Higgs Fields and Harmonic Maps

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Claim 3.10. Let $\tau : Y' \longrightarrow Y$ be either a finite cover or a blowing up. Let $g' : Z' \longrightarrow Y'$ be a desingularization of X'. Then we have an inclusion $\rho : g'_* \omega^{\upsilon}_{Z'/Y'} \longrightarrow \tau^* g_* \omega^{\upsilon}_{Z/Y}$ isomorphic over Y'_0.

Proof. The existence of ρ has been shown in [17] 1.8 and [18], 3.2. ρ is an isomorphism over Y'_0 by 3.9.

For $\upsilon = 1$ 3.7 follows from Kawamata's positivity theorem ([8] or [11]). It says that $g_* \omega_{Z/Y}$ is weakly positive over Y, if there exists some $U \subseteq Y$ such that:

i) $Y - U$ is a normal crossing divisor.
ii) $g^{-1}(U) \longrightarrow U$ smooth
iii) For $k = \dim Z - \dim Y$ the monodromy of $R^k g_* \mathbb{C}_{g^{-1}(U)}$ around the components of $Y - U$ is unipotent.

Those three conditions hold if one replaces Y by a finite cover of a blowing up, and 3.7 follows from 3.10 and 3.4.e.

For $\upsilon > 1$ we have to argue as in [18] §5:

Claim 3.11. Assume that $S^{\mu}(f_* \omega^{\upsilon}_{X/Y} \otimes \mathcal{H}^{\upsilon})$ is globally generated over Y_0 for some $\mu >> 0$. Then $f_* \omega^{\upsilon}_{X/Y} \otimes \mathcal{H}^{\upsilon - 1}$ is weakly positive

Abstract. Yang-Mills-Higgs fields over $\mathbf{R}^3$ satisfying decay conditions yield harmonic (or holomorphic) maps of the 2-sphere at infinity to complex homogeneous manifolds.

1. A smooth connection A and an additional smooth field Φ, a Lie algebra valued Higgs field, give together a gauge invariant field in the three dimensional Yang-Mills gauge theory. For a configuration (A, Φ) the action is defined as $\mathcal{A} = \int_{\mathbf{R}^3}\{|F_A|^2 + |D_A\Phi|^2\}dx$, for F_A, the curvature of A and $D_A\Phi = d\Phi + [A, \Phi]$, the covariant derivative.

The Euler-Lagrange equations of the action are

$$d_A(*F_A) + [\Phi, *D_A\Phi] = 0, \tag{1.1}$$

$$\Delta_A\Phi = 0, \Delta_A = - * d_A * D_A \tag{1.2}$$

(d_A is the covariant exterior derivative and $*$ is the Hodge operator) whose solution is called a Yang-Mills-Higgs field.

We impose the asymptotical condition on a Higgs field, $|\Phi| \to m$, constant > 0 as $r = |x|$ tends to ∞. Here the norm is measured by an adjoint invariant inner product. The value m is called the mass of (A, Φ).

For gauge group $G = SU(2)$ then the normalized Higgs field $|\Phi|^{-1}\Phi$: $S_r^2 = \{x \in \mathbf{R}^3; |x| = r\} \to S = \{X \in su(2); |X| = 1\}$ defines the mapping degree, a topological invariant, which we call (monopole) charge of (A, Φ).

We impose also other decay restriction on the fields $|F_A|$,$|D_A\Phi| = O(1/r^2)$ to ensure that the action is finite.

Amongst other Yang-Mills-Higgs fields there are particular configurations, solutions of the Bogomolnyi equations $F_A = \pm * D_A$. A solution to the equations, called a (magnetic) monopole, minimizes the action in a topological sense. The action has indeed the absolute lower bound; $\mathcal{A} \geq 8\pi k$ in the space of charge $k > 0$ configurations and $\mathcal{A} = 8\pi k$ if and only if a configuration is a charge k monopole.

So, the three dimensional Yang-Mills-Higgs theory would mostly likely correspond to the four dimensional Yang-Mills theory([Itoh 15]). From

Manton's observation this correspondence is actually settled in such a way that a time-direction invariant Yang-Mills field (or instanton) $A = \sum A_i dx^i$ on $\mathbf{R}^4$ is nothing but a Yang-Mills-Higgs field (or monopole) $(\mathbf{A}, \Phi), \mathbf{A} = A_2 dx^2 + A_3 dx^3 + A_4 dx^4, \Phi = A_1$ on $\mathbf{R}^3$, the one dimensional reduction of $\mathbf{R}^4$ ([Manton 19]). S^1 invariant Yang-Mills fields are also interpreted by the conformal invariance as Yang-Mills-Higgs fields on a hyperbolic three space([Atiyah 1],[Braam 3]).

As for the instanton case the twistor method was applied by Hitchin to monopoles to transfer them into holomorphic vector bundles over the space $G(\mathbf{R}^3)$ of all oriented geodesics in $\mathbf{R}^3$ and monopoles of charge k of some decay conditions are characterized as spectral curves in the complex surface $G(\mathbf{R}^3)$ ([Hitchin 10]).

On the other hand, following the Nahm's equations, the time-direction invariant version of the ADHM instanton construction, Donaldson showed that the space of charge k monopoles modulo gauge equivalence is isomorphic to the space of holomorphic maps $f : P_1(\mathbf{C}) \to P_1(\mathbf{C})$, $\deg f = k$, $f(\infty) = 0$([Nahm 21],[Donaldson 5]).

Every holomorphic map and anti-holomorphic map are harmonic with respect to a Hermitian structure on $P_1(\mathbf{C})$ and vice versa([Eells and Wood 6]). So Donaldson's theorem suggests that there be also a close link between Yang-Mills-Higgs theory and theory of harmonic maps on a surface, namely a charge k Yang-Mills-Higgs field to a harmonic map of degree k and a monopole to holomorphic map of corresponding degree.

We would like to give a direct representation of Yang-Mills-Higgs fields into harmonic maps by taking the limit of Higgs field Φ at infinity. Actually Φ is over $\mathbf{R}^3$ a solution of the Laplace equations (1.2) in the presence of the connection A ,and Δ_A has the polar coordinate expression $\Delta_A(.) = -D_r D_r(.) - 2/r D_r(.) + 1/r^2 \Delta_A^r$ for the Laplacian Δ_A^r on the 2-sphere S_r^2 of radius r. So, the limit Φ_∞ gives rise to a harmonic map $S_1^2 = S_\infty^2 \to S = \{X \in su(2); |X| = m\}$ provided some decay rate conditions on A and Φ are satisfied, as is shown in Proposition 1 in section 2.

Charge one monopoles are explicitly described as Prasad-Sommerfield monopole and its Euclidean parallel translations. This PS monopole is spherically symmetric. See for its exact form (2.8) in section 2. Our investigation is guided by the PS monopole for lack of knowledge of decay condition except for field strength decay estimate ([Jaffe and Taubes 17], [Taubes 26] and [Hurtubise 13]).

Our approach might be inside the framework of Hitchin since the 2-sphere at infinity is considered as a subspace P_o in $G(\mathbf{R}^3)$, oriented geodesics through the origin, and to each monopole is associated a configuration of P_o and the spectral curve induced from the monopole.

However our method of taking the Higgs field at infinity is visualized in geometrical sense and is generalized with no difficulty for arbitrary compact simple gauge group G with a Higgs field of some symmetry breaking. The Higgs field Φ_∞ at infinity lies in a hypersphere of $\mathcal{G}$, the Lie algebra of G, as for $G = SU(2)$. The symmetry breaking is then stated as that the image of Φ_∞, $\{\Phi_\infty(\hat{x}); \hat{x} \in S_1^2\}$ sits inside an orbit under the $G-$adjoint action.

So Φ_∞ induces a map $S_1^2 \to G/K \subset \mathcal{G}$ for the isotropy subgroup K at some $\Phi_\infty(\hat{x}_o)$. The homogeneous space G/K admits an invariant complex structure and carries invariant Kähler metrics g as explained in section 3([Itoh 14],[Atiyah 1]).

The homogeneous space G/K being a submanifold of the Euclidean space $\mathcal{G}$ is equipped with the induced invariant metric g_1, Hermitian with respect to the complex structure.

For the simple $G = SU(2)$ case the hypersphere in $\mathcal{G}$, for example, the unit sphere consists of a single adjoint orbit with the complex structure, $\mathrm{ad}(X)$, for each point X and those metrics coincide.

As we will see in Theorem 2, the Higgs fields for general group G yield at infinity harmonic maps from the Riemann sphere S_1^2 to the homogeneous spaces $(G/K, g_1)$ under some decay conditions.

$G-$adjoint orbits in the Lie algebra are generically generalized flag manifolds G/T for maximal tori T associated to elements inside the positive Weyl chamber, contrary to Hermitian symmetric spaces of compact type appearing as "singular" orbits.

Because of the Prasad-Sommerfield limit on a Higgs field Φ, $|\Phi| \to m(r \to \infty)$, the order reduction phenomenon is observed of the Laplace equations to first order equations.

The Laplace equations on Φ reduce at infinity to the first order equations $D_{A_\infty}\Phi_\infty = 0$ for the connection A_∞ defined at infinity, or the Lax type equations $d\Phi_\infty = -[A_\infty, \Phi_\infty]$.

The Lax type equations are key equations from which we can assert for a Yang-Mills-Higgs field that $\Phi_\infty : S_1^2 \to (G/K, g_1)$ is harmonic and further holomorphic, as is shown in Theorem 4 in section 4 and Remark following Theorem 4. This fact shows us that contrary to the Donaldson's correspondence Yang-Mills-Higgs fields yield holomorphic maps. We note that Φ_∞ gives rise to a harmonic map with respect to any Kähler metric, especially to an invariant Kähler metric on G/K.

The $SU(2)$ case is discussed in section 2 and general compact simple group case in section 3. In section 4 the first order equations are derived from the Prasad-Sommerfield limit on the mass.

For general references for this paper the reader sees [Atiyah and Hitchin 2],[Hitchin 10],[Jaffe and Taubes 17] and [Eells and Lemaire 8].

The results obtained in this paper are mainly resumed in [Itoh and Manabe 16].

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2. Higgs field at infinity.

First we investigate the $SU(2)$ gauge group case. Let (A,Φ) be an $SU(2)$ Yang-Mills-Higgs field satisfying the appropriate decay conditions at infinity

$$|\Phi| = m + O(1/r),$$
$$|F_A|, |D_A\Phi| = O(1/r^2) \tag{2.1}$$

for the mass $m > 0$.

We fix a gauge so that the radial component $A_r = A(\partial/\partial r)$ of the connection A vanishes on $\mathbf{R}^3\setminus\{o\}$. Such a radial gauge trivialization always exists by using the parallel transport. There is ambiguity in fixing such a gauge.

Define the limit of the Higgs field Φ by $\Phi_\infty : S_1^2 \to su(2)$, $\Phi_\infty(\hat{x}) = \lim_{t\to\infty}\Phi(t\hat{x})$, $\hat{x} \in S_1^2$. From the decay of the field $|D_A\Phi|$ in (2.1), Φ_∞ is at least C^0. In what follows, however we assume Φ_∞ is of C^2 class.

Since Φ is a solution of the equations $\Delta_A\Phi = 0$ on $\mathbf{R}^3$, the restriction of Φ to S_r^2 satisfies

$$r^{-2}\Delta_A^r(\Phi_{|S_r^2}) = \partial_r^2\Phi + 2r^{-1}\partial_r\Phi, \tag{2.2}$$

for the spherical Laplacian Δ_A^r on S_r^2.

PROPOSITION 1([Manabe 18]). *Let (A,Φ) be an $SU(2)$ Yang-Mills-Higgs field with decay conditions (2.1). If, with a chosen radial gauge trivialization, (A,Φ) satisfies the asymptotical conditions*

$$[[\mathrm{div}A(x), \Phi(x)], \Phi(x)] = o(1/r^2), \tag{2.3}$$

$$\sum_{i=1}^{3}[[A_i(x), [A_i(x), \Phi(x)]], \Phi(x)] = o(1/r^2), \tag{2.4}$$

$\mathrm{div}A = \sum \partial/\partial x^i A_i$, *the divergence, then the Higgs field Φ_∞ at infinity is a charge k harmonic map : $S_1^2 \to S^2 = \{X \in su(2); |X| = m\}$, where k is the charge of (A,Φ).*

PROOF. The map Φ_∞ is considered as a limit of maps $\{\Phi_t : S_1^2 \to su(2)\}$, $\Phi_t(\hat{x}) = \Phi(t\hat{x}), t > 0$ and we see with respect to the ordinary Laplacian

$$\Delta^1\Phi_\infty(\hat{x}) = \lim_{t\to\infty}\Delta^1\Phi_t(\hat{x}) = \lim_{t\to\infty} t^2(\Delta^t\Phi_{|S_t^2})(t\hat{x}). \tag{2.5}$$

On S_t^2 the metric is $t^2(d\theta^2 + \sin^2\theta d\varphi^2)$ so that for $\Psi : S_t^2 \to su(2)$

$$\Delta_A^t \Psi = -1/t^2(D_\theta D_\theta \Psi + \sin^{-2}\theta D_\varphi D_\varphi \Psi).$$

Because of $A_r = 1/r \sum x^i A_i = 0$ this is written as

$$\Delta_A^t \Psi = \Delta^t \Psi - [\mathrm{div} A_{|S_t^2}, \Psi] - \sum [A_i, [A_i, \Psi]]. \tag{2.6}$$

Thus we have for the Higgs field Φ

$$\Delta^t \Phi(x) = [\mathrm{div} A(x), \Phi(x)] + \sum [A_i(x), [A_i(x), \Phi(x)]],$$

$x \in S_t^2$ and hence

$$[\Delta^t \Phi(x), \Phi(x)] = [[\mathrm{div} A(x), \Phi(x)], \Phi(x)] + \sum [[A_i(x), [A_i(x), \Phi(x)]], \Phi(x)]$$

so that from (2.3) and (2.4)

$$[\Delta^1 \Phi_\infty(\hat{x}), \Phi_\infty(\hat{x})] = 0, \tag{2.7}$$

which implies that $\Delta^1 \Phi_\infty$ is orthogonal to $S^2 \subset su(2)$, namely, Φ_∞ is harmonic.

REMARK 1. PS monopole (A, Φ) is the charge one spherically symmetric monopole([Prasad and Sommerfield 22]);

$$A(x) = (1/\sinh r - 1/r)1/r(b_1 i + b_2 j + b_3 k),$$

$$\Phi(x) = -(1/\tanh r - 1/r)1/r(x^1 i + x^2 j + x^3 k), \tag{2.8}$$

$b_1 = x^2 dx^3 - x^3 dx^2, b_2 = x^3 dx^1 - x^1 dx^3, b_3 = x^1 dx^2 - x^2 dx^1$. $\{i, j, k\}$ is the standard basis of $su(2)$. The PS monopole is fixed in a radial gauge and because of the symmetry $\mathrm{div} A = 0$ holds and it satisfies the decay condition (2.4). In fact $\sum [A_i, [A_i, \Phi]] = f(r)\Phi$ for some scalar field $f(r)$ of order $o(1/r^2)$. From the proposition the Higgs field Φ_∞ at infinity must be harmonic. The harmonicity is also seen directly since Φ_∞ is in the PS monopole case the identity map : $S_1^2 \to S^2$ up to constant. By parallel transport the PS monopole exhausts all monopoles of charge one.

REMARK 2. The decay order of the divergence $\mathrm{div} A$ is estimated as follows. For a Yang-Mills-Higgs field with the decay conditions (2.1) the estimate is

$$\mathrm{div} A = O((r^{-1} \log r)^2), \tag{2.9}$$

weaker than (2.3). The half part of the Euler-Lagrange equations $d_A(*F_A) = -[\Phi, *D_A\Phi]$ reduces to

$$\partial/\partial(r^2 \mathrm{div} A) = r^2[\Phi, \partial_r\Phi] + [A_\theta, F_{\theta r}] + \sin^{-2}\theta[A_\varphi, F_{\varphi r}]. \tag{2.10}$$

Integrating this we have (2.9), since $[\Phi, \partial_r\Phi] = O(1/r^2)$ and $F_{\theta r}, F_{\varphi r} = O(1/r)$, and the angular components A_θ, A_φ are at least $O(\log r)$.

<u>Claim 3.9.</u> Let Y be normal, Y' non singular and $\tau : Y' \longrightarrow Y$ a projective generically finite morphism. Assume that 3.7 holds for f'. Then the base change map ([6], III, 9.3.1) $\rho : \tau^* f_* \omega^{\nu}_{X/Y} \longrightarrow f'_* \omega^{\nu}_{X'/Y'}$ is an isomorphism over Y'_0. Moreover 3.7 holds for f.

<u>Proof.</u> Since $f_* \omega^{\nu}_{X/Y}$ is locally free and τ generically flat ρ is injective. If ρ were not surjective over Y'_0 we could find some effective divisor F meeting Y'_0 such that $\tau^* \det(f_* \omega^{\nu}_{X/Y}) \otimes \mathcal{O}_{Y'}(F) = \det(f'_* \omega^{\nu}_{X'/Y'})$. Since F must be an exceptional divisor this contradicts the weak positivity of $\det(f'_* \omega^{\nu}_{X'/Y'})$ over Y'_0. In order to see that 3.7 holds for f we just remark that $f_* \omega^{\nu}_{X/Y}$ is a direct summand of $\tau_* \tau^* f_* \omega^{\nu}_{X/Y} = \tau_* f'_* \omega^{\nu}_{X'/Y'}$. The weak positivity of $f_* \omega^{\nu}_{X/Y}$ over Y_0 follows as in 3.4.e.

Due to 3.8 and 3.9 we may assume Y to be non singular. Moreover, whenever it is convenient, we may replace Y by a generically finite cover. Let $\delta : Z \longrightarrow X$ be a desingularization and $g = f \circ \delta : Z \longrightarrow Y$. Since $f^{-1}(Y_0)$ has rational Gorenstein singularities $\delta_* \omega^{\nu}_Z \longrightarrow \omega^{\nu}_X$ is an isomorphism over $f^{-1}(Y_0)$ and $g_* \omega^{\nu}_{Z/Y} \longrightarrow f_* \omega^{\nu}_{X/Y}$ is an isomorphism over Y_0. g is no longer flat. Nevertheless we get:

3. Symmetry breaking ansatz.

The proposition in section 2 is generalized for an arbitrary compact simple Lie group.

Let G be a compact simple Lie group with Lie algebra $\mathcal{G}$.

Suppose that (A, Φ) is a Yang-Mills-Higgs field of gauge group G. The norm $|\Phi|$ is assumed as for $SU(2)$ to tend to a constant, the mass, $m > 0$ as $r \to \infty$. The norm $|.|$ is measured by a normalized Killing form $-cB$, $c > 0$. The image of the Higgs field Φ_∞ at infinity is further assumed to lie in a $G-$adjoint orbit $\alpha = \{Ad(g)\Phi_\infty(1,0,0); g \in G\}$.

Because of this symmetry breaking ansatz the orbit is written as a homogeneous space G/K through the adjoint representation, where the closed subgroup K is the isotropy at $\Phi_\infty(1,0,0)$.

With no difficulty K is assumed connected, since otherwise we can consider the universal covering G/K_o of G/K (K_o is the identity component of K) ([Taubes 26]).

Before discussing the harmonicity of Φ_∞ we recall invariant geometrical structures which the space G/K carries, i.e., the invariant complex structure and invariant Kähler metrics.

For this we need some knowledge of compact semi-simple Lie algebras ([Humphreys12]).

Let

$$\mathcal{G}^{\mathbf{C}} = \mathcal{H} \oplus \sum_{\alpha \in \Delta^+} \mathbf{C}E_\alpha \oplus \mathbf{C}E_{-\alpha}$$

be the root space decomposition of the complexification $\mathcal{G}^{\mathbf{C}}$ of $\mathcal{G}$ associated to some Weyl basis $\{H_i, E_\alpha, E_{-\alpha}\}$. Δ^+ is the set of all positive roots.

The point $X = \Phi_\infty(1,0,0)$ in the orbit α is assumed in the real Cartan subalgebra $\mathcal{H}_{\mathbf{R}}$, the Lie algebra of a maxial torus. By using the Weyl group argument X can be further assumed in a Weyl chamber as $X = \sum_{i=1}^{l} y^i(\sqrt{-1}H_{(i)})$, $y^i \geq 0$ in terms of fundamental weights $\Lambda_i \in \mathcal{H}^* = \mathrm{Hom}_{\mathbf{C}}(\mathcal{H}, \mathbf{C})$, $i = 1, \ldots, l$. Here $l = \mathrm{rank}\mathcal{G} = \dim_{\mathbf{C}} \mathcal{H}$ and $H_{(i)} \in \mathcal{H}$ is the dual of fundamental weight Λ_i associated to simple root α_i, $i = 1, \ldots, l$.

Denote by Θ the set of simple roots α_i such that $y^i > 0$ and by $\Delta^+(\Theta)$ the set of positive roots $\{\alpha = \sum_{i=1}^{l} n_i\alpha_i \in \Delta^+; n_j > 0 \text{ for some } \alpha_j \in \Theta\}$,

The Lie algebra $\mathcal{K}$ of K is then written as

$$\mathcal{K} = \mathcal{H}_{\mathbf{R}} \oplus \sum_{\alpha \in \Delta^+ \setminus \Delta^+(\Theta)} \mathbf{R}X_\alpha \oplus \mathbf{R}Y_\alpha,$$

$$X_\alpha = E_\alpha + E_{-\alpha}, Y_\alpha = \sqrt{-1}(E_\alpha - E_{-\alpha}). \tag{3.1}$$

Note that $\mathcal{H}_{\mathbf{R}}$ is given by the imaginary vectors $\sum_{i=1}^{l} \mathbf{R}(\sqrt{-1}H_i)$, H_i is the dual of α_i with respect to the Killing form B.

Define a subspace $\mathcal{M}$ in $\mathcal{G}$ by

$$\mathcal{M} = \sum_{\alpha \in \Delta^+(\Theta)} \mathbf{R}X_\alpha \oplus \mathbf{R}Y_\alpha, \tag{3.2}$$

giving the orthogonal decomposition of $\mathcal{G}$; $\mathcal{G} = \mathcal{K} \oplus \mathcal{M}$ relative to B.

The tangent space T_X to the orbit $\alpha = G/K$ at X, in other words, the image of $\mathrm{ad}(X) : \mathcal{G} \to \mathcal{G}$, is identified with the space $\mathcal{M}$ through $\mathrm{ad}(X)$.

Let J be the complex structure on $\sum(\mathbf{C}E_\alpha \oplus \mathbf{C}E_{-\alpha})$ given by $J(E_\alpha) = \sqrt{-1}E_\alpha$, $J(E_{-\alpha}) = -\sqrt{-1}E_{-\alpha}$, $\alpha \in \Delta^+$. Then, since J commutes with $\mathrm{ad}(Z)$,$Z \in \mathcal{K}$,it is a standard argument that the restriction $J_{|\mathcal{M}}$ defines an invariant complex structure on G/K.

It turns out that the orbit $\alpha = G/K$ is a compact, simply connected, complex homogeneous manifold, called a $C-$space due to H.C.Wang [28], which is endowed with invariant Kähler metrics g's parametrized by $\#\Theta$ parameters(see [Itoh 14] for details on invariant Kähler metrics).

Remark that the second homotopy group $\pi_2(G/K)$ is isomorphic to the homology group $\mathrm{H}_2(G/K;\mathbf{Z})$ and then to $\mathbf{Z}^n$,$n = \#\Theta$.

When $X = \sum_i y^i(\sqrt{-1}H_{(i)})$ is with all $y^i > 0$, K is a maximal torus in G and G/K is a generalized flag manifold. When all $y^i = 0$ except one y^i, we have Kähler C-spaces of second Betti number one. All irreducible Hermitian symmetric spaces of compact type are described in this way. The $C-$spaces G/K have also parabolic description as $G^{\mathbf{C}}/P$ for the complexification $G^{\mathbf{C}}$ of G and parabolic subgroups P of $G^{\mathbf{C}}$ containg a Borel subgroup so that $K = G \cap P$.

Each space G/K carries two invariant Hermitian metrics g_1, g_2 other than g. The metric g_1 is induced from the embedding $G/K \subset \mathcal{G}$ and g_2 is the restriction of $-B$ to $\mathcal{M}$. These metrics g, g_1 and g_2 are in general not the same ([Pressley 23 p.561]), whereas $g = g_2 = \mathrm{const}.g_1$ for Hermitian symmetric spaces, because $\mathrm{ad}(X) : \mathcal{M} \to \mathcal{M}$ is then the complex structure J up to constant factor.

The subspace $\mathcal{K}$ in $\mathcal{G}$ being orthogonal to $\mathcal{M}$ gives the normal space $T_X^\perp$ of the orbit G/K in $\mathcal{G}$ at X because $B(Z, \mathrm{ad}(X)Y) = B([Z,X],Y) = 0$ for $Z \in \mathcal{K}, \mathrm{ad}(X)Y \in T_X$.

So, as just for $G = SU(2)$ it is a standard argument in harmonic map theory that a map $\Psi : S_1^2 \to (G/K, g_1)$ is harmonic if and only if the vector $\Delta\Psi(\hat{x})$ in $\mathcal{G}$ is in the normal direction $\mathcal{K}, \hat{x} \in S_1^2$, namely the $\mathcal{G}$-valued function Ψ satisfies

$$[\Delta\Psi(\hat{x}), \Psi(\hat{x})] = 0.$$

We are now in a position to state the harmonicity of Φ_∞ for general compact simple gauge group as

THEOREM 2. *Let G be a compact simple Lie group. Let (A, Φ) be a Yang-Mills-Higgs field of gauge group G with decay conditions; $|\Phi| = m + O(1/r), |F_A|, |D_A\Phi| = O(1/r^2)$. If it satisfies in a fixed radial gauge the asymptotical conditions (2.3), (2.4) and the Higgs field Φ_∞ at infinity has its image inside a G-adjoint orbit $G/K \subset \mathcal{G}$, then $\Phi_\infty : S_1^2 \to (G/K, g_1)$ yields a harmonic map in the homotopy class $[\Phi_\infty] \in \pi_2(G/K)$.*

REMARK. It is stated in [Jaffe and Taubes 17] without proof that the finiteness of the action for a Yang-Mills-Higgs field implies $|\Phi| \to m$, some constant and $|F_A|, |D_A\Phi| = O(1/r^2)$(see also[Taubes 25]).

Proof. We may assume Y_0 to be non singular. Let $\tau : Y' \longrightarrow Y$ be a morphism. We write $f' : X' \longrightarrow Y'$ for the fibre product $X \times_Y Y' \longrightarrow Y'$ and $Y_0' = \tau^{-1}(Y_0)$.

Claim 3.8. We may assume that Y is normal.

Proof. Let $Y' \longrightarrow Y$ be the normalization and $\mathcal{J}$ an ideal sheaf such that $\tau_*\mathcal{J} \subset \mathcal{O}_Y$ and such that the support S of the quotient does not meet Y_0. Using 3.4,a and b, we are allowed to replace Y by a blowing up with center in S and hence we may assume $\mathcal{J}$ to be invertible. By flat base change [6], one has

$$pr_{1*}f'^{*}\mathcal{J} \subset \mathcal{O}_X .$$

This implies that $\tau_*((f'_*\omega^{\upsilon}_{X'/Y'}) \otimes \mathcal{J})$ is contained in $f_*\omega^{\upsilon}_{X'/Y'}$. Let $f^s : X^s = X \times_Y \ldots \times_Y X \longrightarrow Y$ be the s-fold fibre product. f^s is again a Gorenstein morphism and $f^s_*\omega^{\upsilon}_{X^s/Y} = \overset{s}{\otimes} f_*\omega^{\upsilon}_{X/Y}$ ([18], 3.4, for example). Repeating our calculation for X^s instead of X, we obtain $\tau_*((\overset{s}{\otimes} f'_*\omega^{\upsilon}_{X'/Y'}) \otimes \mathcal{J})$ as a subsheaf of $\overset{s}{\otimes} f_*\omega^{\upsilon}_{X/Y}$. The same holds for S^s instead of $\overset{s}{\otimes}$. Choose the ample sheaf $\mathcal{H}$ on Y and $\mathcal{J}$ such that $\tau^*\mathcal{H} \otimes \mathcal{J}$ is ample and $\tau_*\mathcal{O}_{Y'} \otimes \mathcal{H}^b$ generated by its global sections for all $b > 0$. If 3.7 holds for $f' : X' \longrightarrow Y'$, then $S^{2a\cdot b}(f'_*\omega^{\upsilon}_{X'/Y'}) \otimes \tau^*\mathcal{H}^b \otimes \mathcal{J}^b$ is globally generated over Y_0' for some $b >> 0$. Then $S^{2a\cdot b}(f_*\omega^{\upsilon}_{X/Y}) \otimes \mathcal{H}^{2b}$ is as well globally generated over Y_0.

4. The first order equations.

The Higgs field Φ of a Yang-Mills-Higgs field satisfies the Lapalce equations (1.2) and the norm $|\Phi|$ tends to the mass at infinity. Based on these facts we derive the first order equations. Actually from (2.2)

$$\Delta_A^r(\Phi_{|S_r^2}) = r^2\partial_r^2\Phi + 2r\partial_r\Phi,$$

which yields as a distributional limit the Laplace equations over the 2-sphere at infinity

$$\Delta_{A_\infty}\Phi_\infty = 0 \tag{4.1}$$

with respect to the connection A_∞ at infinity.

Since $|\Phi_\infty| = m$ and

$$\Delta(|\Phi_\infty|^2) = 2|D_{A_\infty}\Phi_\infty|^2 + 2 < \Delta_{A_\infty}\Phi_\infty, \Phi_\infty >,$$

Φ_∞ must be covariantly constant as a section of the bundle $\mathcal{G}$ over S^2_∞. In other words, Φ_∞ satisfies

$$d\Phi_\infty = -[A_\infty, \Phi_\infty]. \tag{4.2}$$

For the PS monopole A_∞ is

$$\begin{aligned} A_\infty &= (-\sin\varphi d\theta - \sin\theta\cos\theta\cos\varphi d\varphi)i \\ &+(\cos\varphi d\theta - \sin\theta\cos\theta\sin\varphi d\varphi)j + (\sin^2\theta d\varphi)k. \end{aligned} \tag{4.3}$$

Remark that the notion of homogeneous connection on $\mathbf{R}^3 \setminus \{0\} = S^2_\infty \times \mathbf{R}^+$ appeared in [Hitchin 10] is just the A_∞.

We now assume that A_∞ be a C^1 connection on S^2_∞ to make sense the Laplacian at infinity.

We take the (0,1)-part of (4.2) on the Riemann sphere $S^2_\infty = P_1(\mathbf{C})$ and Lie bracket-multiply it with Φ_∞ to get the $\bar\partial-$equations on Φ_∞

$$[\bar\partial\Phi_\infty, \Phi_\infty] - \sqrt{-1}\mu(\Phi_\infty)\bar\partial\Phi_\infty = -[[A''_\infty, \Phi_\infty], \Phi_\infty] + \sqrt{-1}\mu(\Phi_\infty)[A''_\infty, \Phi_\infty] \tag{4.4}$$

with respect to the smallest positive eigenvalue $\mu(\Phi_\infty)$ of $-\sqrt{-1}(\Phi_\infty) : \mathcal{M} \to \mathcal{M}$.

Let $\zeta = \xi + \sqrt{-1}\eta$ be the affine complex coordinate on $S^2_\infty = P_1(\mathbf{C})$ and $\lambda_t : S^2_\infty = S^2_1 \to \mathbf{R}^3$; $\hat{x} \to t\hat{x}$ be a scale $t(>0)$ embedding.

Since the coordinates (x^1, x^2, x^3) of $\mathbf{R}^3$ pull back by λ_t are written through the stereographic projection as

$$x^1 = t\xi/(|\zeta|^2+1), x^2 = t\eta/(|\zeta|^2+1), x^3 = t(|\zeta|^2-1)/(|\zeta|^2+1),$$

the (0,1)-part A''_∞ is given at $\hat{x} \in S^2_1$ with coordinate $\zeta = \zeta(\hat{x})$ as a limit

$$A''_\infty(\hat{x}) = \lim_{t\to\infty} t\{A_1(t\hat{x}) + \sqrt{-1}A_2(t\hat{x}) + \zeta A_3(t\hat{x})\}(|\zeta|^2+1)^{-1}. \qquad (4.5)$$

We then immediately obtain the following

PROPOSITION 3. *Let (A, Φ) be a configuration of gauge group G satisfying the Yang-Mills-Higgs Euler-Lagrange equations (1.1),(1.2). Assume that (2.1) is satisfied, and in a fixed radial gauge the Higgs field Φ_∞ at infinity lies inside a $G-$adjoint orbit $G/K \subset \mathcal{G}$ and the connection A_∞ at infinity is of C^1 class. Then as a $\mathcal{G}-$valued function Φ_∞ satisfies*

$$[\bar{\partial}\Phi_\infty, \Phi_\infty] = \sqrt{-1}\mu(\Phi_\infty)\bar{\partial}\Phi_\infty \qquad (4.6)$$

($\mu(\Phi_\infty)$ denotes the smallest positive eigenvalue of $-\sqrt{-1}\mathrm{ad}\Phi_\infty$) provided at $\hat{x} \in S^2_1$

$$[[A_1(t\hat{x}) + \sqrt{-1}A_2(t\hat{x}) + \zeta(\hat{x})A_3(t\hat{x}), \Phi(t\hat{x})], \Phi(t\hat{x})]$$
$$- \sqrt{-1}\mu(\Phi_\infty)[A_1(t\hat{x}) + \sqrt{-1}A_2(t\hat{x}) + \zeta(\hat{x})A_3(t\hat{x}), \Phi(t\hat{x})] = o(1/t). \qquad (4.7)$$

The value $\mu(\Phi_\infty)$ is independent of choice of $\hat{x}$ and in the $SU(2)$ case $\mathrm{ad}(\Phi_\infty) = \mu(\Phi_\infty)J$ so that (4.6) is exactly the equations to Φ_∞ being holomorphic and is actually satisfied by the PS monopoles.

In the following theorem we assert that (4.6) gives rise to Φ_∞ being holomorphic for arbitrary compact simple group G.

THEOREM 4. *Let (A, Φ) be a Yang-Mills-Higgs field of gauge group G with decay conditions at infinity $|\Phi| = m + O(1/r), |F_A|, |D_A\Phi| = O(1/r^2)$. Assume with a fixed radial gauge trivialization Φ_∞ lies in a $G-$adjoint orbit $G/K \subset \mathcal{G}$ and the connection A_∞ at infinity is of class C^1.If (4.7) in Proposition 3 is satisfied at each $\hat{x} \in S^2_1$, then $\Phi_\infty : S^2_1 \to G/K$ is a* *holomorphic map and hence yields a harmonic map with respect to any (invariant) Kähler metric g.*

REMARK. The equations (4.6) for a $\mathcal{G}-$valued function Φ_∞ imply also that $\Phi_\infty : S_1^2 \to G/K$ is harmonic with respect to the induced Hermitian metric g_1. In fact, differentiating (4.6) in the ∂ direction we have

$$[\partial\bar{\partial}\Phi_\infty, \Phi_\infty] - [\bar{\partial}\Phi_\infty \wedge \partial\Phi_\infty] - \sqrt{-1}\mu(\Phi_\infty)\partial\bar{\partial}\Phi_\infty = 0, \tag{4.8}$$

and their complex conjugate

$$[\bar{\partial}\partial\Phi_\infty, \Phi_\infty] - [\partial\Phi_\infty \wedge \bar{\partial}\Phi_\infty] + \sqrt{-1}\mu(\Phi_\infty)\bar{\partial}\partial\Phi_\infty = 0 \tag{4.9}$$

so that from $\partial\bar{\partial} + \bar{\partial}\partial = 0$ and $[\partial\Phi_\infty \wedge \bar{\partial}\Phi_\infty] = [\bar{\partial}\Phi_\infty \wedge \partial\Phi_\infty]$ we see $[\partial\bar{\partial}\Phi_\infty, \Phi_\infty] = 0$ which says that Φ_∞ is g_1-harmonic.

PROOF of Theorem 4. From Proposition 3 Φ_∞ satisfies (4.6). The proof is based on the root space decomposition. At each point $\hat{x} \in S_1^2$ $\Phi_\infty(\hat{x})$ is written as $\Phi_\infty(\hat{x}) = \sum_{j=1}^n y^{i_j}\sqrt{-1}H_{(i_j)}, y^{i_j} > 0$ for the set $\Theta = \{\alpha_{i_1}, \ldots, \alpha_{i_n}\}$ of simple roots. So $\mathrm{ad}(\Phi_\infty(\hat{x}))$ acts on $\mathcal{M}^{\mathbf{C}}$, the complexified tangent space $T_X^{\mathbf{C}}, X = \Phi_\infty(\hat{x})$ as

$$\mathrm{ad}(X)E_{\pm\alpha} = \pm 1/2\sqrt{-1}(\sum_{j=1}^n m_{i_j} y^{i_j} |\alpha_{i_j}|^2)E_{\pm\alpha},$$

$\alpha = \sum_{i=1}^l m_i\alpha_i \in \Delta^+(\Theta)$ so that $\mu(\Phi_\infty) = 1/2 \min_j y^{i_j}|\alpha_{i_j}|^2$ because each simple root $\alpha_{i_j} \in \Delta^+(\Theta), j = 1, \ldots, n$.

Fix a $j, 1 \le j \le n$ such that $\mu(\Phi_\infty) = 1/2 y^{i_j}|\alpha_{i_j}|^2$ and define the linear subspaces $\mathcal{M}_j^+, \mathcal{M}_j^-$ by $\mathcal{M}_j^+ = \sum_\alpha \mathbf{C}E_\alpha$, where the summation is over roots $\alpha = \sum_{i=1}^l m_i\alpha_i \in \Delta^+(\Theta)$, $m_{i_j} = 1$ and $m_{i_k} = 0$ for $k \ne j$ and $\mathcal{M}_j^- = \bar{\mathcal{M}}_j^+$. Then $-\sqrt{-1}\mathrm{ad}(X)E_{\pm\alpha} = \pm\mu(\alpha)E_{\pm\alpha}$ with $\mu(\alpha) \ge \mu(\Phi_\infty)$ and $\mu(\alpha) = \mu(\Phi_\infty)$ when $E_{\pm\alpha} \in \mathcal{M}_j^\pm$.

Since $\partial\Phi_\infty/\partial\bar{\zeta}(\hat{x}) = (\Phi_\infty)_*(\partial/\partial\bar{\zeta})$ is a tangent vector, we write it as $\partial\Phi_\infty/\partial\bar{\zeta} = \sum_{\alpha\in\Delta^+(\Theta)} a^\alpha E_\alpha + a^{-\alpha}E_{-\alpha}$ and then have

$$\begin{aligned}\mathrm{ad}(X)\partial\Phi_\infty/\partial\bar{\zeta} &= \sqrt{-1}\mu(\Phi_\infty)\sum\{a^\alpha E_\alpha; E_\alpha \in \mathcal{M}_j^+\} \\ &+ \sqrt{-1}\sum\{\mu(\beta)a^\beta E_\beta; E_\beta \in \mathcal{M}^+ \setminus \mathcal{M}_j^+\} \\ &- \sqrt{-1}\mu(\Phi_\infty)\sum\{a^{-\alpha}E_{-\alpha}; E_{-\alpha} \in \mathcal{M}_j^-\} \\ &- \sqrt{-1}\sum\{\mu(\beta)a^{-\beta}E_{-\beta}; E_{-\beta} \in \mathcal{M}^- \setminus \mathcal{M}_j^-\}.\end{aligned}$$

It follows from (4.6) that $a^\alpha = 0$ for all $\alpha > 0$, that is, $\partial\Phi_\infty/\partial\bar{\zeta} \in \mathcal{M}^-$. This implies the differential $(\Phi_\infty)_*$ commutes with the complex structures of $P_1(\mathbf{C})$ and of G/K, and thus Φ_∞ is holomorphic.

That Φ_∞ is harmonic with respect to a Kähler metric is standard in the harmonic map theory on Kähler manifolds.

REMARK. It is known that there are $SU(2)$ Yang-Mills-Higgs fields on $\mathbf{R}^3$ of any charge which are not monopole ([Taubes 27]), while there are no non minimal harmonic maps: $P_1(\mathbf{C}) \to P_1(\mathbf{C})$ of any given degree ([Eells and Wood 6]). As seen in this section the second order Laplace equations reduce to the first order equations at infinity. This phenomenon explains to a certain extent the different aspects relating to Yang-Mills-Higgs fields on $\mathbf{R}^3$ and harmonic maps of $P_1(\mathbf{C})$ ([Atiyah 1]).

5. Further Remarks.

Theorems 2 and 4 obtained in the previous sections are considered as giving an answer in principle to the harmonic map existence problem : Given a $C-$space G/K, a compact simply connected complex homogeneous manifold, with an invariant Kähler metric g and a given homotopy class γ in $\pi_2(G/K)$, does there exist a harmonic map $P_1(\mathbf{C}) \to (G/K, g)$ representing the homotopy class γ ? See related discussions for this problem [Eells and Wood 7], [Burstall, Rawnsley and Salamon 4] and for a general existence theorem [Sacks and Uhlenbeck 24].

However, for this, two problems remain. One is a problem on the existence of Yand-Mills-Higgs field whose Higgs field at infinity with a symmetry breaking from G to a real parabolic subgroup K represents the homotopy class γ. See [Taubes 25] on the existence of monopoles for general gauge group of nontrivial nonnegative generalized monopole charge. A necessary condition for $m = (m_1, \ldots, m_n) \in \mathbf{Z}^n \cong \pi_2(G/K)$ that there exists a monopole of generalized monopole charge m is that m is nonnegative ([Murray 20]). For argument of generalized monopole charge we refer to [Taubes 26] and [Horvathy and Rawnsley 11].

Another problem is whether a thus derived Yang-Mills-Higgs field satisfies the decay rate at infinity stated in Theorem 4.

We should next comment our theorems in a level of moduli space of Yang-Mills-Higgs fields (or of monopoles).

Denote by $\mathcal{M}(K, \gamma)$ the moduli space of Yang-Mills-Higgs fields whose Higgs field lies in a fixed orbit G/K at infinity and represents a homotopy class $\gamma \in \pi_2(G/K)$. Then the theorems induce a map $f : \mathcal{M}(K, \gamma) \to \mathcal{H}_\gamma(P_1(\mathbf{C}); G/K)$; the moduli space of harmonic (or holomorphic) maps $\varphi : P_1(\mathbf{C}) \to G/K$ representing $\gamma \in \pi_2(G/K)$ modulo isometries. The map f is not necessarily injective. In fact Higgs fields Φ_∞ at infinity coincide for all monopoles (A^a, Φ^a) arising from the PS monopole (A, Φ) by parallel translation $x \mapsto x + a$.

Yang-Mills-Higgs fields are defined also on complete open three dimensional manifolds (see [Floer 9],[Braam 3]). Our investigation is applicable to those manifolds .

for some ν_i which are growing like η. Therefore $S^\eta(T(\mathcal{F} \otimes \mathcal{H}))$ will be globally generated over U for $\eta >> 0$ and 3.3e implies the weak positivity of $T(\mathcal{F} \otimes \mathcal{H})$.

Examples of positive tensor bundles are: $\det(\mathcal{F})$, $S^\nu(\mathcal{F})$ and $\Lambda^\nu(\mathcal{F})$. Especially, if r is the rank of $\mathcal{F}$ and $\mathcal{F}^\vee = \mathcal{H}om(\mathcal{F},\mathcal{O}_Y)$ then $\Lambda^{r-1}(\mathcal{F}) = \mathcal{F}^\vee \otimes \det(\mathcal{F})$ is a positive tensor bundle. Using 3.3b and the equality

$$S^2(\mathcal{F} \oplus \mathcal{F}') = S^2(\mathcal{F}) \oplus S^2(\mathcal{F}') \oplus \mathcal{F} \otimes \mathcal{F}'$$

one sees that weak positivity is compatible with tensor products.

Theorem 3.7. *Let $\nu > 0$ and $f : X \longrightarrow Y$ be a surjective projective flat Gorenstein morphism of reduced quasi projective schemes. Assume that $f_*\omega^\nu_{X/Y}$ is locally free. Let $Y_0 \subseteq Y$ be an open subscheme meeting all components of Y such that $f^{-1}(Y_0)$ is normal with at most rational singularities, for $y_0 \in Y_0$. Then $f_*\omega^\nu_{X/Y}$ is weakly positive over the non singular locus of Y_0.*

Of course 3.7 will be shown by reducing it to the case $\nu = 1$, where it is nothing but the positivity theorem of Kawamata & Fujita (dim $Y = 1$). Since we really need to keep track of the locus where the sheaves are weakly positive we sketch the proof:

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