Generalized graph-manifolds and their effective recognition

S. V. Matveev

1. Introduction.

Let M be a compact orientable irreducible 3-manifold whose boundary is either empty or consists of incompressible tori. It follows from the famous Torus Decomposition Theorem [1, 2] that M contains a canonical finite system of incompressible tori that decomposes M into Seifert manifolds and manifolds that contain no essential tori and annuli. It is convenient to describe the mutual position of these parts by certain graph just as it is made by investigation of graph-manifolds of F. Waldhausen [3] or by construction of topological invariants of integrable Hamiltonian systems [4]. In this sense any 3-manifold is a generalized graph-manifold. The original proofs of this result in [1] and [2] are rather complicated since at the same time they reveal many important properties of characteristic submanifolds. The proof presented here is much simpler and shorter.

The problem of finding out a canonical system of splitting tori admits an algorithmic solution. However, it is based on Haken's theory of normal surfaces and thus has a purely theoretical nature. We propose a partial practical algorithm that is based on the special spine theory [5] and can be realized by a computer (see [6]). The algorithm decomposes a given manifold into Seifert manifolds and into parts that it cannot recognize, and that often turn to be hyperbolic. We describe also a simple method (based on an observation of A. Casson) that in some cases allows to verify that a given unrecognized part contains no essential tori and annuli and thus possesses a hyperbolic structure. In case of successful work of both algorithms, we get an explicit description of M as a sum of Seifert and hyperbolic parts, together with information on how the parts are glued together. Since the decomposition is unique, these data can be considered as a "name" of the manifold that distinguishes it from all other manifolds. This reduces the recognition problem for manifolds to comparing their names.

The author wishes to thank the Russian Fund of Basic Research for financial support, and the Max-Planck-Institute in Bonn, where the paper was written. He thanks also C. Petronio for helpful discussions.

2. Generalized graph-manifolds.

Let us recall some basic notion of 3-manifold topology. A 3-manifold M is called *irreducible* if every 2-sphere $S \subset M$ bounds in M a 3-ball. A surface $F \subset M$ is *compressible* if there is a disc $D^2 \subset M$ such that $D^2 \cap F = \partial D^2$ and the curve ∂D^2 does not bound a disc in F. An *incompressible* surface admits no such discs. Two homeomorphic surfaces $F_1, F_2 \subset M$ are *parallel* if $F_1 \cap F_2 = \partial F_1 = \partial F_2$ and there is a map $\varphi : F_1 \times I \to M$ such that $\varphi(x, 0) = x$ for all $x \in F_1$, $\varphi(x, t) = x$ for all $x \in \partial F_1$ and all $t \in I$, $\varphi(F_1 \times \{1\}) = F_2$, and the restriction of φ onto $(\mathrm{Int}F_1) \times I$ is an embedding. An incompressible torus $T \subset M$ is *essential* if it is not parallel to a boundary torus. An *essential* annulus $A \subset M$ is a proper incompressible annulus in M that is not parallel to an annulus in ∂M.

Definition. *A compact orientable 3-manifold B is called an* elementary block *if one of the following conditions holds:*

(1) B is a Seifert manifold;

(2) B is irreducible , contains no essential tori and annuli, is not a Seifert manifold, and its boundary is either empty or consists of incompressible tori.

Blocks of the above types will be called Seifert *and* conditionally hyperbolic*, respectively.*

Remark. It follows from the proved part of W. Thurston's Geometrization Conjecture [7] that if a conditionally hyperbolic block is sufficiently large (that is, if it contains an incompressible surface, in particular, if its boundary is not empty), then its interior possesses a complete hyperbolic metric of a finite volume. The problem of the existence of conditionally hyperbolic but not hyperbolic blocks is equivalent to the still open part of Thurston's conjecture.

Definition. *A 3-manifold M is called a* generalized graph-manifold *if it can be obtained by pasting together elementary blocks along some homeomorphisms of their boundary tori.*

Note that graph-manifolds of F. Waldhausen [3] are by definition composed of Seifert blocks. Therefore, the generalization consists in involving conditionally hyperbolic blocks.

Theorem 1. *Any compact orientable irreducible 3-manifold M whose boundary is either empty or consists of incompressible tori is a generalized graph-manifold. Moreover, one can decompose M into elementary blocks using only essential tori.*

Proof. Let us cut M along essential tori as long as possible. The process is finite since by Haken Finiteness Theorem [8] any 3-manifold can contain only a finite number of pairwise non-parallel disjoint incompressible surfaces. We get a finite collection of irreducible manifolds that contain no essential tori anymore.

Let M_i be one of them. If M_i contains no essential annuli, then it is either a conditionally hyperbolic block or a closed Seifert manifold without essential tori. The latter is possible only when M fibers over the 2-sphere with ≤ 3 exceptional fibers or over the projective plane with ≤ 1 exceptional fibers. Let us prove that if M_i contains an essential annulus A, then it is a Seifert manifold. Denote by L the connected component of the union $A \cup \partial M_i$ containing A, and by Y^3 a regular neighborhood of L in M_i. Since the boundary circles of A are non-trivial in ∂M_i, the natural fibration of A into circles can be extended to a fibration of L over a graph G consisting of three edges and two vertices of valence 3. It follows that Y^3 also fibers into circles. The base of this fibration is either a disc with two holes or a Möbius strip with one hole (the latter is possible when the coherently oriented boundary circles of A lie on the same boundary torus of M_i, and have opposite orientations there).

Since M_i contains no essential tori, any boundary torus T of Y^3 that lies inside M_i is either parallel to the boundary or compressible. In the second case the torus T, like any other compressible torus in an irreducible manifold, bounds a solid torus in M_i. The meridian of the solid torus is not isotopic to a fiber of the fibration, since otherwise the annulus A would be compressible. It follows that M_i is a Seifert manifold. ∎

Theorem 1 gives a simple method for presenting 3-manifolds: one should present by a graph the gluing scheme of elementary blocks, and describe the elementary blocks themselves as well as gluing homeomorphisms of their boundary tori. Unfortunately, any manifold can be decomposed into blocks in many different ways. In this and the next Sections we show that by amalgamating blocks one can get a minimal decomposition, which is unique up to isotopy.

Definition. *A finite collection $T_1, \dots, T_n$ of disjoint incompressible tori in a 3-manifold M is called a* minimal splitting system *if the following conditions hold:*

1. *The tori decompose M into elementary blocks;*
2. *If a torus T_i separates two (possibly coinciding) Seifert blocks, then for any choice of Seifert fibrations on the blocks the induced circle fibrations of T_i are not isotopic.*

Note that any compact orientable irreducible 3-manifold M whose boundary is either empty or consists of incompressible tori contains a minimal splitting system. To construct it, one should find a decomposition of M into elementary blocks with incompressible boundary (such a decomposition exists by Theorem 1). Then one should simplify it step by step as long as possible. Each step consists in finding out a splitting torus T_i that violates condition 2, and removing it from the splitting system. At last we get a minimal splitting system.

3. Rough and thin tori.

Definition. *An essential torus T in a compact irreducible 3-manifold M is called* thin *if there is a Seifert manifold $Q \subset M$ such that:*

(1) all the boundary tori of Q are incompressible in M;

(2) T lies in Q and is essential in it.

Essential tori in M that are not thin, are called rough. *(The terminology is adopted from Hamiltonian mechanics, see [4]).*

Lemma 2. *For any rough torus $T_1 \subset M$ and any incompressible torus $T_2 \subset M$ there is an isotopy $\varphi_t : M \to M, 0 \leq t \leq 1$, such that $\varphi_1(T_1) \cap T_2 = \emptyset$.*

Proof. Recall that the tori T_1, T_2 are incompressible. Using an innermost circle argument, one can remove all trivial curves in $T_1 \cap T_2$. Then the union $T_1 \cup T_2$ fibers into circles, and the fibration can be extended to a regular neighborhood $N = N(T_1 \cup T_2)$. Some boundary tori of N may be compressible, but then they bound in M solid tori outside N. Meridians of the solid tori are not isotopic to a fiber of N, since otherwise the fiber would be contractible and T_1, T_2 compressible. Therefore, the union Q of N with the solid tori is a Seifert manifold. It follow from the definition of rough torus that the torus $T_1 \subset Q$ must be parallel to the boundary of Q. Hence it can be shifted by an isotopy from Q and thus from $T_2 \subset Q$. ∎

Remark. Of course, any finite collection of disjoint rough tori can be shifted by an isotopy from any other collection of disjoint incompressible tori. Moreover, the isotopy can be chosen to be fixed on those tori of the first collection that are disjoint with the tori of the second collection. The proof remains actually the same.

Definition. *A finite collection of disjoint non-parallel rough tori in a compact orientable irreducible 3-manifold M is called* maximal *if any rough torus in M disjoint with the tori from the collection is parallel to one of them.*

Theorem 3. *For any compact orientable irreducible 3-manifold M a maximal system of rough tori exists and is unique up to isotopy.*

Proof. We shall add to the empty system new rough tori as long as possible, shifting each next torus from the previous ones by Lemma 2. Since by Haken's Finiteness Theorem [8], any manifold can contain only a finite number of disjoint non-parallel incompressible surfaces, the process terminates after a finite number of steps. Therefore, a maximal system does exist.

Let us prove that it is unique. Suppose that $R = \{T_1, \ldots, T_k\}$ and $R' = \{T'_1, \ldots, T'_m\}$ are two maximal systems of rough tori. By Lemma 2, one can assume that the tori of both systems are disjoint. Since the system R is maximal, each torus from the system R' is parallel to a torus of R, and no two tori of R' are parallel to the same torus of R. We may conclude that the system R' is isotopic to a part of R. Since R' is also maximal, this part coincides with R. ∎

Theorem 4. *For any compact orientable irreducible 3-manifold M whose boundary is either empty or consists of incompressible tori, every minimal splitting system of tori that decomposes M into elementary blocks is a maximal system of rough tori.*

Proof. Let $\mathrm{S} = \{T_1, \ldots, T_n\}$ be a minimal splitting system and $\mathrm{R} = \{T_1', \ldots, T_m'\}$ a maximal system of rough tori. Note that each rough torus T_i' can be shifted by an isotopy from the splitting tori, and after that it will be contained in a Seifert or conditionally hyperbolic block B. In both cases it must be parallel to a boundary component of B (by definition of a rough torus in the first case, and by definition of a conditionally hyperbolic block in the second one). It follows that $m \leq n$, and that any rough torus is parallel to a splitting torus.

Let us prove that any splitting torus $T = T_j$ from the minimal system is rough. If T is thin, then it is contained in a Seifert manifold $Q \subset M$ with incompressible boundary, and is essential there. Denote by Q_T the Seifert manifold obtained from Q by cutting along T, and by T_-, T_+ the copies of T in ∂Q_T. Since O_T is not homeomorphic to $T^2 \times I$ or $D^2 \times S^1$, one can find saturated (that is, consisting of fibers) essential annuli A_-, A_+ in Q_T such that $\partial A_- \subset T_-, \partial A_+ \subset T_+$. Using an isotopy of the annuli, remove all trivial circles in their intersection with the splitting tori, and all non-essential annuli in the intersection of A_-, A_+ with blocks. Then A_-, A_+ will intersect the blocks B_-, B_+ adjacent to T along essential annuli. Hence the blocks B_-, B_+ cannot be conditionally hyperbolic. It means that B_-, B_+ are Seifert manifolds. Recall that any essential annulus in a Seifert manifold is saturated with respect to a Seifert structure. Therefore, the Seifert structures on blocks B_-, B_+ can be chosen so that their fibers on T are isotopic. This contradicts to the definition of minimal splitting system. The argument works also in case $B_- = B_+$. Then the block has ≥ 2 boundary components, and contains an essential annulus having both boundary circles in one of them. Any such Seifert manifold admits only one Seifert structure (up to isotopy). ∎

4. Simple and special spines.

Recall that a compact two-dimensional polyhedron P is called *simple* if locally it is modeled on the cone over a complete graph with 4 vertices, see Fig. 1.

It means that P is allowed to have only those local singularities that are present in the cone. To be more precise, the link of each point of P must be homeomorphic to one of the following graphs: (1) a circle; (2) a circle with a diameter; (3) a circle with three radii; (4) a segment; (5) a wedge of three segments with a common endpoint. The union of points having links of types (4) and (5) is a one-dimensional polyhedron, which is called the *boundary* of P and denoted by ∂P. The connected components of the set of type (1) points, that is, of non-singular points, are called *2-components* of P. They are two-dimensional manifolds without boundary. Type (3) points are called *genuine vertices* of P

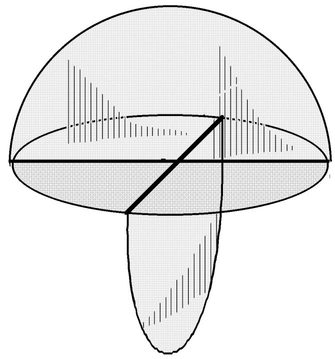

Figure 1: The cone over a complete graph with 4 vertices

(in order to distinguish them, say, from vertices of cell decompositions).

Unfortunately, the property of being simple is not hereditary since a subpolyhedron of a simple polyhedron is not necessarily simple. For example, it may contain a purely one-dimensional part consisting of *main* edges that are adjacent to no 2-components. It is convenient to extend the class of simple polyhedra to a minimal class that contains simple polyhedra and possesses the hereditary property.

Definition. *A two-dimensional polyhedron P is called* almost simple*, if it can be embedded in a simple polyhedron.*

It is easy to see that P is almost simple if and only if the link of any of its points can be embedded into the complete graph with 4 vertices. As above, points, whose links are homeomorphic to the graph, are called *genuine vertices*.

The difference between simple and almost simple polyhedra is not very essential. It is not hard to show that any almost simple polyhedron can be collapsed either onto a graph, or onto a collection of simple polyhedra joined by arcs.

Definition. *A simple polyhedron with empty boundary is called* special *if it has at least one genuine vertex, and all its 2-components are 2-cells.*

Simple and special polyhedra appear in three-dimensional topology in a natural way as spines of 3-manifolds.

Definition. *A subpolyhedron P of a 3-manifold M with non-empty boundary is called a* spine *of M if the difference $M \setminus P$ is homeomorphic to $\partial M \times (0,1]$. By a spine of a closed manifold M we mean a spine of of $M \setminus IntD^3$, where D^3*

is a ball in M.

A spine P is called *simple*, *almost simple* or *special*, if it is a polyhedron of the corresponding type.

It is well known that any compact manifold has a simple (and even a special) spine. Moreover, a 3-manifold can be reconstructed from its special spine in a unique way. For simple spines this is not true, since different 3-manifolds may have homeomorphic simple spines. Nevertheless, simple spines give a convenient way for presenting 3-manifolds. Certainly, they should be equipped with additional information sufficient for recovering the corresponding 3-manifolds.

Let $P \subset M$ be a simple spine presented as a CW-complex. Then M can be identified with a close regular neighborhood of P. Therefore, the cellular structure of P induces a decomposition of the boundary of M into cells. There is a natural map $\varphi : \partial M \to P$ that takes cells to cells and preserves the incidence relation. The inverse image by φ of any 2-cell in P consists of two 2-cells. The inverse image of an edge can consist of 1, 2 or 3 edges while the one of a vertex can contain even 4 points. Information on the cellular structure of ∂M and on the correspondence between 2-cells induced by φ is quite sufficient for recovering M.

Remark. (On presentation of manifolds in computer's memory.) Any 2-dimensional CW-complex can be presented in a computer by storing information on the number of cells, on the incidence relation between the vertices and oriented edges, and on attaching maps of 2-cells. The attaching maps can be presented by strings of integers with signs that show how the boundary curves of the 2-cells run over oriented edges. A presentation of a 3-manifold M with a simple spine P should consist of cellular presentations of P and ∂M, together with the correspondence between their cells. All transformations of manifolds and spines that are described below can be realized on this level.

5. Simplifying simple spines.

The recognition algorithm mentioned in Introduction is based on the following fact (the theoretical background will be given in section 6): the structure of a minimal spine reflects adequately the graph-structure of the corresponding manifold. Let us describe simplifying moves on simple spines. They can be decomposed into three groups:

(A) moves that change only the cellular structure of the spine;

(B) moves that modify the spine but preserve the manifold;

(C) moves that change both the manifold and the spine. In this case one should keep in memory additional information sufficient for recovering the original manifold from its simplified copy.

Let P be a simple spine of a compact orientable 3-manifold M.

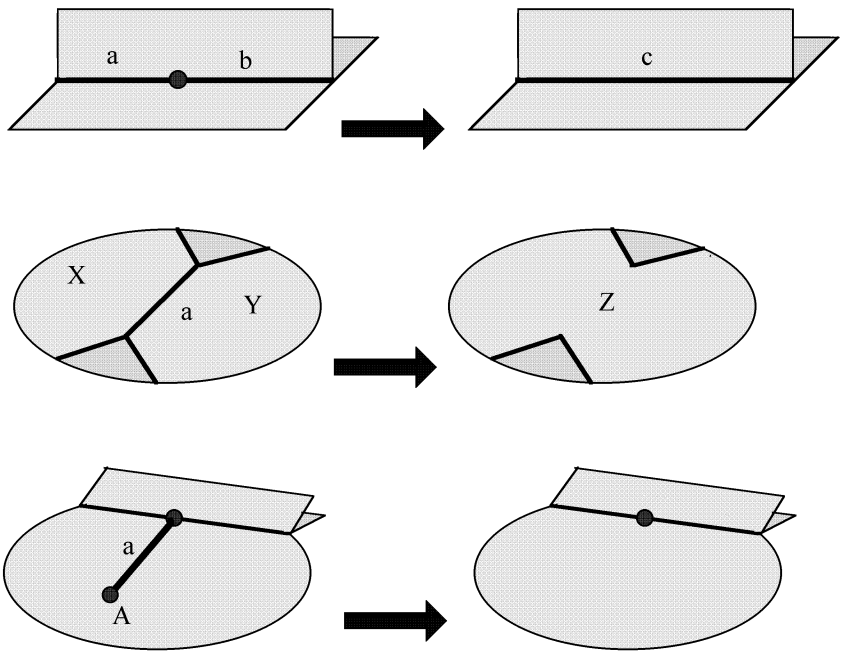

Figure 2: First three simplification moves

Move 1 (removing a vertex of valence 2). Suppose P contains a vertex A adjacent to exactly two different edges a, b. Then we amalgamate the edges to a new edge c by erasing the vertex A.

Move 2 (amalgamating 2-cells). Suppose P contains an edge a adjacent to exactly two different 2-cells X, Y. Then we amalgamate the cells to a new 2-cell Z by erasing the edge a.

Move 3 (removing a needle). Suppose P contains an edge a of valence 2 that has a free vertex. Then we remove the edge a together with the free vertex.

Note that Moves 1 – 3 have type (A), that is, they change neither the spine, nor the manifold, see Fig. 2.

Move 4 (collapsing). Suppose a 2-cell α of P has a free edge a. Then we remove the interior of α together with the free edge.

Remark. Applying Move 4, one should be careful, since one can get an almost simple but not simple spine. It happens when α has another free edge b; after the removal it becomes main. To avoid this, in such cases one should remove only a part of the 2-cell α, retaining a thin strip along the edge b.

Move 5 (admissible 2-cell repasting). Suppose D^2 is a disc in M such that $D^2 \cap P = \partial D^2$ and the curve ∂D^2 is in general position in P. Then D^2

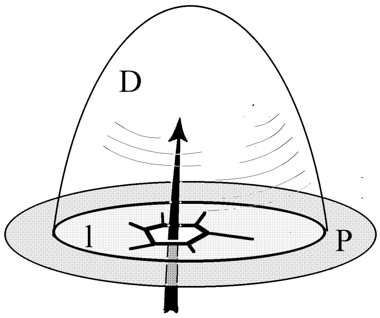

Figure 3: Repasting of a 2-cell

cuts off $M \setminus P$ a ball B^3. Choose in the simple polyhedron $P \cup D^2 \subset M$ a 2-cell α that separates $M \setminus P$ and B^3, and remove it, see Fig. 3. We get a new spine P_1 of M. We say that P_1 is obtained from P by a *2-cell repasting move*. A 2-cell repasting is called *admissible* if the following conditions hold:

(1) the move does not increase the number $v(P)$ of genuine vertices of P, that is, $v(P_1) \leq v(P)$;

(2) $v(P \cup D^2) - v(P) \leq 4$.

Move 5 is a basic one, so we comment on it in more detail.

(i) One should remove the 2-cell α carefully and avoid an appearance of main edges. Just as by Move 4, one should retain thin two-dimensional strips along such edges.

(ii) Let us distinguish an important partial case of Move 5. Suppose a 2-component $\alpha \subset P$ is not a 2-cell. Then it contains a non-trivial orientation-preserving simple closed curve l, which can be shifted into ∂M in two different ways. Suppose that at least one of the circles thus obtained is trivial in ∂M. Then l bounds a disc $D^2 \subset M$ such that $D^2 \cap P = l$, and we can simplify P by Move 5.

(iii) There is an efficient way to find out all discs in M that satisfy the conditions $D^2 \cap P = \partial D^2$ and $v(P \cup D^2) - v(P) \leq 4$. To this end it is sufficient to enumerate all trivial circles in ∂M that intersect the inverse images (by $\varphi : \partial M \to P$) of triple edges of P in no more than 4 points. If l is such a circle, then the curve $\varphi(l) \subset P$ bounds a disc $D^2 \subset M$ such that $D^2 \cap P = \partial D^2$. It may happen that the inequality $v(P \cup D^2) - v(P) \leq 4$ does not hold, because of superfluous vertices that appear in self-intersection points of $\varphi(l)$. Nevertheless, all discs satisfying the inequality can be constructed in this way.

Note that Moves 4, 5 have type (B) since they modify only spines.

Move 6 (cutting a 2-component along a non-trivial circle.) Suppose that a 2-component $\alpha \subset P$ contains a non-trivial orientation-preserving simple closed curve l. Let $A \subset M$ be a proper annulus such that $A \cap \alpha = l$ and the intersection is transverse. Assume the boundary circles of A are non-trivial in ∂M (compare with commentary (ii) above). Removing from P an open regular neighborhood of l in α, and removing from M an open regular neighborhood of A in M, we get a new simple spine P_1 of a new 3-manifolds M_1. In other words, P_1 and M_1 are obtained from P and M by cutting along l and A, respectively. To describe the relation between M_1 and M, denote by L the connected component of $A \cup \partial M$ containing A. Let Y^3 be a regular neighborhood of L in M. Then M can be presented as M_1 with Y^3 attached along one or two boundary tori. Just as in the proof of Theorem 1, one can show that Y^3 fibers into circles over either a disc with two holes or a Möbius strip with one hole. Thus Move 6 consists in splitting off a Seifert block.

Move 7 (cutting a strip). Suppose P contains a simple arc l such that l lies inside a 2-cell, and its endpoints belong to different free edges. The move consists in cutting P along l. To describe the corresponding move on the manifold M, denote by D^2 a proper disc in M such that $D^2 \cap P = l$. Then the new manifold M_1 is obtained by cutting M along the disc D^2. In other words, M can be obtained from M_1 by attaching a handle of index 1. There are three cases:

(i) if ∂D^2 does not decompose ∂M, then $M = (M_1 \cup D^3)\#(D^2 \times S^1)$, where "$\cup D^3$" means that we fill up a spherical component of the boundary by a ball;

(ii) if D^2 does not decompose M, but ∂D^2 decomposes ∂M, then $M = (M_1 \cup D^3)\#(S^2 \times S^1)$;

(iii) if M_1 consists of two connected components M_1', M_1'', then $M = M_1'\#(M_1'' \cup D^3)$.

Move 8 (delicate piercing). Suppose that ∂M consists of at least two components, and at least one of them is a sphere. Then one can find a proper arc $l \subset M$ such that l intersects P in a point inside a 2-cell α, and joints a spherical component of ∂M with another one. Removing from P the 2-cell α (and retaining thin strips along main edges that can appear), we get a simple spine P_1 of a 3-manifold M_1 such that $M_1 = M \cup D^3$. The piercing is called *delicate* since the modification of M is very mild.

Move 9 (rough piercing). We apply this move only in the case when $\partial M = S^2$, and all other simplifying moves has been already exhausted. In particular, P should be a special spine.

Choose a 2-component $\alpha \subset P$ such that after removing α from P and collapsing the resulting polyhedron, we get a simple podpolyhedron $P_1 \subset M$ with the smallest possible number of vertices. Then P_1 is a simple spine of its regular neighborhood M_1 in M. Clearly, $M \setminus \mathrm{Int} M_1$ is a solid torus, that is, M is obtained from M_1 by Dehn filling.

6. The algorithm.

Let M be a compact orientable 3-manifold such that ∂M is either empty or consists of tori. We wish to decompose M into elementary blocks, that is, present it as a generalized graph-manifold.

STEP 1. Construct a simple spine P of M and present it as a CW-complex. Several methods for constructing simple spines are described in [5]. One can start with a Heegaard diagram of M, with a surgery presentation, triangulation, and so on.

STEP 2. Apply Moves 1 – 9 to P as long as possible. Since each move simplifies the spine, the process terminates after a finite number of moves.

We get a collection of special spines of manifolds such that the boundary of every manifold is non-empty and consists of tori. We will consider these manifolds as unrecognized (potentially, hyperbolic) blocks that should be tested for hyperbolicity (see Section 7).

STEP 3. Recall that each time when we applied moves of type (C) that modify the manifold, we stored the information on the modifications. In particular, we know the types of Seifert blocks that have been cut off by Move 6. We know also how they are glued together, and how the solid tori that might appear by Moves 7 and 9 are attached to them. Step 3 consists in amalgamating these blocks into bigger Seifert blocks when it is possible (see Section 2). As a result, we get a decomposition of M into the sum of Seifert blocks and unrecognized pieces given by their special spines.

Let us present a few theoretical and experimental facts that confirm indirectly that the algorithm is effective, i. e., it finds out the canonical decomposition in most cases.

Recall that in complexity theory of 3-manifolds (see [5]), a complexity of compact 3-manifold M is given by formula $c(M) = \min\ v(P)$, where $v(P)$ is the number of genuine vertices of P and the minimum is taken over all almost simple spines. An almost simple spine of a 3-manifold M is called *minimal* if it has exactly $c(M)$ vertices.

Conjecture 1. (**[6]**) *If a special spine of a compact 3-manifold M is not minimal, then the number of vertices can be decreased by admissible disc replacement moves.*

Conjecture 2. (**[6]**) *If a special spine of a closed graph-manifold is minimal, then any rough piercing (Move 9) transforms it into a spine of a graph-manifold.*

Both conjectures were verified by means of a computer for all closed manifolds up to complexity 8.

Theorem 5. *If a minimal almost simple spine of a compact 3-manifold M is special, then M is irreducible, has an incompressible (or empty) boundary and contains no essential annuli.*

For proof see [5, 6].

If follows from Theorem 5 that if Conjecture 1 is true, then all unrecognized blocks produced by the algorithm are irreducible, have incompressible boundary and contain no essential annuli. Unfortunately, it may happen that some of these blocks contain essential tori. A simple example can be obtained by pasting together the complementary space of the Whitehead link and a Seifert manifold whose boundary consists of exactly one incompressible torus.

Nevertheless, if both conjectures are true, then the algorithm would produce the a canonical decomposition for any graph-manifold of Waldhausen.

7. Recognition of hyperbolic blocks.

Consider a special polyhedron P. Denote by $N(V,P)$ a regular neighborhood in P of the set of its vertices. It consists of several copies of the cone over a complete graph with 4 vertices. The cones over open edges of the graph, that is, the connected components of the intersection of $N(V,P)$ with the open 2-cells of P, will be called *wings*. Thus P has $6n$ wings, where n denotes the number of its vertices. Two wings in a neighborhood of a vertex are called *opposite*, if the intersection of their closures consists of only one point (the vertex).

Let us assign to every wing a real number called its *weight* such that the weights of opposite wings are equal. The weights will be considered as variables. Thus there are $3n$ variables. We subject the variables to the following *consistency conditions*:

1. For any vertex of P, the sum of weights of all six wings adjacent to it must be equal to 2π;
2. For any 2-cell of P, the sum of weights of all wings contained in it must be equal to 2π.

The consistency conditions may be written as a system of linear equations. The system will be denoted by $S(P)$.

Remark. Suppose P is a spine a 3-manifold M with boundary. Then it induces a *dual triangulation* of $\mathrm{Int}M$ into 3-simplices with removed vertices. Let us realize the simplices as ideal tetrahedra in the hyperbolic space H^3. If we take the dihedral angles of the tetrahedra as the weights of corresponding wings, then the consistency conditions of the first type will be fulfilled automatically.

Consistency conditions of the second type correspond to the requirement that the sum of all dihedral angles around any edge of the dual triangulation should be equal to 2π. Thus the consistency conditions in our sense form exactly a half of consistency conditions of W. Thurston (see [9]). Instead of conditions on complex parameters of ideal tetrahedra, we impose the corresponding conditions only on their arguments.

Theorem 6. *Let P be a special spine of a 3-manifold M such that the* *system $S(P)$ has a positive solution. Then M is irreducible, ∂M consists of incompressible tori, and M contains no essential tori and annuli.*

By the proved part of Thurston's Geometrization Conjecture, Theorem 6 can be briefly reformulated as follows: if the system $S(P)$ has a positive solution, then M is hyperbolic.

Before proving Theorem 6, we recall some notions of normal surface theory of W. Haken [10]. Let ξ be a handle decomposition of a 3-manifold M consisting of handles of index) 0, 1, and 2. The handles are called *balls, beams* and *plates.* Connected components of intersections of balls with beams are called *islands*, and those with plates are called *bridges.* Define a *lake* to be a connected component of the intersection of balls with the boundary of M.

Definition. *A proper surface $F \subset M$ is called* normal *(with respect to decomposition ξ), if:*

(1) F intersects any plate $D^2 \times I$ along several parallel sheets of type $D^2 \times \{*\}$;

(2) the intersection of F with any beam $I \times D^2$ has the form $I \times L$, where L is a finite collection of disjoint proper arcs in D^2 containing no closed components;

(3) the intersection of F with any ball consists of discs (the discs are called elementary*);*

(4) the intersection of the boundary ∂E of any elementary disc E with any lake and any bridge consists of no more than one segment; moreover, if a lake and bridge are adjacent, then ∂E can intersect no more than one of them;

(5) let ∂E intersects a lake A along an arc l. Then the endpoints of l should be in different connected components of the intersection of A with islands.

We say that an elementary disc E has *type* (m, n) if the curve ∂E intersects m bridges and n lakes.

Recall that any special spine P of M generates a handles decomposition ξ_P of M. To construct it, one should replace vertices, edges, and 2-components of P by balls, beams, and plates, respectively. The boundary of each ball of ξ_P contains four islands, and any two of them are joined by exactly one bridge. It is easy to see that elementary discs for ξ_P can have the following types: (4,0), (3,0), (2,1), (1,2), (0,2), (0,3), (0,4). Each type determines the corresponding disc in a unique way (up to homeomorphisms of the ball taking islands to islands and bridges to bridges), except the type (0,3), which determines two elementary discs, see Fig. 4. It is well known that any closed surface F in M can be transformed to a normal one by isotopies, compressions and removals of local connected components (a surface is *local* if it contained in a ball). Each compression consists in cutting a thin tube along the boundary of its meridional disc, and

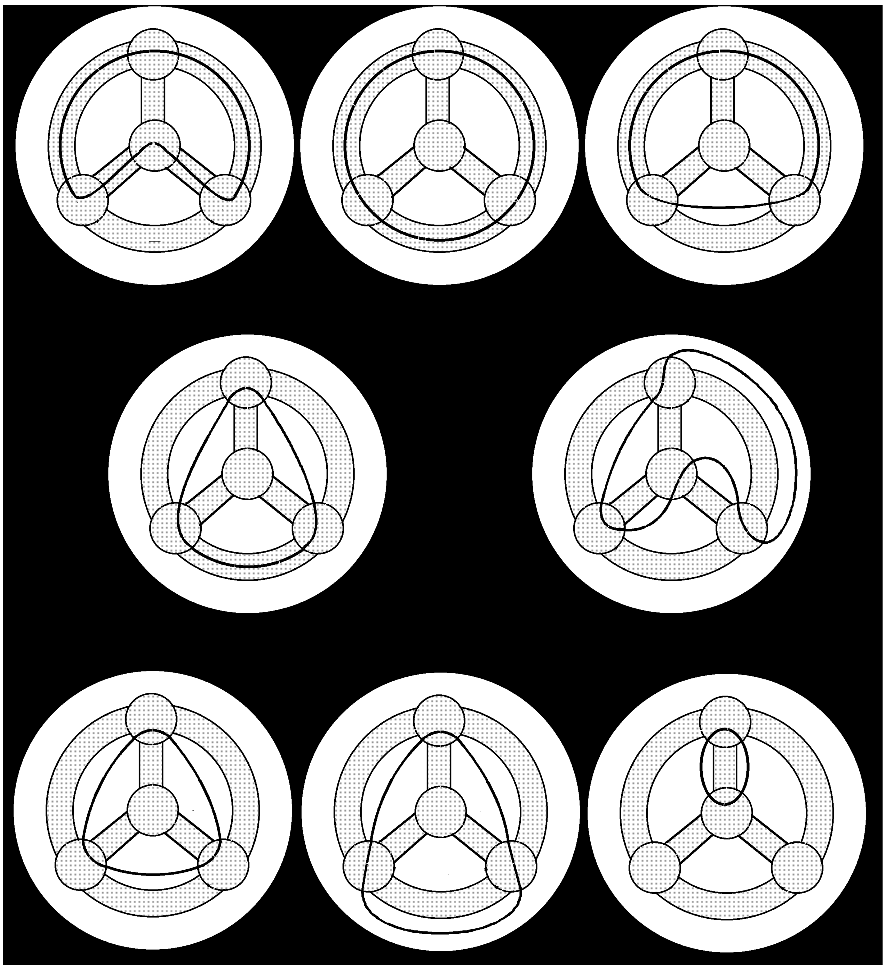

Figure 4: Eight types of elementary discs

gluing up two holes arising in this way by two copies of the disc. If F has a boundary, then one should allow additionally compressions along discs that have one arc of the boundary on F and the other on ∂M. Suppose M contains one of the following surfaces:

(1) a non-trivial sphere;

(2) a non-trivial disc;

(3) an incompressible annulus not parallel to the boundary;

(4) an incompressible torus not parallel to the boundary.

An analysis of the above transformations shows that then at least one of such surfaces can be found among normal ones.

To prove Theorem 6, it is sufficient therefore to show that M contains no normal spheres, discs, annuli, and tori except tori that are parallel to the boundary.

Proof of Theorem 6. Suppose M contains a surface F such that $\chi(F) \geq 0$ and F is normal with respect to ξ_P. Connected components of the intersection of F with balls, beams, and plates we will call *discs, strips, and patches*, respectively. The discs, strips, and patches form a handle decomposition η of F. Connected components of the intersection of patches with regular neighborhoods of discs are called *wings*. The wings of η correspond to the wings of P and thus have positive weights. Denote by k_s the number of discs in η having s attached strips (with multiplicity). Since any type (m, n) disc E has exactly $m + n$ attached strips, the possible values of s are 2, 3, 4. Note also that E is adjacent to m wings, and if $m \leq 3$, then no two of them are opposite with respect to the corresponding vertex of P. Fig. 5 shows an example of a handle decomposition of an annulus having discs of all 9 possible types.

Note that F contains $k_2 + k_3 + k_4$ discs and $(2k_2 + 3k_3 + 4k_4)/2$ strips. Therefore, the Euler characteristic $\chi(F)$ is equal to $-k_3/2 - k_4 + p$, where p is the number of patches. Since $\chi(F) \geq 0$, we get the inequality

$$k_3 + 2k_4 \leq 2p \tag{1}$$

Since the weights of wings form a solution of the system $S(P)$, the total weight of all wings contained in a patch of F is equal to 2π. Thus the total weight of all wings in F equals $2\pi p$. Let us estimate this total weight by considering wings adjacent to each disc E. If E has type (m, n), then it has m adjacent wings. It is easy to see that their total weight w can be estimated as follows:

(1) $w < 2\pi$ if $(m, n) = (4, 0)$;

(2) $w = \pi$ if $(m, n) = (3, 0)$;

(3) $w < \pi$ if $m + n = 3$ and $m > 0$;

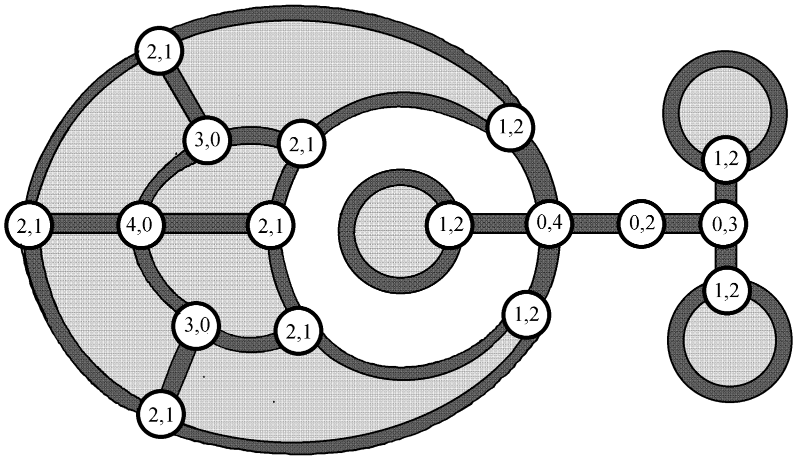

Figure 5: A decomposition of an annulus into discs, strips and patches

(4) $w = 0$ in all the other cases when $m = 0$ and thus there are no wings.

Summing up over all discs, we get the inequality

$$\pi k_3 + 2\pi k_4 \geq 2\pi p \tag{2}$$

Moreover, the equality is possible only if all discs in F have type (3,0). Comparing inequalities 1 and 2, we can conclude that they are actually equalities. It means that $\chi(F) = 0$ and elementary discs in the intersection of F with balls have type (3,0). Thus F is a torus parallel to the boundary. ∎

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Chelyabinsk State University
matveev@csu.ac.ru
matveev@cgu.chel.su