sqrt{13})$

We will now consider the example where $k = \mathbb{Q}(\sqrt{13})$ and $d_0(\Lambda) = 3$. This is one of the cases with relatively small hyperbolic volume of the fundamental domain. Since $h(k) = 1$ in this case, there is only one maximal order in A up to isomorphism (by Eichler's norm theorem and lemma III.5.6 in [11]).

We consider first discrete groups of $\mathrm{Aut}(\mathcal{H}\times\mathcal{H})$ extending Γ. Note that every element in $A^+ = \{a \in A \mid \mathrm{nr}(a) \text{ is totally positive}\}$ acts on $\mathcal{H}\times\mathcal{H}$, giving a map $\varrho : A^+ \to \mathrm{Aut}(\mathcal{H}\times\mathcal{H})$. The totally positive elements of the normaliser of Λ,

$$N^+(\Lambda) = \{n \in A^+ \mid n\Lambda n^{-1} = \Lambda\},$$

naturally act on X. Let $\Gamma_{\mathrm{II}} = \varrho(N^+(\Lambda))$. In our case, we have that Γ_{II} is given as follows: Let $v = 4 + \sqrt{13}$, so $3 = v\overline{v}$. There exists $\lambda_x \in \Lambda$ such that $\mathrm{nr}(\lambda_x) = x$ for $x = v, \overline{v}$ and 3, and we let $\gamma_x = \varrho(\lambda_x) : \mathcal{H}\times\mathcal{H} \to \mathcal{H}\times\mathcal{H}$. This gives three well defined involutions on X, which we denote ι_3, ι_v and $\iota_{\overline{v}}$. We define $\Gamma_{\mathrm{I}} = \Gamma \cup \gamma_3\Gamma$, and we have in fact that $\Gamma_{\mathrm{II}} = \Gamma_{\mathrm{I}} \cup \gamma_v\Gamma_{\mathrm{I}}$.

We will now define an extension of Γ_{II}. Let us fix an element $\beta_{13} \in L$ with $q(\beta_{13}) = 13$, and define a map $T_0 : \mathcal{H}\times\mathcal{H} \to \mathcal{H}\times\mathcal{H}$, by

$$T_0(z_1, z_2) = (\beta_{13}^* z_2, \beta_{13} z_1).$$

We have that T_0^2 is the identity, and one can check that $T_0\Gamma = \Gamma T_0$, so T_0 induces an involution on X, which we denote T. We have $T\iota_x = \iota_{\overline{x}}T$, for $x \in \{1, v, \overline{v}, 3\}$. Let $\Gamma_{\mathrm{III}} = \Gamma_{\mathrm{II}} \cup T_0\Gamma_{\mathrm{II}}$. This group is independent of the choice of the element β_{13}. Hence, we have a tower of discrete groups acting on $\mathcal{H}\times\mathcal{H}$

$$\Gamma \subset \Gamma_{\mathrm{I}} \subset \Gamma_{\mathrm{II}} \subset \Gamma_{\mathrm{III}}. \tag{8}$$

We will consider the 4 quotient surfaces X, $X_{\mathrm{I}} = \mathcal{H}\times\mathcal{H}/\Gamma_{\mathrm{I}}$, $X_{\mathrm{II}} = \mathcal{H}\times\mathcal{H}/\Gamma_{\mathrm{II}}$ and $X_{\mathrm{III}} = \mathcal{H}\times\mathcal{H}/\Gamma_{\mathrm{III}}$. We let Y, Y_{I}, Y_{II} and Y_{III} respectively denote the canonical minimal resolution of the corresponding quotient surface. We let $G_{\mathrm{I}} = \{\iota_1, \iota_3\}$, $G_{\mathrm{II}} = \{\iota_1, \iota_v, \iota_{\overline{v}}, \iota_3\}$ and $G_{\mathrm{III}} = G_{\mathrm{II}} \cup TG_{\mathrm{II}}$ denote the corresponding groups acting on X, so $G_{\mathrm{III}} \cong D_4$. We have that G_{I} is the center of G_{III}, so the filtration (8) is in some sense canonical. Our main result is:

Theorem 8. *(See [5], thm. 12.1) Y is a minimal surface of general type, Y_I is a $K3$-surface blown up 4 times, Y_{II} is an Enriques surface blown up 2 times and Y_{III} is a rational surface with Euler number $e = 12$.*

Calculation of the quadratic form q in this case gives:

$$q = 2{t_0}^2 + t_0 t_1 + 5{t_1}^2 - 13({t_2}^2 - t_2 t_3 + {t_3}^2) \tag{9}$$

By theorem 4, the number of components of F_N is

$$f_N = \begin{cases} 0 & \text{if } \left(\frac{N}{13}\right) = 1 \text{ or } 9 \mid N, \\ 2 & \text{if } 13^2 \mid N \text{ and } 9 \nmid N, \\ 1 & \text{otherwise.} \end{cases} \tag{10}$$

If β is a primitive element of L, then we have, by theorem 2, that $d_0(\Lambda_\beta)$ is the least common multiple of $q(\beta)$ and 3. By proposition 3, we have

$$\kappa(N) = \begin{cases} 2 & \text{if } 13 \mid N, \\ 1 & \text{otherwise.} \end{cases}$$

The action of G_{III} on the curves are given by

Lemma 9. *(See [5], lem. 12.4) For every positive integer N represented by q, we have that $\iota_3(F_N) = F_N$ and $T(F_N) = F_N$. If $3 \nmid N$, then $\iota_x(F_N) = F_{3N}$ and $\iota_x(F_{3N}) = F_N$, for $x = v$ and $x = \overline{v}$.*

Now, X has 8 elliptic points of order 2, and 4 of order 3, corresponding to elliptic elements $\epsilon_1, \dots, \epsilon_8$ and $\rho_1, \dots, \rho_4$ respectively. The elements of order 2 give rise to (-2)-curves on Y, ρ_1, ρ_2 to two intersecting (-2)-curves each and ρ_3, ρ_4 to (-3)-curves. In addition to these curves, we will consider the 6 irreducible curves F_2, F_6, F_5, F_{15}, F_{13} and F_{39}. The binary forms associated to the elliptic points, and the branches of the above curves that pass through them, are given by table 1. All other intersection points of these curves are given in table 2. We get that F_2 and F_6 are non-singular rational curves, F_{13} and F_{39} are non-singular elliptic curves and F_5 and F_{15} are elliptic curves with two nodes each.

Determining the fixed points of the elements of G_{III} requires explicit calculations. The isotropy groups of all points with non-trivial isotropy group $\Gamma_{\mathrm{II},z}$ are given in table 3. F_{13} and F_{39} are pointwise fixed by T and $T\iota_3$ respectively. All other fixed points of elements in $G_{\mathrm{II}}T$ are already fixed points of elements of G_{II}.

We now present the configurations of curves on the different surfaces. We start with the surface Y and proceed by a sequence of blow ups and quotient constructions. We will use the convention that whenever we have two surfaces Z and $\widetilde{Z}$, then $\widetilde{Z}$ is isomorphic to the surface Z blown up in a finite set of points. For easier navigation among the surfaces, we first

Elliptic element	Order	Binary form	Branches
$\epsilon_1,\ \epsilon_2$	2	$[13,0,13]$	$F_{13},\ F_{13}$
$\epsilon_3,\ \epsilon_4$	2	$[39,0,39]$	$F_{39},\ F_{39}$
$\epsilon_5,\ \epsilon_6$	2	$[2,2,7]$	F_2
$\epsilon_7,\ \epsilon_8$	2	$[6,6,21]$	F_6
$\rho_1,\ \rho_2$	3	$[13,-13,13]$	$F_{13},\ F_{39}$
ρ_3	3	$[2,1,5]$	$F_2,\ F_6,\ F_5,\ F_{15}$
ρ_4	3	$[2,-1,5]$	$F_2,\ F_6,\ F_5,\ F_{15}$

Table 1: Elliptic points

Binary form	Branches	$s(\varphi)$
$[2,0,39]$	$F_2,\ F_{39}$	$h(-24)=2$
$[6,0,13]$	$F_6,\ F_{13}$	$h(-24)=2$
$[5,-3,5]$	$F_5,\ F_5,\ F_{13}$	$2h(-7)=2$
$[15,-9,15]$	$F_{15},\ F_{15},\ F_{39}$	$2h(-7)=2$
$[5,2,8]$	$F_5,\ F_{15}$	$h(-12)=1$
$[5,-2,8]$	$F_5,\ F_{15}$	$h(-12)=1$
$[5,0,39]$	$F_5,\ F_{39}$	$h(-60)=2$
$[15,0,13]$	$F_{15},\ F_{13}$	$h(-60)=2$
$[5,-5,11]$	$F_5,\ F_{39}$	$h(-15)=2$
$[7,-1,7]$	$F_{15},\ F_{13}$	$h(-15)=2$
$[13,0,39]$	$F_{13},\ F_{39}$	$h(-156)/2=2$

Table 2: Intersection points

	Γ_z	$\Gamma_{\mathrm{I},z}$	$\Gamma_{\mathrm{II},z}$	$\Gamma_{\mathrm{III},z}$
$[2,-2,7]$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/4$
$[2,-2,7]$	$\mathbb{Z}/2$			
$[6,-6,21]$	$\mathbb{Z}/2$	$\mathbb{Z}/2$		
$[6,-6,21]$	$\mathbb{Z}/2$			
$[13,0,13]$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2 \times \mathbb{Z}/2$
$[13,0,13]$	$\mathbb{Z}/2$			
$[39,0,39]$	$\mathbb{Z}/2$	$\mathbb{Z}/2$		
$[39,0,39]$	$\mathbb{Z}/2$			
$[2,1,5]$	$\mathbb{Z}/3$	$\mathbb{Z}/6$	$\mathbb{Z}/6$	$\mathbb{Z}/12$
$[2,-1,5]$	$\mathbb{Z}/3$	$\mathbb{Z}/6$		
$[13,-13,13]$	$\mathbb{Z}/3$	$\mathbb{Z}/6$	$\mathbb{Z}/6$	D_6
$[13,-13,13]$	$\mathbb{Z}/3$	$\mathbb{Z}/6$		
$[13,0,39]$	(1)	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2 \times \mathbb{Z}/2$
$[13,0,39]$	(1)	$\mathbb{Z}/2$		
$[5,2,8]$	(1)	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/4$
$[5,-2,8]$	(1)	$\mathbb{Z}/2$		

Table 3: Overview of points z with non-trivial isotropy group $\Gamma_{\mathrm{II},z}$

summarise in a diagram the sequence of surfaces that we will construct:

$$\begin{tikzcd}
& \widetilde{Y} \arrow[dl, "\text{blow up}"'] \arrow[dr, "2{:}1"] & & & & & \\
Y & & Y_{\mathrm{I}} \arrow[dr, "2{:}1"] & & \widetilde{Y}_{\mathrm{II}} \arrow[dl, "\text{blow up}"] \arrow[dr, "2{:}1"] & & \\
& & & Y_{\mathrm{II}} & & \widetilde{Y}_{\mathrm{III}} \arrow[dr, "\text{blow down}"] & \\
& & & & & & Y_{\mathrm{III}}
\end{tikzcd} \tag{11}$$

We have drawn the configuration of curves on Y in figure 1. Here all curves drawn with thick lines are exceptional curves coming from the canonical resolution of the singularities of X. If the self-intersection number of such a curve is not explicitly given in the figure, then it is -2. Furthermore, the rational curves F_2 and F_6 have self-intersection -2 and the elliptic curves F_{13} and F_{39} have self-intersection -4. The nodal elliptic curves F_5 and F_{15} have self-intersection 2.

The involution ι_3 has 8 fixed points on X. On the surface Y, the fixed point locus of the lifted involution consists of the two (-3)-curves and the 10

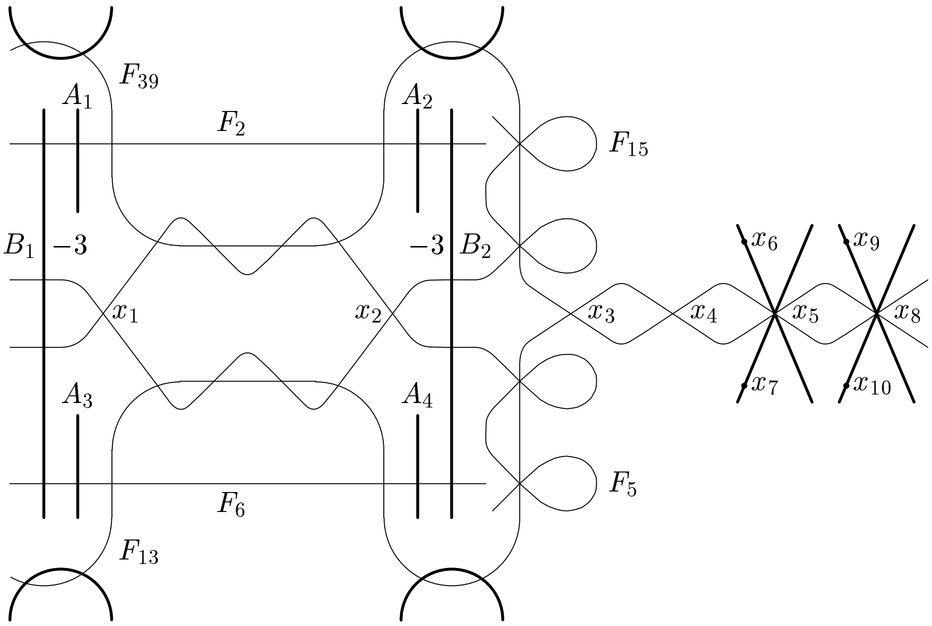

Figure 1: Y

isolated points $x_1, \ldots, x_{10}$, which are indicated in figure 1. Blowing up these 10 points on Y to the surface $\widetilde{Y}$, and thereafter taking the quotient, we get a surface which, in fact, is the minimal desingularisation Y_{I} of X_{I}. We indicate the configuration of curves in figure 2. We have that $F_2^2 = F_6^2 = -1$, that F_{13} and F_{39} are non-singular rational curves with $F_{13}^2 = F_{39}^2 = -4$, and that F_5 and F_{15} are nodal rational curves with $F_5^2 = F_{15}^2 = 0$.

We let ι_{I} denote the involution on Y_{I} induced by the group Γ_{II}. It is fixed point free. If N is not divisible by 3, then ι_{I} maps F_N to F_{3N} and vice versa. We denote their common images in Y_{II} by F_N. We have of course that the self-intersection of F_2, F_{13} and F_5 respectively, are the same as for the corresponding curves on Y_{I}. The configuration on Y_{II} is drawn in figure 3.

The involution on Y_{II} induced by Γ_{III} has the curve F_{13} and the 6 isolated points $y_1, \ldots, y_6$ as fixed point locus. We blow up these points and then we take the quotient. The surface we get is Y_{III} blown up 5 times, see figure 4. On this surface, we have that F_2, F_{13} and F_5 are non-singular rational curves and we have $F_2^2 = -1$, $F_{13}^2 = -8$ and $F_5^2 = 0$.

We blow down the exceptional curve configurations which meet F_{13} and we get the surface Y_{III}. See figure 5.

Since $\zeta_k(-1) = \frac{1}{6}$, we get that $\int_X \omega = \frac{4}{3}$ by equation (2). Furthermore $e_2 = 8$ and $e_3 = 4$, so by (1), we get $e(X) = 8$. It is now straightforward to compute the numerical invariants of all surfaces. We summarise the result in the following table:

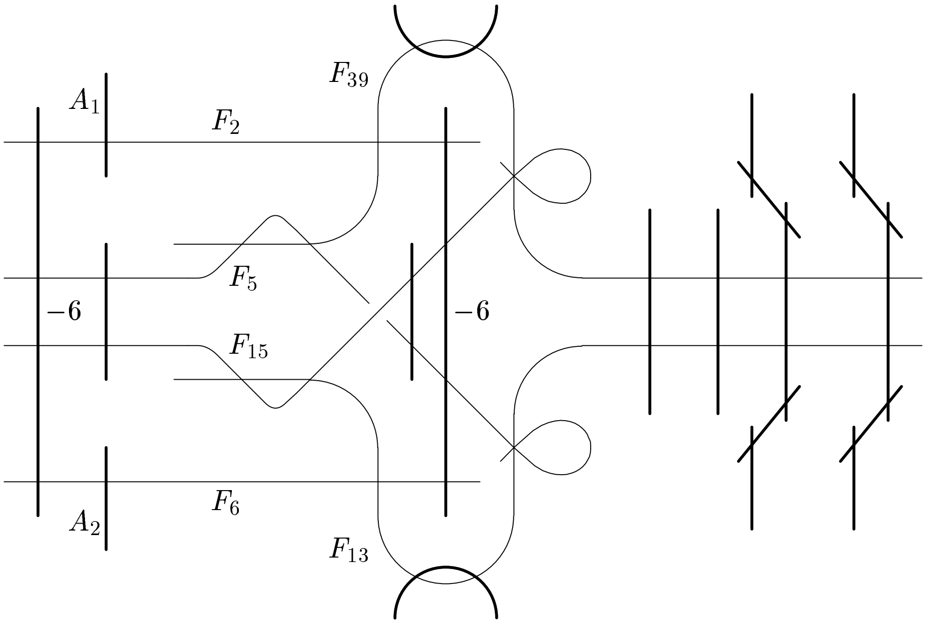

Figure 2: Y_{I}

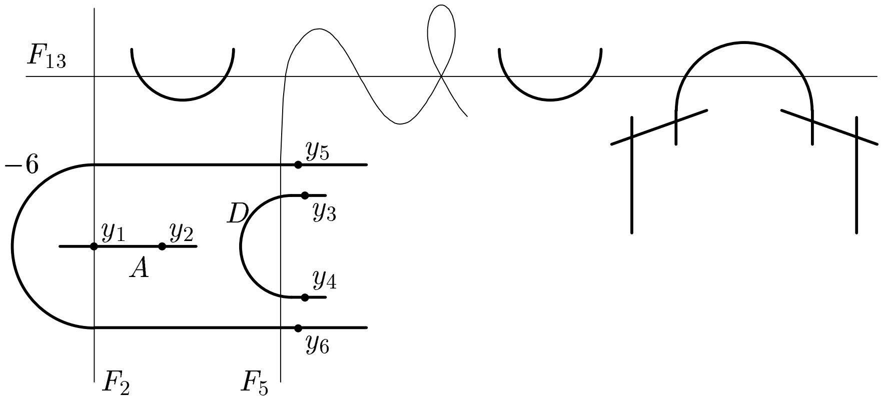

Figure 3: Y_{II}

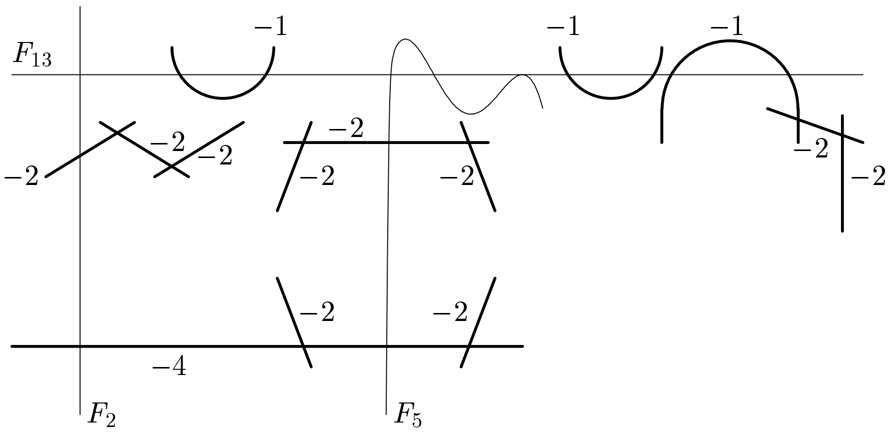

Figure 4: $\widetilde{Y}_{\mathrm{III}}$

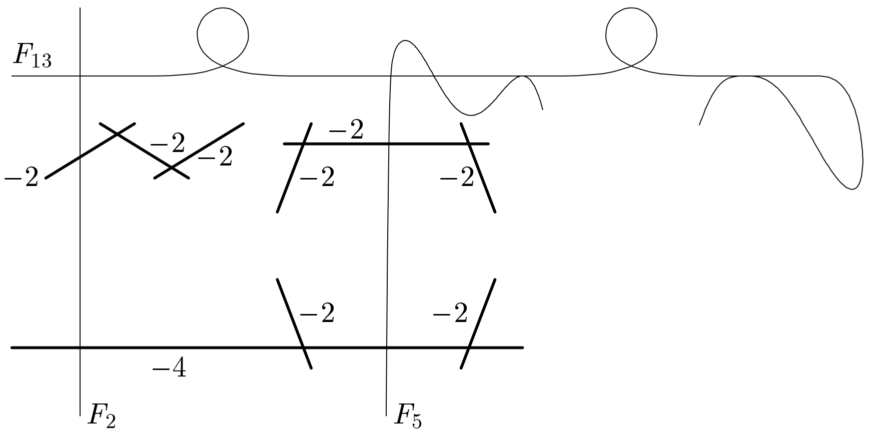

Figure 5: Y_{III}

	Y	Y_{I}	Y_{II}	Y_{III}
e	22	28	14	12
χ	2	2	1	1
K^2	2	-4	-2	0
q	0	0	0	0
p_g	1	1	0	0

We have the following corollary of Castelnuovo's criterion for rationality of a surface (see [6], corollary 1, p. 255):

Proposition 10. *Let Z be a non-singular surface with $q = 0$. If Z contains a non-singular rational curve D with $D^2 \geq 0$, then Z is rational.*

This directly applies to give that Y_{III} is a rational surface. Namely, if we blow down the connected configuration consisting of F_2 and three (-2)-curves, then the image of the (-4)-curve is a non-singular rational curve with self-intersection 0.

Studying the intersection matrix of the divisors generated by the curves in figure 4, we find that these curves generate a sublattice of $\mathrm{Pic}(\widetilde{Y}_{\mathrm{III}})$ of index 2. Using the adjunction formula, one gets the following representation of the canonical divisor of $\widetilde{Y}_{\mathrm{III}}$:

$$2K = 4F_2 - F_{13} + 3C_{14} + 2C_{15} + C_{16}. \tag{12}$$

Using (12), we can trace $2K$ backwards and get the following relation on Y_{II}:

$$2K = 4F_2 + 2A. \tag{13}$$

Let now Z denote the surface we get if we blow down first F_2 and then A on Y_{II}. By (13), we have that $2K = 0$ on Z. Furthermore, $e(Z) = 12$ and $q(Z) = 0$. This shows that Z is an Enriques surface (see [1], chapter VI). Since the map $Y_{\mathrm{I}} \to Y_{\mathrm{II}}$ is unramified, we immediately get that Y_{I} is a $K3$-surface blown up 4 times with

$$K = 2F_2 + A_1 + 2F_6 + A_2. \tag{14}$$

This implies that the canonical divisor on Y is given by

$$K = 2F_2 + 2F_6 + \sum A_i + \sum B_i.$$

We conclude that Y is a minimal surface (since if E an exceptional divisor on Y, then $KE = -1$, which is impossible since K is effective and does not contain any exceptional component). Furthermore, since K is effective and $K^2 > 0$, we have that Y is of general type (see [1]).

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