Does one need resurgent equations for exact semi-classical asymptotics?

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Abstract

In this paper, we present a method of constructing exact semi-classical asymptotics, which is free of usage of apparatus of resurgent equations. In particular, the univaluedness conditions are written in terms of algebraic equations. The latter equatons allow one also to investigate the Stokes phenomenon, at least, in generic position.

Introduction

In this paper, we shall consider the problem of constructing exact asymptotic expansions to differential equations with a small parameter. To begin with, we describe the general scheme (see, for example, the book [1]). For brevity, we shall carry out our considerations on the example of ordinary differential equations.

So, consider an $1/h$-differential equation [2] of the form

$$H\left(x,-ih\frac{d}{dx}\right)=\sum_{j=0}^{n}A_j\left(x\right)\left(-ih\frac{d}{dx}\right)^j u\left(x,h\right)=0, \qquad (1)$$

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where

$$H(x,p) = \sum_{j=0}^{n} A_j(x) p^j$$

is a symbol of the operator involvede into the latter equation. The coefficients $A_j(x)$ of the considered equation are supposed to be entire functions of the variable x (one can even consider the case when all the A_js are polynomials in x). We are intended to construct the asymptotic solutions to equation (1) of the form

$$u(x,h) \simeq e^{\frac{i}{h}S_1(x)} \sum_{k=0}^{\infty} (-ih)^k a_k^{(1)}(x) + e^{\frac{i}{h}S_2(x)} \sum_{k=0}^{\infty} (-ih)^k a_k^{(2)}(x) + \ldots \qquad (2)$$

for complex values of x, where the number of exponentials $\exp\left\{\frac{i}{h}S_j(x)\right\}$ depends, in general, on the point x. Expansion (2) must be valid up to the terms of arbitrary exponential decrease.

Remark 1 More exactly, one should consider the results of resummation of power series in h on the right in (2) rather than these series themselves. The necessity of resummation of power series is due to the fact that these series are, as a rule, divergent. The reader can find the corresponding theory, for example, in the above cited book.

The construction of the asymptotic solution to equation (1) of the form (2) goes within the following two steps:

1. First, one has to construct formal WKB-elements which are em formal solutions to equation (1) up to the arbitrary power of h. As it is well-known, to perform this step one has to solve the Hamilton-Jacobi equation for the action $S(x)$ and the recurrent system of the transport equations for the amplitude functions $a_j(x)$. It is known that the solutions $S(x)$ and $a_j(x)$ of these equations are, in general, ramifying functions of the variable x. The set of ramification points of these functions is called the *set of focal points* of the corresponding asymptotic solution, and we denote this set by $\mathcal{F}$.

2. Second, one has to *resummate* the obtained formal solutions thus constructing the exact asymptotic solution to the considered equation. On this step one must take into account the so-called *Stokes phenomenon*. The meaning of this phenomenon is as follows:

Since the formal solution of the considered equation is a ramifying function (with the set of focal points as the ramification set) of the variable x whereas the exact solution is, on the contrary, a univalued function, one should not expect that the resummation of the formal solution (or of the linear combination of these solutions) gives everywhere a solution to the differential equation. In fact, the whole plane $\mathbf{C}_x$ can be divided on the so-called *Stokes regions* such that in each Stokes region the solution can be constructed as the resummation of some linear combination of the formal solutions. These linear combinations are different in the different Stokes regions, so that the asymptotics of the constructed solution has jumps on the union of boundaries of the Stokes regions (this latter set is called a *Stokes set* of the solution in question; we denote this set by $\mathcal{S}$). Thus, to construct the solution of the form (2) one should determine which linear combinations of formal solutions have to be considered in each Stokes region in order to construct a univalued solution (2).

The first step of the above described scheme is well-investigated, so in this paper we concentrate on the second one. As it can be seen from the above considerations, on this second step one must compare the monodromy properties of the formal solution (*formal monodromy*) and the monodromy properties of the real solution (*real monodromy*). This latter monodromy must be trivial since equations in question have only univalued solutions.

The usual tool of investigating of the Stokes phenomenon is the so-called *resurgent equations* (see, for example, [3], [4], [5], [6], [1]). These equations, based on the notion of *alient derivative* which is a derivation of the resurgent functions algebra, are far from trivial. The question arises whether these complicated notions are nesessary when constructing exact asymptotics. In this paper, we shall show that, at least in the case of the generic position, the *investigation of the Stokes phenomenon in the theory of differential equations can be carried out in a purely algebraic way without using the technique of resurgent equations.*

1 Formal monodromy

Let us first describe in more detail the formal monodromy of solutions to (1). Let us consider a formal solution to equation (1) of the form

$$u(x,h) = e^{\frac{i}{h}S(x)} \sum_{k=0}^{\infty} (-ih)^k a_k(x). \qquad (3)$$

Denote by L the Lagrangian manifold corresponding to the action $S(x)$. This manifold may have singularities over focal points of the considered formal solution. We assume that:

1. *All roots $p = S'(x)$ of the Hamilton-Jacobi equation*

$$H(x, S'(x)) = 0$$

are simple ones. In particular, this means that the projection of the manifold L on the space $\mathbf{C}_x$ forms an n-sheeted covering over $\mathbf{C}_x \setminus \mathcal{F}$. Here, as above, $\mathcal{F}$ is a set of focal points of the correspondind WKB-element.

2. *For any point $x^* \in \mathcal{F}$, each connected component of the Lagrangian manifold L over some neighborhood of this point is an irreducible analytic set. This means that each mentioned component is a connected set over a deleted neighborhood of the point x^*.*

Now let us perform the analytic continuation of the considered solution along various paths in $\mathbf{C}_x \setminus \mathcal{F}$. The set of possible values of the function $S(x)$ at some point $x_0 \in \mathbf{C}_x$ obtained as a result of this continuation has the form

$$S_k(x_0) + \sum_j c_j^k \alpha_j, \tag{4}$$

where $S_k(x_0)$ are values of the function $S(x)$ corresponding to different points of the Lagrangian manifold L lying over x_0, c_j^k are integers, and α_j are periods of the form $p\,dx$ on the manifold L, that is,

$$\alpha_j = \int\limits_{\gamma_j} p\,dx,$$

and $\{\gamma_j\}$ is the basis of one-dimensional homology of the manifold L. In general, the set of points s given by formula (4) can even form a dense set in the complex plain $\mathbf{C}_s$ (this happens when the α_js are independent over the field $\mathbf{Q}$). However, the set of points (3) obtained by the analytic continuation along paths of the given length in the plane $\mathbf{C}_x$ is evidently finite.

In what follows, we shall be interested in the properties of the analytic continuation of WKB-element (3) along paths lying in a deleted neighborhood U of some focal point x^* of this element (we suppose that this neighborhood does not contain other focal points of this element). Let x_0 be some point of U. Then it is evident that analytic continuation along loops lying in U with endpoints at x_0 determines a finite number of germs of WKB-elements

$$u_j(x,h) = e^{\frac{i}{h}S_j(x)} \sum_{k=0}^{\infty} (-ih)^k a_k^{(j)}(x),\ j = 1,2,\ldots,N \tag{5}$$

at x_0 (we denoted by $u_1(x,h)$ the WKB-element given by (3)). Now it is clear that the analytic continuation along the loop l encircling the point x^* acts as a substitution on set (5) of WKB-elements.

For some point $x_0 \in U$, denote by $E_{x_0}^f$ (f = "formal") the space of linear combinations of germs of elements (5) at the point x_0

$$v \in E_{x_0}^f \Leftrightarrow v = \sum_{j=1}^{N} c_j u_j(x,h) \tag{6}$$

with complex coefficients c_j. Then the action of the loop l on the set (5) of WKB-elements induces a linear mapping on the space $E^f_{x_0}$. We denote this mapping by

$$\mathcal{M} : E^f_{x_0} \to E^f_{x_0}.$$

This mapping is an exact description of the monodromy of formal solutions to equation (1). By $\hat{M}$ we denote the matrix of this mapping with respect to the fixed basis (5). The mapping $\mathcal{M}$ we shall call the *formal monodromy*, and the matrix $\hat{M}$ is called the *formal monodromy matrix*. Clearly, the matrix $\hat{M}$ depends on the concrete choice of the basis (5).

Let us introduce one more object describing the formal monodromy of WKB-solutions to equation (1). Namely, the union of all spaces $E^f_{x_0}$ over all points $x_0 \in U$ forms a vector bundle over U; we denote this bundle by $\mathcal{M}^f_{x\bullet}$. A trivialization of $\mathcal{M}^f_{x\bullet}$ over a neighborhood of any point $x_0 \in U$ is constructed in the following way:

Let

$$\{u_1(x,h), \ldots, u_N(x,h)\}$$

be a basis of the space $E^f_{x_0}$ at the point x_0. Then $u_j(x,h)$, $j = 1, \ldots, N$ are germs of formal series of the form (5) at x_0. Denote by the same symbol $u_j(x,h)$ representatives of these germs, that is, $u_j(x,h)$ are formal series of the form (5) determined in neighborhoods U_j of the point x_0. Then the analytic functions $u_j(x,h)$ (with values in the space of formal power WKB-expansions) determine a basis at any point x of the intersection $\cap U_j$ and, hence, determine a trivialization of the bundle in question over this intersection.

This bundle can be also described with the help of gluing functions in the following way. Let $\mathcal{S}$ be a cut with origin at the point x^* such that the set $U \setminus \mathcal{S}$ is simply connected. One can choose the global basis of the form (5) over $U \setminus \mathcal{S}$ and, hence, obtain the trivialization of $\mathcal{M}^f_{x\bullet}$ over this set as it was described above. Then the matrix $\hat{M}$, considered on the above mentioned cut, is evidently the gluing function of this bundle. Since this matrix is, clearly, a constant along the cut $\mathcal{S}$, the constructed bundle is a *locally plain* one. Locally constant sections of this bundle over any open subset V of U are exactly WKB-elements on V having the form (6) at any point of V with respect to some trivialization of the above described type. These sections correspond to one and the same constants c_j at any point $x \in V$.

Examples.

1. Let us first illustrate the introduced notions on the simplest, but very important, example of the Airy equation

$$h^2 \frac{d^2 u}{dx^2} + xu = 0. \tag{7}$$

The straightforward computations show that one of the formal solution to equation (7) of WKB-form is given by

$$u_1(x,h) = x^{-1/4}\exp\left[\frac{2i}{3h}x^{3/2}\right]\sum_{n=0}^{\infty}(-ih)^n\frac{3^n\Gamma(5/6+n)}{2^{2n}n!\Gamma(5/6-n)}x^{-\frac{3n}{2}}. \tag{8}$$

The analytic continuation of element (8) along the loop l encircling the origin (it is easy to see that the origin is the only focal point of the Airy equation) takes (8) into the WKB-element

$$u_2(x,h) = -ix^{-1/4}\exp\left[-\frac{2i}{3h}x^{3/2}\right]\sum_{n=0}^{\infty}(ih)^n\frac{3^n\Gamma(5/6+n)}{2^{2n}n!\Gamma(5/6-n)}x^{-\frac{3n}{2}}. \tag{9}$$

If we perform the analytic continuation along l once more, we shall come again to element (8) but with the opposite sign. Thus, the space $E^f_{x_0}$, in this case, is a two-dimensional space with generators (8) and (9) and the formal monodromy matrix have the form

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{10}$$

The latter expression for the formal monodromy matrix shows that the bundle $\mathcal{M}^f_{x^*}$ (for which this matrix is a gluing function) is a nontrivial two-dimensional bundle over a deleted neighborhood of the origin $x^* = 0$. The nontriviality of this bundle can be understood from the fact that matrix (10) changes the orientation of the fiber.

2. Our second example is the equation for cylinder-parabolic functions (Weber functions):

$$h^2\frac{d^2u}{dx^2} + x^2u = 0.$$

We remark that the second of the above requirements on the Lagrangian manifold L *is not fulfilled* for the Weber equation. Actually, this manifold is described by

$$-p^2 + x^2 = 0$$

and, hence, it splits into two irreducible components over any neighborhood of the origin (which is the only focal point of this equation). In the next subsection we shall see, to which effect this fact leads. The two WKB-elements which are formal solutions to this equation are:

$$u_{1,2}(x,h) = x^{-1/2}\exp\left[\pm\frac{i}{2h}x^2\right]\sum_{n=0}^{\infty}(\pm ih)\frac{\Gamma(3/4+n)}{n!\Gamma(3/4-n)}x^{-2n}. \tag{11}$$

On the contrary to the previous example, these two formal solutions cannot be obtained from each other with the help of the analytic continuation along the loop l encircling the origin (similar to the previous example, the only focal point for the considered equation is the origin).

It is easy to see that, during the analytic continuation along l, both u_1 and u_2 change their sign. Hence, the space E_{x_0} is, again, a two-dimensional linear space, and the formal monodromy matrix is

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The reader can see that the bundle $\mathcal{M}^f_{x^*}$, in this case, is a direct sum of the two Möbius bundles, that is, one-dimensional bundles for which the orientation of a fiber changes along the loop encircling the origin $x^* = 0$.

2 Real monodromy

In this section, we shall try to construct a univalued exact asymptotic solutions to equation (1) with the help of resummation of the formal WKB-solutions considered in the previous section. To do this, we suppose that formal series (5) determine resurgent functions, that is, that the formal Borel transforms

$$U_j(x,s) = \mathcal{B}_f\left[u_j(x,h)\right]$$

in the variable $1/h$ can be analytically continued up to *endlessly continuable functions* [1], [3], [4] in variable s for each $x \in U$. Moreover, we assume[1] that *the support of the obtained resurgent functions consists of the set of values $\{S_1(x), \dots, S_N(x)\}$ for any given value of $x \in U$ and that all these functions coincide with one another at the point x^*: $S_j(x^*) = S_k(x^*)$ for any j, k.*

Consider N functions

$$\tilde{u}_j(x,h) = \sigma\left[e^{\frac{i}{h}S_j(x)} \sum_{k=0}^{\infty} (-ih)^k a_k^{(j)}(x)\right], \; j = 1, 2, \dots, N, \tag{12}$$

where σ is a *resummation operator* (see [1], [3], [4]). For any point $x_0 \in U$, we denote by $E^r_{x_0}$ ($r =$ "real") the space of linear combinations of germs of functions (12)

$$\tilde{v} \in E^r_{x_0} \;\Leftrightarrow\; \tilde{v} = \sum_{j=1}^{N} c_j \tilde{u}_j(x,h) \tag{13}$$

[1] This condition is rather strong, but it can be weakend in such a way that the corresponding theory can be applicable to most of problems of asymptotic theory of differential equations. Such generalization will be described in what follows.

with complex coefficients c_j. We emphasize that, in spite of the fact that the definition of the resummation operator σ is ambiguous at points of the Stokes set $\mathcal{S}$, the space $E^r_{x_0}$ is determined uniquely at such points.

Due to the above assumptions, the analytic continuations of elements from $E^r_{x_0}$ along paths in U from x_0 to any point $x \in U$ belong to the space E^r_x. As above, we see that the union of spaces $E^r_{x_0}$ over all points $x_0 \in U$ forms a locally plain bundle over U; we denote this bundle by $\mathcal{M}^r_{x^\bullet}$. Locally constant sections of this bundle over any open subset V of U are exactly resurgent functions on V having the form (13) at any point of V. However, due to the Stokes phenomenon, *locally constant sections of this bundle are not determined by constant coefficients c_j in the whole set V*. In other words, representation (13) does not determine a trivialization of the bundle $\mathcal{M}^r_{x^\bullet}$ over the neighborhood of any point $x \in U$.

Let us consider the situation in more detail.

If $V \subset U$ is an open subset in U such that V lies as a whole in some Stokes region of the considered equation, then the function

$$\tilde{v}\,(x,h) = \sum_{j=1}^{N} c_j \tilde{u}_j\,(x,h) \tag{14}$$

determines an analytic function in V for any choice of constants c_j. Quite opposite situation takes place if the set V is divided into (say) two parts V_1 and V_2 by the Stokes set $\mathcal{S}$. In this case, the function given by (14) has a jump on the set $\mathcal{S}$ and, hence, does not determine an analytic function in the whole set V. To construct an analytic function $\tilde{u}$ on the whole V which has the form (14) at each point of $V \setminus \mathcal{S}$, one should consider two functions of the form (14)

$$\tilde{v}_1\,(x,h) = \sum_{j=1}^{N} c_j^{(1)} \tilde{u}_j\,(x,h) \tag{15}$$

and

$$\tilde{v}_2\,(x,h) = \sum_{j=1}^{N} c_j^{(2)} \tilde{u}_j\,(x,h) \tag{16}$$

in V_1 and V_2, correspondingly, such that functions (15) and (16) are tied together with the help of the connection homomorphism τ (see [1]). This observation gives rise to the following description of the bundle $\mathcal{M}^r_{x^\bullet}$:

Denote by $U_1, \ldots, U_M$ the connected components of the set $U \setminus \mathcal{S}$ (in other words, $\mathcal{S}$ divides U into the connected components U_k, $k = 1, \ldots, M$). Then locally constant sections of the bundle $\mathcal{M}^r_{x^\bullet}$ over any open set $V \subset U$ can be described as the tuple of

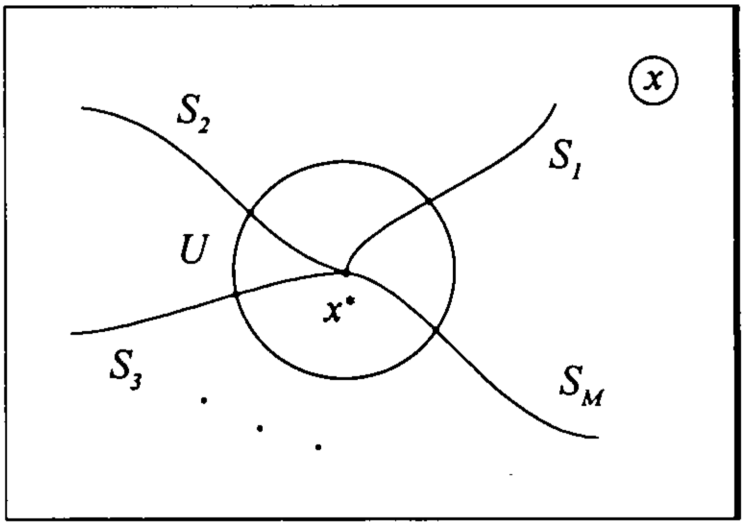

Figure 1: Geometry of the Stokes lines.

functions

$$\tilde{v}_k(x,h) = \sum_{j=1}^{N} c_j^{(k)} \tilde{u}_j(x,h), \; k = 1,\ldots,M$$

given in each intersection $V \cap U_k$ such that any pair of these functions $\tilde{v}_{k'}$, $\tilde{v}_{k''}$ are taught on the component $U_{k'} \cap U_{k''}$ of the Stokes set S by the corresponding connection homomorphism$\mathcal{T}$.

This description can be written down also in the "coordinate" form. To do this, we denote by $\mathcal{S}_1,\ldots,\mathcal{S}_M$ the connected components of the Stokes set (see Figure 1) and choose the component $\mathcal{S}_1$ as the cut mentioned in the previous section. Then the set $U \setminus \mathcal{S}_1$ is simply connected, and one can enumerate the WKB-elements $u_j(x,h)$ in one and the same way at all points of this set. Now, elements of fibers of the bundle $\mathcal{M}^{f}_{x^\bullet}$ can be identified with N-vectors $\bar{c} = (c_1,\ldots,c_N)$ in accordance to formula (6), and the action of the connection homomorphism τ on each $\mathcal{S}_j$ can be described with the help of a Stokes matrix

$$\hat{S}_j = \begin{pmatrix} S_{11}^{(j)} & S_{12}^{(j)} & \ldots & S_{1N}^{(j)} \\ S_{21}^{(j)} & S_{22}^{(j)} & \ldots & S_{2N}^{(j)} \\ \ldots & \ldots & \ldots & \ldots \\ S_{N1}^{(j)} & S_{N2}^{(j)} & \ldots & S_{NN}^{(j)} \end{pmatrix}.$$

Let us consider the elements $S_{kl}^{(j)}$ of the Stokes matrix. As it is known from the resurgent functions theory, at points of the Stokes set S, the connection homomorphism takes any WKB-element $u_j(x,h)$ into the sum of this WKB-element and the linear combination of the elements corresponding to singularities of continuation of the Borel transform $\mathcal{B}^f[u_j(x,h)]$ lying on the ray emanated from the point of support of $u_j(x,h)$ in the direction of the positive imaginary axis. Due to the above assumptions, all these singularities lie at points $s = S_k(x)$ determined by different values of the action function $S(x)$. From the other hand, if all the roots of the Hamilton-Jacobi equation are simple, then each formal solution of equation (1) corresponding to the action $S_k(x)$ equals the corresponding WKB-element $u_k(x,h)$ multiplied by some resurgent function of the variable h which does not depend on x. Thus, the elements of the matrix $\hat{S}_j$ are, in general, resurgent functions in h.

However, if the WKB-elements $u_j(x,h)$ are constructed as analytic continuations of one and the same WKB-element (it is possible, of course, only in the case when the corresponding Lagrangian manifold is connected over the deleted neighborhood of the point x^*), then WKB-elements determined by the analytic continuation of the Borel transform $\mathcal{B}^f[u_j(x,h)]$ coincide (up to the sign) with the WKB-elements $u_k(x,h)$ supported at the corresponding points $S_k(x)$. In this case, the elements $S^{(j)}_{kl}$ of the Stokes matrix are integers.

Thus, under the above assumptions, the bundle $\mathcal{M}^r_{x^*}$ can be described as a bundle corresponding to the covering $\{U_1,\dots,U_N\}$ with gluing functions equal to $\hat{S}_j$ on any component $\mathcal{S}_j$, $j=2,\dots,N$ and to $\hat{S}_1\hat{M}^{-1}$ on the component $\mathcal{S}_1$ of the Stokes set $\mathcal{S}$.

The aim of this section is to formulate the conditions implied on the matrices $\hat{S}_j$, $j = 1,\dots,N$ and $\hat{M}$ by the fact that equation (1) has only holomorphic solutions. This means that any element of the form (13) can be continued up to a holomorphic function in a (non deleted) neighborhood of the focal point x^*. The following affirmation is, in essence, the direct subsequence of the definition of the bundle $\mathcal{M}^r_{x^*}$:

Theorem 1 *If elements* (5) *are constructed as formal WKB-solutions to equation* (1), *then the bundle $\mathcal{M}^r_{x^*}$ is trivial.*

The above coordinate description of the bundle $\mathcal{M}^r_{x^*}$ allows us to reformulate the condition of triviality of the bundle $\mathcal{M}^r_{x^*}$ in terms of the formal monodromy matrix and the Stokes matrices.

Theorem 2 *The bundle $\mathcal{M}^r_{x^*}$ is trivial if and only if the relation*

$$\hat{S}_M\hat{S}_{M-1}\dots\hat{S}_1 = \hat{M} \tag{17}$$

is valid.

In the examples below we shall show what information about the elements of the Stokes matrices can be obtained from relation (17), provided that the formal monodromy matrix $\hat{M}$ is known.

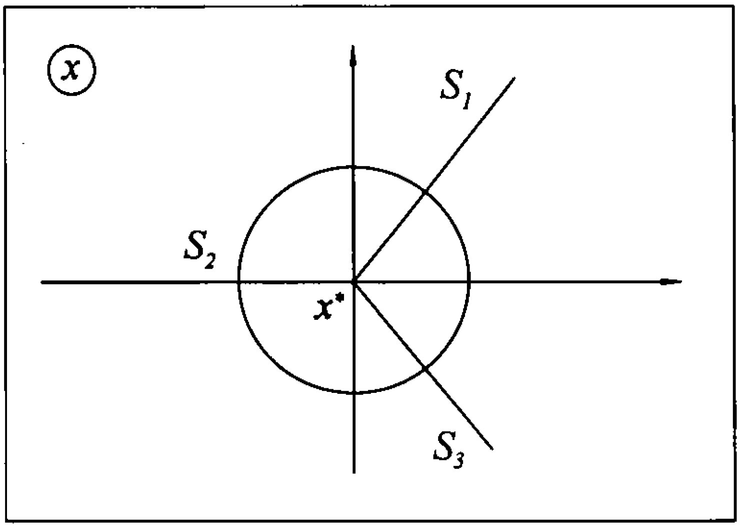

Figure 2: Stokes lines for the Airy equation.

Examples.

1. Let us continue consideration of the Airy equation. As it was already mentioned, the action function for this equation is

$$S(x) = \frac{2}{3}x^{3/2}.$$

Denote by $S_\pm$ the two branches of this function: $S_\pm(x) = \pm 2/3x^{3/2}$. Then, the Stokes lines are given by $\operatorname{Re} S_+(x) = \operatorname{Re} S_-(x)$, or

$$\begin{aligned} \mathcal{S}_1 &= \{\arg x = \pi/3\}, \\ \mathcal{S}_2 &= \{\arg x = \pi\}, \\ \mathcal{S}_3 &= \{\arg x = 5\pi/3\} \end{aligned}$$

(see Figure 2). Since at points of the Stokes line $\mathcal{S}_1$ the component $u_1(x,h)$ is a recessive one, the Stokes matrix on this line has the form

$$\begin{pmatrix} 1 & A_1 \\ 0 & 1 \end{pmatrix}$$

with some unknown A_1. Similar, Stokes matrices at points of the Stokes lines $\mathcal{S}_2$ and $\mathcal{S}_3$ are

$$\begin{pmatrix} 1 & 0 \\ A_2 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & A_3 \\ 0 & 1 \end{pmatrix},$$

respectively. Hence, condition (17) reads

$$\begin{pmatrix} 1 & A_3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ A_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & A_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(see the computation of the formal monodromy matrix in the first example of the previous section). This provides us with the following four equations for the unknowns A_1, A_2, and A_3:

$$\begin{cases} 1 + A_2A_3 = 0, \\ A_1 + A_3 + A_1A_2A_3 = 1, \\ A_2 = -1, \\ 1 + A_1A_2 = 0. \end{cases}$$

Evidently, we have $A_1 = 1$, $A_2 = -1$, $A_3 = 1$ and, hence, all the three Stokes matrices in question can be determined from condition (17). We remark that, in this case, the elements of the Stokes matrices are integers, as it was already mentioned above.

2. Let us turn our mind to the consideration of the Weber equation. In this case, we have

$$S_\pm(x) = \pm\frac{x^2}{2},$$

and, hence, the equations of the Stokes lines are

$$\begin{aligned} \mathcal{S}_1 &= \{\arg x = \pi/4\}, \\ \mathcal{S}_2 &= \{\arg x = 3\pi/4\}, \\ \mathcal{S}_3 &= \{\arg x = 5\pi/4\}, \\ \mathcal{S}_4 &= \{\arg x = 7\pi/4\} \end{aligned}$$

(see Figure 3). Similar to the previous example, the Stokes matrices corresponding to these Stokes lines have the form

$$\begin{pmatrix} 1 & A_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ A_2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & A_3 \\ 0 & 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & 0 \\ A_4 & 1 \end{pmatrix},$$

correspondingly. Hence, condition (17) is written down in the form

$$\begin{pmatrix} 1 & 0 \\ A_4 & 1 \end{pmatrix} \begin{pmatrix} 1 & A_3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ A_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & A_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

(the formal monodromy matrix was computed in the second example of the previous section). Thus, we obtain the following four equations:

$$\begin{cases} 1 + A_2A_3 = -1, \\ A_1 + A_3 + A_1A_2A_3 = 0, \\ A_2 + A_4 + A_2A_3A_4 = 0, \\ 1 + A_1A_4 + A_1A_2 + A_3A_4 + A_1A_2A_3A_4 = -1. \end{cases} \tag{18}$$

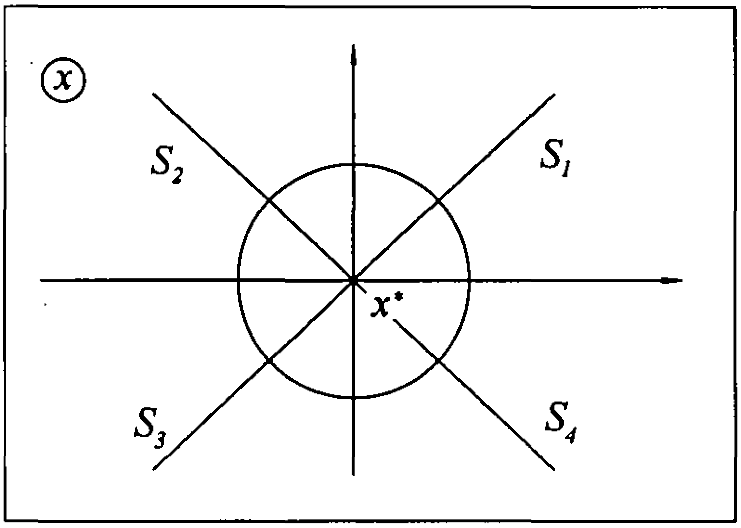

Figure 3: Stokes lines for the Weber equation.

From these equations, we obtain relations for elements A_j of the Stokes matrices:

$$\begin{cases} A_1 = A_3, \\ A_2 = A_4, \\ A_1 A_2 = -2, \end{cases}$$

which are nesessary and sufficient conditions for system (18) to be valid. One can see that, in the case of the Weber equation, the elements of the Stokes matrices are not uniquely determined by relation (17). This happens due to the fact that the corresponding Lagrangian manifold is not a connected one and, hence, the WKB-elements (11) are not uniquely determined by each other. From the other hand, the multiplication of one of this elements by some number (or even by a resurgent function of the variable h, independent of x) leads to the multiplication of the numbers A_1 and A_2 (and, simultaneously, A_3 and A_4) by mutually inverse numbers, so that the ambiguity in the choice of the elements of the Stokes matrices, for Weber equation, is quite natural.

3 General case

In all the above considerations, we have supposed that all values of the action function take part in forming the Stokes lines in a neighborhood of the given focal point x^*. In particular, this assumption leads to the fact that the action function $S(x)$ is a finitevalued one in a neighborhood of x^* and, hence, all the above considered bundles have finite-dimensional fibers. However, in problems of real interest in the mathematical physics, the function $S(x)$ has infinite number of values (see, for example, [7], [8]). In spite of this fact, one can avoid the consideration of the bundles with infinite-dimensional fibers due to the following reasons:

1. Certainly, it is a priori possible that for some point $x^* \in S$ there exist an infinite number of vertical lines in the space $\mathbf{C}_s$ such that each of these lines contains at least two points of singularity of the Borel transform of the considered WKB-element. However, in this case the corresponding Stokes matrices split into the direct sum of matrices corresponding to each of these lines, and, hence, all the problem reduces to the investigation of the Stokes matrices corresponding to one vertical line of the mentioned type.

2. It is also possible that, for $x^* \in \mathbf{C}_s$, the corresponding vertical line in $\mathbf{C}_s$ contains the infinite number of values of the action function $S(x)$. However, the set of points of singularity of the formal Borel transform of the considered WKB-element visible from its support is a discrete set on this line. So, if one considers the asymptotic expansions up to terms of finite exponential order $O\left(e^{-c/h}\right)$ rather than up to rapidly decreasing functions (that is, functions of order $O\left(e^{-c/h}\right)$ with *arbitrary* value of c), the problem reduces to that with finite number of values of action $S(x)$ and we arrive again to considerations of finite-dimensional bundles.

Clearly, the factorization by the space of functions of order $O\left(e^{-c/h}\right)$ with reasonable value of the constant c allows one to compute only the elements of the Stokes matrix corresponding to those values $S_k(x)$ of the action which coinside with the support of the initial WKB-element at the considered focal point x^* (if we factorize by functions of exponential decrease with larger values of c, then the corresponding part of the Lagrangian manifold can become a disconnected one; then one cannot find exact values of the Stokes matrices due to the reasons similar to those explained in the example 2 above). However, with the help of analytic continuation, one can overcome this difficulty (see, for example, [8]).

4 Concluding remarks

Let us formulate the above considerations in the form of the following algorythm of constructing exact semi-classical approximations of solutions to differential equations.

1. Formal WKB-solutions to the equation in question are constructed.
2. The formal monodromy operator is calculated.
3. Stokes domains and the form of Stokes matrices on each component of the Stokes

set are determined.

4. Formal solutions are resummated in each Stokes domain with the help of the Borel resummation method.

5. The elements of (finite) Stokes matrices are computed with the help of the condition of univaluedness of solutions in a neighborhood of the set of focal points. The solvability of the corresponding system of algebraic equations is the consequence of univaluedness of solutions to the original problem.

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