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Polynomial and Poisson Dependence in Free Poisson Algebras and Free Poisson Fields

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Abstract

We prove that any two Poisson dependent elements in a free Poisson algebra and a free Poisson field of characteristic zero are algebraically dependent, thus answering positively a question from [8]. We apply this result to give a new proof of the tameness of automorphisms for free Poisson algebras of rank two (see [9, 10]).

1 Introduction

The free Poisson algebras were first considered in [12]. They are naturally and closely related to polynomial algebras, free associative algebras, and free Lie algebras. For example, the free Poisson algebra and Poisson brackets were used in [13, 14] to prove that the Nagata automorphism of the polynomial algebra of rank three is wild.

A systematic study of free Poisson algebras was started in [8] where several open questions on their structure were formulated. It was proved in [8] that the centralizer of a nonconstant element of a free Poisson algebra in the case of characteristic zero is a polynomial algebra in a single variable; this is an analogue of the famous Bergman Centralizer Theorem [1]. Then in [10] it was proved that locally nilpotent derivations of free Poisson algebra of rank two in the case of characteristic zero are triangulable and that automorphisms of these algebras are tame; these are analogues of the well-known Rentschler Theorem [11] and Jung Theorem [5] respectively. Finally, in [9] the Freiheitssatz was proved for free Poisson algebras over a field of characteristic zero.

In this paper we continue the study of free Poisson algebras and solve positively a question formulated in [8] by proving that every two Poisson dependent elements in a free Poisson algebra over a field of characteristic zero are algebraically dependent. In fact we prove a bit more: any two Poisson dependent elements which are rational over a free Poisson algebra are algebraically dependent.

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As a corollary, we give another proof of the result from [10] that the automorphisms of free Poisson algebra of rank two in the case of characteristic zero are tame (see also [9]).

2 The main part

Below $\mathcal{P} = \mathcal{P}\langle X\rangle$ denotes a free Poisson algebra on a set of generators $X = \{x_1, \dots, x_n\}$ over a field F of characteristic zero. Recall (see, for example [12]) that $\mathcal{P}$ is isomorphic to $S(Lie\langle X\rangle)$, where $Lie\langle X\rangle$ is the free Lie algebra over X and $S(V)$ means the symmetric algebra over a vector space V. Denote by $y_1 = x_1, y_2 = x_2, \dots, y_n = x_n, y_{n+1} = \{x_1, x_2\}, \dots$ a basis of $Lie\langle X\rangle$ consisting of the Lie monomials which are ordered by increasing Lie degree (and arbitrary for monomials of the same degree). Hence as a commutative algebra $\mathcal{P}$ is a polynomial algebra $F[y_1, y_2, \dots]$ with infinitely many generators. For elements $f, g \in \mathcal{P}$ we denote by fg their product as elements of $F[y_1, y_2, \dots]$ and by $\{f, g\}$ their Poisson product (the Poisson bracket) which is defined on $y_1, y_2, \dots$ as elements of $Lie\langle X\rangle$, extended on monomials of $\mathcal{P}$ by the Leibnitz law, and then on $\mathcal{P}$ by linearity.

A family of polynomial *weight* degree functions can be defined on $F[y_1, y_2, \dots]$ by giving arbitrary real weights $w_i = w(y_i)$ to the generators and extending it on monomials $M = y_1^{j_1} y_2^{j_2} \dots$ by $w(M) = \sum_i j_i w(y_i)$. Then for $f \in F[y_1, y_2, \dots]$ degree can be defined as $D(f) = \max(w(M) | M \in f)$, i.e. maximum by all monomials contained in f with non-zero coefficients. Of course not all of these functions make sense for $\mathcal{P}$ as a Poisson algebra. We say that a weight degree function D on $\mathcal{P}$ is compatible with the Poisson structure if it satisfies the following natural condition:

for any two monomials $M_1,\ M_2 \in \mathcal{P}$ (as a polynomial algebra) the bracket $\{M_1, M_2\}$ is D-homogeneous.

For example the weight which is defined on a Lie monomial y as the number of appearances of a free generator x_k in y defines a compatible degree function d_{x_k}. It is easy to check that in order to define a compatible degree function the weight should be given on a Lie monomial y by

$$w(y) = \sum_i (w(x_i) - c) d_{x_i}(y) + c$$

where $w(x_i)$ and c are arbitrary real numbers. To see it define $\Delta(i, j) = w(y_i) + w(y_j) - D(\{y_i, y_j\})$ for two different Lie monomials $y_i,\ y_j$. Take $\{y_i y_j, y_k\} = \{y_i, y_k\} y_j + y_i \{y_j, y_k\}$ where $y_i,\ y_j,\ y_k$ are pairwise different Lie monomials. Then $D(\{y_i, y_k\} y_j) = D(y_i \{y_j, y_k\})$, $\Delta(i, k) = \Delta(j, k)$, and $\Delta(i, j) = c$ is a constant. Therefore weight w is completely determined by $w_i = w(x_i)$ and $c = \Delta(i, j)$.

Examples of compatible degree functions are d_{x_k} defined above and the Poisson degree which corresponds to $w(x_1) = \dots = w(x_n) = 1,\ c = 0$ (i. e. to the Lie degree). Total polynomial degree deg on $F[y_1, y_2, \dots]$ is also compatible and corresponds to $w(x_1) = \dots = w(x_n) = c = 1$.

Recall that $\deg(\{f, g\}) = \deg(f) + \deg(g) - 1$ for homogeneous f and g if $\{f, g\} \neq 0$. Similar relation is true for any compatible weight degree function: $D(\{f, g\}) = D(f) + D(g) - c$ if $\{f, g\} \neq 0$ and f and g are D-homogeneous.

Below we will consider only the weights for which all parameters are integers.

Elements $a_1, \ldots, a_m$ of a Poisson algebra $\mathcal{S}$ are called *Poisson dependent* if there exists a non-zero element $p(x_1, \ldots, x_m)$ in the free Poisson algebra $\mathcal{P}\langle x_1, \ldots, x_m\rangle$ such that $p(a_1, \ldots, a_m) = 0$ in $\mathcal{S}$; the elements $a_1, \ldots, a_m$ are called *algebraically dependent* if there exists a non-zero polynomial $f(x_1, \ldots, x_m) \in F[x_1, \ldots, x_m]$ such that $f(a_1, \ldots, a_m) = 0$.

Element u is called *algebraic* over a Poisson algebra $\mathcal{S}$ if u belongs to a commutative algebra $\mathcal{T}$ containing $\mathcal{S}$ (as a commutative subalgebra) and $p(u) = 0$ in $\mathcal{T}$ for a non-zero polynomial $p \in \mathcal{S}[t]$. If $\mathcal{S}$ is a domain the bracket can be extended uniquely from $\mathcal{S}$ to the field $\mathcal{S}(u)$. Indeed, take a non-zero polynomial $p(t) = \sum_i p_i t^i$ where $p_i \in \mathcal{S}$ for which $p(u) = 0$ of the minimal degree possible. If an extension of the bracket exists and we use the same notation for it then $0 = \{f, p(u)\} = \{f, u\}p'(u) + \sum_i \{f, p_i\}u^i$ for any $f \in \mathcal{S}$ which defines $\{f, u\}$ provided $p'(u) \neq 0$, i. e. in the zero characteristic case or for a separable extension. It is a straightforward computation to check that this bracket makes $\mathcal{S}(u)$ a Poisson algebra.

Denote by $\mathcal{Q}$ the field of fractions of $\mathcal{P}$ considered as a commutative polynomial algebra. We can extend the bracket from $\mathcal{P}$ to $\mathcal{Q}$ as we saw above. A compatible weight degree function D can be extended from $\mathcal{P}$ to $\mathcal{Q}$ by $D(\frac{a}{b}) = D(a) - D(b)$. We will call $\mathcal{Q}$ a free Poisson field.

Lemma 1 *Let $f,\ g$ be elements algebraic over a free Poisson algebra $\mathcal{P}$. If $f,\ g \in \mathcal{P}[f, g]$ are Poisson dependent and $r_1(x_1, x_2),\ r_2(x_1, x_2) \in F(x_1, x_2)$ are rational functions then $r_1(f, g),\ r_2(f, g) \in \mathcal{P}(f, g)$ are also Poisson dependent.*

Proof. Elements $f,\ g$ are Poisson dependent if the basic Lie monomials of $f,\ g$ are algebraically dependent. Denote by $y_1, \ldots, y_{N(a)}$ the set of all basic Lie monomials with $d(y_j) \leq a$. Consider the smallest A for which $y_1(f, g), y_2(f, g), \ldots y_{N(A)}(f, g)$ are algebraically dependent. It is easy to check using induction on $a_i = d(y_i(x_1, x_2))$ that $y_i(r_1(f, g), r_2(f, g)) \in F(f, g)[y_3(f, g), \ldots y_{N(a_i)}(f, g)]$. Hence there is an algebraic dependence between $y_1(r_1(f, g), r_2(f, g)), \ldots, y_{N(A)}(r_1(f, g), r_2(f, g))$. □

For $f \in \mathcal{Q}$ denote by $\mathrm{supp}\,(f)$ the minimal set of polynomial variables on which f depends.

Lemma 2 *Let $f,\ g \in \mathcal{Q}$ be elements which are algebraically independent. Then for a given polynomial weight degree function D there exists an element $h \in F[f, g]$ such that the leading forms $f_D,\ h_D$ are algebraically independent.*

Proof. A standard proof of this fact would be based on the notion of Gelfand-Kirillov dimension (see [4]) and is well-know for the polynomial case. We give a proof using Poisson brackets which is possible in the case of zero characteristic.

Consider $\mathrm{supp}\,(f) \bigcup \mathrm{supp}\,(g) = \{y_{i_1}, \ldots, y_{i_k}\}$. Since $f,\ g$ are algebraically independent we may assume without loss of generality that $f, g, y_{i_3}, \ldots, y_{i_k}$ are algebraically independent and introduce on $F(y_{i_1}, \ldots, y_{i_k})$ a deficiency function (somewhat similar to the one introduced in [7]) by

$$\mathrm{def}(f, h) = D(\mathrm{J}_{y_{i_1}, \ldots, y_{i_k}}(f, h, y_{i_3}, \ldots, y_{i_k})) - D(h)$$

where $\mathrm{J}_{y_{i_1}, \ldots, y_{i_k}}(f, h, y_{i_3}, \ldots, y_{i_k})$ is the Jacobian of $f, h, y_{i_3}, \ldots, y_{i_k}$, i.e. the determinant of the corresponding Jacobi matrix. This function is defined and has values in $\mathbb{Z}$ when $\mathrm{J}_{y_{i_1}, \ldots, y_{i_k}}(f, h, y_{i_3}, \ldots, y_{i_k}) \neq 0$.

Since $\mathrm{J}_{y_{i_1},\ldots,y_{i_k}}(f,p(f,g),y_{i_3},\ldots,y_{i_k}) = \mathrm{J}_{y_{i_1},\ldots,y_{i_k}}(f,g,y_{i_3},\ldots,y_{i_k})\frac{\partial p}{\partial g}$ for any $p \in F[f,g]$ and $\mathrm{J}_{y_{i_1},\ldots,y_{i_k}}(f,g,y_{i_3},\ldots,y_{i_k}) \neq 0$ function def is defined on any algebraically independent pair from $F[f,g]$.

Observe that

$$\begin{aligned} \operatorname{def}(f,h^k) &= \operatorname{def}(f,h), \\ \operatorname{def}(f,hr(f)) &= \operatorname{def}(f,h),\ r(f) \in F(f) \setminus 0 \\ \operatorname{def}(f,h) &\leq D(f) - c, \end{aligned}$$

where c is a constant which depends only on degrees of $f,\ y_{i_1},\ldots,y_{i_k}$.

If f_D and g_D are algebraically dependent then there exists a non-zero polynomial $q = \sum_{i=0}^k q_i(x)y^i \in F[x,y]$ for which all monomials with non-zero coefficients have the same D degree ($D(x) = D(f),\ D(y) = D(g)$), $k = \deg_y(q)$ is minimal possible, and $q(f_D, g_D) = 0$. In our setting elements $f,\ g' = q(f,g)$ are algebraically independent. Denote in this Lemma only $\mathrm{J}_{y_{i_1},\ldots,y_{i_k}}(f,h,y_{i_3},\ldots,y_{i_k})$ by $\{f,h\}$.

We have

$$\operatorname{def}(f,g') = D(\{f,g'\}) - D(g') = D(\sum_i \{f, q_i(f)g^i\}) - D(g') >$$
$$D(\{f,q_k(f)g^k\}) - D(q_k(f)g^k) = \operatorname{def}(f,g^k) = \operatorname{def}(f,g)$$

since $D(g') < D(q_k(f)g^k)$ while $D(\{f,q_k(f)g^k\}) = D(kq_k(f)\ g^{k-1}) + D(\{f,g\}) =$ $D(\sum_i iq_i(f)\ g^{i-1}) + D(\{f,g\}) = D(\sum_i \{f,q_i(f)g^i\})$ (recall that $\sum_i iq_i(f_D)\ g_D^{i-1} \neq 0$). If $f_D,\ g'_D$ are algebraically dependent, we repeat the procedure and obtain a pair $f,\ g''$ with $\operatorname{def}(f,g'') > \operatorname{def}(f,g')$. Since $\operatorname{def}(f,h) \leq D(f) - c$ for any h and $\operatorname{def}(f,h) \in \mathbb{Z}$, the process will stop after a finite number of steps and we will get an element $h \in F[f,g]$ for which h_D is algebraically independent with f_D.

□

Lemma 3 *Let $f,\ g \in \mathcal{Q}$ be elements which are Poisson dependent but not algebraically dependent. Then there exists a pair of elements which are homogeneous relative to any compatible weight degree function D with the same property.*

Proof. Denote by h_D the leading form of $h \in \mathcal{Q}$ relative to D. From the definition of compatibility $y_i(f,g)_D = y_i(f_D,g_D)$ if $y_i(f_D,g_D) \neq 0$. Since $P(f,g) = 0$ for a Poisson polynomial, $P_D(f_D,g_D) = 0$ for a polynomial P_D consisting of monomials M of P for which $D(M(f_D,g_D))$ is maximal possible. Hence $f_D,\ g_D$ are Poisson dependent.

If f_D and g_D are algebraically dependent then we can use Lemma 2 to find an element $h \in F[f,g]$ such that h_D and f_D are algebraically independent and to obtain a D-homogeneous pair of Poisson dependent elements which are algebraically independent. The space of compatible weights is finite dimensional lattice, hence we can obtain a pair of Poisson dependent elements which are algebraically independent and are homogeneous relative to all compatible degree functions. □

We will call elements which are homogeneous relative to all compatible degree functions *completely* homogeneous.

Lemma 4 *Let $f, g \in \mathcal{Q}$ be a Poisson dependent pair and x be the smallest element in supp(f). Write $f = x^n f_x + \cdots$, $g = x^m g_x + \cdots$, where f_x, g_x do not contain x and dots stand for terms with smaller (polynomial) degrees in x. Then the pair f_x, g_x is Poisson dependent as well.*

Proof. Consider the Poisson polynomial $P(x_1, x_2)$ for which $P(f, g) = 0$; it is a sum of monomials of the type

$$u = y_1^{k_1} y_2^{k_2} \cdots y_s^{k_s},$$

where y_i are Lie monomials in x_1, x_2. We have

$$\begin{aligned} \{f, g\} &= x^{n+m}\{f_x, g_x\} + \ldots, \\ y_i(f, g) &= x^{N_i} y_i(f_x, g_x) + \cdots, \ N_i = n d_{x_1}(y_i) + m d_{x_2}(y_i), \\ u(f, g) &= x^{N(u)} u(f_x, g_x) + \cdots, \ N(u) = \sum_i k_i N_i, \end{aligned}$$

where again dots mean terms of smaller degree in x. Observe that x cannot appear in $\{f_x, g_x\}$ or in $y_i(f_x, g_x)$ when $d(y_i) > 1$ since for any $y \in \text{supp}\,(f)$, $z \in \text{supp}\,(g)$ we have $\{y, z\} > y \geq x$. Therefore,

$$0 = P(f, g) = Q(f_x, g_x) x^N + \cdots,$$

where $N = \max\{N(u) \mid u \text{ monomial in } P(x_1, x_2)\}$, $Q(x_1, x_2) = \sum_{N(u)=N} u(x_1, x_2)$. Since all monomials u in $P(x_1, x_2)$ are linearly independent, we have $Q(x_1, x_2) \neq 0$ and hence f_x, g_x are Poisson dependent. □

Lemma 5 *In the conditions of Lemma 4, assume that $f_x = 1$, $f = x^n + \alpha x^{n-1} + \cdots$. Then the pair $nx + \alpha$, g_x is Poisson dependent.*

Proof. Let us check by induction on the Poisson degree that $y_i(f, g) = x^{N_i} y_i(nx + \alpha, g_x) + \cdots$ for $i > 1$, where $N_i = (n-1)d_{x_1}(y_i) + m d_{x_2}(y_i)$ for any Lie monomial with $i > 1$ and dots stand for the terms of smaller degree in x (recall that $\deg_x(g) = m$). The base of induction for $y_2(f, g) = g$ is clear. A Lie monomial $y_k(f, g)$ with $k > 2$ can be presented as either $\{y_l(f, g), f\}$ or $\{y_l(f, g), g\}$ where y_l is a monomial with a smaller Poisson degree. If $l = 1$ then $k = 3$ is the only interesting case and $y_3(f, g) = \{f, g\} = \{x^n + \alpha\, x^{n-1} + \cdots, x^m g_x + \cdots\} = n x^{n-1+m}\{x, g_x\} + x^{n-1+m}\{\alpha, g_x\} + \cdots = x^{n-1+m}\{nx + \alpha, g_x\} + \cdots$. If $l > 1$ then by induction $y_l(f, g) = x^{N_l} y_l(nx + \alpha, g_x) + \cdots$ and similar computations verify the claim. It is essential that x is the smallest element in supp$\,(f)$ because supp$\,(\{y, z\})$ does not contain x if $y \geq x$ and no additional powers of x may appear as results of Poisson brackets.

Therefore for $u = y_1^{k_1} y_2^{k_2} \cdots y_s^{k_s}$ the leading form of $u(f, g)$ relative to x is $x^{nk_1 + N_u} y_2(nx + \alpha, g_x)^{k_2} \cdots y_s(nx + \alpha, g_x)^{k_s}$ where $N_u = (n-1)d_{x_1}(y_2^{k_2} \cdots y_s^{k_s}) + m d_{x_2}(y_2^{k_2} \cdots y_s^{k_s})$. Hence different monomials of $P(f, g)$ cannot cancel in the x-leading form of $P(f, g)$ and the elements $nx + \alpha$, g_x are Poisson dependent. □

Consider now a pair of algebraically independent elements f, $g \in \mathcal{Q}$. By the Shirshov-Witt theorem a subalgebra of a free Lie algebra is a free Lie algebra (see [15, 17]) so the elements of $\mathrm{supp}\,(f)\bigcup \mathrm{supp}\,(g)$ generate a free Lie algebra $\mathcal{L}$ with the free basis which contains two smallest elements $x,\ y$ of $\mathrm{supp}\,(f)\bigcup \mathrm{supp}\,(g)$. Elements x and y are different since otherwise $\mathrm{supp}\,(f)\bigcup \mathrm{supp}\,(g) = x$ and $f,\ g$ are algebraically dependent. If $\mathcal{P}$ is the free Poisson algebra which correspond to $\mathcal{L}$ and $\mathcal{Q}$ is the field of fractions of $\mathcal{P}$ then $f,\ g \in \mathcal{Q}$. Though $f,\ g$ are possibly written through different generators, the size of $\mathrm{supp}\,(f)\bigcup \mathrm{supp}\,(g)$ did not change.

Assume that there exists a pair of algebraically independent Poisson dependent elements in a free Poisson field $\mathcal{Q}$. Then we can find a pair which is minimal in the following sense: the size $|f,g|$ of $\mathrm{supp}\,(f)\bigcup \mathrm{supp}\,(g)$ is minimal possible, $\mathcal{Q}$ is generated by $\mathrm{supp}\,(f)\bigcup \mathrm{supp}\,(g)$, elements f and g are completely homogeneous.

As we observed $|f,\ g|$ does not change when we replace the original Poisson field with the "minimal" one. The elements may stop being completely homogeneous but by Lemma 3 we can produce a completely homogeneous pair which belongs to $F[f,g]$, hence the union of supports of these two elements belongs to the union of supports of the original elements. Since $|f,\ g|$ is minimal it implies that the size cannot become smaller, so the union of supports of a completely homogeneous pair is the same as for the original pair.

Recall that if two homogeneous polynomials $f,g \in F[X]$ are algebraically dependent then there exists a homogeneous polynomial $h \in F[X]$ such that $f = \alpha h^k,\ g = \beta h^l$ for some $\alpha,\beta \in F$ and natural numbers k,l (see, for example, [2]). Similar statement is true for two algebraically dependent homogeneous rational functions $f,g \in F(X)$ if one of them, say f, has a non-zero degree. Indeed, since we may assume that $D(f^ig^j)$ is the same for all monomials of q

$$q(f,g) = \sum_{iD(f)+jD(g)=d} q_{ij} f^i g^j = 0.$$

If $D(g) = 0$ then $q(f,g) = f^a q_a(g),\ q_a(g) = 0$ and $g \in F$. If $D(g) \neq 0$ then $q(f,g) = f^a g^b \widetilde{q}(f^\rho g^{-\sigma})$ where ρ,σ are relatively prime integers for which $\rho D(f) = \sigma D(g)$ and $\widetilde{q}(x) \in F[x]$. Hence $q(f,g) = f^a g^b \prod (f^\rho g^{-\sigma} - c_i)$ where $c_i \in \overline{F}$, an algebraic closure of F. So $f^\rho - cg^\sigma = 0$ for some $c \in \overline{F}$. Furthermore, if $r\rho + s\sigma = 1,\quad r,\ s \in \mathbb{Z}$, and $h = f^s g^r$ then h^σ is proportional to f and h^ρ is proportional to g. If we also assume that h is not a proper power (of a rational function) then all rational functions which are algebraically dependent with h belong to $F(h)$.

Lemma 6 *Let $f,g \in \mathcal{Q}$ be a minimal pair. If x is the smallest element in supp $(f)\bigcup$ supp (g), then there exists a minimal pair $\widetilde{f} = x + f_1,\ \widetilde{g}$ where supp $(f_1)\bigcup$ supp $(\widetilde{g}) \not\ni x$.*

Proof. Write $f = x^n f_x + \cdots,\ g = x^m g_x + \cdots$, where f_x, g_x do not contain x and dots stand for terms with smaller (polynomial) degrees in x. Then the pair f_x, g_x is Poisson dependent by Lemma 4 and is algebraically dependent since $|f_x, g_x| < |f,g|$. If $D(f_x) = 0$ for any compatible degree function consider the second smallest element $y \in \mathrm{supp}\,(f)\bigcup \mathrm{supp}\,(g)$ and present $f = y^{n_1} f_y + \cdots$ where f_y does not contain y and dots stand for terms with smaller degrees in y. If $D(f_y) = 0$ for any compatible degree function then $D(x^n) = D(y^{n_1})$ for any compatible degree. But $x,\ y$ are elements of a free basis, so the Poisson degrees d_x and d_y are compatible degree functions and either $x = y$ which is impossible or $n = n_1 = 0$. Similar considerations for g show that either $D(g_x)$ or $D(g_y)$ is not identically zero or $m = m_1 = 0$. If $n = m = 0$ consider polynomial dependence q

between $f_x,\ g_x$ and a minimal pair $f,\ g_1 = q(f,g)$. Then $g_1 = x^k g_{1x} + \dots$ where $k < 0$. So either $D(g_{1x})$ or $D(g_{1y})$ is not identically zero. Since $x,\ y$ are elements of a free basis we can reorder them as well as $f,\ g_1$ and assume that $D(f_x) \neq 0$ for some compatible degree function. Then by remarks above there exists a completely homogeneous element $h \in \mathcal{Q}$ such that $f_x = c_1 h^a,\ g_x = c_2 h^b$ where $c_1,\ c_2 \in F \setminus 0$, $D(h) \neq 0$ for some compatible degree function, and $a \neq 0$. Without loss of generality we may assume that $c_1 = c_2 = 1$. Hence $f = x^n h^a + \cdots,\ g = x^m h^b + \cdots$. The pair $f^b g^{-a},\ f \in \mathcal{Q}$ is Poisson dependent by Lemma 1. We can write $f^b g^{-a} = x^{bn-am} + \alpha\, x^{nb-am-1} + \dots$. Hence by Lemma 5 the pair $(bn - am)x + \alpha,\ h^a$ is Poisson dependent. Recall that $\mathrm{supp}\,(\alpha) \bigcup \mathrm{supp}\,(h) \not\ni x$. Therefore $(bn - am)x + \alpha$ and h are algebraically independent if $(bn - am) \neq 0$. So if $(bn - am) \neq 0$ we proved the Lemma.

If $bn - am = 0$ then algebraically independent rational functions $f,\ g$ have algebraically dependent leading forms relative to polynomial $\deg_x$. According to Lemma 2 ring $F[f,g]$ contains an element g' such that $\deg_x$-leading forms of f and g' are algebraically independent. Since $\mathrm{supp}\,(g') \subset \mathrm{supp}\,(f) \bigcap \mathrm{supp}\,(g)$ the pair $f,\ g'$ is minimal and we can use it to prove the lemma. □

Theorem 1 *Every two Poisson dependent elements in the free Poisson field $\mathcal{Q}$ are algebraically dependent.*

Proof. Assume that the theorem is not true. Then by the previous lemmas there exists a completely homogeneous Poisson dependent algebraically independent pair $f = x + f_1, g \in \mathcal{Q}$, where the size $|f,g|$ is minimal possible, x is the minimal element in $\mathrm{supp}\,(f) \bigcup \mathrm{supp}\,(g)$ and $\mathrm{supp}\,(f_1) \bigcup \mathrm{supp}\,(g) \not\ni x$, and x is an element of the free basis of $\mathcal{P}$.

Consider the smallest element $y \in \mathrm{supp}\,(g)$ and write $f = y^n f_y + \dots,\ g = y^m g_y + \dots$ where $\mathrm{supp}\,(f_y) \bigcup \mathrm{supp}\,(g_y) \not\ni y$. Elements f_y and g_y should be Poisson dependent by Lemma 4 and algebraically dependent since $|f_y, g_y| < |f,g|$.

If $n = 0$ then $f_y = x + f_{1y}$ and g_y are algebraically dependent and $d_x(f_y) = 1$. Hence $g_y = c f_y^b$ and $b = 0$ since otherwise $\mathrm{supp}\,(g_y) \ni x$. If furthermore $m = 0$ then $g = c + \dots$ where $c \in F$ and we will replace g by $\widetilde{g} = g - c$. Then $\widetilde{g} = y^{\widetilde{m}} \widetilde{g}_y + \dots$ where $\widetilde{m} \neq 0$. Furthermore, $f_y = x + f_{1y}$ and $\widetilde{g}_y$ are Poisson and algebraically dependent, which as above is possible only if $\widetilde{g}_y \in F$. Since y is a Lie monomial and $y \neq x$ there is an element $z \neq x$ in the free basis for which $d_z(y) \neq 0$ and $d_z(\widetilde{g}) = \widetilde{m} d_z(y) \neq 0$. But $D(\widetilde{g}) = 0$ for any compatible weight degree function. Therefore $\widetilde{g}_y \notin F$, $\widetilde{g}_y = c f_y^b$ where b is a nonzero integer, and $\mathrm{supp}\,(\widetilde{g}_y) \ni x$, a contradiction. We can conclude that $m \neq 0$ and that $f_y, my + m\beta$ are Poisson dependent by Lemma 5. Since $f_y = x + f_{1y}$ where all elements of $\mathrm{supp}\,(f_{1y})$ are larger than x and all elements of $\mathrm{supp}\,(\beta)$ are larger than y we can see that $y_i(x + f_{1y}, y + \beta) = y_i(x,y) + \dots$ for any Lie monomial y_i where $\dots$ stand for Lie monomials larger than $y_i(x,y)$. Hence these elements are Poisson independent and $n = 0$ is impossible.

Since x is an element of the free basis it follows from the complete homogeneity that $0 = d_z(x) = d_z(y^n f_y)$. Therefore $d_z(f_y) = -n d_z(y) \neq 0$. Elements $f_y,\ g_y$ are algebraically dependent, hence $f_y = c_1 h^a,\ g_y = c_2 h^b$ for some element h where $a \neq 0$ and we may assume that $c_1 = c_2 = 1$.

If $b = 0$ and $m \neq 0$ then $g = y^m + \dots$ and f_y and $my + \beta$ are Poisson dependent. They are algebraically independent since $y \notin \mathrm{supp}\,(f_y),\ y \in \mathrm{supp}\,(my + \beta)$. But

$\mathrm{supp}\,(f_y)\bigcap \mathrm{supp}\,(g_y) \not\ni x$ and we have a contradiction with the minimality of the pair $f,\ g$. If $m=0$ consider $\widetilde{g}=g-1$. Then $\widetilde{g}=y^{\widetilde{m}}\widetilde{g}_y+\dots$ where $\widetilde{m}\neq 0$ and $\widetilde{g}_y=h^{\widetilde{b}}$ because $f_y=h^a$ where $a\neq 0$. Now, $d_x(f)=d_x(x)=1=d_x(y^nf_y),\ d_z(f)=d_z(x)=0=d_z(y^nf_y)$ and $d_x(f_y)=1-nd_x(y),\ d_z(f_y)=-nd_z(y)$. Since $d_x(y^{\widetilde{m}}h^{\widetilde{b}})=0,\ d_z(y^{\widetilde{m}}h^{\widetilde{b}})=0$ and $d_x(f_y)=1-nd_x(y),\ d_z(f_y)=-nd_z(y)$ both $\widetilde{m}=\widetilde{b}=0$, which is impossible. Therefore $b\neq 0$.

Replace now g by $\widetilde{g}=g^{-a}f^b$. Then $\widetilde{g}=y^k+y^{k-1}\widetilde{g}_1+\dots$. The case $k=0$ could be brought to a contradiction just as the case $b=m=0$ above. Therefore $k\neq 0$.

Elements $ky+\widetilde{g}_1,\ f_y$ are algebraically independent since $\mathrm{supp}\,(f_y)\not\ni y$ and $k\neq 0$. Since

$$\mathrm{supp}\,(ky+\widetilde{g}_1)\bigcup \mathrm{supp}\,(f_y)\subseteq \mathrm{supp}\,(f)\bigcup \mathrm{supp}\,(g)$$

we should have $\mathrm{supp}\,(ky+\widetilde{g}_1)\bigcup \mathrm{supp}\,(f_y)=\mathrm{supp}\,(f)\bigcup \mathrm{supp}\,(g)$ by the minimality condition and $x\in \mathrm{supp}\,(\widetilde{g}_1)$ ($x\notin \mathrm{supp}\,(f_y)$ since $n>0$). Recall that $\widetilde{g}=g^{-a}f^b$ and therefore $x\in \mathrm{supp}\,(\widetilde{g}_1)$ only if $n=1$, i. e. if $f=yh^a+(x+\delta)+\dots$ where $\dots$ stand for the terms with negative powers in y. Hence $\widetilde{g}=\frac{(yh^a+(x+\delta)+\dots)^b}{(y^mh^b+\varepsilon y^{m-1}+\dots)^a}=y^k+[b(x+\delta)h^{-a}-a\varepsilon h^{-b}]y^{k-1}+\dots$ and $\widetilde{g}_1=b(x+\delta)h^{-a}-a\varepsilon h^{-b}$.

The elements $(ky+\widetilde{g}_1)f_y,\ f_y$ are Poisson dependent by Lemma 1. Hence $[ky+b(x+\delta)h^{-a}-a\varepsilon h^{-b}]h^a=b(x+\delta)-a\varepsilon h^{a-b}+kyh^a$ and h^a are Poisson dependent.

It is clear that $\mathrm{supp}\,(h^a)$ is a proper subset of $\mathrm{supp}\,(g)$. So we may apply induction on the size of $\mathrm{supp}\,(g)$ to prove the Theorem. The base of induction when $|g|=1$ corresponds to $g=y^m,\ m\neq 0$. As we have seen above in order to avoid a contradiction we should have $f=y^nf_y+\dots$ where $n\neq 0,\ f_y\notin F$, and $g_y\notin F$. But $g_y=1$ and we have a contradiction which proves the theorem.

□

It was shown in [8] that $f,\ g\in\mathcal{P}$ is algebraically dependent if and only if $\{f,g\}=0$ i.e. f and g are *Poisson commuting.* Of course if two elements $f,\ g$ of a Poisson algebra which is a domain are algebraically dependent they Poisson commute: $p(f,g)=0$ implies that $p_g(f,g)\{f,g\}=0$. The Theorem shows that for $f,\ g\in\mathcal{Q}$ Poisson commuting implies an algebraic dependence. Hence for the pairs from $\mathcal{Q}$ the notions of Poisson commuting, algebraic dependence, and Poisson dependence are equivalent.

Corollary 1 *Let $f,g\in\mathcal{Q},\ \{f,g\}\neq 0$. Then f,g generate a free Lie algebra with respect to the bracket $\{,\}$, and they generate a free Poisson subalgebra in $\mathcal{Q}$ in complete analogy to the case of free associative algebras.*

Observe that the theorem is evidently not true for more than two elements: the elements $x_1,x_2,\{x_1,x_2\}$ are Poisson dependent but are algebraically independent. It is not true as well if $char\,F=p>0$; the elements x_1,x_2^p are algebraically independent but $\{x_1,x_2^p\}=p\,x_2^{p-1}\{x_1,x_2\}=0$.

3 Application to automorphisms

It is well known [3, 5, 16, 6] that the automorphisms of polynomial algebras and free associative algebras in two variables are tame. The automorphisms of free Poisson algebras in two variables over a field of characteristic zero are also tame [10]. In [9] this result was obtained as a corollary of the Freiheitssatz for Poisson algebras. Here we show that the result follows from our theorem as well.

Theorem 2 *[10] Automorphisms of the free Poisson algebra $\mathcal{P}\langle x, y\rangle$ of rank two over a field F of characteristic 0 are tame.*

Proof. Let α be an automorphism of $\mathcal{P}_2 = \mathcal{P}\langle x, y\rangle$. Since any (tame) automorphism of $F[x, y]$ can be lifted to a (tame) automorphism of $\mathcal{P}_2$, we can assume without loss of generality that the abelianization of α (that is, its homomorphic image under the natural epimorphism $Aut(\mathcal{P}_2) \twoheadrightarrow Aut(F[x, y])$) is the identity automorphism of $F[x, y]$. It remains to show that then α is the identity automorphism of $\mathcal{P}_2$.

Let $\alpha(x) = f$, $\alpha(y) = g$. Assume that either $f \neq x$ or $g \neq y$. If we take weights $w(x) = \rho$, $w(y) = 1$ where $\rho \geq 0$ then $f_{\sim} = x$ and $g_{\sim} = y$ where $f_{\sim}$ and $g_{\sim}$ are the lowest Poisson forms of f and g with respect to w. If we start now to decrees ρ then for some non-positive value of ρ either $f_{\sim} \neq x$ or $g_{\sim} \neq y$ for the corresponding $f_{\sim}$ and $g_{\sim}$. Let us take the largest ρ with this property. Then $f_{\sim}$ and $g_{\sim}$ are Poisson w-homogeneous, $d_w(f_{\sim}) = \rho$, $d_w(g_{\sim}) = 1$, $f_{\sim} = x + f_1$, $g_{\sim} = y + g_1$, where at least one of f_1, g_1 is nonzero and their abelianizations in $F[x, y]$ are both zero. Clearly, $f_{\sim}$ and $g_{\sim}$ are Poisson independent.

Let $x = X(f, g)$ for some Poisson polynomial $X(x_1, x_2)$, then $x = (X(f, g))_{\sim} = X_{\sim}(f_{\sim}, g_{\sim})$ since $f_{\sim}$ and $g_{\sim}$ are Poisson independent. Similarly, y belongs to the Poisson subalgebra generated by $f_{\sim}$ and $g_{\sim}$. Therefore, the w-homogeneous Poisson forms $f_{\sim}$, $g_{\sim}$ generate $\mathcal{P}_2$.

Consider now the Poisson leading forms $\widetilde{(f_{\sim})}$ and $\widetilde{(g_{\sim})}$ of $f_{\sim}$ and $g_{\sim}$ with respect to the Poisson degree, when $d(x) = d(y) = 1$. If they were Poisson independent, then as above they would generate $\mathcal{P}_2$. But this is impossible since otherwise their abelianizations, the images under the epimorphism $\mathcal{P}_2 \twoheadrightarrow F[x, y]$, would generate $F[x, y]$, while at least one of them is 0.

Next we can use our Theorem and conclude that $\widetilde{(f_{\sim})}$ and $\widetilde{(g_{\sim})}$ are algebraically dependent. Therefore up to scalars they are h^a, h^b for a certain Poisson-homogeneous element $h \in \mathcal{P}_2$ and non-negative integers a, b. Then we have $ad_w(h) = \rho$, $bd_w(h) = 1$ where $\rho < 0$, which is impossible. $\square$

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