SPECTRAL CURVES AND PARAMETERIZATIONS OF A DISCRETE INTEGRABLE 3-DIMENSIONAL MODEL

S. PAKULIAK† AND S. SERGEEV†‡

ABSTRACT. We consider a discrete classical integrable model on the 3-dimensional cubic lattice. The solutions of this model can be used to parameterize the Boltzmann weights of the different 3-dimensional spin models. We have found the general solution of this model constructed in terms of the theta-functions defined on an arbitrary compact algebraic curve. The imposing of the periodic boundary conditions fixes the algebraic curve. We have shown that in this case the curve coincides with the spectral one of the auxiliary linear problem. In the case when the curve is a rational one, the soliton solutions have been constructed.

1. INTRODUCTION

The paper is devoted to a description of the periodic and soliton solutions of some generic classical 3-dimensional discrete integrable model. We will see that this description can be presented in the completely analogous way as the description of the finite-gap solutions of the hierarchy of the continuous integrable equations, although the way we get the system of equation is unusual and motivated by the approach developed in [1] for discrete integrable spin models. First, we construct the discrete equations of motion from an equivalence of the linear systems (which replace zero-curvature condition for Lax operators in usual formulation) and after that prove the integrability by the counting the independent number of the integrals of motion.

Several types of the boundary conditions may be considered on the cubic lattice (the open boundary, periodical boundary conditions in chosen directions, or the completely periodical boundary conditions). This choice leads to several dynamical interpretation of the model: as a Cauchy problem, as a Bäcklund transformation, or as an analogue of the standing vibrations on the discrete $3d$ torus.

Starting from the discrete equations of motion we introduce a change of variables through triple of the Legendre variables, which transforms the equations of motion into the three-linear form. These three-linear equations appear to be a generalization of the famous Hirota bi-linear discrete equation. Then we observe that these three-linear equations can be formally solved with the help of Fay's identity for a theta-function on an arbitrary algebraic curve.

Some facts observed in this paper are the demonstration of the general statements proved more than two decades ago in [2] – any discrete integrable system can be solved using the algebraic-geometry methods. In this paper we develop further an alternative approach to 3-dimensional discrete integrable systems [1] which does not use the notion of Lax operators

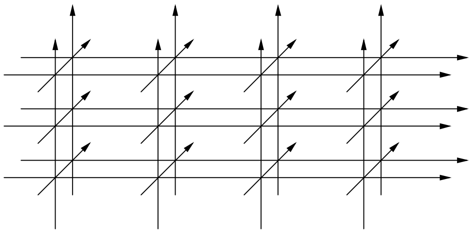

FIGURE 1. Example of cubic lattice of the size $3 \times 2 \times 4$.

and can be applied equivalently to quantum (spin) or classical integrable systems associated with several 3-dimensional lattices. Instead of the Lax operators we use the notion of the linear system defined on auxiliary planes. In this paper we consider the cubic lattice, although these methods may be applied to arbitrary 3-dimensional lattices formed by a set of planes.

2. Classical discrete integrable system on the cubic lattice

Let the vertices of the cubic lattice spanned by orthogonal basis $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ be marked by the vector

$$\mathbf{n} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3 \ . \tag{1}$$

The cubic lattice is formed by three sets of parallel planes or, equivalently, by three sets of parallel lines (see Fig. 1). Let us associate to each edge of the cubic lattice of a given type α, ($\alpha = 1, 2, 3$ corresponds to three orthogonal directions of the lattice) the dynamical variables as shown in Fig. 2. Namely, the pairs of dynamical variables $u_{\alpha,\mathbf{n}}$, $w_{\alpha,\mathbf{n}}$, $\alpha = 1, 2, 3$, are associated with the edges incoming to the oriented vertex $\mathbf{n}$, while $u_{\alpha,\mathbf{n}+\mathbf{e}_\alpha}$, $w_{\alpha,\mathbf{n}+\mathbf{e}_\alpha}$ are associated to the outgoing edges. Fig. 2 shows also two auxiliary planes intersecting the incoming and outgoing edges. Each of these planes cuts seven of eight octants around the vertex of the cubic lattice numbered by the vector $\mathbf{n}$.

Let us stress that our considerations at the moment are local. Our goal is to obtain the relations between dynamical variables surrounding given vertex. After this we extend these relations to the whole lattice and so obtain the discrete dynamical system.

Each of the triangles on the auxiliary planes is formed by three lines obtained by the intersection of this plane with planes forming the vertex $\mathbf{n}$ of the cubic lattice. In this way the vertices of the auxiliary triangles are associated to the edges of the cubic lattice and so to a pair of the dynamical variables. Let us consider two linear problems attached to the auxiliary triangles via the following rules. First, we introduce the linear variables Φ_a, Φ_b, Φ_c and Φ_d around the vertex on the auxiliary plane according to Fig. 3. The linear problems for each vertex on the auxiliary planes will be always of the form

$$0 = \Phi_a - \Phi_b \cdot u + \Phi_c \cdot w + \Phi_d \cdot \kappa \; u \; w \ . \tag{2}$$

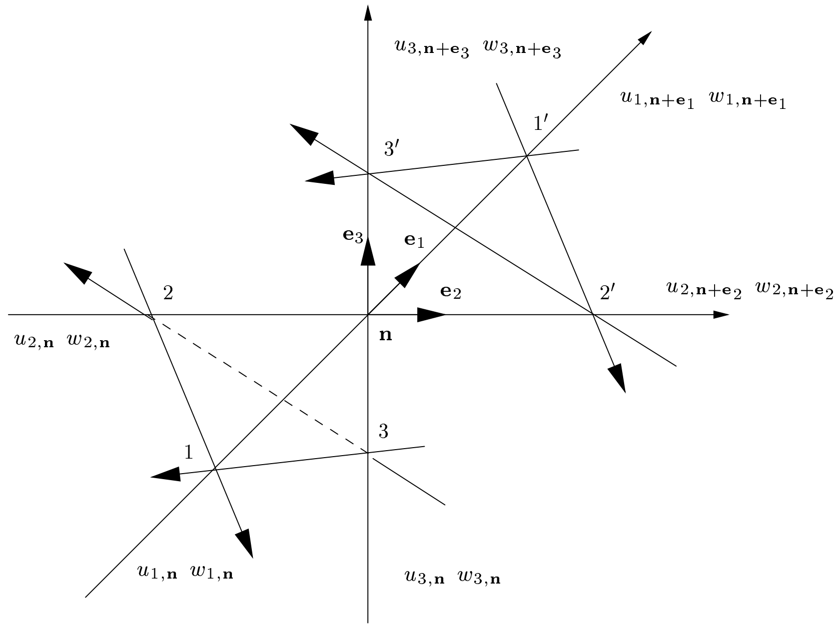

FIGURE 2. Association of the dynamical variables to the edges of the cubic lattice. Two intersecting planes form two triangles 123 and $1'2'3'$.

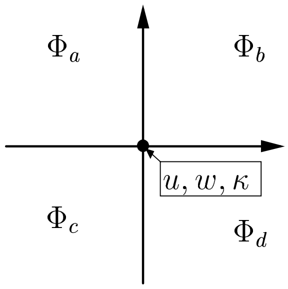

FIGURE 3. The linear problem for the vertex with associated dynamical variables and additional parameter κ.

Here $\kappa \in \mathbb{C}$ is an additional parameter associated with the line of the cubic lattice. Note that the coefficients of the linear form (2) are fixed by the orientation of the lines which form the vertex on the auxiliary plane.

The enumeration of the linear variables on the auxiliary triangles is shown in Fig. 4. It is clear that these linear variables are associated with internal part of the cubes in the cubic lattice. According to (2) we can write the system of the linear equations

$$\left\{\begin{array}{lcl} 0 &=& \Phi_{\mathbf{n}+\mathbf{e}_2} - \Phi_{\mathbf{n}+\mathbf{e}_2+\mathbf{e}_3} u_{1,\mathbf{n}} + \Phi_{\mathbf{n}} w_{1,\mathbf{n}} + \Phi_{\mathbf{n}+\mathbf{e}_3} \kappa_{1:\mathbf{n}} u_{1,\mathbf{n}} w_{1,\mathbf{n}} \ , \\ 0 &=& \Phi_{\mathbf{n}} - \Phi_{\mathbf{n}+\mathbf{e}_3} u_{2,\mathbf{n}} + \Phi_{\mathbf{n}+\mathbf{e}_1} w_{2,\mathbf{n}} + \Phi_{\mathbf{n}+\mathbf{e}_1+\mathbf{e}_3} \kappa_{2:\mathbf{n}} u_{2,\mathbf{n}} w_{2,\mathbf{n}} \ , \\ 0 &=& \Phi_{\mathbf{n}+\mathbf{e}_2} - \Phi_{\mathbf{n}} u_{3,\mathbf{n}} + \Phi_{\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2} w_{3,\mathbf{n}} + \Phi_{\mathbf{n}+\mathbf{e}_1} \kappa_{3:\mathbf{n}} u_{3,\mathbf{n}} w_{3,\mathbf{n}} \end{array}\right. \tag{3}$$

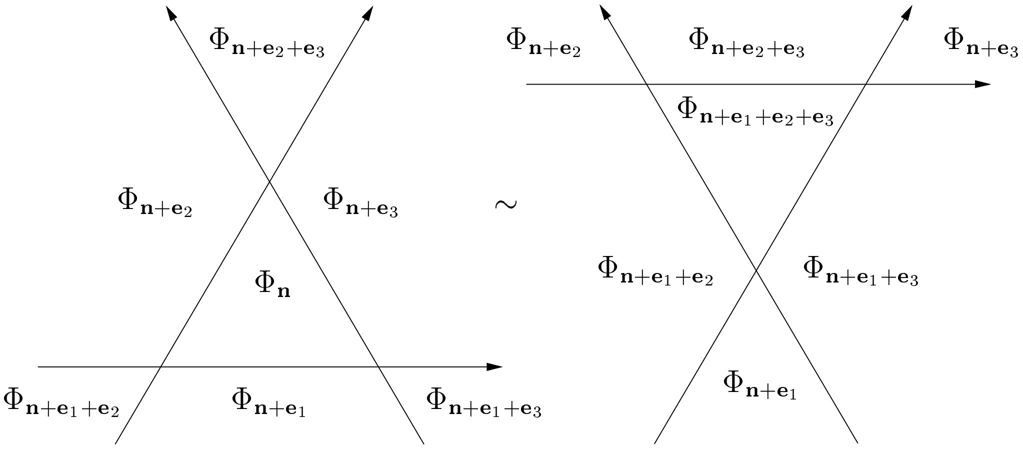

FIGURE 4. Parameterizations of the classical linear variables $\Phi_{\mathbf{n}}$ in cubic geometry.

for left triangle shown in Fig. 4 and the linear system

$$\left\{\begin{array}{rcl} 0 & = & \Phi_{\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2} - \Phi_{\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3} u_{1,\mathbf{n}+\mathbf{e}_1} + \Phi_{n+\mathbf{e}_1} w_{1,\mathbf{n}+\mathbf{e}_1} + \\ & & + \ \Phi_{\mathbf{n}+\mathbf{e}_1+\mathbf{e}_3} \kappa_{1,\mathbf{n}} u_{1,\mathbf{n}+\mathbf{e}_1} w_{1,\mathbf{n}+\mathbf{e}_1} \ , \\ 0 & = & \Phi_{\mathbf{n}+\mathbf{e}_2} - \Phi_{\mathbf{n}+\mathbf{e}_2+\mathbf{e}_3} u_{2,\mathbf{n}+\mathbf{e}_2} + \Phi_{n+\mathbf{e}_1+\mathbf{e}_2} w_{2,\mathbf{n}+\mathbf{e}_2} + \\ & & + \ \Phi_{\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3} \kappa_{2,\mathbf{n}} u_{2,\mathbf{n}+\mathbf{e}_2} w_{2,\mathbf{n}+\mathbf{e}_2} \ , \\ 0 & = & \Phi_{\mathbf{n}+\mathbf{e}_2+\mathbf{e}_3} - \Phi_{\mathbf{n}+\mathbf{e}_3} u_{3,\mathbf{n}+\mathbf{e}_3} + \Phi_{n+\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3} w_{3,\mathbf{n}+\mathbf{e}_3} + \\ & & + \ \Phi_{\mathbf{n}+\mathbf{e}_2+\mathbf{e}_3} \kappa_{3,\mathbf{n}} u_{3,\mathbf{n}+\mathbf{e}_3} w_{3,\mathbf{n}+\mathbf{e}_3} \end{array}\right. \tag{4}$$

for the right one.

Supposing that dynamical variables do not vanish identically on any edge of the lattice we may exclude from the systems (3) and (4) the linear variables $\Phi_{\mathbf{n}}$ and $\Phi_{\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3}$. Let us require that the systems (3) and (4) (each contains now only two linear relations for the rest 6 linear variables $\Phi_{\mathbf{n}+\mathbf{e}_1}$, $\Phi_{\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2}$, etc.) are equivalent. This means that the matrix elements of these linear systems which are rational functions of all dynamical variables around given vertex coincide identically. Moreover, let us require also that each type of the parameters $\kappa_{\alpha,\mathbf{n}}$ is conserved along the corresponding direction (which is already taken into account in (4))

$$\kappa_{\alpha,\mathbf{n}} = \kappa_{\alpha,\mathbf{n}+\mathbf{e}_\alpha} \ , \quad \alpha = 1, 2, 3. \tag{5}$$

As the result, we obtain the recurrent relations for dynamical variables $u_{\alpha,\mathbf{n}}$, $w_{\alpha,\mathbf{n}}$, $u_{\alpha,\mathbf{n}+\mathbf{e}_\alpha}$ and $w_{\alpha,\mathbf{n}+\mathbf{e}_\alpha}$:

$$\left\{\begin{array}{rcl} u_{1,\mathbf{n}+\mathbf{e}_1} & = & \dfrac{\kappa_{2,\mathbf{n}}\, u_{1,\mathbf{n}}\, u_{2,\mathbf{n}}\, w_{2,\mathbf{n}}}{\kappa_{1,\mathbf{n}}\, u_{1,\mathbf{n}}\, w_{2,\mathbf{n}} + \kappa_{3,\mathbf{n}}\, u_{2,\mathbf{n}}\, w_{3,\mathbf{n}} + \kappa_{1,\mathbf{n}}\, \kappa_{3,\mathbf{n}}\, u_{1,\mathbf{n}}\, w_{3,\mathbf{n}}} \ , \\ & & \\ w_{1,\mathbf{n}+\mathbf{e}_1} & = & \dfrac{w_{1,\mathbf{n}}\, w_{2,\mathbf{n}} + u_{3,\mathbf{n}}\, w_{2,\mathbf{n}} + \kappa_{3,\mathbf{n}}\, u_{3,\mathbf{n}}\, w_{3,\mathbf{n}}}{w_{3,\mathbf{n}}} \ , \end{array}\right. \tag{6}$$

$$\left\{\begin{aligned} u_{2,\mathbf{n}+\mathbf{e}_2} &= \frac{u_{1,\mathbf{n}}\,u_{2,\mathbf{n}}\,u_{3,\mathbf{n}}}{u_{2,\mathbf{n}}\,u_{3,\mathbf{n}} + u_{2,\mathbf{n}}\,w_{1,\mathbf{n}} + \kappa_{1,\mathbf{n}}\,u_{1,\mathbf{n}}\,w_{1,\mathbf{n}}}\,,\\ w_{2,\mathbf{n}+\mathbf{e}_2} &= \frac{w_{1,\mathbf{n}}\,w_{2,\mathbf{n}}\,w_{3,\mathbf{n}}}{w_{1,\mathbf{n}}\,w_{2,\mathbf{n}} + u_{3,\mathbf{n}}\,w_{2,\mathbf{n}} + \kappa_{3,\mathbf{n}}\,u_{3,\mathbf{n}}\,w_{3,\mathbf{n}}}\,, \end{aligned}\right. \tag{7}$$

$$\left\{\begin{aligned} u_{3,\mathbf{n}+\mathbf{e}_3} &= \frac{u_{2,\mathbf{n}}\,u_{3,\mathbf{n}} + u_{2,\mathbf{n}}\,w_{1,\mathbf{n}} + \kappa_{1,\mathbf{n}}\,u_{1,\mathbf{n}}\,w_{1,\mathbf{n}}}{u_{1,\mathbf{n}}}\,,\\ w_{3,\mathbf{n}+\mathbf{e}_3} &= \frac{\kappa_{2,\mathbf{n}}\,u_{2,\mathbf{n}}\,w_{2,\mathbf{n}}\,w_{3,\mathbf{n}}}{\kappa_{1,\mathbf{n}}\,u_{1,\mathbf{n}}\,w_{2,\mathbf{n}} + \kappa_{3,\mathbf{n}}\,u_{2,\mathbf{n}}\,w_{3,\mathbf{n}} + \kappa_{1,\mathbf{n}}\,\kappa_{3,\mathbf{n}}\,u_{1,\mathbf{n}}\,w_{3,\mathbf{n}}}\,. \end{aligned}\right. \tag{8}$$

where due to (5) we have

$$\kappa_{1,\mathbf{n}} = \kappa_{1:n_2,n_3}\,,\quad \kappa_{2,\mathbf{n}} = \kappa_{2:n_1,n_3}\,,\quad \kappa_{3,\mathbf{n}} = \kappa_{3:n_1,n_2}\,. \tag{9}$$

One may ask why the recurrent relations (6)–(8) are the dynamical system. Let us introduce the following Poisson bracket for the edges incoming to the vertex with some fixed $\mathbf{n}$:

$$\{u_{\alpha,\mathbf{n}}, w_{\alpha,\mathbf{n}}\} = u_{\alpha,\mathbf{n}} w_{\alpha,\mathbf{n}} \tag{10}$$

and any other bracket for incoming edges vanishes. The transformation (6)–(8) is the canonical one, i.e. from (10) it follows that

$$\{u_{\alpha,\mathbf{n}+\mathbf{e}_\alpha}, w_{\alpha,\mathbf{n}+\mathbf{e}_\alpha}\} = u_{\alpha,\mathbf{n}+\mathbf{e}_\alpha} w_{\alpha,\mathbf{n}+\mathbf{e}_\alpha} \tag{11}$$

and any other bracket for outgoing edges vanishes. In general, the auxiliary triangles shown in Fig. (2) are the fragments of two adjacent space-like surfaces, and the set of 3d vertices between these surfaces provides by means of (6)–(8) the canonical transformation of the set of dynamical pairs from incoming surface to the outgoing one. We call this transformation the local equation of motion in the direction perpendicular to the chosen space-like surface. Besides the local equations of motion (6)–(8), a complete formulation of a dynamical system needs the specification of boundary conditions.

The type of the boundary conditions as well as the global characteristics of the model depend on the choice of the space-like surface. The auxiliary lattices described are appropriate for the Cauchy problem for the cubic lattice. In Section 6 we will choose n_1 as the discrete time, while n_2 and n_3 will be the discrete space coordinates. At this choice the system (6) will describe the "time" evolution of the variables $u_{1,\mathbf{n}}$, $w_{1,\mathbf{n}}$ and the systems (7)–(8) – the space distribution of the "auxiliary" for this evolution variables $u_{2,\mathbf{n}}$, $w_{2,\mathbf{n}}$, $u_{3,\mathbf{n}}$ and $w_{3,\mathbf{n}}$. All the dynamical quantities, such as integrals of motion, spectral curves etc., should be calculated in this case for the system (6). This will be subject of the Section 6.

3. Legendre transform

It follows from (6)–(8)

$$
\begin{aligned}
w_{1,\mathbf{n}}w_{2,\mathbf{n}} &= w_{1,\mathbf{n}+\mathbf{e}_1}w_{2,\mathbf{n}+\mathbf{e}_2}\,, \quad u_{2,\mathbf{n}}u_{3,\mathbf{n}} = u_{2,\mathbf{n}+\mathbf{e}_2}u_{3,\mathbf{n}+\mathbf{e}_3}\,,\\
\frac{u_{1,\mathbf{n}}}{w_{3,\mathbf{n}}} &= \frac{u_{1,\mathbf{n}+\mathbf{e}_1}}{w_{3,\mathbf{n}+\mathbf{e}_3}}\,.
\end{aligned}
\tag{12}
$$

Relations (12) provide in general the possibility of the following change of the variables:

$$
\begin{aligned}
u_{1,\mathbf{n}} &= u^{(0)}_{1:n_2,n_3}\frac{\tau_{2,\mathbf{n}}}{\tau_{2,\mathbf{n}+\mathbf{e}_3}}\,, & w_{1,\mathbf{n}} &= w^{(0)}_{1:n_2,n_3}\frac{\tau_{3,\mathbf{n}+\mathbf{e}_2}}{\tau_{3,\mathbf{n}}}\,,\\
u_{2,\mathbf{n}} &= u^{(0)}_{2:n_1,n_3}\frac{\tau_{1,\mathbf{n}}}{\tau_{1,\mathbf{n}+\mathbf{e}_3}}\,, & w_{2,\mathbf{n}} &= w^{(0)}_{2:n_1,n_3}\frac{\tau_{3,\mathbf{n}}}{\tau_{3,\mathbf{n}+\mathbf{e}_1}}\,,\\
u_{3,\mathbf{n}} &= u^{(0)}_{3:n_1,n_2}\frac{\tau_{1,\mathbf{n}+\mathbf{e}_2}}{\tau_{1,\mathbf{n}}}\,, & w_{3,\mathbf{n}} &= w^{(0)}_{3:n_1,n_2}\frac{\tau_{2,\mathbf{n}}}{\tau_{2,\mathbf{n}+\mathbf{e}_1}}\,.
\end{aligned}
\tag{13}
$$

Eqs. (6)–(8) with the substitution (13) become

$$
\begin{aligned}
&r_{\alpha,\mathbf{n}}\tau_{\alpha,\mathbf{n}+\mathbf{e}_\beta+\mathbf{e}_\gamma}\tau_{\beta,\mathbf{n}}\tau_{\gamma,\mathbf{n}} = \tau_{\alpha,\mathbf{n}}\tau_{\beta,\mathbf{n}+\mathbf{e}_\gamma}\tau_{\gamma,\mathbf{n}+\mathbf{e}_\beta} +\\
&\quad + s_{\beta,\mathbf{n}}\tau_{\alpha,\mathbf{n}+\mathbf{e}_\beta}\tau_{\beta,\mathbf{n}+\mathbf{e}_\gamma}\tau_{\gamma,\mathbf{n}} + s^{-1}_{\gamma,\mathbf{n}}\tau_{\alpha,\mathbf{n}+\mathbf{e}_\gamma}\tau_{\beta,\mathbf{n}}\tau_{\gamma,\mathbf{n}+\mathbf{e}_\beta}\,,
\end{aligned}
\tag{14}
$$

where (α,β,γ) is any *cyclic permutation* of the indices $(1,2,3)$, and the coefficients in (14) are

$$
\begin{aligned}
s_{1,\mathbf{n}} &= \frac{\kappa_{3:n_1,n_2}\,w^{(0)}_{3:n_1,n_2}}{w^{(0)}_{2:n_1,n_3}}\,, & r_{1,\mathbf{n}} &= \frac{u^{(0)}_{1:n_2,n_3}\,u^{(0)}_{3:n_1,n_2}}{w^{(0)}_{1:n_2,n_3}\,u^{(0)}_{2:n_1,n_3}}\,,\\
s_{2,\mathbf{n}} &= \frac{u^{(0)}_{3:n_1,n_2}}{w^{(0)}_{1:n_2,n_3}}\,, & r_{2,\mathbf{n}} &= \frac{\kappa_{2:n_1,n_3}\,u^{(0)}_{2:n_1,n_3}\,w^{(0)}_{2:n_1,n_3}}{\kappa_{1:n_2,n_3}\,\kappa_{3:n_1,n_2}\,u^{(0)}_{1:n_2,n_3}\,w^{(0)}_{3:n_1,n_2}}\,,\\
s_{3,\mathbf{n}} &= \frac{u^{(0)}_{2:n_1,n_3}}{\kappa_{1:n_2,n_3}\,u^{(0)}_{1:n_2,n_3}}\,, & r_{3,\mathbf{n}} &= \frac{w^{(0)}_{1:n_2,n_3}\,w^{(0)}_{3:n_1,n_2}}{w^{(0)}_{2:n_1,n_3}\,u^{(0)}_{3:n_1,n_2}}\,.
\end{aligned}
\tag{15}
$$

This change of variables, $\tau_{\alpha,\mathbf{n}}$ instead of $u_{\alpha,\mathbf{n}}$ and $w_{\alpha,\mathbf{n}}$, has the following interpretation. Equations (6)–(8) are the kind of the Hamiltonian equations of motion for the classical discrete system. Substitution (13) therefore is the Legendre transformation. And finally equations (14) are the Lagrangian equations of motion. We will call $\tau_{\alpha,\mathbf{n}}$ as the Legendre variables, and the coefficients $u^{(0)}_{\alpha:n_\beta,n_\gamma}$ and $w^{(0)}_{\alpha:n_\beta,n_\gamma}$ as the pre-exponents.

Equations (14) generalize in some sense the famous bi-linear Hirota equation [3]. It can be obtained from (14) in the special limit when all parameters $\kappa_{\alpha:n_\beta,n_\gamma}$ tends to zero. Indeed, let the parameters $r_{2,\mathbf{n}}$, $s_{3,\mathbf{n}}$ and $s^{-1}_{1,\mathbf{n}}$ which contains κ-s tend to infinity

$$
r_{2,\mathbf{n}} \sim \varepsilon^{-2} + \frac{1}{2}\varepsilon^{-1}\,, \quad s_{3,\mathbf{n}} \sim \varepsilon^{-2} - \frac{1}{2}\varepsilon^{-1}\,, \quad s^{-1}_{1,\mathbf{n}} \sim \varepsilon^{-1}
\tag{16}
$$

when $\varepsilon \to 0$. Equating in (14) with $\alpha = 2$, $\beta = 3$ and $\gamma = 1$ the coefficients at ε^{-2} and ε^{-1} we obtain respectively

$$\tau_{3,\mathbf{n}} = \tau_{2,\mathbf{n}+\mathbf{e}_3} \ , \quad \tau_{1,\mathbf{n}} = \tau_{2,\mathbf{n}+\mathbf{e}_1} \ . \tag{17}$$

Note that in the limit (16) $\kappa_{1,\mathbf{n}} \sim \varepsilon^2$, $\quad \kappa_{2,\mathbf{n}} \sim \kappa_{3,\mathbf{n}} \sim \varepsilon$. Imposing further one more relation $r_{1,\mathbf{n}} = s_{2,\mathbf{n}} r_{3,\mathbf{n}}$, one can see that the rest two equations of (14) coincide and can be written as a single difference equation for the Legendre variable $\tau_{2,\mathbf{n}}$

$$r_{1,\mathbf{n}} \tau_{2,\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3} \tau_{2,\mathbf{n}} = \tau_{2,\mathbf{n}+\mathbf{e}_1} \tau_{2,\mathbf{n}+\mathbf{e}_2+\mathbf{e}_3} + s_{2,\mathbf{n}} \tau_{2,\mathbf{n}+\mathbf{e}_3} \tau_{2,\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2} \ .$$

In the homogeneous limit for the parameters $r_{1,\mathbf{n}} = r_1$ and $s_{2,\mathbf{n}} = s_2$ the latter equation can be rewritten in the canonical Hirota's form after obvious re-enumeration of discrete variables. Due to this analogy we will call sometimes the Legendre variables $\tau_{j,\mathbf{n}}$ as the triplet of τ-functions.

We conclude this section with a description of the discrete gauge invariance of the linear systems (3) and (4). Let us require that these systems are invariant under simultaneous shift of the linear variables

$$\Phi_{\mathbf{n}} \mapsto \xi_{\mathbf{n}} \Phi_{\mathbf{n}} \ , \tag{18}$$

where the set of $\xi_{\mathbf{n}} \in \mathbb{C}$. The invariance requires the corresponding change of dynamical variables

$$u_{1,\mathbf{n}} \mapsto \frac{\xi_{\mathbf{n}+\mathbf{e}_2}}{\xi_{\mathbf{n}+\mathbf{e}_2+\mathbf{e}_3}} u_{1,\mathbf{n}} \ , \quad w_{1,\mathbf{n}} \mapsto \frac{\xi_{\mathbf{n}+\mathbf{e}_2}}{\xi_{\mathbf{n}}} w_{1,\mathbf{n}} \ , \quad \text{etc.}$$

Now we have to request that gauge transformation of the parameters $\kappa_{\alpha,\mathbf{n}}$, $\alpha = 1,2,3$ in both linear systems (3) and (4), coincide:

$$\kappa_{1,\mathbf{n}} \mapsto \frac{\xi_{\mathbf{n}} \xi_{\mathbf{n}+\mathbf{e}_2+\mathbf{e}_3}}{\xi_{\mathbf{n}+\mathbf{e}_2} \xi_{\mathbf{n}+\mathbf{e}_3}} \kappa_{1,\mathbf{n}} = \frac{\xi_{\mathbf{n}+\mathbf{e}_1} \xi_{\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3}}{\xi_{\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2} \xi_{\mathbf{n}+\mathbf{e}_1+\mathbf{e}_3}} \kappa_{1,\mathbf{n}} \ , \tag{19}$$

and similarly for the parameters $\kappa_{2,\mathbf{n}}$ and $\kappa_{3,\mathbf{n}}$. Any of this requirements leads the single restriction

$$\xi_{\mathbf{n}} \xi_{\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2} \xi_{\mathbf{n}+\mathbf{e}_1+\mathbf{e}_3} \xi_{\mathbf{n}+\mathbf{e}_2+\mathbf{e}_3} = \xi_{\mathbf{n}+\mathbf{e}_1} \xi_{\mathbf{n}+\mathbf{e}_2} \xi_{\mathbf{n}+\mathbf{e}_3} \xi_{\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3} \ . \tag{20}$$

The last equation can be easily solved by the ansatz

$$\xi_{\mathbf{n}} = \xi_{1:n_2,n_3} \xi_{2:n_1,n_3} \xi_{3:n_1,n_2} \tag{21}$$

and explains appearing of all ξ-dependent factors in the statement of the following Proposition 1.

4. General solution of the classical equations of motion

As we have seen, equations (14) generalize the famous Hirota equation. Actually the structure of (14) is the same as the structure of the Hirota equation. In this section we present a general solution of these classical equation of motion when each three-linear equation (14) is reduced to a pair of bi-linear equations. This form is suitable for imposing the periodic boundary condition. The bi-linear relations become a celebrated Fay's identities for the Θ-functions associated with an algebraic curve of a finite genus. Periodic boundary condition will fix the generic algebraic curve in an unique way. The curve may depend on the type of the boundary conditions. We claim that for the periodic boundary conditions the found solution of the three-linear equations (14) is the most general. We start with formulation of the necessary algebraic geometry objects (see e.g. [4, 5]).

Let Γ be an arbitrary algebraic curve of genus g, and $\omega = (\omega_1, \omega_2, ...\omega_g)$ be the vector of g holomorphic differentials, normalized as usual:

$$\oint_{a_j} \omega_k \;=\; \delta_{k,j}\;,\quad \oint_{b_j} \omega_k \;=\; \Omega_{k,j}\;, \tag{22}$$

where a_j, b_j, $j = 1, \dots, g$ are the sets of canonical cycles on Γ.

Let $\mathbf{I}$: $\Gamma^{\otimes 2} \mapsto \mathrm{Jac}(\Gamma)$ be the Jacobi transform

$$X, Y \;\in\; \Gamma \;\mapsto \mathbf{I}(X,Y) \;=\; \int_X^Y \omega \;\in \mathrm{Jac}(\Gamma)\;. \tag{23}$$

Let further $\Theta_\epsilon(\mathbf{v})$, $\mathbf{v} \in \mathbb{C}^g$, $\epsilon = (\epsilon_1, \epsilon_2)$, $\epsilon_i \in \mathbb{C}^g$ be the theta-function with characteristic ϵ on the Jacobian $\mathrm{Jac}(\Gamma)$,

$$\Theta_\epsilon(\mathbf{v}) = \sum_{\mathbf{m}\in\mathbb{Z}^g} \exp\left(i\pi(\mathbf{m}+\epsilon_1, \Omega\mathbf{m}+\epsilon_1) + 2i\pi(\mathbf{m}+\epsilon_1, \mathbf{v}+\epsilon_2)\right), \tag{24}$$

and $E(X,Y) = -E(Y,X)$ be the prime form $X, Y \in \Gamma$, so that the cross-ratio

$$\frac{E(X,Y)\,E(X',Y')}{E(X,Y')\,E(X',Y)} = \frac{\Theta_{\epsilon_{\mathrm{odd}}}(\mathbf{I}(X,Y))\Theta_{\epsilon_{\mathrm{odd}}}(\mathbf{I}(X',Y'))}{\Theta_{\epsilon_{\mathrm{odd}}}(\mathbf{I}(X,Y'))\Theta_{\epsilon_{\mathrm{odd}}}(\mathbf{I}(X',Y))} \tag{25}$$

is well defined quasi-periodical function on $\Gamma^{\otimes 4}$. ϵ_{odd} is a non-singular odd theta characteristic such that $\Theta_{\epsilon_{\mathrm{odd}}}(\mathbf{0}) = 0$. We denote by $\Theta(\mathbf{v})$ the theta-function with zero characteristic.

There is a famous identity on $\Gamma^{\otimes 4} \otimes \mathrm{Jac}(\Gamma)$, so called bi-linear Fay's identity, which can be written in the form:

$$\begin{aligned} &\Theta(\mathbf{v})\;\Theta(\mathbf{v}+\mathbf{I}(B+D,A+C)) = \\ &\quad= \frac{E(A,B)E(D,C)}{E(A,C)E(D,B)}\,\Theta(\mathbf{v}+\mathbf{I}(D,A))\;\Theta(\mathbf{v}+\mathbf{I}(B,C)) + \\ &\quad\quad+ \frac{E(A,D)E(C,B)}{E(A,C)E(D,B)}\,\Theta(\mathbf{v}+\mathbf{I}(B,A))\;\Theta(\mathbf{v}+\mathbf{I}(D,C))\;, \end{aligned} \tag{26}$$

where one should understand $\mathbf{I}(B+D,A+C) = \mathbf{I}(B,A)+\mathbf{I}(D,C) = \mathbf{I}(D,A)+\mathbf{I}(B,C)$.

Let $\mathbf{v} \in \mathrm{Jac}(\Gamma)$, X_{n_1}, X'_{n_1}, Y_{n_2}, Y'_{n_2}, Z_{n_3}, Z'_{n_3} ($n_\alpha \in \mathbb{Z}$) and P, Q be arbitrary distinct points on the algebraic curve Γ. We have

Proposition 1. *Any solution of local equations of motion (14) is a particular case of the following:*

$$\begin{aligned} \tau_{1,\mathbf{n}} &= \xi_{1:n_2,n_3}\,\Theta(\mathbf{I_n}+\mathbf{I}(Q,X_{n_1}))\;, \\ \tau_{2,\mathbf{n}} &= \xi_{2:n_1,n_3}\,\Theta(\mathbf{I_n}+\mathbf{I}(Q,Y_{n_2}))\;, \\ \tau_{3,\mathbf{n}} &= \xi_{3:n_1,n_2}\,\Theta(\mathbf{I_n}+\mathbf{I}(Q,Z_{n_3}))\;, \end{aligned} \tag{27}$$

where

$$\mathbf{I_n} \;=\; \mathbf{v} \;+\; \sum_{m_1=0}^{n_1-1}\mathbf{I}(X'_{m_1},X_{m_1}) + \sum_{m_2=0}^{n_2-1}\mathbf{I}(Y'_{m_2},Y_{m_2}) + \sum_{m_3=0}^{n_3-1}\mathbf{I}(Z'_{m_3},Z_{m_3})\;, \tag{28}$$

and the parameters ξ *and points* $X_{n_1},...,Z'_{n_3}$ *enter the parameterizations of* $\kappa_{\alpha,\mathbf{n}}$ *and pre-exponents as follows:*

$$\begin{aligned} \kappa_{1:n_2,n_3} &= -\,\frac{\xi_{1:n_2,n_3}\,\xi_{1:n_2+1,n_3+1}}{\xi_{1:n_2+1,n_3}\,\xi_{1:n_2,n_3+1}}\,\frac{E(Y'_{n_2},Z_{n_3})\,E(Y_{n_2},Z'_{n_3})}{E(Y'_{n_2},Z'_{n_3})\,E(Y_{n_2},Z_{n_3})}\;, \\ \kappa_{2:n_1,n_3} &= -\,\frac{\xi_{2:n_1+1,n_3}\,\xi_{2:n_1,n_3+1}}{\xi_{2:n_1,n_3}\,\xi_{2:n_1+1,n_3+1}}\,\frac{E(X_{n_1},Z_{n_3})\,E(X'_{n_1},Z'_{n_3})}{E(X'_{n_1},Z_{n_3})\,E(X_{n_1},Z'_{n_3})}\;, \\ \kappa_{3:n_1,n_2} &= -\,\frac{\xi_{3:n_1,n_2}\,\xi_{3:n_1+1,n_2+1}}{\xi_{3:n_1+1,n_2}\,\xi_{3:n_1,n_2+1}}\,\frac{E(X'_{n_1},Y_{n_2})\,E(X_{n_1},Y'_{n_2})}{E(X_{n_1},Y_{n_2})\,E(X'_{n_1},Y'_{n_2})}\;, \end{aligned} \tag{29}$$

and

$$\begin{aligned} \frac{u^{(0)}_{3:n_1,n_2}}{w^{(0)}_{1:n_2,n_3}} &= -\,\frac{\xi_{1:n_2,n_3}\,\xi_{3:n_1,n_2+1}}{\xi_{1:n_2+1,n_3}\,\xi_{3:n_1,n_2}}\,\frac{E(Y'_{n_2},Z_{n_3})\,E(Y_{n_2},X_{n_1})}{E(Y'_{n_2},X_{n_1})\,E(Y_{n_2},Z_{n_3})}\;, \\ \frac{u^{(0)}_{2:n_1,n_3}}{u^{(0)}_{1:n_2,n_3}} &= \frac{\xi_{1:n_2+1,n_3+1}\,\xi_{2:n_1,n_3}}{\xi_{1:n_2+1,n_3}\,\xi_{2:n_1,n_3+1}}\,\frac{E(Y'_{n_2},Z_{n_3})\,E(X_{n_1},Z'_{n_3})}{E(Y'_{n_2},Z'_{n_3})\,E(X_{n_1},Z_{n_3})}\;, \\ \frac{w^{(0)}_{3:n_1,n_2}}{w^{(0)}_{2:n_1,n_3}} &= \frac{\xi_{2:n_1+1,n_3}\,\xi_{3:n_1,n_2+1}}{\xi_{2:n_1,n_3}\,\xi_{3:n_1+1,n_2+1}}\,\frac{E(Y'_{n_2},X'_{n_1})\,E(X_{n_1},Z_{n_3})}{E(Y'_{n_2},X_{n_1})\,E(X'_{n_1},Z_{n_3})}\;. \end{aligned} \tag{30}$$

Proof: In order to prove that substitution (13) with τ-functions given by (27) solves (14) one should use repeatedly the Fay identity. For example, equation (14) for the choice $\alpha=1$, $\beta=2$ and $\gamma=3$ is the consequence of two Fay identities (26) taken for two sets of divisors (A,B,C,D) and (A',B',C',D') with identification $A=A'=X_{n_1}$, $B=B'=Y_{n_2}$, $D=D'=Y'_{n_2}$, $C=Z'_{n_3}$ and $C'=Z_{n_3}$. The appearing of the ratios containing parameters $\xi_{\alpha:n_\beta,n_\gamma}$ are due to gauge invariance of the linear system (3) or (4). Gauge parameters $\xi_{\alpha:n_\beta,n_\gamma}$, the divisors X_{n_1}, X'_{n_1}, etc. as well as period matrix Ω and the point on the Jacobian $\mathbf{v}$ are free parameters of the general solution. By imposing the periodic boundary condition (which means in particular the fixing of the size of the system) they may be related to the values of the integrals of motion. ■

One may solve expressions (30) in order to avoid the rations of the pre-exponents. To do this we need to introduce several extra parameters A_{n_1}, B_{n_2}, C_{n_3}. Then, one obtains for the "first" block:

$$
\begin{aligned}
u_{1,\mathbf{n}} &= \frac{\xi_{1:n_2+1,n_3}}{\xi_{1:n_2+1,n_3+1}} \frac{E(Y'_{n_2},Z'_{n_3})}{E(Y'_{n_2},Z_{n_3})} \frac{E(C_{n_3},Z_{n_3})}{E(C_{n_3},Z'_{n_3})} \frac{\tau_{2,\mathbf{n}}}{\tau_{2,\mathbf{n}+\mathbf{e}_3}}\,,\\
w_{1,\mathbf{n}} &= -\frac{\xi_{1:n_2+1,n_3}}{\xi_{1:n_2,n_3}} \frac{E(Z_{n_3},Y_{n_2})}{E(Z_{n_3},Y'_{n_2})} \frac{E(B_{n_2},Y'_{n_2})}{E(B_{n_2},Y_{n_2})} \frac{\tau_{3,\mathbf{n}+\mathbf{e}_2}}{\tau_{3,\mathbf{n}}}\,,\\
\kappa_{1:n_2,n_3} &= -\frac{\xi_{1:n_2,n_3}\,\xi_{1:n_2+1,n_3+1}}{\xi_{1:n_2+1,n_3}\,\xi_{1:n_2,n_3+1}} \frac{E(Y'_{n_2},Z_{n_3})\,E(Y_{n_2},Z'_{n_3})}{E(Y'_{n_2},Z'_{n_3})\,E(Y_{n_2},Z_{n_3})}\,,
\end{aligned}
\tag{31}
$$

for the "second" block:

$$
\begin{aligned}
u_{2,\mathbf{n}} &= \frac{\xi_{2:n_1,n_3}}{\xi_{2:n_1,n_3+1}} \frac{E(X_{n_1},Z'_{n_3})}{E(X_{n_1},Z_{n_3})} \frac{E(C_{n_3},Z_{n_3})}{E(C_{n_3},Z'_{n_3})} \frac{\tau_{1,\mathbf{n}}}{\tau_{1,\mathbf{n}+\mathbf{e}_3}}\,,\\
w_{2,\mathbf{n}} &= -\frac{\xi_{2:n_1,n_3}}{\xi_{2:n_1+1,n_3}} \frac{E(Z_{n_3},X'_{n_1})}{E(Z_{n_3},X_{n_1})} \frac{E(A_{n_1},X_{n_1})}{E(A_{n_1},X'_{n_1})} \frac{\tau_{3,\mathbf{n}}}{\tau_{3,\mathbf{n}+\mathbf{e}_1}}\,,\\
\kappa_{2:n_1,n_3} &= -\frac{\xi_{2:n_1+1,n_3}\,\xi_{2:n_1,n_3+1}}{\xi_{2:n_1,n_3}\,\xi_{2:n_1+1,n_3+1}} \frac{E(X_{n_1},Z_{n_3})\,E(X'_{n_1},Z'_{n_3})}{E(X'_{n_1},Z_{n_3})\,E(X_{n_1},Z'_{n_3})}\,,
\end{aligned}
\tag{32}
$$

and the "third" block finally:

$$
\begin{aligned}
u_{3,\mathbf{n}} &= \frac{\xi_{3:n_1,n_2+1}}{\xi_{3:n_1,n_2}} \frac{E(X_{n_1},Y_{n_2})}{E(X_{n_1},Y'_{n_2})} \frac{E(B_{n_2},Y'_{n_2})}{E(B_{n_2},Y_{n_2})} \frac{\tau_{1,\mathbf{n}+\mathbf{e}_2}}{\tau_{1,\mathbf{n}}}\,,\\
w_{3,\mathbf{n}} &= -\frac{\xi_{3:n_1,n_2+1}}{\xi_{3:n_1+1,n_2+1}} \frac{E(Y'_{n_2},X'_{n_1})}{E(Y'_{n_2},X_{n_1})} \frac{E(A_{n_1},X_{n_1})}{E(A_{n_1},X'_{n_1})} \frac{\tau_{2,\mathbf{n}}}{\tau_{2,\mathbf{n}+\mathbf{e}_1}}\,,\\
\kappa_{3:n_1,n_2} &= -\frac{\xi_{3:n_1,n_2}\,\xi_{3:n_1+1,n_2+1}}{\xi_{3:n_1+1,n_2}\,\xi_{3:n_1,n_2+1}} \frac{E(X'_{n_1},Y_{n_2})\,E(X_{n_1},Y'_{n_2})}{E(X_{n_1},Y_{n_2})\,E(X'_{n_1},Y'_{n_2})}\,,
\end{aligned}
\tag{33}
$$

where $\tau_{\alpha,\mathbf{n}}$ are given by (27). In addition, the discrete Baker-Akhiezer function $\Phi_{\mathbf{n}}$, obeying the whole set of (3) (and therefore, (4)) and normalized by $\Phi_{\mathbf{0}} = \xi_{\mathbf{0}}$, is given by

$$
\Phi_{\mathbf{n}} = \Phi_{\mathbf{n}}(P) = \xi_{\mathbf{n}}\Phi_{\mathbf{n}}^{(0)}(P)\,\frac{\Theta(\mathbf{v}+\mathbf{I}(Q,P)+\mathbf{I}_{\mathbf{n}})}{\Theta(\mathbf{v}+\mathbf{I}(Q,P))}\,,
\tag{34}
$$

where

$$\begin{aligned}\Phi_{\mathbf{n}}^{(0)}(P) &= \prod_{m_1=0}^{n_1-1} \frac{E(P,X_{m_1})E(A_{m_1},X'_{m_1})}{E(P,X'_{m_1})E(A_{m_1},X_{m_1})}\times \\ &\times \prod_{m_2=0}^{n_2-1} \frac{E(P,Y_{m_2})E(B_{m_2},Y'_{m_2})}{E(P,Y'_{m_2})E(B_{m_2},Y_{m_2})} \prod_{m_3=0}^{n_3-1} \frac{E(P,Z_{m_3})E(C_{m_3},Z'_{m_3})}{E(P,Z'_{m_3})E(C_{m_3},Z_{m_3})}.\end{aligned} \tag{35}$$

5. Finite lattice with open boundary

As one could see, all the considerations until now were local and did not take into account the size and the boundary of the cubic lattice. To describe the global characteristic, such as integrals of motion, we should specify this external data.

The most natural is the case of the finite cubic lattice with open boundary conditions and the corresponding Cauchy problem. For the cubic lattice of the size $N_1 \times N_2 \times N_3$ we fix the coordinates n_1, n_2, n_3 of 3d vector $\mathbf{n}$ (1) as follows:

$$0 \le n_\alpha < N_\alpha\ , \quad \alpha = 1,2,3\ . \tag{36}$$

$\Delta = N_1N_2 + N_2N_3 + N_3N_1$ edges incoming to the cubic lattice correspond to the 2Δ initial data

$$\begin{aligned} u_{1:n_2,n_3} &\equiv u_{1,0\mathbf{e}_1+n_2\mathbf{e}_2+n_3\mathbf{e}_3}\ , \quad w_{1:n_2,n_3} \equiv w_{1,0\mathbf{e}_1+n_2\mathbf{e}_2+n_3\mathbf{e}_3}\ , \\ u_{2:n_1,n_3} &\equiv u_{2,n_1\mathbf{e}_1+0\mathbf{e}_2+n_3\mathbf{e}_3}\ , \quad w_{2:n_1,n_3} \equiv w_{2,n_1\mathbf{e}_1+0\mathbf{e}_2+n_3\mathbf{e}_3}\ , \\ u_{3:n_1,n_2} &\equiv u_{3,n_1\mathbf{e}_1+n_2\mathbf{e}_2+0\mathbf{e}_3}\ , \quad w_{3:n_1,n_2} \equiv w_{3,n_1\mathbf{e}_1+n_2\mathbf{e}_2+0\mathbf{e}_3}\ . \end{aligned} \tag{37}$$

Equations of motion (6)–(8), being applied recursively, defines the transformation from the initial data (37) to the 2Δ final data

$$\begin{aligned} u'_{1:n_2,n_3} &\equiv u_{1,N_1\mathbf{e}_1+n_2\mathbf{e}_2+n_3\mathbf{e}_3}\ , \quad w'_{1:n_2,n_3} \equiv w_{1,N_1\mathbf{e}_1+n_2\mathbf{e}_2+n_3\mathbf{e}_3}\ , \\ u'_{2:n_1,n_3} &\equiv u_{2,n_1\mathbf{e}_1+N_2\mathbf{e}_2+n_3\mathbf{e}_3}\ , \quad w'_{2:n_1,n_3} \equiv w_{2,n_1\mathbf{e}_1+N_2\mathbf{e}_2+n_3\mathbf{e}_3}\ , \\ u'_{3:n_1,n_2} &\equiv u_{3,n_1\mathbf{e}_1+n_2\mathbf{e}_2+N_3\mathbf{e}_3}\ , \quad w'_{3:n_1,n_2} \equiv w_{3,n_1\mathbf{e}_1+n_2\mathbf{e}_2+N_3\mathbf{e}_3}\ . \end{aligned} \tag{38}$$

Suppose that the initial data (37) are generic. The natural question arises. How one can invert the parameterizations described by the formulas (31)–(33) in order to restore the algebraic geometry data in terms of (37) and the parameters $\kappa_{\alpha:n_\beta,n_\gamma}$ (the total number of parameters is equal to 3Δ).

In general the solution of this problem is not unique. Evidently, the same initial data may be parameterized by the infinite number of the different sets of algebraic geometry data with sufficiently high genus. But there exists one preferred compact Riemann surface, which is minimal in the set of all possible curves parameterizing the dynamics (6-8).

In the Cauchy problem with (37) and (38) this curve appears as follows. Let the initial data are parameterized in the terms of some Γ. One may write

$$u_{\alpha:n_\beta,n_\gamma} \;=\; u_{\alpha:n_\beta,n_\gamma}(\mathbf{v})\ , \quad w_{\alpha:n_\beta,n_\gamma} \;=\; w_{\alpha:n_\beta,n_\gamma}(\mathbf{v})\ .$$

Then due to (27,28) the solution of the Cauchy problem is

$$u'_{\alpha:n_\beta,n_\gamma} = u_{\alpha:n_\beta,n_\gamma}(\mathbf{v}+\mathbf{T}_\alpha)\,, \quad w_{\alpha:n_\beta,n_\gamma} = w_{\alpha:n_\beta,n_\gamma}(\mathbf{v}+\mathbf{T}_\alpha)\,,$$

where

$$\mathbf{T}_1 = \sum_{n_1=0}^{N_1} \mathbf{I}(X'_{n_1}, X_{n_1})\,, \quad \mathbf{T}_2 = \sum_{n_2=0}^{N_2} \mathbf{I}(Y'_{n_2}, Y_{n_2})\,, \quad \mathbf{T}_3 = \sum_{n_3=0}^{N_3} \mathbf{I}(Z'_{n_3}, Z_{n_3})\,. \tag{39}$$

The transformation from the initial data to the final ones is the *evolution* if

$$\mathbf{T}_1 = \mathbf{T}_2 = \mathbf{T}_3 = \mathbf{T} \mod \mathbb{Z}^g + \Omega\mathbb{Z}^g. \tag{40}$$

In what follows, all the relations of the type (40) will be understood modulo $\mathbb{Z}^g + \Omega\mathbb{Z}^g$.

The evolution conditions (40) mean that due to the Abel theorem there exist two meromorphic functions $\lambda(P)$ and $\mu(P)$, $P \in \Gamma$, with the divisors

$$\begin{aligned} (\lambda) &= \sum_{n_1\in\mathbb{Z}_{N_1}} X_{n_1} - \sum_{n_1\in\mathbb{Z}_{N_1}} X'_{n_1} - \sum_{n_3\in\mathbb{Z}_{N_3}} Z_{n_3} + \sum_{n_3\in\mathbb{Z}_{N_3}} Z'_{n_3}\,, \\ (\mu) &= \sum_{n_2\in\mathbb{Z}_{N_2}} Y_{n_2} - \sum_{n_2\in\mathbb{Z}_{N_2}} Y'_{n_2} - \sum_{n_3\in\mathbb{Z}_{N_3}} Z_{n_3} + \sum_{n_3\in\mathbb{Z}_{N_3}} Z'_{n_3}\,. \end{aligned} \tag{41}$$

This means that Γ is the compact Riemann surface given by a polynomial equation $J_\triangle(\lambda,\mu) = 0$ (see e.g. Theorem 10-23 in [6]). Moreover, (41) fix the generic structure of $J_\triangle(\lambda,\mu)$ (later we will give it, see (48)). The key observation is that due to the evolution conditions (40) $J_\triangle(\lambda,\mu)$ as a functional of the initial data should produce the set of invariants of the evolution, so the curve is the spectral one.

$J_\triangle$ may be derived with the help of the linear system of the type (3) written for the whole auxiliary plane. Let us fix the position of the auxiliary planes corresponding to the initial and final data. The auxiliary plane for the initial data crosses all the incoming edges of the cubic lattice while for the final data it intersects all the outgoing edges. These auxiliary planes plays the role of the two-dimensional space-like surfaces and discrete evolution corresponds to the translation of the auxiliary plane into direction perpendicular to this space-like surfaces.

All the objects, namely the dynamical and auxiliary linear variables on the space-like surface, can be numbered by the two-dimensional discrete index. Let us choose the numbering for the linear variables in the form similar to numbering of the initial (37) or final (38) data:

$$\begin{aligned} &\Phi_{0\mathbf{e}_1+n_2\mathbf{e}_2+n_3\mathbf{e}_3} = \Phi_{1:n_2,n_3}\,, \quad \Phi_{n_1\mathbf{e}_1+0\mathbf{e}_2+n_3\mathbf{e}_3} = \Phi_{2:n_1,n_3}\,, \\ &\Phi_{n_1\mathbf{e}_1+n_2\mathbf{e}_2+0\mathbf{e}_3} = \Phi_{3:n_1,n_2}\,, \quad 0 \le n_i \le N_i\,, \quad i = 1,2,3\,. \end{aligned} \tag{42}$$

An example of such enumeration on the auxiliary plane in the simple case $N_1 = N_2 = N_3 = 2$ is shown in Fig. 5.

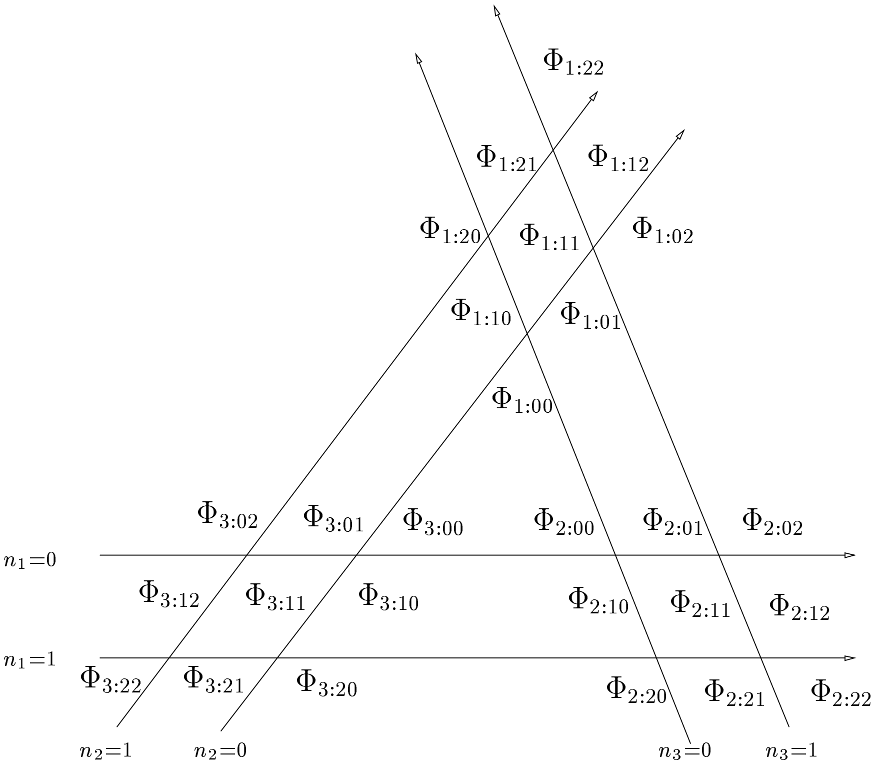

FIGURE 5. Example of enumeration of the linear variables.

Let $\mathcal{L}(\lambda,\mu)$ be the matrix of the linear system

$$
(43)\qquad \left\{\begin{array}{rcl}
0 & = & \Phi_{1:n_2+1,n_3} - \Phi_{1:n_2+1,n_3+1}u_{1:n_2,n_3} + \Phi_{1:n_2,n_3}w_{1:n_2,n_3} + \\
 & & + \ \Phi_{n_2,n_3+1}\kappa_{1:n_2,n_3}u_{1:n_2,n_3}w_{1:n_2,n_3}\ , \\
0 & = & \Phi_{2:n_1,n_3} - \Phi_{2:n_1,n_3+1}u_{2:n_1,n_3} + \Phi_{2:n_1+1,n_3}w_{2:n_1,n_3} + \\
 & & + \ \Phi_{2:n_1+1,n_3+1}\kappa_{2:n_1,n_3}u_{2:n_1,n_3}w_{2:n_1,n_3}\ , \\
0 & = & \Phi_{3:n_1,n_2+1} - \Phi_{3:n_1,n_2}u_{3:n_1,n_2} + \Phi_{3:n_1+1,n_2+1}w_{3:n_1,n_2} + \\
 & & + \ \Phi_{3:n_1+1,n_2}\kappa_{3:n_1,n_2}u_{3:n_1,n_2}w_{3:n_1,n_2}
\end{array}\right.
$$

written in the matrix form $0 = \Phi\cdot\mathcal{L}(\lambda,\mu)$, where the linear variables satisfy the identification conditions

$$
(44)\qquad \Phi_{1:0,n_3} = \Phi_{2:0,n_3}\ ,\quad \Phi_{1:n_2,0} = \Phi_{3:0,n_2}\ ,\quad \Phi_{2:n_1,0} = \Phi_{3:n_1,0}\ ,
$$

and the quasi-periodicity conditions

$$
(45)\qquad \frac{\Phi_{3:N_1,n_2}}{x} = \frac{\Phi_{1:n_2,N_3}}{z}\ ,\quad \frac{\Phi_{3:n_1,N_2}}{y} = \frac{\Phi_{2:n_1,N_3}}{z}\ ,\quad \frac{\Phi_{1:N_2,n_3}}{y} = \frac{\Phi_{2:N_1,n_3}}{x}
$$

for the boundary domains on the auxiliary plane. Parameters

$$
(46)\qquad \lambda = \frac{x}{z}\ ,\qquad \mu = \frac{y}{z}
$$

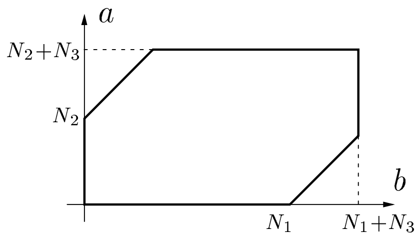

FIGURE 6. Newton's polygon.

are the complex numbers and we call them the spectral parameters. Due to (44) and (45) it is clear that the total number of the independent linear variables in the system (43) is Δ and $\mathcal{L}(\lambda,\mu)$ is $\Delta\times\Delta$ square matrix.

Define

$$J_{\triangle}(\lambda,\mu)=\det\mathcal{L}(\lambda,\mu)\left(\prod_{n_2,n_3}u_{1:n_2,n_3}\right)^{-1}. \tag{47}$$

(47) is a normalized Laurent polynomial ($J_{0,0}=1$) of the spectral parameters λ and μ,

$$J_{\triangle}=\sum_{a,b\in\Pi}\lambda^a\mu^{-b}J_{a,b}\,,\quad \Pi\ :\ \left\{\begin{array}{l}0\leq a\leq N_2+N_3\\ 0\leq b\leq N_1+N_3\\ -N_1\leq a-b\leq N_2\end{array}\right. . \tag{48}$$

Domain Π (Newton's polygon of $J_{\triangle}$) is shown in Fig. 6. One may prove (see [1]) that the coefficients of this Laurent polynomial are invariants of the evolution, i.e they are the same for the initial (37) and final (38) data and can be served as a source of the algebraic geometry data for the solution of these equations with open boundary conditions.

The requirement that the linear system (43) has non-trivial solution is equivalent to the fact that the spectral parameters belongs to the algebraic curve:

$$P=(\lambda,\mu)\in\Gamma_{\triangle}\quad\Leftrightarrow\quad J_{\triangle}(\lambda,\mu)=0\,. \tag{49}$$

Assuming that all the incoming data together with parameters $\kappa_{\alpha:n_\beta,n_\gamma}$ are in general position, one may calculate the genus of the curve (49) using Newton's polygon. For the cubic lattice of the size $N_1\times N_2\times N_3$ such that $N_1\geq N_2\geq N_3$ Newton's polygon associated with this curve (49) is shown in Fig. 6. The genus of the curve $\Gamma_{\triangle}$ is equal to the number of internal points of this polygon

$$g_{\triangle}=(N_1N_2+N_1N_3+N_2N_3)-(N_1+N_2+N_3)+1\,. \tag{50}$$

g coefficients $J_{a,b}$ in the Laurent polynomial $J_{\triangle}(\lambda,\mu)$ corresponding to the internal points of the Newton's polygon are related to the moduli of the algebraic curve $\Gamma_{\triangle}$ and so give the period matrix $\Omega_{\triangle}$. Coefficients $J_{a,b}$ corresponding to the perimeter of the polygon are related to the divisors of the meromorphic on $\Gamma_{\triangle}$ functions λ,μ and so give the set of $X_{n_1},\ldots,Z'_{n_3}$.

From (41) and (34) we may get explicitly

$$
\begin{aligned}
\lambda &= \prod_{n_1=0}^{N_1-1} \frac{E(P,X_{n_1})E(A_{n_1},X'_{n_1})}{E(P,X'_{n_1})E(A_{n_1},X_{n_1})} \prod_{n_2=0}^{N_2-1} \frac{E(P,Y'_{n_2})E(B_{n_2},Y_{n_2})}{E(P,Y_{n_2})E(B_{n_2},Y'_{n_2})}\;, \\
\mu &= \prod_{n_2=0}^{N_2-1} \frac{E(P,Y_{n_2})E(B_{n_2},Y'_{n_2})}{E(P,Y'_{n_2})E(B_{n_2},Y_{n_2})} \prod_{n_3=0}^{N_3-1} \frac{E(P,Z'_{n_3})E(C_{n_3},Z_{n_3})}{E(P,Z_{n_3})E(C_{n_3},Z'_{n_3})}\;.
\end{aligned} \tag{51}
$$

Previously in [7] a bit different approach was used. Due to the Theorem 2 of [7], the solution $\Phi(\lambda,\mu)$ of the linear problem $\Phi\cdot\mathcal{L}(\lambda,\mu)=0$ as the vector of meromorphic functions on the curve $\Gamma_\triangle$ is given by (34) on $\mathrm{Jac}(\Gamma_\triangle)$. With the help of (34) the parameterizations (27-30) may be restored uniquely.

We conclude the discussion on the Cauchy problem by the calculation of the degrees of freedom of the algebraic geometry parameterizations. With $\Delta = N_1N_2+N_2M_3+N_3N_1$, $\Delta' = N_1+N_2+N_3$ and $g=\Delta-\Delta'+1$, we have g moduli of $\Gamma_\triangle$, g complex numbers $\mathbf{v}\in\mathrm{Jac}(\Gamma_\triangle)$, g independent cross-ratios of $\xi_{\mathbf{n}}$, $2\Delta'-2$ independent divisors $X_{n_1},...,Z'_{n_3}$ and Δ' arbitrary divisors A_{n_1},B_{n_2},C_{n_3} *minus* one degree of freedom corresponding to the arbitrariness of Q: totally 3Δ parameters. It proves the completeness of the parameterizations in the terms of $\Gamma_\triangle$.

6. Bäcklund transformation

Curve $\Gamma_\triangle$ corresponds to the generic position of the initial data and open boundary conditions. For the application to the integrable spin models the periodical boundary conditions $\mathbf{T}=0$, i.e.

$$
\sum_{n_1\in\mathbb{Z}_{N_1}} \mathbf{I}(X'_{n_1},X_{n_1}) \;=\; \sum_{n_2\in\mathbb{Z}_{N_2}} \mathbf{I}(Y'_{n_2},Y_{n_2}) \;=\; \sum_{n_3\in\mathbb{Z}_{N_3}} \mathbf{I}(Z'_{n_3},Z_{n_3}) \;=\; 0 \tag{52}
$$

are important. Periodical boundary conditions reduces the spectral curve $\Gamma_\triangle$. Namely, due to (52), the parameters x,y,z in (45) become the meromorphic functions with divisors $(x)=\sum_{n_1}X_{n_1}-\sum_{n_1}X'_{n_1}$ and (58). Since now x,y,z are meromorphic functions (not only their ratios) the curve can be defined by an algebraic equations for any pair from x,y,z:

$$
J_1(y,z) \;=\; J_2(x,z) \;=\; J_3(x,y) \;=\; 0\;. \tag{53}
$$

The Laurent polynomial $J_\triangle(x/z,y/z)$ is not a generic one and $J_\triangle=0$ is a consequence of the algebraic relations (53).

In this section we describe explicitly the reduced spectral curve and the meromorphic functions, which uniformize it for another choice of the auxiliary plane. The interpretation of the problem now differs from the Cauchy one for the open cubic lattice. The discrete evolution can be identified now with a sequence of the Bäcklund transforms for the square auxiliary plane.

First we define the new position of the auxiliary plane. Let it crosses the ingoing edges for the vertices of the 3-dimensional lattice with coordinates $n_2\mathbf{e}_2 + n_3\mathbf{e}_3$. With this choice the role of the variables u_α and w_α, $\alpha = 1,2,3$ becomes different: $u_{2,\mathbf{n}}$, $w_{2,\mathbf{n}}$, $u_{3,\mathbf{n}}$, $w_{3,\mathbf{n}}$ are auxiliary, while $u_{1,\mathbf{n}}, w_{1,\mathbf{n}}$ are dynamical. The evolution will describe the change of these dynamical variables with respect to the "time" n_1 with periodic boundary conditions implied in the second and third directions:

$$u_{1:n_2+N_2,n_3} = u_{1:n_2,n_3+N_3} = u_{1:n_2,n_3}\ , \quad w_{1:n_2+N_2,n_3} = w_{1:n_2,n_3+N_3} = w_{1:n_2,n_3}\ . \tag{54}$$

Of course, the auxiliary for this evolution variables are also different from layer to layer according to the discrete equations (7)–(8). We will keep the open boundary conditions in the first direction.

Consider one layer of the cubic lattice corresponding to $n_1 = 0$. Equations of motion (6) and the periodical boundary condition (54) yields an implicit transformation

$$u_{1:n_2,n_3}, w_{1:n_2,n_3} \mapsto u'_{1:n_2,n_3}, w'_{1:n_2,n_3}. \tag{55}$$

According to (27-30), one may define

$$u_{1:n_2,n_3} = u_{1:n_2,n_3}(\mathbf{v})\ , \quad w_{1:n_2,n_3} = w_{1:n_2,n_3}(\mathbf{v})\ , \tag{56}$$

where $\mathbf{v} \in \mathrm{Jac}(\Gamma)$ for some algebraic curve Γ. Then the transformation (55) can be written in the form

$$u'_{1:n_2,n_3} = u_{1:n_2,n_3}(\mathbf{v} + \mathbf{I}(X'_0, X_0))\ , \quad w'_{1:n_2,n_3} = w_{1:n_2,n_3}(\mathbf{v} + \mathbf{I}(X'_0, X_0))\ . \tag{57}$$

The periodicity conditions (54) in the parameterizations (27-30) means $\sum_{n_2\in\mathbb{Z}_{N_2}} \mathbf{I}(Y'_{n_2} Y_{n_2}) = \sum_{n_3\in\mathbb{Z}_{N_3}} \mathbf{I}(Z'_{n_3}, Z_{n_3}) = 0$, (see (52)). Repeating the consideration from page 12, we may conclude that there exist two meromorphic functions y and z on Γ such that

$$(y) \;=\; \sum_{n_2} Y_{n_2} - Y'_{n_2}\ , \quad (z) \;=\; \sum_{n_3} Z_{n_3} - Z'_{n_3}\ , \tag{58}$$

and a compact Riemann surface Γ may be defined by a polynomial equation $J_\square(y,z) = 0$. Moreover, the structure of (y) and (z) fixes the algebraic form of Γ (see (61) later).

We call transformation (55) the Bäcklund transformation for the two-dimensional square lattice, since (55) is a canonical transformation and it conserve the integrals of motion. In order to specify the integrals of motion we again consider the linear system for the chosen auxiliary plane. This linear system is a system of N_2N_3 equations

$$j_{n_2,n_3} = 0, \quad \Phi_{-1,n_3} = y^{-1}\Phi_{N_2-1,n_3}\ , \quad \Phi_{n_2,N_3} = z\Phi_{n_2,0}\ , \quad \Phi_{-1,N_3} = y^{-1}z\Phi_{N_2-1,0}\ , \tag{59}$$

where $0 \le n_2 < N_2$, $0 \le n_3 < N_3$ and linear form j_{n_2,n_3} is defined by the first line in (3) with slightly shifted enumeration of the auxiliary linear variables Φ_{n_2,n_3}

$$\begin{aligned} j_{n_2,n_3} \;=\; & \Phi_{n_2,n_3} - \Phi_{n_2,n_3+1}u_{1:\,n_2,n_3} + \Phi_{n_2-1,n_3}w_{1:\,n_2,n_3} + \\ & + \Phi_{n_2-1,n_3+1}\kappa_{1:\,n_2,n_3}u_{1:\,n_2,n_3}w_{1:\,n_2,n_3}\ . \end{aligned} \tag{60}$$

Equations in (59) which includes the spectral parameters y and z describe quasi-periodical boundary conditions for the linear variables Φ_{n_2,n_3}.

Let $L(y,z)$ be the complete matrix of the coefficients of the system (59), $j = \Phi\cdot L(y,z)$. Define a Laurent polynomial $J_\square(y,z)$

$$J_\square(y,z) = \det\ L(y,z) = \sum_{a=0}^{N_3}\sum_{b=0}^{N_2} J_{a,b}\, y^{-a}\, z^{b}\ , \quad J_{0,0}\ =\ 1\ . \tag{61}$$

$J_{a,b}$ in this case are the invariants of the Bäcklund transform (55), see [1]. In order the system $\Phi(P)\cdot L(y,z) = 0$ has a nontrivial solution, the spectral parameters y and z should belong to the spectral curve

$$P = (y,z) \in \Gamma_\square \quad \Leftrightarrow \quad J_\square(y,z) = 0\ . \tag{62}$$

Assuming again that the dynamical variables $u_{1:n_2,n_3}$, $w_{1:n_2,n_3}$ and parameters $\kappa_{1:n_2,n_3}$ are generic, the genus of the spectral curve (62) is equal to $g_\square = (N_2-1)(N_3-1)$. Recall, the form (61) of the polynomial $J_\square$ follows uniquely from the conditions $(y) = \sum Y_{n_2} - Y'_{n_2}$, $(z) = \sum Z_{n_3} - Z'_{n_3}$.

In the backward direction, Theorem 2 from [7] says that the non-normalized solution of $\Phi(P)\cdot L(y,z) = 0$ as the meromorphic function of $\Gamma_\square$ is given by

$$\begin{aligned}\Phi_{n_2,n_3}(P) \ &=\ \prod_{m_2=0}^{n_2-1}\frac{E(P,Y_{m_2})}{E(P,Y'_{m_2})}\frac{E(B_{m_2},Y'_{m_2})}{E(B_{m_2},Y_{m_2})}\ \prod_{m_3=0}^{n_3-1}\frac{E(P,Z_{m_3})}{E(P,Z'_{m_3})}\frac{E(C_{m_3},Z'_{m_3})}{E(C_{m_3},Z_{m_3})}\times\\ &\times\Theta\left(\mathbf{v}+\mathbf{I}(Q,P)+\sum_{m_2=0}^{n_2-1}\mathbf{I}(Y'_{m_2},Y_{m_2})+\sum_{m_3=0}^{n_3-1}\mathbf{I}(Z'_{m_3},Z_{m_3})\right),\end{aligned} \tag{63}$$

where theta-functions are constructed by means of the period matrix $\Omega_\square$ of the algebraic curve $\Gamma_\square$. Using formula (63) we may find the corresponding expressions for the dynamical variables $u_{1:n_2,n_3}$ and $w_{1:n_2,n_3}$ to observe that they coincide with those given by the Proposition 1 and arbitrary curve Γ being identified with the spectral curve $\Gamma_\square$. Explicit form of the spectral parameters uniformizing the spectral curve $\Gamma_\square$ are

$$\begin{aligned} z(P) \ &=\ \prod_{n_3\in\mathbb{Z}_{N_3}}\frac{E(P,Z_{n_3})}{E(P,Z'_{n_3})}\ \frac{E(C_{n_3},Z'_{n_3})}{E(C_{n_3},Z_{n_3})}\ ,\\ y(P) \ &=\ \prod_{n_2\in\mathbb{Z}_{N_2}}\frac{E(P,Y_{n_2})}{E(P,Y'_{n_2})}\ \frac{E(B_{n_2},Y'_{n_2})}{E(B_{n_2},Y_{n_2})}\ .\end{aligned} \tag{64}$$

In case when the algebraic curve used for the construction of the general solution becomes a rational one, the one step of this evolution $n_1 \mapsto n_1+1$ with open b.c. in the 1-st direction can be identified with creation of additional soliton. This will be explicitly demonstrated in the next section.

In this Section we have established the origin of the curve $\Gamma_\square$ appearing when the periodical b.c. are imposed in 2-nd and 3-rd directions. Evidently, $J_\square(y,z)$ should be identified with $J_1(y,z)$ in (53). In the same way the periodical b.c. may be imposed in any other pair of directions, and the corresponding J_2 and J_3 also are the spectral determinants. If the periodical b.c. are imposed in all three directions, then any of three relations $J_1 = 0$, $J_2 = 0$ and $J_3 = 0$ should define the same curve, and since the principle of generic data in this case has been lost, the genus of the curve may be established as

$$g \;\leq\; \text{min of } \;(N_1-1)(N_2-1)\;,\;\;(N_2-1)(N_3-1)\;,\;\;(N_3-1)(N_1-1)\;. \tag{65}$$

7. Rational limit

Let us again forget on the boundary conditions and consider a rational limit of the algebraic geometry solutions to the discrete equation of motion (14). This rational limit for the Θ-function (24) on a Jacobian of an algebraic curve corresponds to (see [8])

$$\mathbf{e}^{i\pi\Omega_{n,n}+2i\pi v_n} \;=\; -f_n\;, \tag{66}$$

and

$$\mathbf{e}^{i\pi\Omega_{n,n}} \;\mapsto\; 0\;,\quad \mathbf{e}^{i\pi\Omega_{k,n}} \;\mapsto\; \frac{(q_k-q_n)\,(p_k-p_n)}{(q_k-p_n)\,(p_k-q_n)} \;\stackrel{def}{=}\; d_{k,n}\;. \tag{67}$$

Thus the set of $d_{k,n}$ is the reminder of the period matrix, while the set of f_n is the reminder of $\mathbf{v}$. The prime forms in the rational limit are the prime forms on the sphere:

$$\frac{E(A,B)E(C,D)}{E(A,D)E(C,B)} \;=\; \frac{(A-B)(C-D)}{(A-D)(C-B)}\;. \tag{68}$$

For $g = 1,2,\ldots$ and the sequence of the parameters

$$p_0,q_0,f_0;\;\; p_2,q_2,f_2;\;\; \ldots\;\; p_{g-1},q_{g-1},f_{g-1} \;=\; \{p_k,q_k,f_k\}_{k=0}^{g-1} \tag{69}$$

we introduce the rational limit of the Θ-function (see the appendix of [8])

$$H^{(g)}(\{p_k,q_k,f_k\}_{k=0}^{g-1}) \;=\; \frac{\det|q_j^i \;-\; f_j\,p_j^i|_{i,j=0}^{g-1}}{\displaystyle\prod_{i>j}(q_i\;-\;q_j)}\;. \tag{70}$$

We set $H^{(0)} \equiv 1$ by definition. Note that if all parameters f_k vanish $H^{(g)}(\{p_k,q_k,0\}_{k=0}^{g-1}) = 1$ as well. Let the function $\sigma_k(z)$ be

$$\sigma_k(z) \;=\; \frac{p_k\;-\;z}{q_k\;-\;z}\;. \tag{71}$$

When $g = 0$ the equations (14) in the rational limit have the simple solution $\tau_{\alpha,\mathbf{n}} = 1$ due to identity

$$r_\alpha = 1 + s_\beta + s_\gamma^{-1}\;, \tag{72}$$

where

$$
\begin{aligned}
r_\alpha &= -\frac{X'_\beta - X'_\gamma}{X_\beta - X_\gamma}\,\frac{X_\alpha - X_\beta}{X_\alpha - X_\gamma}\,\frac{X_\alpha - X'_\beta}{X_\alpha - X'_\gamma}\,, \\
s_\alpha &= -\frac{X_\alpha - X_\gamma}{X_\alpha - X_\beta}\,\frac{X'_\alpha - X_\beta}{X'_\alpha - X_\gamma}
\end{aligned} \tag{73}
$$

and (α,β,γ) is any even permutation of the set $(1,2,3)$. The notations in (72) and (73) are related to those in (15) and the Proposition 1 as follows: $r_\alpha = r_{\alpha,\mathbf{n}}$, $s_\alpha = s_{\alpha,\mathbf{n}}$; $X_1 = X_{n_1}$, $X'_1 = X'_{n_1}$, $X_2 = Y_{n_2}$, $X'_2 = Y'_{n_2}$, $X_3 = Z_{n_3}$ and finally $X'_3 = Z'_{n_3}$.

Proposition 2. *The soliton solutions of the equation (14) are given by the formulas*

$$
\begin{aligned}
\tau_{1,\mathbf{n}} &= H^{(g)}\left(\left\{\frac{I_{\mathbf{n}:k}}{\sigma_k(X_{n_1})}\right\}_{k=0}^{g-1}\right)\,, \\
\tau_{2,\mathbf{n}} &= H^{(g)}\left(\left\{\frac{I_{\mathbf{n}:k}}{\sigma_k(Y_{n_2})}\right\}_{k=0}^{g-1}\right)\,, \\
\tau_{3,\mathbf{n}} &= H^{(g)}\left(\left\{\frac{I_{\mathbf{n}:k}}{\sigma_k(Z_{n_3})}\right\}_{k=0}^{g-1}\right)\,,
\end{aligned} \tag{74}
$$

where $I_{\mathbf{n}:k}$, $n_1,n_2,n_3 \in \mathbb{Z}$

$$
I_{\mathbf{n}:k} = f_k\left(\prod_{m_1=0}^{n_1-1}\frac{\sigma_k(X'_{m_1})}{\sigma_k(X_{m_1})}\right)\left(\prod_{m_2=0}^{n_2-1}\frac{\sigma_k(Y'_{m_2})}{\sigma_k(Y_{m_2})}\right)\left(\prod_{m_3=0}^{n_3-1}\frac{\sigma_k(Z'_{m_3})}{\sigma_k(Z_{m_3})}\right)\,, \tag{75}
$$

and parameters $r_{\alpha,\mathbf{n}}$ and $s_{\alpha,\mathbf{n}}$ are defined by the formulas (73).

The proof of this Proposition is based on the rational variant of the Fay's identity

$$
\begin{aligned}
&(A-D)(C-B)\,H^{(g)}(\{f_k\frac{\sigma_k(A)}{\sigma_k(B)}\}_{k=1}^{g})\,H^{(g)}(\{f_k\frac{\sigma_k(C)}{\sigma_k(D)}\}_{k=1}^{g}) + \\
&(A-B)(D-C)\,H^{(g)}(\{f_k\frac{\sigma_k(A)}{\sigma_k(D)}\}_{k=1}^{g})\,H^{(g)}(\{f_k\frac{\sigma_k(C)}{\sigma_k(B)}\}_{k=1}^{g}) = \\
&(A-C)(D-B)\,H^{(g)}(\{f_k\}_{k=1}^{g})\,H^{(g)}(\{f_k\frac{\sigma_k(A)\sigma_k(C)}{\sigma_k(B)\sigma_k(D)}\}_{k=1}^{g})
\end{aligned} \tag{76}
$$

described in [8].

So far the parameters p_k, q_k and f_k (69) are arbitrary complex parameters and the solution given by the Proposition 2 is relevant for lattice infinite in all directions. Now we would like to impose periodic boundary conditions (54) in directions 2 and 3 and interpret the evolution along the direction 1 as a sort of Bäcklund transformation which create the solitons. To do this we first note that boundary conditions (54) are equivalent to the following algebraic relations for the parameters p and q:

$$\prod_{n_2=0}^{N_2-1} \frac{\sigma(Y'_{n_2})}{\sigma(Y_{n_2})} = \prod_{n_3=0}^{N_3-1} \frac{\sigma(Z'_{n_3})}{\sigma(Z_{n_3})} = 1 \ . \tag{77}$$

One may verify that for the parameters $Y_{n_2}, \ldots, Z'_{n_3}$ in general position the system of equations (77) has exactly $g = (N_2-1)(N_3-1)$ non-equivalent solutions (equivalence means that if (p,q) is a solution of (77), then (q,p) is also the solution). Let us choose this set of solutions as sequence (69) leaving parameters f_k, $k = 1, \ldots, g$ to be free.

Using this freedom let us redefine the 'amplitudes' f_k as follows:

$$f_k = F_k \cdot \sigma_k(X_k), \tag{78}$$

where we recall that parameters p_k and q_k of the functions (71) are already fixed by the system of equations (77), while parameters X_k are still free. Let us consider the solutions for the tau-functions $\tau_{2,\mathbf{n}}$ and $\tau_{3,\mathbf{n}}$ given by the Proposition 2 and with redefined amplitudes (78) at the value of the discrete coordinate $n_1 = 0$. Using the freedom in parameters X_k we send

$$X_k \mapsto p_k \ . \tag{79}$$

It is clear that in this limit all components of the set $\{I_{\mathbf{n}:k}\}_{k=0}^{g-1}$ vanish and according to definition (70) we will have $\tau_{2,\mathbf{n}} = \tau_{3,\mathbf{n}} = 1$. In other words, we obtained for these tau-functions at $n_1 = 0$ a zero-soliton solution. Let us repeat the same procedure at $n_1 = 1$. Namely, again consider generic solution for these tau-functions given by the Proposition 2 and then make a limit (79). It is clear that now in the set $\{I_{\mathbf{n}:k}\}_{k=0}^{g-1}$ one element which correspond to $k = 0$ will survive and as result we obtain one-soliton solution. Increasing n_1 results in increasing the number of solitons. One may see that the maximal number of solitons which can be reached by this procedure is equal to $g = (N_2-1)(N_3-1)$. See detailed description of this phenomena in the simplest situation discussed in the paper [8].

But before conclude this section we would like to give one more explanation of the solitons creation procedure. Let us consider the equation (14) at $n_1 = 0$ for $\alpha, \beta, \gamma = 1, 2, 3$ and homogeneous or zero-soliton τ-functions $\tau^{(0)}_{2:n_2,n_3} = \tau^{(0)}_{3:n_2,n_3} = 1$. This is a difference with respect to "space" coordinates n_2 and n_3 linear equation for the function $\tau_{1:n_2,n_3}$. Using simple algebra one may verify that besides trivial solution $\tau_{1:n_2,n_3} = 1$ this equation has a solution of the form $\tau_{1:n_2,n_3} = \prod_{m_2=0}^{n_2-1} \frac{\sigma_0(Y'_{m_2})}{\sigma_0(Y_{m_2})} \prod_{m_3=0}^{n_3-1} \frac{\sigma_0(Z'_{m_3})}{\sigma_0(Z_{m_3})}$, where parameter p_0 of the function σ_0 is identified with X_0. The complete solutions of linear difference equations is

$$\tau^{(1)}_{1:n_2,n_3} = 1 - F_0 \prod_{m_2=0}^{n_2-1} \frac{\sigma_0(Y'_{m_2})}{\sigma_0(Y_{m_2})} \prod_{m_3=0}^{n_3-1} \frac{\sigma_0(Z'_{m_3})}{\sigma_0(Z_{m_3})} \tag{80}$$

with arbitrary F_0, Now solving the equation (14) for $\alpha, \beta, \gamma = 2, 3, 1$ and $3, 1, 2$ with found $\tau^{(1)}_{1:n_2,n_3}$ we can find the values of the τ-functions $\tau_{2:n_2,n_3}$ and $\tau_{3:n_2,n_3}$ at the discrete time

$n_1 = 1$. They will have similar to (80) one-soliton form. Using these solutions again in the equation (14) with $\alpha, \beta, \gamma = 1, 2, 3$ we will find two-soliton solution for the function $\tau_{1:n_2,n_3}$ and then two-soliton solutions for the functions $\tau_{2:n_2,n_3}$ and $\tau_{3:n_2,n_3}$ at the next value of the discrete time $n_2 = 2$. It is clear that this procedure can be continued and demonstrate the equivalence of the discrete evolution parameter n_1 to the number of solitons. This simple explanation finally justify that investigated in this paper the discrete dynamic given by the equations of motion (6)–(8) is a set of consecutive Bäcklund transformations.

8. Discussion

As it is well known [1], the dynamics of parameters of 3d spin models is equivalent to the dynamics (6-8). So the classical solutions discussed in this paper can be used for the spin models of different types. The solutions with periodic boundary conditions can be utilized for construction generic spin model, such that their Boltzmann weights are parameterized by the theta-functions of the higher genus. The integrability of such spin models is based on the modified tetrahedron equation [10]. This should include the different generalization of the famous Chiral Potts model.

On the other hand the solitonic solutions are convenient for completely inhomogeneous ZBB model and open a way to develop the quantum separation of variables for ZBB model [9]. In particular, these solutions with parameters $f_k \neq 0$ allows one to construct the complete family of isospectral deformations of ZBB transfer-matrix (see also [8] for realization of this program in case of relativistic Toda chain model with spin degrees of freedom).

9. Acknowledgment

This work was supported in part by the grants INTAS OPEN 00-00055 and CRDF RM1-2334-MO-02. S.P.'s work was supported by the grant RFBR 01-01-00539, grant for support of scientific schools RFBR 00-15-96557 and grant of Heisenberg-Landau Program HLP-2002-11. S.S's work was supported by the grant RFBR 01-01-00201 and by Grant-in-Aid for scientific research of Japan Society for the Promotion of Science.

References

[1] S. Sergeev. A three-dimensional quantum integrable mapping, *Theoretical and Mathematical Physics*, **118** (1999) 479-487; Quantum 2+1 evolution model, *J. Phys. A: Math. Gen.* **32** (1999) 5693-5714; Solitons in a 3d integrable model, *Phys. Lett.* **A 265** (2000) 364-368; Auxiliary transfer matrices for three dimensional integrable models, *Theoretical and Mathematical Physics*, **124** (2000) 1187-1201; Complex of three dimensional integrable models, *J. Phys. A: Math. Gen.* **34** (2001) 10493-10503.

[2] I. Krichever. Algebraic curves and nonlinear difference equations. *Russian Math. Survey* **33** (1978) 215–216.

[3] R. Hirota. Discrete analogue of a generalized Toda equation", *J. Phys. Soc. Jpn.* **50** (1981) 3785-3791.

[4] D. Mumford. Tata lectrures on Theta I,II. Boston-Basel-Stuttgart, Birkhäuser, 1983, 1985.

[5] J.D. Fay. Theta Functions on Riemann Surfaces. *Lect. Notes in Math.*, **352** (1973).

[6] G. Springer. Introduction to Riemann Surfaces. Chelsea Pub. Co. New-York, sc. ed., 1981.

[7] S. Sergeev. On exact solution of a classical 3d integrable model. *Journal of Nonlinear Mathematical Physics*, **27** (2000) 57-72.

[8] S. Pakuliak, S. Sergeev. Quantum relativistic toda chain at root of unity: isospectrality, modified Q-operators and functional Bethe ansatz. Preprint MPI-2002-45 (nlin.SI/0205037).

[9] S. Sergeev. Functional equations and separation of variables for 3d spin models, Preprint MPI-2002-46.

[10] G. von Gehlen, S. Pakuliak, S. Sergeev. Explicit free parametrization of the modified tetrahedron equation. In preparation.

† Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna 141980, Moscow reg., Russia

‡ Max Plank Institute of Mathematics, Vivatsgasse 7, D-53111, Bonn, Germany
E-mail address: sergeev@mpim-bonn.mpg.de, pakuliak@thsun1.jinr.ru