ON COMPLETE MANIFOLDS WITH NONNEGATIVE RICCI CURVATURE

by

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Complete open Riemannian manifolds (M^n, g) with nonnegative sectional curvature are well understood. The basic theorems are the Toponogov Splitting Theorem and the Soul Theorem [CG1]. The Splitting Theorem has been extended to manifolds of nonnegative Ricci curvature [CG2]. On the other hand the Soul Theorem does not extend even topologically according to recent examples in [GM2]. A different method to construct manifolds which carry a metric with Ric > 0, but no metric with nonnegative sectional curvature, has been given by L. Bérard-Bergery [BB]. This leads to the question (c.f. also [Y1]): Is there any finiteness result for complete Riemannian manifolds with Ric ≥ 0 ? The answer is certainly affirmative in the low-dimensional special cases $n = 2$, where all notions of curvature coincide, and $n = 3$, where nonnegative Ricci curvature has been studied by means of stable minimal surfaces [MSY], [SY]. On the other hand, J.P. Sha and D.G. Yang [ShY] have constructed complete manifolds with strictly positive Ricci curvature in higher dimensions. For example they can choose the underlying space to be $\mathbb{R}^4 \times S^3$ with infinitely many copies of $S^3 \times \mathbb{CP}^2$ attached to it by surgery. It is therefore clear that any finiteness result for arbitrary dimensions requires additional assumptions.

The purpose of this paper is to establish the following main result.

Theorem A

Let M^n be a complete open Riemannian manifold with Ric ≥ 0. Suppose that M^n has diameter growth of order $o(r^{1/n})$. Then M^n is homotopy equivalent to the interior of a compact manifolds with boundary, provided the sectional curvature is bounded away from $-\infty$.

The notion of <u>diameter growth</u> requires a precise definition. Roughly speaking, we would like to measure the diameters of the "essential components" of the distance spheres $S(p_0,r)$ w.r.t. the intrinsec metric in $M^n \setminus B(p_0,\zeta \cdot r)$ where $\frac{1}{2} < \zeta < 1$ is a fixed number. Given any open set $\Omega \subset M^n$, not necessarily connected, we shall write $\dim(\Sigma,\Omega)$ for the diameter of any connected subset $\Sigma \subset \Omega$ measured w.r.t. the intrinsec distance function of the open submanifold Ω. Let $C(p_0,r)$ denote the union of the unbounded connected components of $M^n \setminus \overline{B(p_0,r)}$. We set:

$$(0.1) \qquad \mathrm{diam}(p_0;r) := \quad \sup \ \mathrm{diam}(\Sigma_k, C(p_0,\zeta r)),$$

where the supremum is taken over all components of Σ_k of $\partial C(p_0,r)$.

<u>Definition.</u> Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a monotonic function. A Riemannian manifold M^n with base point p_0 is said to have <u>diameter growth of order</u> $o(f)$ [resp. $\mathcal{O}(f)$], if and only if $f(r)^{-1} \cdot \mathrm{diam}(p_0;r)$ converges to zero as $r \to \infty$ • [resp. remains bounded].

This definition will be discussed further in section 1. Here we would just like to point out that the details have been arranged in such a way that the diameter growth condition in Theorem A is as little a restriction as possible. The reason for taking the supremum in formula (0.1) rather than a sum or any other norm becomes even more clear when we present our result in a slightly more general context. Quite in contrast to the Splitting Theorem in [CG2]. Theorem A extends to manifolds with <u>asymptotically nonnegative Ricci curvature</u> thus going beyond a rigidity result.

<u>Theorem B:</u>

Let M^n be a complete open Riemannian manifold with base point p_0, and let $r_0(q) = d(p_0,q)$ for all $q \in M^n$. Suppose that

0) there is a non-increasing function $\lambda : [0,\infty) \to [0,\infty)$ such that $C_0(\lambda) = \int_0^\infty r \cdot \lambda(r)dr$ converges and $Ric_q \geq -(n-1)\cdot \lambda \circ r_0(q)$ at all points $q \in M^n$,

i) the sectional curvatures are uniformly bounded from below by some (negative) constant K_0, and

ii) M^n has diameter growth of order $o(r^{1/n})$ with respect to p_0.

Then all critical points of the distance function r_0 lie inside some large ball $B(p_0,R)$, which therefore is a deformation retract of M^n, and M^n is homotopy equivalent to the interior of a compact manifold with boundary.

Let us illustrate our results in one <u>example</u>. Let $M(d_1,d_2)$ be the connected sum of infinitely many copies of $S^{d_1} \times S^{d_2}$, where $1 \leq d_1 \leq d_2$. (see Fig. 1)

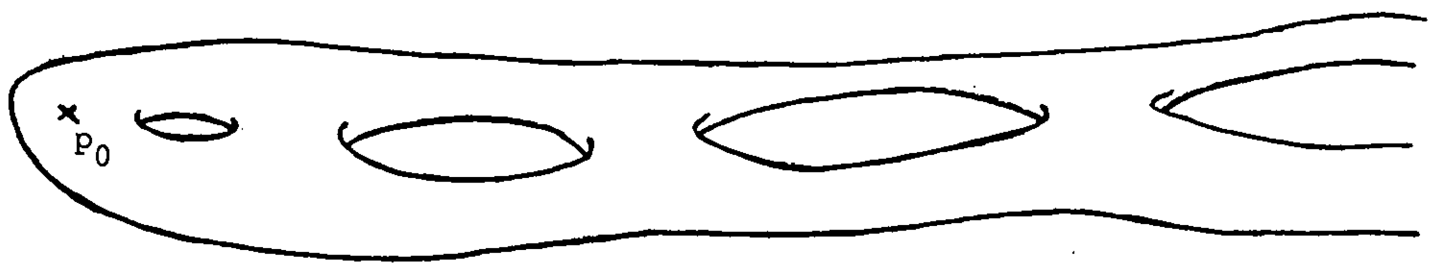

Figure 1

If $d_1 = 1$, the fundamental group grows exponentially and there cannot be a complete metric with $Ric \geq 0$ (c.f. Proposition 1). Nothing can be said - using such a classical argument - for metrics with asymtotically nonnegative Ricci curvature. If $d_1 \geq 2$ it has not been known so far whether or not $M(d_1, d_2)$ can carry any metric with $Ric \geq 0$ at all. It is easy to put complete Riemannan metrics on the manifolds $M(d_1, d_2)$ such that their diameter growth is of order $\mathcal{O}(1)$ ("bounded diameter"). By Theorem B these metrics cannot even have asymtotically nonnegative Ricci curvature, unless possibly their sectional curvature K is not bounded away from $-\infty$.

Let us now discuss the additional hypothesis in Theorems A and B. Bounding the diameter growth seems to be a very natural condition. In fact, it is this condition which is violated in the Sha-Yang examples. On these manifolds of infinite type the metric can be chosen to have diameter growth of order at most $O(r^{1/2})$. The condition also does not hold for the Bérard-Bergery examples (finite homotopy type, diameter growth $\geq O(r^{2/3})$). However, it does hold in the large class of the Gromoll-Meyer examples. They all have even bounded diameter.

All these examples have sectional curvature bounded away from $-\infty$. Indeed this hypothesis appears to be a fairly weak assumption; it enters our arguments only in an integrated form (c.f. Lemma 4.2).

In both theorems we have only claimed finite homotype type for every <u>single</u> M^n, but <u>not</u> a <u>uniform</u> bound for a whole class of manifolds. Such a bound does not even exist for the numbers of homotopy types of compact manifolds with positive sectional curvature as the examples by Wallach show [AW].

Nevertheless - as a consequence of Gromov's Betti numbers theorem (c.f. [A], [G]) - a uniform bound does exist for the homotopy types with coefficients in any field. This holds even for non-compact spaces with asymptotically nonnegative sectional curvature. However, such an estimate cannot hold for the class of compact manifolds with strictly positive Ricci curvature, according to examples in [ShY]. We do not know whether or not in our context a fixed lower sectional curvature bound K_0 gives rise to an a priori estimate for all the Betti numbers.

Many results on manifolds with $\mathrm{Ric} \geq 0$ are proven by volume comparison (c.f. section 1). These arguments are not sufficient to prove Theorems A and B. We need much stronger bounds for the distance function. In fact, the main result in section 2 is a lower bound on the height of thin triangles involving just the lengths of their edges and a lower bound for the Ricci curvature (c.f. Proposition 2.3 and Corollary 2.4). Here Toponogov's triangle comparison theorem is not required.

Our argument is modelled on the basic step in the proof of the Splitting Theorem; we calculate a bound on the Laplacian of certain distance functions and apply the maximum principle. In the case of the Splitting Theorem this bound is always zero; in our non-rigid situation the bound can - and will - take different values. This problem is dealt with in Theorem 2.1, which seems to be a new estimate on "subharmonic" Lipschitz functions.

In section 3 we compute (as far as needed) the explicite bounds for the thin triangles. In particular, we analyse the asymptotic curvature condition so that in section 4 we will be prepared to prove a new critical point lemma and deduce Theorem A and Theorem B.

1. Diameter Growth and Volume Growth.

Our first goal is to show that both notions, volume growth and diameter growth, can be used equally well to distinguish qualitatively between hyperbolic spaces and manifolds with nonnegative Ricci curvature. It is a direct consequence of the definitions that hyperbolic space has exponential volume growth as well as exponential diameter growth. Notice that we are considering the quantities $\operatorname{diam}(S(p_0,r), M^n \setminus B(p_0,\zeta\cdot r))$, i.e. we have defined the relevant distance between two points $q_1, q_2 \in S(p_0,r)$ as the infimum over the lengths of only those curves from q_1 to q_2 which lie inside $M^n \setminus B(p_0,\zeta\cdot r)$.

Proposition 1.1. (linear diameter growth)

Any complete Riemannian manifold with $\operatorname{Ric} \geq 0$ has diameter growth of order $\mathcal{O}(r)$ with respect to any point $p_0 \in M^n$.

This proposition is a direct consequence of Lemma 1.4 below. In order to make our point clear, let us state the corresponding result for volume growth next.

Proposition 1.2. (polynomial volume growth)

Let M^n be a complete non-compact Riemannian manifold with $\operatorname{Ric} \geq 0$, let $p_0 \in M^n$ be arbitrary. Then

i) $\operatorname{vol} B(p_0,r) \leq \omega_n \cdot r^n$ for $r > 0$, and

ii) $\operatorname{vol} B(p_0,R) \geq \frac{1}{2} (\frac{R}{r} - 1) \cdot \operatorname{vol} B(p_0,r)$ for $0 < r < R$.

Here ω_n stand for the volume of the euclidean unit ball $B^n(1)$. This proposition completes our elementary comparison of volume and diameter growth. The second inequality is due to E. Calabi and S.T. Yau [CGT], [Y2]. Since both statements are actually fairly direct consequence of the well-known relative volume comparison theorem, it is in fact easy to extend them - of course only up to some positive factors - to manifolds with asymptotically nonnegative Ricci curvature as we have defined them. (Notice that our condition is stronger than the condition of almost nonnegative Ricci curvature at infinity, which has been introduced in section 4 of [CGT]). A result which does not extend is the following

Proposition 1.3. (polynomial growth of π_1)
Let M^n be a complete Riemannian manifold with $Ric \geq 0$. Then

$$\#(\alpha \in \pi_1(M^n) \mid \|\alpha\|_{geo} \leq r) \leq const \cdot r^n \quad .$$

In particular, the first Betti number $b_1(M^n, \mathbb{R})$ is bounded from above by n.

Here $\|\alpha\|_{geo}$ stands for the geometric norm taken w.r.t. some base point $\tilde{p}_0$ in the universal covering $\tilde{M}^n$, i.e. $\|\alpha\|_{geo} = d(\tilde{p}_0, \tilde{\alpha} \cdot \tilde{p}_0)$, where $\tilde{\alpha}$ is the decktransformation representing $\alpha \in \pi_1(M^n)$. The proposition is proved by looking at the Dirichlet cell $\tilde{D}$ around $\tilde{p}_0$ and the action of the decktransformation group. Given $\rho_0 > 0$, one compares the volume of $\tilde{D} \cap B(\tilde{p}_0, \rho_0)$ to the volume of large balls $B(\tilde{p}_0, r)$; c.f. [CG2], [M], and also [An] for further results.

Remark. Working with manifolds of asymptotically nonnegative Ricci curvature, one can in general at best pass to some finite covering, and this already weakens most decay conditions in relation to the degree of the covering. This makes it clear where the proof of Proposition 1.3 breaks down, when turning to manifolds with asymptotically nonnegative Ricci curvature. Of course, it is also easy to give a direct counterexample.

Before we begin with the proof of Proposition 1.1, let us recall the basic tool:

<u>Relative Volume Comparison Theorem.</u> (R. Bishop [BS] and M. Gromov [GLP], [MS])
Let M^n be a complete Riemannian manifold with $\mathrm{Ric} \geq (n-1)\cdot\kappa$, and let $q \in M^n$ be arbitrary. Then

$$(1.1) \qquad \frac{\mathrm{vol}\ B(q,r)}{\mathrm{vol}\ B(q,R)} \geq \frac{\mathrm{vol}\ B_\kappa(r)}{\mathrm{vol}\ B_\kappa(R)}, \quad \text{provided} \quad 0 \leq r \leq R.$$

Here $B_\kappa(r)$ denotes a ball of radius r in the simply connected model space of constant curvature κ.

<u>Lemma 1.4.</u>
Let M^n be a complete Riemannian manifold with $\mathrm{Ric} \geq 0$, and let $p_0 \in M^n$. Then for all $r > 0$,

$$(1.2) \qquad \mathrm{diam}(p_0,r) \leq 4\xi\cdot(1 + \tfrac{2}{\xi})\cdot r \quad \text{where} \quad \xi = \tfrac{1}{2}(1 - \zeta).$$

<u>Proof.</u> Pick a maximal family of points $q_j \in S(p_0,r)$ such that the balls $B_j = B(q_j,\xi\cdot r)$ are disjoint. As $B_j \subset B(p_0,(1+\xi)\cdot r) \subset B(q_j,(2+\xi)\cdot r)$, it is standard to conclude - using the hypothesis $\mathrm{Ric} \geq 0$ via the relative volume comparison theorem - that for all j,

$$(\frac{\xi}{2+\xi})^n \text{ vol } B(p_0,(1+\xi)r) \leq \text{vol } B_j \leq \text{vol } B(p_0,(1+\xi)r) \ ,$$

and hence

$$(1.3) \qquad \#(q_j) \leq (1 + \frac{2}{\xi})^n \ .$$

The balls $B(q_j,2\xi r)$ cover $S(p_0,r)$, but they still do not intersect $B(p_0,\xi r)$. In particular, if $B(q_j,2\xi r) \cap B(q_{j'},2\xi r) \neq \phi$, then the minimining geodesic joining q_j and $q_{j'}$, has length less than $4\xi r$ and hence does not intersect $B(p_0,\zeta\cdot r)$ either. Therefore the lemma follows directly by counting the number of balls $B(q_j,2\xi r)$, as in inequality (1.3).

□

The proofs of Proposition 1.3 and Lemma 1.4 illustrate how one can get some length control from volume estimates. This works since the standard volume estimates are for metric balls and involve the radius which is already a one-dimensional quantity. We have actually proved more: if M^n is a complete Riemannian manifold with $\text{Ric} \leq 0$, then for all $p_0 \in M^n$, $r > 0$, and all $\xi \in (0,\frac{1}{2})$ the following inequality holds:

$$(1.4) \qquad \inf_{\Sigma} \sum_j \text{diam}(\Sigma_j, M^n \backslash B(p_0,(1-2\xi)r) \leq 4\xi(1+\frac{2}{\xi})^n \cdot r \ .$$

Here the infimum is taken over all countable coverings $\Sigma = (\Sigma_j)$ of the distance sphere $S(p_0,r)$. It is necessary to allow that a single Σ_j may consist of several connected components of $S(p_0,r)$. In this paper we are not going to compare diameter growth w.r.t. different base points in detail.

One should certainly not expect a better statement than for volume growth; this notion is known to be independant of the base point only if the volume does not grow superexponentially. Without referring to Theorem B we do not know how to prove, in the case of asymptotically nonnegative Ricci curvature, that the diameter growth does not depend on the base point.

2. Thin Triangles.

In this section we present an inequality for thin triangles which requires only a lower bound for the Ricci curvature and allows to generalize the basic argument in the proof of the Cheeger-Gromoll theorem [CG2].

We begin with a fundamental estimate on "subharmonic" Lipschitz functions $f : M^n \to \mathbb{R}$. Bounds for the Laplacian of such a function will be formulated in terms of upper and lower barriers, just as in the proof of the splitting theorem given by J. Eschenburg and E. Heintze [EH]. An upper (lower) barrier for f at a point q in the interior of the domain of f is by définition a C^2-function f_q defined on a given neighborhood U_q of q that $f_q \geq f$ (resp. $f_q \leq f$) on U_q and $f_q(q) = f(q)$.

This analytic result already requires the lower bound for the Ricci curvature. We use comparison with the standard model spaces M^n_κ of constant curvature; in polar coordinates these spaces are usually described in terms of the functions:

$$s_\kappa(t) = \begin{cases} \frac{1}{\sqrt{\kappa}} \sin \sqrt{\kappa}\, t & ; \kappa > 0 \\ t & ; \kappa = 0 \\ \frac{1}{\sqrt{-\kappa}} \sinh \sqrt{-\kappa}\, t & ; \kappa < 0 \end{cases} \quad \text{and} \quad c_\kappa(t) = \begin{cases} \cos \sqrt{\kappa}\, t & ; \kappa > 0 \\ 1 & ; \kappa = 0 \\ \cosh \sqrt{-\kappa}\, t & ; \kappa < 0 \, . \end{cases}$$

Our estimates in particular will involve the expression

$$\varphi_{n,\kappa}(\rho,\ell) = \iint_{\rho \leq t \leq \tau \leq \ell} \left(\frac{s_\kappa(\tau)}{s_\kappa(t)}\right)^{n-1} d\tau \, dt, \tag{2.1}$$

which is defined for $0 < \rho \le \ell$, provided $\kappa\ell \le \pi^2$. Note that the radially symmetric function $\bar{h}(\bar{q}) := \varphi_{n,\kappa}(d(\bar{p},\bar{q}),\ell)$ on the punctured ball $B(\bar{p},\ell)\setminus\{\bar{p}\}$ in the model space M^n_κ satisfies:

$$(2.2)\quad \begin{array}{llll} \text{i)} & \Delta\bar{h} = 1 & \text{on} & B(\bar{p},\ell)\setminus\{\bar{p}\}\ , \\ \text{ii)} & \bar{h}(\bar{q})=0,\quad \text{grad}\ \bar{h}|_{\bar{q}} = 0 & \text{for} & \bar{q} \in \partial B(\bar{p},\ell)\ . \end{array}$$

These two properties determine the function $\varphi_{n,\kappa}$.

Theorem 2.1.

Let M^n be a complete Riemannian manifold, and let

$f : B(p,R) \subset M^n \to [0,\infty)$ be a Lipschitz function. Suppose that

i) $\text{Ric} \ge (n-1)\cdot\kappa$ on $B(p,R)$

ii) $\text{dil}\ f \le C_1$

iii) $\Delta f \le C_2$ in the sense that for all $q \in B(p,R)$ and all $\epsilon > 0$ there exists an upper barrier $f_{q,\epsilon}$ for f such that $\Delta f_{q,\epsilon}(q) \le C_2 + \epsilon$

iv) f has a zero z at distance $\ell := d(p,z) < R$

Then:

$$(2.3)\quad f(p) \le \inf_{\rho\in(0,\ell)} C_1\rho + C_2\varphi_{n,\kappa}(\rho,\ell) =: \Phi_n(\kappa,C_1,C_2,\ell)$$

Remarks.

i) $C_1 \ge 0$. Considering the zero z of f, it is clear that $C_2 \ge 0$ as well.

ii) $\Phi_n(\kappa,C_1,0,\ell) = \Phi_n(\kappa,0,C_2,\ell) = 0$

iii) Myers' theorem states that $\kappa\cdot\ell^2 \le \pi^2$. This inequality is precisely the condition under which $\Phi_n(\kappa,C_1,C_2,\ell)$ is well defined and depends continuously on its parameters.

Proof: Suppose the theorem is false. Using the continuity of Φ_n we can pick $\tilde{\kappa} < \kappa$ such that

$$(2.4) \qquad f(p) > \Phi_n(\tilde{\kappa},C_1,C_2,\ell) \ge 0 \quad \text{and} \quad \tilde{\kappa}\ell^2 < \pi^2 .$$

Similarly these inequalities persist when C_1,C_2 and ℓ are replaced by $\tilde{C}_1 = C_1 + \epsilon$, $\tilde{C}_2 = C_2 + \epsilon$ and $\tilde{\ell} = \ell + \epsilon$, provided $\epsilon \in (0,R-\ell)$ is sufficiently small. We shall give a lower bound $h : B(p,R) \to [0,\infty)$ for f such that h is strictly positive on $B(p,\tilde{\ell})$. In particular, this yields $f(z) \ge h(z) > 0$, contradicting hypothesis (iv).

In order to define h let us consider the piecewise C^2-functions $\tilde{\varphi}_\rho : [0,R) \to [0,\infty)$ defined by

$$(2.5) \qquad \tilde{\varphi}_\rho(d) = \begin{cases} \tilde{C}_1\cdot(\rho-d) + \tilde{C}_2\cdot\varphi_{n,\tilde{\kappa}}(d,\tilde{\ell}) & ; \ 0 \le d \le \rho \\ \tilde{C}_2\cdot\varphi_{n,\tilde{\kappa}}(d,\tilde{\ell}) & ; \ \rho \le d \le \tilde{\ell} \\ 0 & ; \ \tilde{\ell} \le d < R \end{cases}$$

Since the map $d \mapsto \tilde{C}_2\cdot\varphi_{n,\tilde{\kappa}}(d,\tilde{\ell})$ is strictly convex, there is precisely one $\rho_0 \in (0,\tilde{\ell}]$ such that the function $\tilde{\varphi}_{\rho_0}$ is of class C^1. Clearly $\tilde{\varphi}_{\rho_0}(0) = \Phi_n(\tilde{\kappa},\tilde{C}_1,\tilde{C}_2,\tilde{\ell}) < f(p)$. We set

$$(2.6) \qquad h(q) = \tilde{\varphi}_{\rho_0}(d(p,q)) \quad \text{for} \quad q \in B(p,R).$$

It is clear that $\tilde{C}_1, \tilde{C}_2 > 0$, so h is strictly positive in $B(p,\tilde{\ell})$ and vanishes outside this ball. It remains to show that

$$(2.7) \qquad f(q) \geq h(q) \qquad \text{for all} \quad q \in B(p,\tilde{\ell}).$$

Since $f(p) > h(p)$, it follows directly from hypothesis (ii) that inequality (2.7) holds on $\overline{B(p,\rho_0)}$. In the annulus $A = B(p,\tilde{\ell})\setminus\overline{B(p,\rho_0)}$ one can apply the maximum principle: If $f - h$ had a local minimum at some $q \in A$, then its upper barrier $f_{q,\epsilon/2} - h_q$ would have a local minimum at q at well. Here h_q denotes the lower barrier for h constructed in Lemma 2.2 below. Therefore, $\Delta(f_{q,\epsilon/2} - h_q)(q) \leq C_2 + \frac{\epsilon}{2} - \tilde{C}_2 < 0$, a contradiction which shows that a local minimum of $f - h$ cannot exist in A.

□

<u>Lemma 2.2</u>

At any $q \in A$ the function h defined in formula (2.6) has a lower barrier h_q such that $\Delta h_q(q) \geq \tilde{C}_2$.

<u>Proof:</u> Given $q \in A$, we pick a minimizing geodesic γ from p to q. Let $d = d(p,q)$ denote its length. For $\delta \in (0,d)$ we set

$$(2.8) \qquad h_\delta^\gamma(x) := \tilde{C}_2 \cdot \varphi_{n,\kappa}(\delta + d(x,\gamma(\delta)),\tilde{\ell}) \qquad \text{for} \quad x \in B(\gamma(\delta),\tilde{\ell}-\delta).$$

The map $\rho \longmapsto C_2 \cdot \varphi_{n,\tilde{\kappa}}(\rho,\tilde{\ell})$, $0 < \rho < \tilde{\ell}$, is decreasing. The triangle inequality implies that

$$(2.9)\qquad \begin{aligned} h^\gamma_\delta(q) &= h(q) && \text{and} \\ h^\gamma_\delta(x) &\le h(x) && \text{for} \quad x \in B(\gamma(\delta), \tilde{\ell} - \delta) \ . \end{aligned}$$

Since γ is minimizing, its restriction to $[\delta, d]$ remains minimizing, even when it is extended a little beyond the endpoint $q = \gamma(d)$. Therefore the distance function $d_{\gamma(\delta)}(x) = d(\gamma(\delta), x)$ is differentiable in a neighborhood U^γ_δ of $\gamma(\delta, d]$, and so is the function h^γ_δ. It is a standard fact that

$$(2.10)\qquad \begin{aligned} \|\operatorname{grad}\, d_{\gamma(\delta)}\| &= 1 && \text{and} \\ \Delta\ d_{\gamma(\delta)} &\le (n-1)\, \frac{c_\kappa}{s_\kappa} \circ d_{\gamma(\delta)} && \text{on} \quad U^\gamma_\delta \ . \end{aligned}$$

We compute

$$(2.11)\qquad \begin{aligned} \Delta\, h^\gamma_\delta(q) &= \tilde{C}_2 \cdot \frac{\partial^2}{\partial\rho^2}\, \varphi_{n,\tilde{\kappa}}(\rho, \tilde{\ell})\, |_{\rho=d} \cdot \|\operatorname{grad}\, d_{\gamma(\delta)}|_q\|^2 \\ &\quad + \tilde{C}_2 \cdot \frac{\partial}{\partial\rho}\, \varphi_{n,\tilde{\kappa}}(\rho, \tilde{\ell})\, |_{\rho=d} \cdot \Delta\, d_{\gamma(\delta)}(q) \\ &\ge \tilde{C}_2 + (n-1)\tilde{C}_2 \cdot \left(\frac{c_{\tilde{\kappa}}(d)}{s_{\tilde{\kappa}}(d)} - \frac{c_\kappa(d-\delta)}{s_\kappa(d-\delta)} \right) \cdot \int_d^{\ell} \left(\frac{s_{\tilde{k}}(r)}{s_{\tilde{k}}(d)} \right)^{n-1} dr \ . \end{aligned}$$

Since $\tilde{\kappa} < \kappa$, we can pick $\delta \in (0, d)$ so small that the expression on the right-hand side is $\ge \tilde{C}_2$. Because of formula (2.9) the function $h_q = h^\gamma_\delta | U^\gamma_\delta$ is the desired lower barrier at q with $\Delta\, h_q(q) \ge \tilde{C}_2$.

□

Theorem 2.1 has a direct geometric application. Let γ be a minimizing geodesic joining two points $p_0, p_1 \in M^n$. Given a third point $p \in M^n$ we set (c.f. Fig. 2.):

$$
\begin{aligned}
&r_i(p) = d(p, p_i) && , \quad i = 0,1, \\
(2.12) \qquad &\ell(p) = d(p, \gamma) && , \quad \text{and} \\
&e(p) = r_0(p) + r_1(p) - d(p_0, p_1) && \quad \text{(the "excess function").}
\end{aligned}
$$

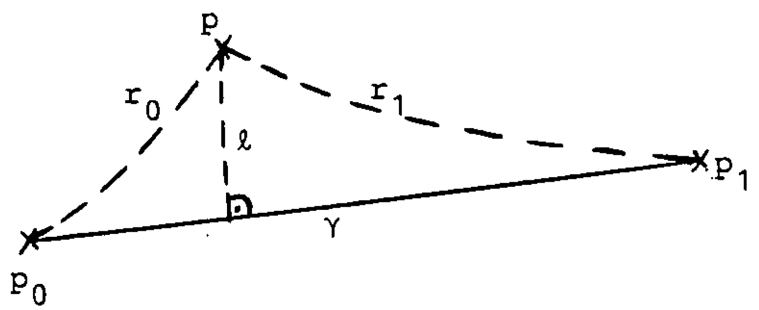

Figure 2

By the triangle inequality,

$$(2.13) \qquad 0 \le e(p) \le 2 \cdot \ell(p) \quad .$$

We are going to improve this inequality in the region where $\ell(p)$ is small, i.e. at points p which are close to γ .

Proposition 2.3

Let p, p_0, p_1, and γ be as above, and let $R > \ell(p)$. Suppose that $Ric \ge (n-1)\kappa$ on $B(p,R)$. Moreover, we assume that the Laplacian of the excess function e is bounded by some constant $C_2(R)$, in the sense that for all $q \in B(p,R)$ and all $\epsilon > 0$ there is an upper barrier $e_{q,\epsilon}$ with $\Delta e_{q,\epsilon}(q) \le C_2(R) + \epsilon$. Then,

$$(2.14)\quad e(p) \le \inf_{0<\rho<\ell(p)} \{2\rho + C_2(r)\cdot \iint_{\rho\le t\le\tau\le\ell(p)} \left(\frac{s_\kappa(\tau)}{s_\kappa(t)}\right)^{n-1} d\tau\, dt\} < 2\ell(p).$$

In particular, when $\kappa \le 0$,

$$(2.15)\qquad e(p) \le \begin{cases} 2\cdot\frac{n-1}{n-2}\cdot(\frac{1}{2}\, C_3\ell^n)^{1/n-1}\cdot\ell & ;\ n \ge 3\ , \\ C_3\ell^2\cdot\left(\frac{1}{1+\sqrt{1+C_3^2\ell^2}} + \ell n\,\frac{1+\sqrt{1+C_3^2\ell^2}}{C_3\ell}\right) & ;\ n = 2. \end{cases}$$

Here we have set $\ell = \ell(p)$ and $C_3 = \frac{1}{n}\, s_{\kappa\ell^2}(1)\cdot C_2(R)$.

Corollary 2.4

Let p, p_0, p_1 and γ be as above. Assume that M^n is a complete Riemannian manifold with $\mathrm{Ric} \ge 0$. If $\ell(p) < \min(r_0(p), r_1(p))$, then inequality (2.15) holds with

$$(2.16)\qquad C_3 = \frac{n-1}{n}\cdot\left(\frac{1}{r_0(p)-\ell(p)} + \frac{1}{r_1(p)-\ell(p)}\right)\ .$$

On the right-hand side of (2.15) we see the factor $\ell^{n/n-1}$. The exponent $\frac{n}{n-1}$ occurs in the borderline Sobolev embedding $L_1^1(\mathbb{R}^n) \to L^{n/n-1}(\mathbb{R}^n)$ - for the very same reason - ; it makes both inequalities scale invariant.

When $n = 2$, the exponent $\frac{n}{n-1}$ takes the value 2. However, there is a logarithmic factor which makes our estimate (near $\ell = 0$) even qualitatively weaker than the bound obtained from Toponogov's theorem. But when $n \ge 3$ and we assume only that $\mathrm{Ric} \ge 0$, Toponogov's theorem does not apply.

Proof of Proposition 2.3

Since $\operatorname{dil} e \leq 2$, and since the excess function e vanishes at the footpoint z of p on γ, i.e. at a point in $B(p,R)$, inequality (2.14) is a direct consequence of Theorem 2.1. The proof of (2.15) is just computational. Using $t \cdot s_{\kappa t^2}(1) = s_\kappa(t)$ and $1 \leq s_{\kappa t^2}(1) \leq s_{\kappa t^2}(1) \leq s_{\kappa \ell^2}(1)$, we calculate that

$$2\rho + C_2(R) \cdot \iint\limits_{\rho \leq t \leq \tau \leq \ell(p)} \left(\frac{s_\kappa(\tau)}{s_\kappa(t)}\right)^{n-1} d\tau \, dt$$

$$(2.17) \qquad \leq 2\rho + s_{\kappa\ell^2}(1)^{n-1} \cdot C_2(r) \cdot \int_\rho^\ell \int_t^\ell \left(\frac{\tau}{t}\right)^{n-1} d\tau \, dt$$

$$\leq 2\rho + \frac{1}{2} C_3 \cdot \left(\rho^2 - \ell^2 + 2\ell^n \cdot \int_\rho^\ell t^{1-n} dt\right) .$$

We regard the above right-hand side as a function $\Psi(\rho)$. It follows from inequality (2.14) that $e(p) \leq \inf\{\Psi(\rho) \mid 0 < \rho < \ell\}$. The function Ψ is convex, and the infimum is achieved at the unique $\rho_0 \in (0,\ell)$ with $\Psi'(\rho_0) = 0$, or, more explicitly

$$(2.18) \qquad \rho_0^{n-1} = \frac{1}{2} C_3 \cdot (\ell^n - \rho_0^n) \leq \frac{1}{2} C_3 \ell^n .$$

When $n \geq 3$, we conclude that

$$e(p) \le \Psi(\rho_0) = 2\rho_0 + \frac{1}{2} C_3 \cdot \left(\rho_0^2 - \frac{n}{n-2} \ell^2 + \frac{2}{n-2} \ell^n \rho_0^{2-n}\right)$$

$$(2.19) \qquad = 2 \frac{n-1}{n-2} \rho_0 + \frac{1}{2} \frac{2}{n-2} C_3 \cdot (\rho_0^2 - \ell^2)$$

$$\le 2 \frac{n-1}{n-2} (\tfrac{1}{2} C_3 \ell^n)^{1/n-1}.$$

When $n = 2$, we find

$$(2.20) \qquad e(p) \le \Psi(\rho_0) = \rho_0 + C_3 \ell^2 \cdot \ell n \frac{\ell}{\rho_0} .$$

In this case (2.15) follows by eliminating ρ_0 from the right hand side using the quadratic equation in (2.18).

□

<u>Proof of Corollary 2.4:</u> Given $\ell(p) < R < \min(r_0(p), r_1(p))$, we merely need to show that in $B(p,R)$ the Laplacian of the excess function e is bounded by $C_2(R) = \frac{n-1}{r_0(p)-R} + \frac{n-1}{r_1(p)-R}$ in the sense of Proposition 2.3. In (2.15) we can then pass to the limit $R \to \ell(p)$. So let us pick minimizing geodesics γ_0, γ_1 from p_0, p_1 to some point $q \in B(p,R)$. We set

$$(2.21) \qquad e_q^\delta(x) = 2\delta + d(\gamma_0(\delta),x) + d(\gamma_1(\delta),x) - d(p_0,p_1) \qquad \text{for } x \in M^n,$$

where δ varies between 0 and $\min(r_0(p), r_1(p)) - R$. each function e_q^δ is differentiable when restricted to a suitable neighborhood U_q^δ of q. Indeed, $e_q^\delta \mid U_q^\delta$ is an upper barrier for e at q such that

$$(2.22) \qquad \Delta e_q^{\delta}(x) \le \frac{n-1}{r_0(q)-\delta} + \frac{n-1}{r_1(q)-\delta} < \frac{n-1}{r_0(p)-R-\delta} + \frac{n-1}{r_1(p)-R-\delta} .$$

c.f. formula (2.10). Thus given $\epsilon > 0$, we can choose $\delta(\epsilon) > 0$ so small that $\Delta e_q^{\delta(\epsilon)}(q) \le C_2(R)$, as required.

□

3. Explicite Estimates for Manifolds with Asymtotically Nonnegative Ricci curvature.

In this section we are going to determine explicit bound for thin triangles in our more general situation.

Proposition 3.1.

Let M^n be a complete Riemannian manifold of dimension $n \geq 3$, and let p, p_0, p_1 and γ be as in (2.12); c.f. Fig 2. Suppose , $L := d(p_0, p_1) \geq 2 \cdot r_0(p)$ and, moreover, that there exists a non-increasing function $\lambda : [0,\infty) \to [0,\infty)$ such that

$C_0(\lambda) = \int_0^\infty r\,\lambda(r)dr$ converges and $Ric|_q \geq -(n-1) \cdot \lambda \circ r_0(q)$ at all points $q \in M^n$. Then the height of the triangles can be bounded from below in terms of $r_0(p)$ and the excess $e(p)$:

$$(3.1)\quad d(p,\gamma) \geq \min\left(\frac{1}{6} r_0(p),\ \frac{r_0(p)}{\sqrt{1+8C_0(\lambda)}},\ C_4 \cdot r_0(p)^{1/n} \cdot (2e(p))^{1-1/n}\right),$$

where $C_4 = C_4(n,\lambda) = \frac{4}{17}\,\frac{n-2}{n-1}\left(\frac{5}{1+\sqrt{1+8C_0(\lambda)}}\right)^{1/n}$.

Remarks

i) For manifolds with nonnegative Ricci curvature we have - as a direct consequence of Corollary 2.4 - the stronger estimate:

$$(3.1')\qquad d(p,\gamma) \geq \frac{1}{2} \min\left((r_0(p),\ \frac{n-2}{n-1} \cdot r_0(p)^{1/n} \cdot e(p)^{1-1/n}\right)$$

ii) Since $C_4(2,\lambda) = 0$, the Proposition holds trivially for 2-manifolds so that we need not explicitly exclude this case in subsequent applications. It has already been explained after Corollary 2.4, that more reasonable estimates in the two-dimensional case should be based on Toponogov's theorem; we are not going to state them here.

The convergence of the integral $C_0(\lambda)$, which is a hypothesis of the proposition, is essentially a decay condition on the lower curvature bound. Roughly speaking, this bound must tend to zero a little quicker than $\text{const}\cdot r_0(p)^{-2}$. More precisely:

Lemma 3.2.

Let $\lambda : [0,\infty) \to [0,\infty)$ be a montonic function such that

$C_0(\lambda) = \int_0^\infty r\,\lambda(r)dr$ converges. Then the monotonic functions

$$(3.2)\qquad \lambda_1(r) = \int_r^\infty \lambda(t)dt \qquad \text{and} \qquad \lambda_2(r) = \int_r^\infty \lambda_1(t)dt$$

exist, and moreover:

$$(3.3)\qquad r^2\lambda \le 2C_0(\lambda) \qquad \text{and} \qquad r\lambda_1(r) \le C_0(\lambda) \quad \text{for all} \quad r > 0.$$

This lemma, which has been proved in chapter II of [A], will be useful in deducing Proposition 3.1 from Proposition 2.3. However, before we can actually give this argument, we need to know more about the analysis of the decay condition. Let us consider the Riccati equation

$$u'(r) = u(r)^2 - \lambda(r) \quad . \tag{3.4}$$

For any $L > 0$ there are unique solutions $u_L^0 : (0,L] \to [0,\infty)$ such that $u_L^0(L) = 0$ and $u_L^\infty : (0,L) \to (0,\infty)$ such that $u_L(r) \to +\infty$ for $r \to L$.

Lemma 3.3.

If $C_0(\lambda)$ converges, then:

i) The solutions $u_L^0(r)$ of (3.4) depend monotonically on r and on the parameter L. They are bounded by $\min(\lambda_1(r),\sqrt{\lambda(r)})$, and hence in the limit $L \to \infty$, they converge to a non-increasing solution $u_\infty : (0,\infty) \to [0,\infty)$.

ii) The solutions $u_L^\infty(r)$ also converge monotonically in L and uniformly on compact subsets of $(0,\infty)$ to the solution u_∞. When $0 < r < L$, the following inequalities hold:

$$u_\infty(r) < u_L^\infty(r) \le u_\infty(r) + \frac{1}{L-r} \le \min(\lambda_1(r),\sqrt{\lambda(r)}) + \frac{1}{L-r}. \tag{3.5}$$

Proof: Part i) of this lemma has also been proved in chapter II of [A], where the condition $C_0(\lambda) < \infty$ has been analyzed in detail. Anyway the common upper bound for the functions u_L^0 as well as their monotonicity is obtained by a simple comparison of first order differential equations. In order to prove part (ii), let us substitute $u(r) = u_\infty(r) + v(r)^{-1}$ into equation (3.4). We see that the function v satisfies

$$v'(r) = -1 - 2u_\infty(r)\cdot v(r). \tag{3.6}$$

Since any positive initial value v_0 decays to zero within finite time, we conclude that any solution $u(r)$ which exceeds $u_\infty(r)$ at some point cannot exist globally on $(0,\infty)$ and is in fact some u_L^∞. Equation (3.6) also implies that $u_\infty(r) < u_L^\infty(r) \le u_\infty(r) + \frac{1}{L-r}$.

□

For any $\delta \ge 0$ let $w_\delta^\infty : (\delta,\infty) \to (0,\infty)$ be the unique non-increasing solution of

$$w'(r) + w(r)^2 - \lambda(r) = 0, \tag{3.7}$$

with initial data given by $\lim_{r\to\delta} w(r) = +\infty$.

Lemma 3.4:

Suppose that $C_0(\lambda)$ converges. Then for all $r > 0$ and all $\epsilon > 0$ there is some $\delta(\epsilon,r) > 0$ such that

$$w_\delta^\infty(r) \le \frac{1}{2r} \cdot (1 + \sqrt{1+8C_0(\lambda)}) + \epsilon \quad \text{for} \quad 0 \le \delta \le \delta(\epsilon,r). \tag{3.8}$$

Proof: Set $a = \frac{1}{2}(1 + \sqrt{1+8C_0(\lambda)})$. By Lemma 3.2 it it clear that $\lambda(r) \le -a(1-a)r^{-2}$ for all $r > 0$, and so $v'(r) + v(r)^2 + a(1-a)r^{-2} = 0$ is a comparison equation for (3.7). Its generic solution is

$$v_\delta^\infty(r) = \frac{a}{r} + \frac{2a-1}{r}\,\frac{\delta^{2a-1}}{r^{2a-1}-\delta^{2a-1}} . \tag{3.9}$$

Standard comparison arguments yield $w_\delta^\infty(r) \le v_\delta^\infty(r)$ for $r > \delta \ge 0$.

□

In subsequent arguments the Riccati equations (3.4) and (3.7), which have been analyzed above, will be used as (one-dimensional) comparison equations in the following geometric context: let $d_{\tilde{p}}$ denote the distance function to some point $\tilde{p} \in M^n$, and let c be a unit speed geodesic of finite length which begins at $c(0) = \tilde{p}$ and which does not intersect the cut locus $C_{\tilde{p}}$ of $\tilde{p}$; then $d_{\tilde{p}}$ is differentiable along c except at $\tilde{p}$ itself, and its Hessian, viewed as a symmetric 1,1-tensor, satisfies

$$\nabla_{c'}\text{Hess}\, d_{\tilde{p}} + (\text{Hess}\, d_{\tilde{p}})^2 + R(.,c')c' = 0, \tag{3.10}$$

and hence the differential inequality

$$d_{c'}\left(\frac{1}{n-1}\Delta d_{\tilde{p}}\right) + \left(\frac{1}{n-1}\Delta d_{\tilde{p}}\right)^2 + \frac{1}{n-1}\langle \text{Ric}\, c',c'\rangle \le 0. \tag{3.10'}$$

<u>Proof of Proposition 3.1.</u> It is sufficient to consider the case where $\ell(p) < \tilde{r}_0(p) = \min(\frac{1}{6}r_0(p), \frac{r_0(p)}{\sqrt{1+8C_0(\lambda)}})$. Let us choose $\ell(p) < R < \tilde{r}_0(p)$. Our goal is to apply Proposition 2.3 to the triangle p_0, p_1, p. The lower bound κ on the Ricci curvature in the ball $B(p,R)$ can be controlled by means of Lemma 3.3; it follows that

$$(3.11) \qquad s_{\kappa\ell(p)^2}(1) \le \frac{5}{6}\,\frac{\sqrt{1+8C_0(\lambda)}}{\sqrt{2C_0(\lambda)}}\,\sinh\frac{6}{5}\,\frac{\sqrt{2C_0(\lambda)}}{\sqrt{1+8C_0(\lambda)}}$$

$$< \frac{5}{3}\sinh\frac{3}{5} < \frac{17}{16} \quad .$$

In order to be able to use Proposition 2.3, it is therefore sufficient to give a weak upper bound for the Laplacian of the excess function e on the ball $B(p,R)$. Upper barriers at some point $q \in B(p,R)$ can be defined as in the proof of Corollary 2.4; we select minimizing unit speed geodesics γ_0, γ_1 from p_0, p_1 to q. For small $\delta > 0$ and all $x \in B(p,R)$ we define:

$$(3.12) \qquad e_q^\delta(x) = 2\delta + d(\gamma_0(\delta),x) + d(\gamma_1(\delta),x) - L,$$

Again, the point $q = \gamma_0(r_0(q)) = \gamma_1(r_1(q))$ lies neither on the cut locus of $\gamma_0(\delta)$ nor on the cut locus of $\gamma_1(\delta)$. The distance functions $d_{\gamma_0(\delta)}$ and $d_{\gamma_1(\delta)}$ are differentiable along the curves $\delta_0 \mid (\delta, r_0(q)]$ and $\gamma_1 \mid (\delta, r_1(q)]$, respectively. In particular, the differential inequality (3.10′) holds along both these geodesics.

Since $Ric|_{\gamma_0(r)} \ge -(n-1)\lambda(r)$, comparison of (3.10′) with the Riccati equation (3.7) yields

$$(3.13) \qquad \Delta d_{\gamma_0(\delta)} \le (n-1)w_\delta^\infty(r_0(q)).$$

Since λ is supposed to be non-increasing, it follows from the triangle inequality that $Ric|_{\gamma_1(L-r)} \ge -(n-1)\lambda(r)$. As the parametrization has been reversed, the differential inequality for $\frac{1}{n-1}\Delta d_{\gamma_1(\delta)}$ must be compared to the Riccati equation (3.4) rather than (3.7). We conclude that

$$\Delta d_{\gamma_1(\delta)}(q) \le (n-1)\cdot u^{\infty}_{L-\delta}(L - r_1(q)). \tag{3.14}$$

Our estimates above verify that each function e^{δ}_{q} is an upper barrier for the excess function e at q when restricted to a suitable neighborhood U^{δ}_{q} of this point. It satisfies

$$\Delta e^{\delta}_{q}(q) \le (n-1)\cdot\left[w^{\infty}_{\delta}(r_0(q)) + u^{\infty}_{L-\delta}(L-r_1(q))\right] . \tag{3.15}$$

In the limit $\delta \to 0$ the right-hand side of (3.15) converge to $(n-1)[w^{\infty}_{0}(r_0(q)) + u^{\infty}_{L}(L-r_1(q))]$. Therefore Proposition 2.3 yields

$$\begin{aligned} &e(p) \le 2\,\frac{n-1}{n-2}\cdot(\tfrac{1}{2}C_3\ell(p)^n)^{1/n-1}, \quad \text{where} \\ &C_3 = \frac{n-1}{n}\cdot(\tfrac{17}{16})^{n-1}\cdot\left[w^{\infty}_{0}(r_0(p) - R) + \sup_{|r|\le R} u^{\infty}_{L}(L-r_1(p)-r)\right] . \end{aligned} \tag{3.16}$$

Of course, this estimate can be slightly improved by taking the limit $R \to \ell(p)$. From 3.2 and 3.3 we conclude that

$$\sup_{|r|\le\ell} u^{\infty}_{L}(L-r_1(p)-r) \le \sup_{|r|\le\ell} (u_{\infty}(L-r_1(p)-r) + \frac{1}{r_1(p)+r})$$

$$\leq \quad \sup_{|r|\leq \ell} \left(\frac{\sqrt{2C_0(\lambda)}}{L-r_1(p)-r} + \frac{1}{r_1(p)+r}\right)$$

$$\text{(3.17)} \qquad \leq \quad \sup_{|r|\leq \ell} \left(\frac{3}{2} \cdot \frac{\sqrt{1+8C_0(\lambda)}}{2r_0(p)-3r} + \frac{1}{r_0(p)+r}\right)$$

$$\leq \quad \left(\frac{6}{7} + \sqrt{1+8C_0(\lambda)}\right) \frac{1}{r_0(p)}.$$

Here we have used the inequalities $L - r_1(p) \geq r_0(p) - 2\ell(p) \geq \frac{2}{3}r_0(p)$ and $r_1(p) \geq L - r_0(p) \geq r_0(p)$, and the assumption $\ell(p) < \frac{1}{6}\, r_0(p)$ itself. Similarly, Lemma 3.4 yields that

$$\text{(3.18)} \qquad w_0^\infty(r_0(p) - \ell(p)) \leq \frac{3}{5}(1 + \sqrt{1+8C_0(\lambda)})\, \frac{1}{r_0(p)} .$$

Combining (3.16) through (3.18), we obtain

$$r_0(p)\cdot(2e(p))^{n-1} \leq \left(\frac{17}{4}\, \frac{n-1}{n-2}\ell(p)\right)^n \cdot \frac{8}{5} \cdot \frac{2}{17}\, \frac{n-2}{n}\, (1+\sqrt{1+8C_0(\lambda)}) \quad ,$$

and the Proposition is proved.

□

4. A New Critical Point Lemma.

Before we can establish our main theorem, we need to recall another concept:

Fix a point $p_0 \in M^n$ and consider the distance function $r_0(p) = d(p_0,p)$ on M. A point $p \in M^n$ is a critical point of r_0, if and only if for any non-zero tangent vector $v \in T_pM^n$ there is a minimizing geodesic γ_0 to p_0 such that $\sphericalangle(\gamma_0'(0),v) \le \frac{\pi}{2}$. It is easy to define a continuous gradient-like vector field $\tilde{v}$ on the complement of the set of critical points of r_0, which gives rise to the

Isotopy Lemma: (c.f. [GS],[G])

Let $0 < \rho_1 < \rho_2$, and C a connected component of $\overline{B(p_0,\rho_2)}\backslash B(p_0,\rho_1)$. Let U be an open neighborhood of C. Suppose that C contains no critical point of r_0. Then there exists

i) an isotopy from $\overline{B(p_0,\rho_2)}$ to $\overline{B(p_0,\rho_2)}\backslash C$ which is the identity map outside of U, and

ii) an isotopy from $M^n\backslash B(p_0,\rho_2)$ to $(B(p_0,\rho_2) \cup C)$ which is the identity map outside U.

The hypthesis on sectional curvature enters the proof of Theorem B through the following

Lemma 4.1. (Critical Point Lemma)

Let M^n be a complete Riemannian manifold with base point p_0, and let $p \in M^n$ be a critical point of r_0. Suppose that:

0) there is a non-increasing function $\lambda : [0,\infty) \to [0,\infty)$ such that $C_0(\lambda) = \int_0^\infty r\lambda(r)dr$ converges and that $\mathrm{Ric}|_q \geq -(n-1)\cdot\lambda\circ r_0(q)$ at all points $q \in M^n$,

i) the sectional curvatures of M^n are bounded from below by $-\Lambda^2$ where Λ is some positive constant, and

ii) $$r_0(p) \geq R_0 := \max\left(\frac{3}{2\Lambda}, \frac{4}{17\Lambda}\left(1 + \sqrt{1+8C_0(\lambda)}\right)\right).$$

Then any minimizing geodesic from p_0 to a point in $B(p, C_4\Lambda^{-1+1/n}r_0(p)^{1/n})$, when extended beyond its endpoint, will meet the cut locus C_{p_0} of p_0, before its length exceeds $2r_0(p)$.

Here $C_4 = C_4(n,\lambda) = \frac{4}{17}\frac{n-2}{n-1}\left(\frac{5}{\sqrt{1+8C_0(\lambda)}}\right)^{1/n}$ as in Proposition 3.1.

<u>Proof:</u> Assume on the contrary that there is a minimizing geodesic γ from p_0 to some point $p_1 \in S(p_0, 2r_0(p))$ such that $d(p,\gamma) \leq C_4\Lambda^{-1+1/n}r_0(p)^{1/n}$. Because of hypothesis (ii) we know that

$$\min\left(\frac{1}{6}r_0(p), \frac{r_0(p)}{\sqrt{1+8C_0(\lambda)}}\right) \geq C_4 \cdot \Lambda^{-1+1/n} \cdot r_0(p)^{1/n}. \tag{4.1}$$

Therefore Proposition 3.1. implies that $e(p) \leq \frac{1}{2\Lambda}$.

On the other hand, we can reason as in the proof of the standard critical point lemma: Let γ_1 be a minimizing unit speed geodesic from p to p_1. Since p is a critical point of r_0, there exists a minimizing geodesic γ_0 from p to p_0 such that $\sphericalangle(\gamma_0'(0), \gamma_1'(0)) \leq \frac{\pi}{2}$. Let us consider the points $\tilde{p}_0 = \gamma_0(\rho)$ and $\tilde{p}_1 = \gamma_1(\rho)$, where $\rho = \frac{5}{4\Lambda}$. The triangle inequality implies $e(p) = 2\rho - d(\tilde{p}_0,\tilde{p}_1)$. Applying Toponogov's theorem and the Law of Cosines to the isosceles triangle $\tilde{p}_0,p\tilde{p}_1$, we obtain that $\cosh \Lambda d(p_0,p_1) \le \cosh^2 \Lambda\rho$. Altogether:

$$(4.2) \qquad e(p) \ge 2\rho - \Lambda^{-1} \operatorname{arcosh}(\cosh^2 \Lambda\rho) > \frac{1}{2\Lambda} \quad .$$

This contradicts the upper bound for the excess obtained from Proposition 3.1, and the lemma is proved.

□

Remarks: i) Recall that the standard critical point lemma is proved by applying Toponogov's theorem twice (c.f. [G], [GS]). We have replaced one of these steps by our estimate in Proposition 3.1. This way we can make use of a lower bound for Ricci curvature, which in our case is quantitatively considereably stronger than the lower bound for sectional curvature. The price paid for working with the weaker notion of curvature is that we can only control the height $d(p,\gamma)$ of the triangle p_0,p,p_1 from below, rather than its angle at p_0.
ii) Since the function $2\rho - \Lambda^{-1}\cdot \operatorname{arccosh}(\cosh^2 \Lambda\rho)$ is monotonically increasing in ρ and bounded by $\Lambda^{-1}\ell n(2)$, it is clear that we are not losing much when choosing ρ to be $\frac{5}{4\Lambda}$ in the proof of the lemma. We emphasize that Toponogov's theorem is only needed to get a uniform estimate for the excess of the a priori bounded triangles $\tilde{p}_0,p\tilde{p}_1$. This suggests that a lower bound for sectional curvature which we have required in Lemma 4.1 might just be a technical hypothesis. It is an open question whether there is a critical point lemma which involves only a lower bound on the Ricci curvature.

Roughly speaking, Lemma 4.1 confines the sice of the set of critical points. This restriction, which is non-trivial on all complete Riemannian manifolds, can be made more explicit for spaces satisfying a suitable diameter growth condition.

Proposition 4.2:

Let M^n be a complete Riemannian manifold with base point p_0. Suppse that

0) there is a non-increasing function $\lambda : [0,\infty) \to [0,\infty)$ such that $C_0(\lambda) = \int_0^\infty r\lambda(r)dr$ converges and $Ric|_q \geq -(n-1)\lambda \circ r_0(q)$ at all points $q \in M^n$,

i) the sectional curvatures are uniformly bounded from below by $-\Lambda^2$ where Λ is some positive constant, and

ii) there exists $R_1 > 0$ such that $diam(p_0,r) < C_4(n,\lambda)\Lambda^{-1+1/n}r^{1/n}$ for all $r > R_1$.

Then, all critical points of r_0 are contained in the union of $\overline{B(p_0,R_2)}$ and all bounded components K of $M^n\setminus\overline{B(p_0,R_2)}$, where $R_2 = \max(R_1, \frac{3}{2\Lambda}, \frac{4}{17\Lambda}(1+\sqrt{1+8C_0(\lambda)}))$.

Notice that the constant $C_4(n,\lambda)$ contains a factor $\frac{n-2}{n-1}$, and so hypothesis (ii) implies that M^n has dimension $n \geq 3$.

Proof: Assume on the contrary that there is a critical point p of r_0 which lies in an unbounded component C of $M^n\setminus\overline{B(p_0,R_2+\delta)}$, for some $\delta > 0$. By the Hopf-Rinow theorem there exists a sequence of points $p_j \in C$,

$1 \le j < \infty$, such that $r_0(p_j) \to \infty$ in the limit $j \to \infty$. Let γ_j be a minimizing geodesic from p_0 to p_j. It is a standard fact that a subsequence of these geodesics γ_j converges towards a ray γ emanating from p_0.

Clearly $\gamma[R_2 + \delta, \infty] \subset C$. On the other hand our critical point lemma implies that $d(p,\gamma) \ge C_4(n,\lambda) \cdot \Lambda^{-1+1/n} r_0(p)^{1/n}$. Using hypothesis (ii) we conclude that p and $\gamma \circ r_0(p)$ lie in <u>different</u> connected components of the distance sphere $S(p_0, r_0(p))$.

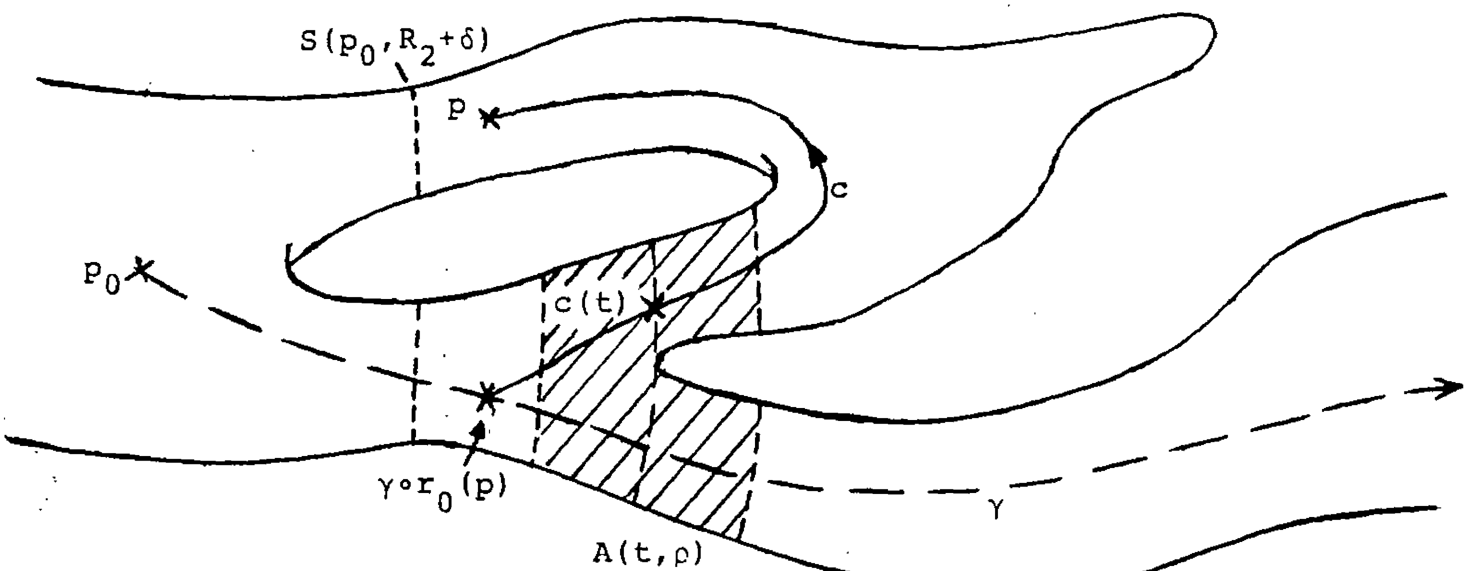

Figure 3

This figure depicts the basic problem which has been taken care of in the proof of Proposition 4.2. Our reasoning is essentially that on the one hand the annulus $A(t,\rho)$ must contain a critical point of r_0, since the curve $c(t)$ leaves the component $\Sigma(t)$ of the distance sphere $S(p_0, r_0 \circ c(t))$, while on the other hand it cannot contain such a point by the estimate given in Lemma 4.1.

From the way the ray γ has been constructed, it is clear that there is a rectifiable curve $c : [0,1] \to C$ such that $c(0) = \gamma \circ r_0(p)$ and $c(1) = p$. Let $\Sigma(t)$ be the connected component of $S(p_0, r_0(c(t)))$ which contains $\gamma \circ r_0(c(t))$. Consider the set

(4.3) $$A = \{t \in [0,1] \mid c(t) \in \Sigma(t)\}.$$

Now $0 \in A$, $1 \notin A$, and $A \subset [0,1]$ is a closed subset. Our indirect proof will be accomplish by deriving the contradiction that A is an open subset of $[0,1]$ as well. For this purpose let us pick some $t \in A$. By hypothesis (ii), there is an ϵ-neighborhood $U_\epsilon\Sigma(t) = \{x \in M^n \mid d(x,\Sigma(t)) \le \epsilon\}$ of $\Sigma(t)$ such that

(4.4) $$\operatorname{diam} U_\epsilon\Sigma(t) < C_4(n,\lambda)\cdot\Lambda^{-1+1/n} \cdot (r_0 \circ c(t) - \epsilon)^{1/n}.$$

Choosing $\rho > 0$ sufficiently small, we may assume that the intersection of the annulus $A(t,\rho) = \overline{B(p_0, r_2 \circ c(t)+\rho)} \setminus B(p_0, r_0 \circ c(t)-\rho)$ and $U_\epsilon\Sigma(t)$ is a connected component of $A(t,\rho)$. In light of Lemma 4.1, inequality (4.4) implies that the component $A(t,\rho) \cap U_\epsilon\Sigma(t)$ contains no critical point of r_0, and therefore the isotopy lemma applies to this piece of the annulus. In this context let us consider an open neighborhood $U(t)$ of t in $[0,1]$ such that $\Sigma(t') \subset A(t,\rho) \cap U_\epsilon\Sigma(t)$ for all $t' \in U(t)$. The isotopies of the set $A(t,\rho) \cap U_\epsilon\Sigma(t)$ in its neighborhood $U_\epsilon\Sigma(t)$, which we have obtained above, show that $c(t') \in \Sigma(t')$ for all $t' \in U(t)$. Hence $U(t) \subset A$, i.e. t is an interior point of the subset $A \subset [0,1]$.

□

Proof of Theorem B:

In dimension $n = 2$, we are just dealing with asymptotically nonnegative sectional curvature, and Theorem B turns out to be an easy Corollary to the proof of the Betti numbers theorem as given in [A].

In the general case when M^n has dimension $n \geq 3$ it is evident that there exists some radius $R_1 > 0$ such that $\mathrm{diam}(p_0, r) < C_4(n,\lambda)\Lambda^{-1+1/n} r^{1/n}$ for all $r > R_1$, simply because we are assuming that M^n has diameter growth of order $o(r^{1/n})$. Hence it follows from Proposition 4.2 that <u>all critical points</u> of r_0 are contained in some large ball $B(p_0,R)$. Notice that we do <u>not</u> claim that $M^n \backslash \overline{B(p_0,R_2)}$ has only finitely many bounded connected components K; this is only true for the complement of a generic closed ball. Anyway, all but finitely many of the connected components K are contained in $B(p_0,2R_2)$, and this is all we have used.

Since M^n is connected, the other assertions in Theorem B follow now by standard isotopy arguments.

□

Theorem A is a special case of Theorem B, and so we have proved it as well. Finally let us point out that, in case one only wants to deal with manifolds of nonnegative Ricci curvature, the isotopy arguments in the proof of Proposition 4.2 are not needed. Instead we could refer to the following

<u>Proposition 4.3.</u>

Let M^n be a complete Riemannian manifold with $\mathrm{Ric} \geq 0$. Then

$$\#\mathrm{im}(j_*^{\Omega} : \pi_1(M^n) \to \pi_1(M^n/(M^n \backslash \Omega)) \leq 2 \tag{4.5}$$

for any bounded domain $\Omega \subset M^n$. Moreover, given any ball $B(p_0,r) \subset M^n$, the boundary of each component of the complement $M^n \backslash B(p_0,r)$ must be connected.

<u>Proof:</u> As any complete manifold with two or more ends contains a line, we conclude from the Cheeger-Gromoll splitting theorem that the universal

covering $\tilde{M}^n$ has at most two ends. Now (4.5) follows by counting the preimages in $\tilde{M}^n$ of the point $[M^n \setminus \Omega]$ in the quotient space $M^n/(M^n \setminus \Omega)$, using the commutative diagram:

$$\begin{array}{ccc} \tilde{M}^n & \xrightarrow{\ \tilde{j}^\Omega\ } & \widetilde{M^n/(M^n \setminus \Omega)} \\ \downarrow & & \downarrow \\ M^n & \xrightarrow{\ j^\Omega\ } & M^n/(M^n \setminus \Omega) \end{array} .$$

Suppose there is a ball $B(p_0,r)$ such that the boundary of $M^n \setminus B(p_0,r)$ has two or more connected components. Then $\pi_1(M^n)$ contains an infinite cyclic group by van Kampen's theorem. Now a contradiction to inequality (4.5) arises, since j_*^Ω is injective on this infinite subgroup of $\pi_1(M^n)$, provided Ω is chosen sufficiently large.

□

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