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Abstract

We establish the linear independence over $\mathbb{Q}$, in both qualitative and quantitative forms, of the four numbers 1, $\mathrm{Li}_1(1/z) = -\log(1-1/z)$, $\mathrm{Li}_2(1/z)$ and $\mathrm{Li}_2(1/(1-z))$, for all integers $z \ge 9$ or $z \le -8$ and for rationals $z = s/r$ or $z = 1 - s/r$ with $1 < r < s$, where s is large in comparison with r.

1 Introduction

Problems of irrationality and linear independence of values of the dilogarithmic function

$$\mathrm{Li}_2(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^2}, \qquad |x| < 1,$$

have a long history. The latest news in this direction is given in the work [RV] of G. Rhin and the first-named author, where the best records of irrationality of the dilogarithm at positive rational points are established, in both qualitative and quantitative forms. The results in [RV] are more general, and can be interpreted as linear independence results over $\mathbb{Q}$ for the set 1, $\mathrm{Li}_1(x) := \sum_{n=1}^{\infty} x^n/n = -\log(1-x)$ and $\mathrm{Li}_2(x)$, for suitable $x \in \mathbb{Q}$, $x > 0$.

In [RV], Rhin and the first-named author adapted their permutation group method successfully used earlier to obtain record irrationality measures of $\zeta(2)$ and $\zeta(3)$. The method allowed them to improve some earlier results of M. Hata [Ha1], in particular to show that 1, $\mathrm{Li}_1(1/z)$ and $\mathrm{Li}_2(1/z)$ are linearly independent over $\mathbb{Q}$ for any integer $z \ge 6$. The principal players in [RV] are the following integrals:

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$$I_z^{(0)}(h,j,k,l,m) = z^{-l-m} \int_0^1\int_0^1 \frac{x^j(1-x)^h y^k(1-y)^l}{(x(1-y)+yz)^{j+k-m+1}}\,\mathrm{d}x\,\mathrm{d}y,$$

$$I_z^{(1)}(h,j,k,l,m) = z^{-l-m} \int_0^1 \Bigg(\frac{1}{2\pi i} \oint\limits_{\left|y-\frac{x}{x-z}\right|=\varrho} \frac{x^j(1-x)^h y^k(1-y)^l}{(x(1-y)+yz)^{j+k-m+1}}\,\mathrm{d}y\Bigg)\mathrm{d}x,$$

$$I_z^{(2)}(h,j,k,l,m) = \frac{z^{-l-m}}{2\pi i} \oint\limits_{|x-z|=\sigma} \Bigg(\frac{1}{2\pi i} \oint\limits_{\left|y-\frac{x}{x-z}\right|=\varrho} \frac{x^j(1-x)^h y^k(1-y)^l}{(x(1-y)+yz)^{j+k-m+1}}\,\mathrm{d}y\Bigg)\mathrm{d}x,$$

for any $\varrho, \sigma > 0$, and

$$I_z(h,j,k,l,m) = I_z^{(0)}(h,j,k,l,m) - (\log z)\, I_z^{(1)}(h,j,k,l,m), \tag{1.1}$$

where z is assumed to be real and greater than 1, and where the parameters h, j, k, l, m are non-negative integers. In (1.1), the role of the double integral $I_z^{(1)}(h,j,k,l,m)$ of mixed type (i.e., made over the real interval $(0,1)$ in x and over a complex contour in y) is essential to separate linear forms in 1 and $\mathrm{Li}_2(1/z)$ from those in 1 and $\mathrm{Li}_1(1/z)$ (see [RV, Theorem 2.1]). A similar structure occurs in the present paper, and to achieve this result we build upon Theorem 2.1 of [RV] (see the proof of Lemma 2.1 below).

In his doctorate thesis [Mi, Chap. 4], M.-A. Miladi establishes the linear independence over $\mathbb{Q}$ of

$$1, \quad \mathrm{Li}_1\Big(\frac{1}{z}\Big) = -\,\mathrm{Li}_1\Big(\frac{1}{1-z}\Big), \quad \mathrm{Li}_2\Big(\frac{1}{z}\Big) \quad \text{and} \quad \mathrm{Li}_2\Big(\frac{1}{1-z}\Big),$$

for any integer $z \ge 11$ (hence for any integer $z \le -10$ as well). His construction was highly influenced by the works [Ha1, Ha3] of M. Hata, though he required different techniques to evaluate the asymptotics of numerical linear forms so constructed. The Padé-type approximations Miladi constructed depend on two parameters and have a symmetry under the involution $z \leftrightarrow 1-z$.

The structure of Miladi's Padé-type approximations is similar to the one in [RV] reproduced above, except that his integrands (in his notation) are

$$\frac{x^{n_2}(1-x)^{n_2} y^{n_1}(1-y)^{n_2-n_1}}{(x(1-y)+yz)^{n_2+1}}\,(1-y+yz)^{n_1},$$

where $n_2 \ge n_1 > 0$ are integers.

The principal aim of this paper is to extend the Rhin–Viola permutation group method to integrals of Miladi's type, which will depend below on six rather than two parameters, in order to establish new linear independence results over $\mathbb{Q}$ for the set 1, $\mathrm{Li}_1(1/z)$, $\mathrm{Li}_2(1/z)$, $\mathrm{Li}_2(1/(1-z))$, in both qualitative and quantitative forms. We shall prove the following

Main theorem. *For any integer $z \geq 9$ or $z \leq -8$, the numbers*

$$1,\ \mathrm{Li}_1(1/z),\ \mathrm{Li}_2(1/z) \text{ and } \mathrm{Li}_2(1/(1-z))$$

are linearly independent over $\mathbb{Q}$. Furthermore, for any $\varepsilon > 0$ and any quadruple $(a_0, a_1, a_2, a_3) \in \mathbb{Z}^4 \setminus \{\mathbf{0}\}$, the inequality

$$|a_0 + a_1 \mathrm{Li}_1(1/z) + a_2 \mathrm{Li}_2(1/z) + a_3 \mathrm{Li}_2(1/(1-z))| > C(\varepsilon, z)\, A^{-\mu(z)-\varepsilon}$$

holds for some explicitly given exponent $\mu(z) > 0$, where

$$A = \max\{|a_0|, |a_1|, |a_2|, |a_3|\}$$

and the constant $C(\varepsilon, z) > 0$ does not depend on the quadruple.

We remark that Miladi was unable to get the full power even of his two-parametric construction, because of quite involved computation of the asymptotics of the approximations; specifically, he uses $n_1 = \lfloor n/3 \rfloor$ and $n_2 = n - n_1$, where $n > 0$ is the increasing integer parameter. Miladi himself states on pp. 86–87 of [Mi] a 'Remarque Importante', where he claims that his result about the linear independence of 1, $\mathrm{Li}_1(1/z)$, $\mathrm{Li}_2(1/z)$ and $\mathrm{Li}_2(1/(1-z))$ over $\mathbb{Q}$ might be improved to the range $z \geq 8$ by choosing $n_1 = \lfloor 3n/8 \rfloor$ and $n_2 = n - n_1$. This is partly true: we now confirm the range $z \geq 9$ under the choice.

D. V. and G. V. Chudnovsky announce in [CC] the same linear independence result as Miladi's, for the same range $z \geq 11$, though they give no details of their construction besides a lengthy recursion satisfied by approximations. This recursion in fact allows one to check that their approximations coincide with those of Miladi for the choice $n_1 = \lfloor n/3 \rfloor$ and $n_2 = n - n_1$.

We obtain the proof of the Main theorem stated above as a special case of our Proposition 6.1, which yields linear independence results of 1, $\mathrm{Li}_1(1/z)$, $\mathrm{Li}_2(1/z)$, $\mathrm{Li}_2(1/(1-z))$ not only for integers z, but also for rationals $z = s/r$, with integers $1 < r < s$ where s is large in comparison with r. Specifically, as an instance of this, we prove the $\mathbb{Q}$-linear independence, also in quantitative forms, of 1, $\mathrm{Li}_1(r/s)$, $\mathrm{Li}_2(r/s)$, $\mathrm{Li}_2(-r/(s-r))$ for $r = 2$ and $s \geq 143$, for $r = 3$ and $s \geq 742$, and for $r = 4$ and $s \geq 2355$. Similarly to [RV, formulae (5.27) and (5.29)], such results for $r \geq 2$ rely on the formula

$$\delta - \alpha - \beta = h + j - l \tag{1.2}$$

(see (2.1) below) relating the degree δ of the polynomials occurring in the linear forms with the degree $\alpha+\beta$ of the factor $z^\alpha(1-z)^\beta$, repeatedly used in our results yielding the arithmetical structure of the linear forms, e.g., in Lemma 2.1. As a consequence of (1.2), the constant c_3 used in our Proposition 6.1 turns out to be independent of the degree δ of the polynomials involved, and this independence is crucial to treat the cases $z = s/r$ with $r \geq 2$.

The present paper is organised as follows. In Section 2 we define, for $z > 1$, the double integrals $J_z^{(0)}$, $J_z^{(1)}$ and $J_z^{(2)}$, similar to the above integrals $I_z^{(0)}$, $I_z^{(1)}$ and $I_z^{(2)}$ but containing the factor $(1-y+yz)^{j+q-m}$ of Miladi's type, and we prove an arithmetical structure result (Lemma 2.1) for such integrals. We also apply to $J_z^{(\mu)}$ $(\mu = 0, 1, 2)$ a birational change of variables ((2.5) below) which simplifies the analytic structure of the integrals, and allows us to define $J_z^{(0)}$ also for $z < 0$. The symmetry of $J_z^{(1)}$ and of $J_z^{(2)}$ under the involution $z \leftrightarrow 1-z$ is also proved (see Lemmas 2.2 and 2.3), while the symmetry property for $J_z^{(0)}$ under $z \leftrightarrow 1-z$ is subtler, and involves a new double integral $K_z^{(0)}$ of mixed type (Lemma 2.4).

In Section 3 we show that the permutation group method applies to the integrals $J_z^{(\mu)}$ and $K_z^{(0)}$, and we do this by means of a further birational change of variables ((3.2) below), and by the hypergeometric integral transformation already used in [RV, Section 3].

In Section 4 we apply Hata's saddle point method in $\mathbb{C}^2$ and we get the required asymptotic formulae for $J_z^{(\mu)}$ and $K_z^{(0)}$, where the integer parameters h, j, k, l, m, q are replaced by hn, jn, kn, ln, mn, qn with $n \to \infty$.

In Section 5 we introduce some arithmetical lemmas yielding the linear independence over $\mathbb{Q}$ and a $\mathbb{Q}$-linear independence measure of $1, \gamma_1, \ldots, \gamma_S$, where $\gamma_1, \ldots, \gamma_S \in \mathbb{R}$, under suitable asymptotic formulae, as $n \to \infty$, for linear forms

$$q_n\gamma_\mu - p_n^{(\mu)} \in \mathbb{Z}\gamma_\mu + \mathbb{Z} \qquad (\mu = 1, \ldots, S)$$

and for their common coefficients q_n.

Finally, in Section 6 we combine the permutation group method developed in Section 3 with the arithmetical lemmas in Section 5, together with the asymptotic formulae in Section 4, and we obtain the $\mathbb{Q}$-linear independence results stated above for 1, $\mathrm{Li}_1(1/z)$, $\mathrm{Li}_2(1/z)$ and $\mathrm{Li}_2(1/(1-z))$, in qualitative and quantitative forms.

It is our pleasure to thank G. Rhin for providing us with the text of Miladi's thesis [Mi] (for which he happened to be in the committee) and G. F. Gronchi for creating for us Figure 1.

2 Rational approximations to Li_1 and Li_2

Let $z \in \mathbb{R}$, $z > 1$.

2.1

For integer parameters h, j, k, l, m, $q \geq 0$ such that $j+k-m$, $j+q-m$, $h+m-k$ and $h+q-k$ are also ≥ 0, define the double integrals

$$
\begin{aligned}
J_z^{(0)} &= z^{k-l-q} \int_0^1\!\!\int_0^1 \frac{x^j(1-x)^h\, y^k(1-y)^l}{\big(x(1-y)+yz\big)^{j+k-m+1}}\,(1-y+yz)^{j+q-m}\,\mathrm{d}x\,\mathrm{d}y, \\
J_z^{(1)} &= z^{k-l-q} \\
&\times \int_0^1 \Bigg(\frac{1}{2\pi i} \oint\limits_{\left|y-\frac{x}{x-z}\right|=\varrho} \frac{x^j(1-x)^h\, y^k(1-y)^l}{\big(x(1-y)+yz\big)^{j+k-m+1}}\,(1-y+yz)^{j+q-m}\,\mathrm{d}y\Bigg)\mathrm{d}x
\end{aligned}
$$

and

$$
\begin{aligned}
J_z^{(2)} &= z^{k-l-q} \\
&\times \frac{1}{2\pi i} \oint\limits_{|x-z|=\sigma} \Bigg(\frac{1}{2\pi i} \oint\limits_{\left|y-\frac{x}{x-z}\right|=\varrho} \frac{x^j(1-x)^h\, y^k(1-y)^l}{\big(x(1-y)+yz\big)^{j+k-m+1}}\,(1-y+yz)^{j+q-m}\,\mathrm{d}y\Bigg)\mathrm{d}x,
\end{aligned}
$$

where $\varrho, \sigma > 0$ are arbitrary, and also set

$$J_z = J_z^{(0)} - (\log z) J_z^{(1)},$$

where $\log z$ is the real value of the logarithm. The quantities so defined clearly generalise (apart from the normalisation factor z^{k-l-q} in place of z^{-l-m}) the corresponding families $I_z^{(\mu)}$ ($\mu = 0, 1, 2$) and I_z in [RV], which are reproduced in the introduction.

In what follows, the notation d_n is used for the least common multiple of $1, 2, \ldots, n$.

Lemma 2.1. *Let*

$$
\begin{aligned}
H &= \max\{l+q,\ h+m-k,\ h+j-l,\ j+k-m\}, \\
H' &= \max\{l+q,\ \min\{h+m-k,\ h+j-l\},\ j+k-m\}, \\
\alpha &= \max\{0,\ l+q-k,\ l+q-m\}, \qquad (2.1)\\
\beta &= \max\{0,\ k+l-h\}, \\
\delta &= \alpha+\beta+h+j-l.
\end{aligned}
$$

Then

$$(2.2)\qquad \begin{aligned} d_H d_{H'} z^\alpha (1-z)^\beta\, J_z &= P(z) - Q(z)\,\mathrm{Li}_2(1/z),\\ d_H d_{H'} z^\alpha (1-z)^\beta\, J_z^{(1)} &= R(z) - Q(z)\,\mathrm{Li}_1(1/z), \end{aligned}$$

where

$$P(z),\, Q(z),\, R(z) \in \mathbb{Z}[z], \qquad \max\{\deg P(z),\, \deg Q(z),\, \deg R(z)\} \le \delta.$$

Moreover

$$(2.3)\qquad d_H d_{H'} z^\alpha (1-z)^\beta\, J_z^{(2)} \,=\, Q(z).$$

Proof. Applying the binomial theorem, in two different ways, to the factor

$$((1-y)+yz)^{j+q-m} = (1+y(z-1))^{j+q-m}$$

in the definitions of $J_z^{(\mu)}$ $(\mu = 0,1,2)$, we obtain

$$\begin{aligned} J_z^{(\mu)} &= \sum_{\lambda=0}^{j+q-m} \binom{j+q-m}{\lambda} z^{j+k+\lambda} I_z^{(\mu)}(h,\, j,\, k+\lambda,\, j+l+q-m-\lambda,\, m+\lambda)\\ &= \sum_{\lambda=0}^{j+q-m} \binom{j+q-m}{\lambda} z^{k+m-q+\lambda}(z-1)^\lambda I_z^{(\mu)}(h,\, j,\, k+\lambda,\, l,\, m+\lambda), \end{aligned}$$

where the integrals $I_z^{(\mu)}$ are defined in the introduction. We now apply Theorem 2.1 of [RV]: either of the above representations of $J_z^{(\mu)}$ can be used to get the expressions for H and H'; the former representation leads to the expression for α, while the latter yields the formula for β.

Concerning δ, let α_1, β_1 and δ_1 be the integers defined in [RV, formula (2.9)] and denoted therein by α, β and δ, respectively. By [RV, Lemma 2.8] we have in any case

$$(2.4)\qquad \delta_1 - \alpha_1 - \beta_1 = \, h - k - l.$$

From the latter representation of $J_z^{(\mu)}$ we get

$$d_H d_{H'} z^\alpha (1-z)^\beta J_z^{(\mu)} \,=$$
$$\sum_{\lambda=0}^{j+q-m} (-1)^\lambda \binom{j+q-m}{\lambda} d_H d_{H'} z^{k+m-q+\lambda+\alpha}(1-z)^{\lambda+\beta} I_z^{(\mu)}(h,\, j,\, k+\lambda,\, l,\, m+\lambda),$$

and applying again Theorem 2.1 of [RV] to $I_z^{(\mu)}(h,\,j,\,k+\lambda,\,l,\,m+\lambda)$ and using (2.4), we see that the degrees of the polynomials $P(z)$, $Q(z)$ and $R(z)$ do not exceed

$$\max_{0\le\lambda\le j+q-m}(k+m-q+\lambda+\alpha+\lambda+\beta+h-k-\lambda-l)\;=\;\alpha+\beta+h+j-l\;=\;\delta.$$

□

2.2

In what follows, we use the notation Z to denote either the number $z>1$, or the number $1-z<0$. Our next goal is to extend the families of integrals defined in Subsection 2.1 to the latter case $Z=1-z<0$.

Clearly, the quantities

$$\begin{aligned} J_Z^{(1)} \;&=\; Z^{k-l-q} \\ &\times\int_0^1\Bigg(\frac{1}{2\pi i}\oint\limits_{\left|y-\frac{x}{x-Z}\right|=\varrho}\frac{x^j(1-x)^h\,y^k(1-y)^l}{\big(x(1-y)+yZ\big)^{j+k-m+1}}\,(1-y+yZ)^{j+q-m}\,\mathrm{d}y\Bigg)\mathrm{d}x \end{aligned}$$

and

$$\begin{aligned} J_Z^{(2)} \;&=\; Z^{k-l-q} \\ \times\frac{1}{2\pi i}&\oint\limits_{|x-Z|=\sigma}\Bigg(\frac{1}{2\pi i}\oint\limits_{\left|y-\frac{x}{x-Z}\right|=\varrho}\frac{x^j(1-x)^h\,y^k(1-y)^l}{\big(x(1-y)+yZ\big)^{j+k-m+1}}\,(1-y+yZ)^{j+q-m}\,\mathrm{d}y\Bigg)\mathrm{d}x \end{aligned}$$

are well defined for both $Z>1$ and $Z<0$. This is not the case of $J_z^{(0)}$: if $Z=1-z<0$ we cannot define $J_Z^{(0)}$ in a similar manner, because for $Z<0$ the denominator $x(1-y)+yZ$ vanishes along a segment of hyperbola inside the unit square $(0,1)\times(0,1)\subset\mathbb{R}^2$. In order to avoid this difficulty, we use a different representation for the double integrals above. We apply the change of variables

$$\left\{\begin{aligned} x &= \xi \\ y &= \frac{\eta}{\eta-Z}\,. \end{aligned}\right. \tag{2.5}$$

We begin with $J_z^{(0)}$. For $Z=z>1$, (2.5) changes the integration path $[0,1)$ for y to $[0,-\infty)$ for η. Thus we get

$$J_z^{(0)}=(-1)^{k+l+q}\int_0^1\xi^j(1-\xi)^h\Bigg(\int_0^{-\infty}\frac{\eta^k(1-\eta)^{j+q-m}}{(\xi-\eta)^{j+k-m+1}(\eta-z)^{l+q+1}}\,\mathrm{d}\eta\Bigg)\mathrm{d}\xi.$$

Take any $\zeta \in \mathbb{C}$ such that $|\zeta| = 1,\ 0 < \arg\zeta < 2\pi$. For every $0 < \xi \leq 1$, the inner integral is unchanged if we rotate the integration path $[0, -\infty)$ by moving it to the half-line $[0, \zeta\infty)$ going from 0 to ∞ through ζ. To see this, let for brevity

$$\varphi_\xi(\eta) := \frac{\eta^k (1-\eta)^{j+q-m}}{(\xi-\eta)^{j+k-m+1}(\eta - z)^{l+q+1}}\,.$$

Since $\xi > 0$ and $z > 0$, the function $\varphi_\xi(\eta)$ is holomorphic for $0 < \arg\eta < 2\pi$, as well as in a neighbourhood of $\eta = 0$. By Cauchy's theorem we get, for any $\varrho > 0$,

$$\int_0^{-\varrho} \varphi_\xi(\eta)\,\mathrm{d}\eta = \int_0^{\varrho\zeta} \varphi_\xi(\eta)\,\mathrm{d}\eta + \int_{\mu_\varrho} \varphi_\xi(\eta)\,\mathrm{d}\eta,$$

where μ_ϱ is the arc $\{|\eta| = \varrho,\ \arg\eta\ \text{ from }\ \arg\zeta \text{ to } \pi\}$. As $\varrho \to +\infty$ we have

$$\text{(2.6)} \qquad \left|\int_{\mu_\varrho} \varphi_\xi(\eta)\,\mathrm{d}\eta\right| \leq \frac{\varrho^k(\varrho+1)^{j+q-m}}{(\varrho-1)^{j+k-m+1}(\varrho - z)^{l+q+1}} \cdot 2\pi\varrho \ll \varrho^{-l-1} \longrightarrow 0,$$

whence

$$\int_0^{-\infty} \varphi_\xi(\eta)\,\mathrm{d}\eta = \int_0^{\zeta\infty} \varphi_\xi(\eta)\,\mathrm{d}\eta.$$

Thus, for any ζ as above,

$$\text{(2.7)} \quad J_z^{(0)} = (-1)^{k+l+q} \int_0^1 \xi^j (1-\xi)^h \left(\int_0^{\zeta\infty} \frac{\eta^k (1-\eta)^{j+q-m}}{(\xi-\eta)^{j+k-m+1}(\eta - z)^{l+q+1}}\,\mathrm{d}\eta \right) \mathrm{d}\xi.$$

We will now show that the double integral (2.7) converges absolutely, so that the integrations in ξ and η can be interchanged. We split the half-line $[0, \zeta\infty)$ as $[0, \zeta] \cup (\zeta, \zeta\infty)$, whence

$$\int_0^1 \xi^j (1-\xi)^h \left(\int_0^{\zeta\infty} |\varphi_\xi(\eta)|\,|\mathrm{d}\eta| \right) \mathrm{d}\xi$$

$$= \int_0^1 \xi^j (1-\xi)^h \left(\int_0^{\zeta} |\varphi_\xi(\eta)|\,|\mathrm{d}\eta| \right) \mathrm{d}\xi + \int_0^1 \xi^j (1-\xi)^h \left(\int_\zeta^{\zeta\infty} |\varphi_\xi(\eta)|\,|\mathrm{d}\eta| \right) \mathrm{d}\xi.$$

Let $0 \le \xi \le 1,\ \eta \in [0,\zeta]$, and let $\vartheta = \arg\zeta$. If $\pi/2 \le \vartheta \le 3\pi/2$ we clearly have $|\xi-\eta| \ge \max\{\xi, |\eta|\}$, and if $0 < \vartheta < \pi/2$ or $3\pi/2 < \vartheta < 2\pi$ a picture shows that $|\xi-\eta| \ge \max\{\xi\,|\sin\vartheta|, |\eta|\,|\sin\vartheta|\}$. Since $\vartheta = \arg\zeta$ is fixed and different from 0 and 2π, we get in any case, writing $|\eta| = u$,

$$
\begin{aligned}
(2.8)\qquad & \int_0^1 \xi^j(1-\xi)^h\Big(\int_0^\zeta \big|\varphi_\xi(\eta)\big|\,|\mathrm{d}\eta|\Big)\mathrm{d}\xi \\
&\ll \int_0^1\!\!\int_0^1 \frac{\xi^j(1-\xi)^h\,u^k(u+1)^{j+q-m}}{(\max\{\xi,u\})^{j+k-m+1}(z-u)^{l+q+1}}\,\mathrm{d}\xi\,\mathrm{d}u \\
&\ll \int_0^1\!\!\int_0^1 \frac{\xi^j u^k}{(\max\{\xi,u\})^{j+k-m+1}}\,\mathrm{d}\xi\,\mathrm{d}u \\
&= \int_0^1 \frac{\xi^j}{\xi^{j+k-m+1}}\Big(\int_0^\xi u^k\mathrm{d}u\Big)\mathrm{d}\xi + \int_0^1 \frac{u^k}{u^{j+k-m+1}}\Big(\int_0^u \xi^j\mathrm{d}\xi\Big)\mathrm{d}u \\
&= \frac{1}{k+1}\int_0^1 \xi^m\mathrm{d}\xi + \frac{1}{j+1}\int_0^1 u^m\mathrm{d}u \ll 1.
\end{aligned}
$$

For $0 \le \xi \le 1,\ \eta \in (\zeta, \zeta\infty)$, we have $\min\{|\xi-\eta|,\ |\eta-z|\} \ge |\eta|$ if $\pi/2 \le \vartheta \le 3\pi/2$, and $\min\{|\xi-\eta|,\ |\eta-z|\} \ge |\eta|\,|\sin\vartheta|$ if $0 < \vartheta < \pi/2$ or $3\pi/2 < \vartheta < 2\pi$. In any case

$$
\begin{aligned}
\int_\zeta^{\zeta\infty} \big|\varphi_\xi(\eta)\big|\,|\mathrm{d}\eta| &\ll \int_\zeta^{\zeta\infty} \frac{|\eta|^k(|\eta|+1)^{j+q-m}}{|\eta|^{j+k-m+l+q+2}}\,|\mathrm{d}\eta| \\
&= \int_1^{+\infty} \frac{u^k(u+1)^{j+q-m}}{u^{j+k-m+l+q+2}}\,\mathrm{d}u \ll \int_1^{+\infty} u^{-l-2}\mathrm{d}u = \frac{1}{l+1},
\end{aligned}
$$

whence

$$
\int_0^1 \xi^j(1-\xi)^h\Big(\int_\zeta^{\zeta\infty} \big|\varphi_\xi(\eta)\big|\,|\mathrm{d}\eta|\Big)\mathrm{d}\xi \ll 1.
$$

This proves the absolute convergence of the double integral (2.7) for any $\zeta \in \mathbb{C}$ satisfying $|\zeta| = 1,\ 0 < \arg\zeta < 2\pi$.

For any such ζ, the right-hand side of (2.7), viewed as a function of the complex variable z, is plainly holomorphic in the cut plane

$$\Pi_\zeta := \mathbb{C}\setminus[0,\zeta\infty).$$

In particular, for any $Z<0$ and any $\zeta\in\mathbb{C}$ with $|\zeta|=1$ and $\operatorname{Im}\zeta\neq 0$, we can define

$$(2.9)\qquad J_Z^{(0)} = (-1)^{k+l+q}\int_0^1 \xi^j(1-\xi)^h\left(\int_0^{\zeta\infty}\frac{\eta^k(1-\eta)^{j+q-m}}{(\xi-\eta)^{j+k-m+1}(\eta-Z)^{l+q+1}}\,\mathrm{d}\eta\right)\mathrm{d}\xi,$$

since $Z\in\Pi_\zeta$. From (2.7) we have

$$\overline{J_z^{(0)}} = (-1)^{k+l+q}\int_0^1 \xi^j(1-\xi)^h\left(\int_0^{\bar\zeta\infty}\frac{\eta^k(1-\eta)^{j+q-m}}{(\xi-\eta)^{j+k-m+1}(\eta-\bar z)^{l+q+1}}\,\mathrm{d}\eta\right)\mathrm{d}\xi.$$

Thus for every $Z<0$ we get two values of $J_Z^{(0)}$, conjugate to each other, one given by (2.9) for $\operatorname{Im}\zeta>0$, and the other for $\operatorname{Im}\zeta<0$. If $\operatorname{Im}\zeta>0$ (resp. $\operatorname{Im}\zeta<0$), by analytic continuation we can move continuously in (2.7) from $z=1-Z>1$ to $z=Z<0$ along a path contained in the half-plane $\operatorname{Im} z<0$ (resp. $\operatorname{Im} z>0$), since such a path does not cross the cut $[0,\zeta\infty)$. Accordingly, for $Z<0$ we define

$$J_Z \;=\; J_Z^{(0)} - (\log Z)J_Z^{(1)},$$

where, by analytic continuation, we take

$$(2.10)\qquad \log Z \;=\begin{cases}\log|Z|-\pi i, & \text{if in } J_Z^{(0)} \text{ we have } \operatorname{Im}\zeta>0\\ \log|Z|+\pi i, & \text{if in } J_Z^{(0)} \text{ we have } \operatorname{Im}\zeta<0.\end{cases}$$

We now turn to $J_Z^{(1)}$ and $J_Z^{(2)}$. In order to apply the substitution (2.5) to such integrals, where Z can be either >1 or <0, we interchange the integrations in x and y. For $J_Z^{(1)}$, let λ denote a fixed contour in $\mathbb{C}$ enclosing the open real interval $(0,\,1/(1-Z))$ if $Z<0$, or $(1/(1-Z),\,0)$ if $Z>1$, and passing through the endpoints 0 and $1/(1-Z)$. Then λ encloses the point $x/(x-Z)$ for any x such that $0<x<1$. Therefore

$$\begin{aligned} J_Z^{(1)} &= Z^{k-l-q}\int_0^1\left(\frac{1}{2\pi i}\oint_\lambda \frac{x^j(1-x)^h\,y^k(1-y)^l}{\big(x(1-y)+yZ\big)^{j+k-m+1}}\,(1-y+yZ)^{j+q-m}\,\mathrm{d}y\right)\mathrm{d}x\\ &= Z^{k-l-q}\,\frac{1}{2\pi i}\oint_\lambda\left(\int_0^1 \frac{x^j(1-x)^h\,y^k(1-y)^l}{\big(x(1-y)+yZ\big)^{j+k-m+1}}\,(1-y+yZ)^{j+q-m}\,\mathrm{d}x\right)\mathrm{d}y.\end{aligned}$$

The open interval of endpoints 0 and $1/(1-Z)$ for y corresponds through (2.5) to the open interval $(0,1)$ for η. Applying (2.5), the contour λ is transformed into a contour for η, which we denote by $\mathit{\Gamma}_{0,1}$, enclosing $(0,1)$ and passing through the endpoints 0 and 1. We easily get

$$(2.11)\qquad J_Z^{(1)} = (-1)^{k+l+q}\,\frac{1}{2\pi i}\oint\limits_{\mathit{\Gamma}_{0,1}}\Bigg(\int\limits_0^1 \frac{\xi^j(1-\xi)^h\,\eta^k(1-\eta)^{j+q-m}}{(\xi-\eta)^{j+k-m+1}(\eta-Z)^{l+q+1}}\,\mathrm{d}\xi\Bigg)\mathrm{d}\eta.$$

The absolute convergence of the double integral (2.11) in the neighbourhood of the points $\xi=\eta=0$ and $\xi=\eta=1$ where the factor $(\xi-\eta)^{j+k-m+1}$ vanishes, and hence the above interchange of integrations, are justified by the argument used in (2.8), provided the tangents to the contour $\mathit{\Gamma}_{0,1}$ at the points $\eta=0$ and $\eta=1$ are distinct from the real line.

For a fixed contour $\mathit{\Gamma}_{0,1}$ as above, the right-hand side of (2.11) is clearly a one-valued analytic function of the complex variable Z, holomorphic in the open part of $\mathbb{C}$ not enclosed by $\mathit{\Gamma}_{0,1}$. Taking $\mathit{\Gamma}_{0,1}$ symmetric with respect to the real line, i.e., $\overline{\mathit{\Gamma}_{0,1}}=\mathit{\Gamma}_{0,1}$, from (2.11) we get $\overline{J_Z^{(1)}}=J_{\overline{Z}}^{(1)}$. Hence for any $z>1$ we have $J_z^{(1)}\in\mathbb{R}$ and $J_{1-z}^{(1)}\in\mathbb{R}$.

Concerning $J_Z^{(2)}$, we have $|x-Z|=\sigma$ if and only if $|\,x/(x-Z)-1\,|=|Z|/\sigma$. Hence the contour $|\,y-x/(x-Z)\,|=\varrho$ in the integral defining $J_Z^{(2)}$ can be replaced by $|y-1|=\varrho$ provided $\varrho\sigma>|Z|$, since under this assumption the circumference $|y-1|=\varrho$ encloses the point $x/(x-Z)$ for any x such that $|x-Z|=\sigma$. After this replacement, we may interchange the integrations over $|x-Z|=\sigma$ and $|y-1|=\varrho$. It follows that

$$\begin{aligned} J_Z^{(2)} &= Z^{k-l-q}\\ &\times\frac{1}{2\pi i}\oint\limits_{|y-1|=\varrho>0}\Bigg(\frac{1}{2\pi i}\oint\limits_{|x-Z|=\sigma>|Z|/\varrho}\frac{x^j(1-x)^h\,y^k(1-y)^l}{\big(x(1-y)+yZ\big)^{j+k-m+1}}\,(1-y+yZ)^{j+q-m}\,\mathrm{d}x\Bigg)\mathrm{d}y\\ &= Z^{k-l-q}\\ &\times\frac{1}{2\pi i}\oint\limits_{|y-1|=\varrho}\Bigg(\frac{1}{2\pi i}\oint\limits_{\left|x-\frac{yZ}{y-1}\right|=\sigma}\frac{x^j(1-x)^h\,y^k(1-y)^l}{\big(x(1-y)+yZ\big)^{j+k-m+1}}\,(1-y+yZ)^{j+q-m}\,\mathrm{d}x\Bigg)\mathrm{d}y, \end{aligned}$$

where in the last double contour integral we may take any $\varrho,\sigma>0$. When y describes the circumference $|y-1|=\varrho$ in the positive sense, the point $\eta=yZ/(y-1)$, corresponding to y through (2.5), describes $|\eta-Z|=|Z|/\varrho$ in the negative sense. Thus, applying (2.5) we obtain

$$J_Z^{(2)} = (-1)^{k+l+q+1} \times \frac{1}{2\pi i} \oint_{|\eta-Z|=\varrho_1} \left(\frac{1}{2\pi i} \oint_{|\xi-\eta|=\varrho_2} \frac{\xi^j(1-\xi)^h\,\eta^k(1-\eta)^{j+q-m}}{(\xi-\eta)^{j+k-m+1}(\eta-Z)^{l+q+1}}\,\mathrm{d}\xi \right) \mathrm{d}\eta, \tag{2.12}$$

for any $\varrho_1, \varrho_2 > 0$. By applying twice Cauchy's integral formula in (2.12), first to the integral in ξ and then to the integral in η, we easily get $J_Z^{(2)} \in \mathbb{Z}[Z]$, with $\deg J_Z^{(2)} = h + j - l$ if $h + j - l \geq 0$, or with $J_Z^{(2)}$ identically zero if $h + j - l < 0$.

2.3

We denote the above integrals $J_Z^{(\mu)}$ by $J_Z^{(\mu)}(h, j, k, l, m, q)$ $(\mu = 0, 1, 2)$.

Lemma 2.2. *We have*

$$J_{1-Z}^{(1)}(j,\, h,\, j+q-m,\, l,\, h+q-k,\, q) = (-1)^{l+1} J_Z^{(1)}(h, j, k, l, m, q).$$

Proof. By (2.11),

$$J_{1-Z}^{(1)}(j,\, h,\, j+q-m,\, l,\, h+q-k,\, q) = (-1)^{j+l+m} \frac{1}{2\pi i} \oint_{\Gamma_{0,1}} \left(\int_0^1 \frac{\xi^h(1-\xi)^j\,\eta^{j+q-m}(1-\eta)^k}{(\xi-\eta)^{j+k-m+1}(\eta-(1-Z))^{l+q+1}}\,\mathrm{d}\xi \right) \mathrm{d}\eta,$$

where the contour $\Gamma_{0,1}$ may be taken symmetric about the point $1/2$. We apply the change of variables

$$\begin{cases} \xi = 1 - \widehat{\xi} \\ \eta = 1 - \widehat{\eta}. \end{cases} \tag{2.13}$$

Passing from ξ to $\widehat{\xi}$, (2.13) changes the orientation of $[0, 1]$ and changes $\mathrm{d}\xi$ to $-\mathrm{d}\widehat{\xi}$, and from η to $\widehat{\eta}$ (2.13) preserves the orientation of $\Gamma_{0,1}$ and changes $\mathrm{d}\eta$ to $-\mathrm{d}\widehat{\eta}$. Therefore

$$
\begin{aligned}
&J^{(1)}_{1-Z}(j,\, h,\, j+q-m,\, l,\, h+q-k,\, q)\\
&\quad = (-1)^{j+l+m+1}\,\frac{1}{2\pi i}\oint\limits_{\Gamma_{0,1}}\Bigg(\int_0^1 \frac{\widehat{\xi}^{j}(1-\widehat{\xi})^{h}\,\widehat{\eta}^{k}(1-\widehat{\eta})^{j+q-m}}{(\widehat{\eta}-\widehat{\xi})^{j+k-m+1}(Z-\widehat{\eta})^{l+q+1}}\,\mathrm{d}\widehat{\xi}\Bigg)\mathrm{d}\widehat{\eta}\\
&\quad = (-1)^{k+q+1}\,\frac{1}{2\pi i}\oint\limits_{\Gamma_{0,1}}\Bigg(\int_0^1 \frac{\widehat{\xi}^{j}(1-\widehat{\xi})^{h}\,\widehat{\eta}^{k}(1-\widehat{\eta})^{j+q-m}}{(\widehat{\xi}-\widehat{\eta})^{j+k-m+1}(\widehat{\eta}-Z)^{l+q+1}}\,\mathrm{d}\widehat{\xi}\Bigg)\mathrm{d}\widehat{\eta}\\
&\quad = (-1)^{l+1}J^{(1)}_{Z}(h,j,k,l,m,q). \qquad\Box
\end{aligned}
$$

Similarly,

Lemma 2.3. *We have*

$$
J^{(2)}_{1-Z}(j,\, h,\, j+q-m,\, l,\, h+q-k,\, q) = (-1)^{l}\, J^{(2)}_{Z}(h,j,k,l,m,q).
$$

Proof. By (2.12),

$$
\begin{aligned}
&J^{(2)}_{1-Z}(j,\, h,\, j+q-m,\, l,\, h+q-k,\, q) = (-1)^{j+l+m+1}\\
&\qquad\times\frac{1}{2\pi i}\oint\limits_{|\eta-(1-Z)|=\varrho_1}\Bigg(\frac{1}{2\pi i}\oint\limits_{|\xi-\eta|=\varrho_2}\frac{\xi^{h}(1-\xi)^{j}\,\eta^{j+q-m}(1-\eta)^{k}}{(\xi-\eta)^{j+k-m+1}(\eta-(1-Z))^{l+q+1}}\,\mathrm{d}\xi\Bigg)\mathrm{d}\eta.
\end{aligned}
$$

Again we apply (2.13), which changes the contours $|\eta-(1-Z)|=\varrho_1$ and $|\xi-\eta|=\varrho_2$ respectively to $|\widehat{\eta}-Z|=\varrho_1$ and $|\widehat{\xi}-\widehat{\eta}|=\varrho_2$, with the same orientations. Thus

$$
\begin{aligned}
&J^{(2)}_{1-Z}(j,\, h,\, j+q-m,\, l,\, h+q-k,\, q)\\
&= (-1)^{j+l+m+1}\,\frac{1}{2\pi i}\oint\limits_{|\widehat{\eta}-Z|=\varrho_1}\Bigg(\frac{1}{2\pi i}\oint\limits_{|\widehat{\xi}-\widehat{\eta}|=\varrho_2}\frac{\widehat{\xi}^{j}(1-\widehat{\xi})^{h}\,\widehat{\eta}^{k}(1-\widehat{\eta})^{j+q-m}}{(\widehat{\eta}-\widehat{\xi})^{j+k-m+1}(Z-\widehat{\eta})^{l+q+1}}\,\mathrm{d}\widehat{\xi}\Bigg)\mathrm{d}\widehat{\eta}\\
&= (-1)^{k+q+1}\,\frac{1}{2\pi i}\oint\limits_{|\widehat{\eta}-Z|=\varrho_1}\Bigg(\frac{1}{2\pi i}\oint\limits_{|\widehat{\xi}-\widehat{\eta}|=\varrho_2}\frac{\widehat{\xi}^{j}(1-\widehat{\xi})^{h}\,\widehat{\eta}^{k}(1-\widehat{\eta})^{j+q-m}}{(\widehat{\xi}-\widehat{\eta})^{j+k-m+1}(\widehat{\eta}-Z)^{l+q+1}}\,\mathrm{d}\widehat{\xi}\Bigg)\mathrm{d}\widehat{\eta}\\
&= (-1)^{l}\, J^{(2)}_{Z}(h,j,k,l,m,q). \qquad\Box
\end{aligned}
$$

For $Z=z>1$ or $Z=1-z<0$, the change of variables (2.13) yields also a transformation formula relating $J^{(0)}_{Z}(h,j,k,l,m,q)$ with $J^{(0)}_{1-Z}(j,\, h,\, j+q-m,\, l,\, h+q-k,\, q)$. However this formula involves a further double integral which we denote by $K_Z^{(0)}$, and we define by

$$K_Z^{(0)} = K_Z^{(0)}(h,j,k,l,m,q) = (-1)^{k+l+q} \int_{\gamma_{0,1}} \left(\int_0^1 \frac{\xi^j (1-\xi)^h \, \eta^k (1-\eta)^{j+q-m}}{(\xi-\eta)^{j+k-m+1}(\eta - Z)^{l+q+1}} \, \mathrm{d}\xi \right) \mathrm{d}\eta, \tag{2.14}$$

where $\gamma_{0,1}$ is an arc from 0 to 1 contained either in the upper half-plane $\operatorname{Im}\eta > 0$, or in the lower half-plane $\operatorname{Im}\eta < 0$, with tangents at the endpoints $\eta = 0$ and $\eta = 1$ distinct from the real line. Again by the argument in (2.8), already used to prove the absolute convergence of (2.11), we see that the double integral (2.14) converges absolutely. For $Z \in \mathbb{R}$, $Z > 1$ or $Z < 0$, the two values obtained for $K_Z^{(0)}$, corresponding to $\operatorname{Im}\gamma_{0,1} > 0$ or to $\operatorname{Im}\gamma_{0,1} < 0$, are clearly conjugate to each other. Also, from (2.11) and (2.14) we get, for $Z > 1$ or $Z < 0$,

$$J_Z^{(1)} = \pm \frac{1}{2\pi i}\left(K_Z^{(0)} - \overline{K_Z^{(0)}} \right) = \pm \frac{1}{\pi} \operatorname{Im} K_Z^{(0)}, \tag{2.15}$$

with the + sign (resp. − sign) if in $K_Z^{(0)}$ we have $\operatorname{Im}\gamma_{0,1} < 0$ (resp. $\operatorname{Im}\gamma_{0,1} > 0$).

Lemma 2.4. *The following equality holds:*

$$J_z^{(0)}(h,j,k,l,m,q) + (-1)^l J_{1-z}^{(0)}(j,\, h,\, j+q-m,\, l,\, h+q-k,\, q) = K_z^{(0)}(h,j,k,l,m,q), \tag{2.16}$$

where in $K_z^{(0)}(h,j,k,l,m,q)$ *we take* $\operatorname{Im}\gamma_{0,1} < 0$ *(resp.* $\operatorname{Im}\gamma_{0,1} > 0$*) if in* $J_{1-z}^{(0)}(j,\, h,\, j+q-m,\, l,\, h+q-k,\, q)$ *we have* $\operatorname{Im}\zeta > 0$ *(resp.* $\operatorname{Im}\zeta < 0$*).*

Proof. For $z > 1$ we have, by (2.9),

$$J_{1-z}^{(0)}(j,\, h,\, j+q-m,\, l,\, h+q-k,\, q) = (-1)^{j+l+m} \int_0^1 \xi^h (1-\xi)^j \left(\int_0^{\zeta\infty} \frac{\eta^{j+q-m}(1-\eta)^k}{(\xi-\eta)^{j+k-m+1}(\eta-(1-z))^{l+q+1}} \, \mathrm{d}\eta \right) \mathrm{d}\xi,$$

where we take, e.g., $\operatorname{Im}\zeta > 0$. Applying (2.13) we get

$$\begin{aligned}
&J^{(0)}_{1-z}(j,\, h,\, j+q-m,\, l,\, h+q-k,\, q)\\
&\quad = (-1)^{j+l+m} \int_0^1 \widehat{\xi}^j (1-\widehat{\xi})^h \Big(\int_L \frac{\widehat{\eta}^k (1-\widehat{\eta})^{j+q-m}}{(\widehat{\eta}-\widehat{\xi})^{j+k-m+1} (z-\widehat{\eta})^{l+q+1}} (-\mathrm{d}\widehat{\eta}) \Big) \mathrm{d}\widehat{\xi}\\
&\quad = (-1)^{k+q} \int_0^1 \widehat{\xi}^j (1-\widehat{\xi})^h \Big(-\int_L \frac{\widehat{\eta}^k (1-\widehat{\eta})^{j+q-m}}{(\widehat{\xi}-\widehat{\eta})^{j+k-m+1} (\widehat{\eta}-z)^{l+q+1}} \mathrm{d}\widehat{\eta} \Big) \mathrm{d}\widehat{\xi},
\end{aligned}$$

where L, the image of $(0, \zeta\infty)$ through (2.13), is the half-line from 1 to ∞ parallel to $(0, -\zeta\infty)$ in the lower half-plane $\operatorname{Im}\widehat{\eta} < 0$. By virtue of (2.6) we obtain, for an arc $\gamma_{0,1}$ from 0 to 1 with $\operatorname{Im}\gamma_{0,1} < 0$,

$$\begin{aligned}
&J^{(0)}_{1-z}(j,\, h,\, j+q-m,\, l,\, h+q-k,\, q)\\
&\quad = (-1)^{k+q} \int_0^1 \widehat{\xi}^j (1-\widehat{\xi})^h \Big(\int_{\gamma_{0,1}} \frac{\widehat{\eta}^k (1-\widehat{\eta})^{j+q-m}}{(\widehat{\xi}-\widehat{\eta})^{j+k-m+1} (\widehat{\eta}-z)^{l+q+1}} \mathrm{d}\widehat{\eta}\\
&\qquad\qquad - \int_0^{-\zeta\infty} \frac{\widehat{\eta}^k (1-\widehat{\eta})^{j+q-m}}{(\widehat{\xi}-\widehat{\eta})^{j+k-m+1} (\widehat{\eta}-z)^{l+q+1}} \mathrm{d}\widehat{\eta} \Big) \mathrm{d}\widehat{\xi}\\
&\quad = (-1)^l\, K^{(0)}_z(h,j,k,l,m,q) + (-1)^{l+1} J^{(0)}_z(h,j,k,l,m,q),
\end{aligned}$$

since, for $z > 1$, (2.7) holds independently of the sign of $\operatorname{Im}\zeta$. The result is exactly the required equality. $\square$

Lemma 2.5. *We have*

$$K^{(0)}_{1-Z}(j,\, h,\, j+q-m,\, l,\, h+q-k,\, q) = (-1)^l\, \overline{K^{(0)}_Z(h,j,k,l,m,q)}, \tag{2.17}$$

with the same integration path $\gamma_{0,1}$ on both sides of (2.17).

Proof. We remark that the image through (2.13) of an arc $\gamma_{0,1}$ from 0 to 1 in the half-plane $\operatorname{Im}\eta > 0$ is an arc from 1 to 0 in the half-plane $\operatorname{Im}\widehat{\eta} < 0$. Thus if we apply the substitution (2.13) directly to

$$\begin{aligned}
&K^{(0)}_{1-z}(j,\, h,\, j+q-m,\, l,\, h+q-k,\, q)\\
&\quad = (-1)^{j+l+m} \int_0^1 \xi^h (1-\xi)^j \Big(\int_{\gamma_{0,1}} \frac{\eta^{j+q-m} (1-\eta)^k}{(\xi-\eta)^{j+k-m+1} (\eta-(1-z))^{l+q+1}} \mathrm{d}\eta \Big) \mathrm{d}\xi,
\end{aligned}$$

the same computation as above yields

$$
\begin{aligned}
&K^{(0)}_{1-z}(j,\, h,\, j+q-m,\, l,\, h+q-k,\, q)\\
&\qquad = (-1)^{k+q}\int_0^1 \widehat{\xi}^{j}(1-\widehat{\xi})^{h}\Bigg(\int_{\overline{\gamma_{0,1}}} \frac{\widehat{\eta}^{k}(1-\widehat{\eta})^{j+q-m}}{(\widehat{\xi}-\widehat{\eta})^{j+k-m+1}(\widehat{\eta}-z)^{l+q+1}}\,\mathrm{d}\widehat{\eta}\Bigg)\mathrm{d}\widehat{\xi},
\end{aligned}
$$

and (2.17) follows. □

Finally, we set

$$
\begin{aligned}
K_Z \;=\; & K_Z(h,j,k,l,m,q)\\
&= J_Z(h,j,k,l,m,q) + (-1)^{l}\, J_{1-Z}(j,\, h,\, j+q-m,\, l,\, h+q-k,\, q).
\end{aligned}
$$

By expanding the definitions of J_Z and J_{1-Z} on the right-hand side of this expression in the case $Z = z > 1$, and applying (2.10) and Lemma 2.4, we get the following result.

Lemma 2.6. *For $z > 1$ we have*

$$
\begin{aligned}
K_z(h,j,k,l,m,q) \;=\; & K^{(0)}_z(h,j,k,l,m,q) - (\log z)\, J^{(1)}_z(h,j,k,l,m,q)\\
&- (\log(1-z))\,(-1)^{l}\, J^{(1)}_{1-z}(j,\, h,\, j+q-m,\, l,\, h+q-k,\, q),
\end{aligned}
$$

with the convention

$$
\log(1-z) \;=\; \begin{cases} \log|1-z| - \pi i, & \text{if in } K^{(0)}_z \text{ we have } \operatorname{Im}\gamma_{0,1} < 0\\ \log|1-z| + \pi i, & \text{if in } K^{(0)}_z \text{ we have } \operatorname{Im}\gamma_{0,1} > 0. \end{cases}
$$

3 The permutation group

3.1

Lemma 3.1. *The quantities*

$$
J^{(\mu)}_Z(h,j,k,l,m,q) \quad (\mu = 0,1,2), \qquad K^{(0)}_Z(h,j,k,l,m,q),
$$

$$
K_Z(h,j,k,l,m,q) \;\; \text{and} \;\; J_Z(h,j,k,l,m,q)
$$

are invariant under the action of the permutation

$$
\boldsymbol{\nu} \;=\; (h\;\; j)(k\;\; m)
$$

which acts identically on l and q.

Proof. By absolute convergence, we can write (2.9) in the form

$$(3.1)\quad J_Z^{(0)}(h,j,k,l,m,q) = (-1)^{k+l+q}\int_0^{\zeta\infty} \frac{\eta^k(1-\eta)^{j+q-m}}{(\eta-Z)^{l+q+1}}\left(\int_0^1 \frac{\xi^j(1-\xi)^h}{(\xi-\eta)^{j+k-m+1}}\,\mathrm{d}\xi\right)\mathrm{d}\eta,$$

where $Z \in \Pi_\zeta$ can be either $z>1$ or $1-z<0$, and apply to the inner integral the involution $\xi \leftrightarrow \widetilde{\xi}$ given by

$$(3.2)\qquad \xi = \eta\,\frac{\widetilde{\xi}-1}{\widetilde{\xi}-\eta}\,.$$

For each $\eta \in \mathbb{R}\setminus[0,1]$, (3.2) changes the interval $[0,1]$ for ξ into the interval $[0,1]$ for $\widetilde{\xi}$ (from 1 to 0), and for each $\eta \notin \mathbb{R}$ (3.2) changes the interval $[0,1]$ for ξ into an arc of circumference going from 1 to 0 in the plane of the complex variable $\widetilde{\xi}$. It is easily seen that, for $0<\xi<1$, $\operatorname{Im}\widetilde{\xi}$ and $\operatorname{Im}\eta$ have opposite signs. Thus, for each $\eta \in (0,\zeta\infty)$ in (3.1), after the change of variable (3.2) the integration path for $\widetilde{\xi}$ can be moved by Cauchy's theorem to the real interval $[0,1]$ (from 1 to 0) without encountering η. Hence (3.2) yields

$$\begin{aligned} J_Z^{(0)}(h,j,k,l,m,q) &= (-1)^{l+m+q}\int_0^{\zeta\infty} \frac{\eta^m(1-\eta)^{h+q-k}}{(\eta-Z)^{l+q+1}}\left(\int_0^1 \frac{\widetilde{\xi}^h(1-\widetilde{\xi})^j}{(\widetilde{\xi}-\eta)^{h+m-k+1}}\,\mathrm{d}\widetilde{\xi}\right)\mathrm{d}\eta \\ &= J_Z^{(0)}(j,h,m,l,k,q). \end{aligned}$$

The same argument holds for each $\eta \in \varGamma_{0,1}$ in (2.11), or for each $\eta \in \gamma_{0,1}$ in (2.14). This shows that $J_Z^{(0)}(h,j,k,l,m,q)$, $J_Z^{(1)}(h,j,k,l,m,q)$ and $K_Z^{(0)}(h,j,k,l,m,q)$ are invariant under the action of the permutation $\boldsymbol{\nu}$ from the statement of the lemma. We get the same conclusion for $J_Z^{(2)}(h,j,k,l,m,q)$ given by (2.12), since, for any fixed η with sufficiently small $|\eta-Z|=\varrho_1$, the substitution (3.2) changes the contour $|\xi-\eta|=\varrho_2$, described by ξ in the positive sense, to $|\widetilde{\xi}-\eta|=\widetilde{\varrho}_2$, where $\widetilde{\varrho}_2 = |\eta(1-\eta)|/\varrho_2$, described by $\widetilde{\xi}$ in the negative sense. Thus $J_Z^{(0)}$, $J_Z^{(1)}$, $J_Z^{(2)}$, $K_Z^{(0)}$, and hence also J_Z and K_Z, are invariant under the action of $\boldsymbol{\nu}$. □

To the inner integrals in ξ appearing in (2.11), (2.12), (2.14) and (3.1) we can also apply the hypergeometric transformation as in [RV, Section 3]. The result is

Lemma 3.2. *The quotients*

$$\frac{J_Z^{(\mu)}(h,j,k,l,m,q)}{h!\,j!}\quad (\mu=0,1,2),\qquad \frac{K_Z^{(0)}(h,j,k,l,m,q)}{h!\,j!}\,,$$

and hence

$$\frac{K_Z(h,j,k,l,m,q)}{h!\,j!} \quad \textit{and} \quad \frac{J_Z(h,j,k,l,m,q)}{h!\,j!},$$

are invariant under the action of the permutation

$$\boldsymbol{\varphi} = (h \quad h+m-k)(j \quad j+k-m)(k \;\; m),$$

which also acts identically on l and q.

Proof. By [RV, formula (3.3)] we have

$$\int_0^1 \frac{\xi^j(1-\xi)^h}{(\xi-\eta)^{j+k-m+1}}\,\mathrm{d}\xi = \Big(-\frac{1}{\eta}\Big)^{j+k-m+1} \int_0^1 \frac{\xi^j(1-\xi)^h}{\Big(1-\dfrac{\xi}{\eta}\Big)^{j+k-m+1}}\,\mathrm{d}\xi$$

$$= \Big(-\frac{1}{\eta}\Big)^{j+k-m+1} \frac{h!\,j!}{(h+m-k)!\,(j+k-m)!} \int_0^1 \frac{\xi^{j+k-m}(1-\xi)^{h+m-k}}{\Big(1-\dfrac{\xi}{\eta}\Big)^{j+1}}\,\mathrm{d}\xi$$

$$= \frac{h!\,j!}{(h+m-k)!\,(j+k-m)!}\,(-\eta)^{m-k} \int_0^1 \frac{\xi^{j+k-m}(1-\xi)^{h+m-k}}{(\xi-\eta)^{j+1}}\,\mathrm{d}\xi,$$

and similarly, by [RV, Lemma 3.1],

$$\frac{1}{2\pi i} \oint\limits_{|\xi-\eta|=\varrho_2} \frac{\xi^j(1-\xi)^h}{(\xi-\eta)^{j+k-m+1}}\,\mathrm{d}\xi$$

$$= \frac{h!\,j!}{(h+m-k)!\,(j+k-m)!}\,(-\eta)^{m-k}\,\frac{1}{2\pi i} \oint\limits_{|\xi-\eta|=\varrho_2} \frac{\xi^{j+k-m}(1-\xi)^{h+m-k}}{(\xi-\eta)^{j+1}}\,\mathrm{d}\xi.$$

From (2.11), (2.12) and (3.1) we get

$$J_Z^{(\mu)}(h,j,k,l,m,q) = \frac{h!\,j!}{(h+m-k)!\,(j+k-m)!} \times J_Z^{(\mu)}(h+m-k,\ j+k-m,\ m,\ l,\ k,\ q) \qquad (\mu = 0,1,2),$$

and from (2.14)

$$K_Z^{(0)}(h,j,k,l,m,q) = \frac{h!\,j!}{(h+m-k)!\,(j+k-m)!} \times K_Z^{(0)}(h+m-k,\ j+k-m,\ m,\ l,\ k,\ q).$$

This shows that the quotients in the statement of the lemma are all invariant under the action of the permutation $\boldsymbol{\varphi}$. □

3.2

The following lemma extends Lemma 2.1 to the case $Z = 1 - z < 0$, and also improves the expression for H' by making use of the permutation $\boldsymbol{\nu}$ in Lemma 3.1.

Lemma 3.3. *Let $Z > 1$ or $Z < 0$, and let*

$$\begin{aligned} H &= \max\{l+q,\, h+m-k,\, h+j-l,\, j+k-m\}, \\ H' &= \max\{l+q,\, \max{}'\{h+m-k,\, h+j-l,\, j+k-m\}\}, \\ \alpha &= \max\{0,\, l+q-k,\, l+q-m\}, \\ \beta &= \max\{0,\, k+l-h,\, l+m-j\}, \\ \delta &= \alpha+\beta+h+j-l, \end{aligned} \tag{3.3}$$

where the second successive maximum $\max'$ *of a multi-set is defined as in* [RV, p. 417]. *Then*

$$\begin{aligned} d_H d_{H'} Z^\alpha (1-Z)^\beta J_Z(h,j,k,l,m,q) &= P(Z) - Q(Z)\,\mathrm{Li}_2(1/Z), \\ d_H d_{H'} Z^\alpha (1-Z)^\beta J_Z^{(1)}(h,j,k,l,m,q) &= R(Z) - Q(Z)\,\mathrm{Li}_1(1/Z), \end{aligned}$$

where

$$P(Z),\, Q(Z),\, R(Z) \in \mathbb{Z}[Z], \qquad \max\{\deg P(Z),\, \deg Q(Z),\, \deg R(Z)\} \le \delta.$$

Moreover

$$d_H d_{H'} Z^\alpha (1-Z)^\beta J_Z^{(2)}(h,j,k,l,m,q) \;=\; Q(Z).$$

Proof. From the results in Subsection 2.2 it follows that the integrals $J_Z^{(\mu)}$ ($\mu = 0,1,2$) are analytic functions of the complex variable Z. More precisely, $J_Z^{(0)}$ is holomorphic for $Z \in \Pi_\zeta = \mathbb{C} \setminus [0, \zeta\infty)$, $J_Z^{(1)}$ is holomorphic for $Z \in \mathbb{C} \setminus [0,1]$, and $J_Z^{(2)}$ is an entire function of Z (in fact, a polynomial). Therefore, by analytic continuation, the formulae (2.2) and (2.3) of Lemma 2.1 remain valid in the whole domain of holomorphy of the integrals appearing in each formula.

By Lemma 3.1, J_Z, $J_Z^{(1)}$ and $J_Z^{(2)}$ are invariant under the action of the permutation $\boldsymbol{\nu}$, and plainly the integers in (3.3) are also invariant under the action of $\boldsymbol{\nu}$. Since the integer β appearing in (2.1) does not exceed the β defined in (3.3), Lemma 2.1 implies that the conclusion of Lemma 3.3 holds if we take for H' either the value

$$\max\{l+q,\, \min\{h+m-k,\, h+j-l\},\, j+k-m\}$$

in (2.1), or the value

$$\max\{l+q,\, \min\{j+k-m,\, h+j-l\},\, h+m-k\}$$

obtained by applying $\boldsymbol{\nu}$ to the previous one. Therefore Lemma 3.3 holds with the least of the two values above, i.e., with the H' appearing in (3.3). $\square$

Note that α, β and δ in (3.3) are invariant under the actions of both permutations $\boldsymbol{\nu}$ and $\boldsymbol{\varphi}$, while we have to further modify the definitions of H and H' to have them invariant under $\boldsymbol{\varphi}$, by taking

$$\begin{aligned} H &= \max\{l+q,\, h,\, j,\, h+m-k,\, h+j-l,\, j+k-m\}, \\ H' &= \max\{l+q,\, \max{}'\{h,\, j,\, h+m-k,\, h+j-l,\, j+k-m\}\}. \end{aligned}$$

They happen to be invariant under the permutation

$$(h\;\; j)(k\;\; j+q-m)(m\;\; h+q-k)$$

induced by the transformation $Z \leftrightarrow 1-Z$ in Lemmas 2.2–2.5, and this permutation interchanges the values of α and β in (3.3). By combining the results of Subsections 2.3, 3.1 and Lemma 3.3, as well as the identities

$$\mathrm{Li}_1(1/Z) + \mathrm{Li}_1(1/(1-Z)) = 0 \quad\text{and}\quad \mathrm{Li}_2(1/Z) + \mathrm{Li}_2(1/(1-Z)) = -\tfrac{1}{2}\,\mathrm{Li}_1(1/Z)^2,$$

we arrive at the following general statement.

Proposition 3.1. *Let*

$$\begin{aligned} H &= \max\{l+q,\, h,\, j,\, h+m-k,\, h+j-l,\, j+k-m\}, \\ H' &= \max\{l+q,\, \max{}'\{h,\, j,\, h+m-k,\, h+j-l,\, j+k-m\}\}, \\ \alpha &= \max\{0,\, l+q-k,\, l+q-m\}, \\ \beta &= \max\{0,\, k+l-h,\, l+m-j\}, \\ \delta &= \alpha+\beta+h+j-l. \end{aligned}$$

Then

$$\begin{aligned} d_H d_{H'} z^\alpha (1-z)^\beta J_z(h,j,k,l,m,q) &= P_1(z) - Q(z)\,\mathrm{Li}_2(1/z), \\ (-1)^l d_H d_{H'} z^\alpha (1-z)^\beta & \\ \times J_{1-z}(j,\, h,\, j+q-m,\, l,\, h+q-k,\, q) &= P_2(z) - Q(z)\,\mathrm{Li}_2(1/(1-z)), \\ d_H d_{H'} z^\alpha (1-z)^\beta K_z(h,j,k,l,m,q) &= P(z) - Q(z)\big(-\tfrac{1}{2}\,\mathrm{Li}_1(1/z)^2\big) \end{aligned}$$

and

$$d_H d_{H'} z^\alpha (1-z)^\beta J_z^{(1)}(h,j,k,l,m,q) = R(z) - Q(z)\,\mathrm{Li}_1(1/z),$$

where

$$Q(z) = d_H d_{H'} z^\alpha (1-z)^\beta J_z^{(2)}(h,j,k,l,m,q)$$
$$= (-1)^l d_H d_{H'} z^\alpha (1-z)^\beta J_{1-z}^{(2)}(j,\, h,\, j+q-m,\, l,\, h+q-k,\, q),$$

$P_1(z)$, $P_2(z)$, $P(z) = P_1(z) + P_2(z)$ and $R(z)$ are polynomials with integer coefficients and degrees not exceeding δ. Moreover, the quotients

$$\frac{Q(z)}{h!\,j!}\,, \quad \frac{P_1(z)}{h!\,j!}\,, \quad \frac{P_2(z)}{h!\,j!}\,, \quad \frac{P(z)}{h!\,j!} \quad \textit{and} \quad \frac{R(z)}{h!\,j!}$$

are invariant under the actions of the permutations

$$\boldsymbol{\nu} = (h\ \ j)(k\ \ m) \quad \textit{and} \quad \boldsymbol{\varphi} = (h\ \ h+m-k)(j\ \ j+k-m)(k\ \ m).$$

4 Asymptotics

4.1

Let

(4.1) $$f(\xi,\eta) = f_Z(\xi,\eta) = \frac{\xi^j(1-\xi)^h\,\eta^k(1-\eta)^{j+q-m}}{(\xi-\eta)^{j+k-m}(\eta-Z)^{l+q}}\,.$$

We require information on the asymptotic behaviour of the above double integrals of

$$f(\xi,\eta)^n\,\frac{\mathrm{d}\xi\,\mathrm{d}\eta}{(\xi-\eta)(\eta-Z)}\,,$$

for fixed h, j, k, l, m, q and for $n \to +\infty$. For this purpose we can use Hata's saddle point method in $\mathbb{C}^2$ [Ha4, Section 1]. We seek the stationary points of $f(\xi,\eta)$ satisfying $\xi(1-\xi)\eta(1-\eta) \neq 0$. We have

$$\log f(\xi,\eta) = j\log\xi + h\log(1-\xi) + k\log\eta + (j+q-m)\log(1-\eta)$$
$$- (j+k-m)\log(\xi-\eta) - (l+q)\log(\eta-Z),$$

whence

$$\frac{1}{f}\frac{\partial f}{\partial \xi} = \frac{\partial}{\partial \xi}\log f = \frac{j}{\xi} - \frac{h}{1-\xi} - \frac{j+k-m}{\xi-\eta}\,,$$

$$\frac{1}{f}\frac{\partial f}{\partial \eta} = \frac{\partial}{\partial \eta}\log f = \frac{k}{\eta} - \frac{j+q-m}{1-\eta} + \frac{j+k-m}{\xi-\eta} - \frac{l+q}{\eta-Z}\,.$$

Thus we seek the solutions of the system

$$\text{(4.2)}\qquad \begin{cases} \dfrac{j}{\xi} - \dfrac{h}{1-\xi} = \dfrac{j+k-m}{\xi-\eta} \\[2ex] \dfrac{l+q}{\eta - Z} - \dfrac{k}{\eta} + \dfrac{j+q-m}{1-\eta} = \dfrac{j+k-m}{\xi-\eta}\,. \end{cases}$$

The first equation (4.2) yields η as a function of ξ:

$$\text{(4.3)}\qquad \eta = H(\xi) := \xi\,\frac{(h+m-k)\xi + k - m}{(h+j)\xi - j}\,,$$

with fixed points $\xi = \eta = 0$ and $\xi = \eta = 1$. Subtracting the equations (4.2) and then substituting (4.3), we get a fourth degree equation in ξ.

The first step to apply the $\mathbb{C}^2$-saddle point method consists in solving either the equation $\partial f/\partial\xi = 0$ with respect to ξ, thus locally expressing ξ as a holomorphic function $\Xi(\eta)$ of η in some open region Δ contained in the plane of the complex variable η, or, interchanging the roles of ξ and η, the equation $\partial f/\partial\eta = 0$ with respect to η. Since Z does not appear in the first equation (4.2), it is convenient to solve $\partial f/\partial\xi = 0$ with respect to ξ, i.e. to invert the function (4.3), thus obtaining a suitable function $\xi = \Xi(\eta)$ independent of Z.

The (global) inverse of (4.3) is a two-valued function ξ of η with branch points at

$$\eta_\pm = \frac{h(j+k-m) + j(h+m-k) \pm 2\sqrt{hj(h+m-k)(j+k-m)}}{(h+j)^2}\,,$$

corresponding through (4.3) to the solutions

$$\xi_\pm = \frac{j}{h+j} \pm \frac{\sqrt{hj(h+m-k)(j+k-m)}}{(h+j)(h+m-k)}$$

of $\mathrm{d}H/\mathrm{d}\xi = 0$, precisely such that

$$\eta_+ = H(\xi_+) \quad \text{and} \quad \eta_- = H(\xi_-).$$

We have $\eta_- < \eta_+$, with $\eta_- > 0$ if and only if $k \neq m$, and $\xi_- < \xi_+$, with $\xi_- > 0$ if and only if $k < m$.

If we set

$$\xi = \frac{j}{h+j} + \frac{\sqrt{hj(h+m-k)(j+k-m)}}{(h+j)(h+m-k)}\,e^{i\vartheta} \qquad (0 \le \vartheta \le 2\pi),$$

ξ describes the circumference of diameter $[\xi_-,\, \xi_+]$. Then the corresponding $\eta = H(\xi)$ given by (4.3) is easily seen to be

$$\eta = \frac{h(j+k-m)+j(h+m-k)}{(h+j)^2} + \frac{\sqrt{hj(h+m-k)(j+k-m)}}{(h+j)^2}\,(e^{i\vartheta}+e^{-i\vartheta})$$
$$= \frac{h(j+k-m)+j(h+m-k)+2\sqrt{hj(h+m-k)(j+k-m)}\,\cos\vartheta}{(h+j)^2}\,.$$

Thus the function (4.3) maps both the upper and the lower half-circumference of diameter $[\xi_-,\, \xi_+]$ onto the real interval $[\eta_-,\, \eta_+]$.

The next step in applying the saddle point method, for each of the double integrals (3.1), (2.14), (2.11) and (2.12) which we write here in the form

(4.4) $$J_Z^{(0)} = \pm \int_0^{\zeta\infty} \left(\int_0^1 f(\xi,\eta)^n \frac{\mathrm{d}\xi}{\xi-\eta} \right) \frac{\mathrm{d}\eta}{\eta - Z}\,,$$

(4.5) $$K_Z^{(0)} = \pm \int_{\gamma_{0,1}} \left(\int_0^1 f(\xi,\eta)^n \frac{\mathrm{d}\xi}{\xi-\eta} \right) \frac{\mathrm{d}\eta}{\eta - Z}\,,$$

(4.6) $$J_Z^{(1)} = \pm \frac{1}{2\pi i} \oint_{\Gamma_{0,1}} \left(\int_0^1 f(\xi,\eta)^n \frac{\mathrm{d}\xi}{\xi-\eta} \right) \frac{\mathrm{d}\eta}{\eta - Z}\,,$$

(4.7) $$J_Z^{(2)} = \pm \frac{1}{2\pi i} \oint_{|\eta - Z| = \varrho_1} \left(\frac{1}{2\pi i} \oint_{|\xi-\eta|=\varrho_2} f(\xi,\eta)^n \frac{\mathrm{d}\xi}{\xi-\eta} \right) \frac{\mathrm{d}\eta}{\eta - Z}\,,$$

is to find:

(i) an open region $\varDelta$ in the η-plane;

(ii) a stationary point (ξ_*, η_*) of $f(\xi,\eta)$, with $\eta_* \in \varDelta$, at which

(4.8) $$\frac{\partial^2 f}{\partial \xi^2} \neq 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial \xi^2}\,\frac{\partial^2 f}{\partial \eta^2} \neq \left(\frac{\partial^2 f}{\partial \xi\, \partial \eta} \right)^2;$$

(iii) a local inverse $\xi = \varXi(\eta)$ of (4.3), holomorphic in $\varDelta$, with $\xi_* = \varXi(\eta_*)$,

such that the integration path for η can be transformed by Cauchy's theorem into a new integration path $\varGamma \subset \varDelta$ passing through η_*, for which

(4.9) $$\max_{\eta \in \varGamma} |f(\varXi(\eta),\eta)| \,=\, |f(\xi_*,\eta_*)|,$$

and the maximum on Γ is attained only at $\eta = \eta_*$.

As a final step, for any fixed $\eta \in \Gamma$ we transform the integration path for ξ, again applying Cauchy's theorem, into a new integration path δ_η (in general depending on η) passing through $\Xi(\eta)$, so that

$$\max_{\xi \in \delta_\eta} |f(\xi, \eta)| = |f(\Xi(\eta), \eta)|, \tag{4.10}$$

and the maximum on δ_η is attained only at $\xi = \Xi(\eta)$.

Then, by Hata's theorem [Ha4, Section 1],

$$\lim_{n\to\infty} \frac{1}{n} \log \left| \int_\Gamma \left(\int_{\delta_\eta} f(\xi,\eta)^n \frac{\mathrm{d}\xi}{\xi - \eta} \right) \frac{\mathrm{d}\eta}{\eta - Z} \right| = \log |f(\xi_*, \eta_*)|. \tag{4.11}$$

We remark that the saddle point method is not necessary for the integral (4.4) if $Z = z > 1$, because in this case, going back to the coordinates x, y through the inverse of (2.5), we have

$$J_z^{(0)} = z^{(k-l-q)n} \int_0^1 \int_0^1 \left(\frac{x^j (1-x)^h \, y^k (1-y)^l}{\big(x(1-y) + yz\big)^{j+k-m}} \, (1 - y + yz)^{j+q-m} \right)^n \frac{\mathrm{d}x \, \mathrm{d}y}{x(1-y) + yz}.$$

Inside the unit square $(0,1) \times (0,1)$ the integrand is positive with one stationary point, and vanishes on the boundary. Then the desired

$$\lim_{n\to\infty} \frac{1}{n} \log \big| J_z^{(0)} \big|$$

is obtained by Laplace's elementary asymptotic method. This yields

$$\lim_{n\to\infty} \frac{1}{n} \log \left| \int_0^{-\infty} \left(\int_0^1 f(\xi,\eta)^n \frac{\mathrm{d}\xi}{\xi - \eta} \right) \frac{\mathrm{d}\eta}{\eta - z} \right| = \log |f(\xi_*, \eta_*)|, \tag{4.12}$$

where (ξ_*, η_*) is the stationary point of $f(\xi, \eta)$ with $0 < \xi_* < 1$ and $\eta_* < 0$.

4.2

We show how to work out a concrete example. We choose the numerical values

$$(h, j, k, l, m, q) = (37,\ 37,\ 22,\ 14,\ 23,\ 8). \tag{4.13}$$

With these values we have

$$(h, j, k, l, m, q) = (j,\ h,\ j + q - m,\ l,\ h + q - k,\ q), \tag{4.14}$$

whence, by the results in Subsection 2.3,

$$J_z^{(1)}(37,\ 37,\ 22,\ 14,\ 23,\ 8) = -J_{1-z}^{(1)}(37,\ 37,\ 22,\ 14,\ 23,\ 8), \tag{4.15}$$

$$J_z^{(2)}(37,\,37,\,22,\,14,\,23,\,8) \;=\; J_{1-z}^{(2)}(37,\,37,\,22,\,14,\,23,\,8), \tag{4.16}$$

$$K_z^{(0)}(37,\,37,\,22,\,14,\,23,\,8) \;=\; \overline{K_{1-z}^{(0)}(37,\,37,\,22,\,14,\,23,\,8)} \tag{4.17}$$

and

$$J_z^{(0)}(37,\,37,\,22,\,14,\,23,\,8) + J_{1-z}^{(0)}(37,\,37,\,22,\,14,\,23,\,8) \;=\; K_z^{(0)}(37,\,37,\,22,\,14,\,23,\,8), \tag{4.18}$$

where, e.g., $\operatorname{Im}\zeta > 0$ in $J_{1-z}^{(0)}$ and $\operatorname{Im}\gamma_{0,1} < 0$ in $K_z^{(0)}$.

The function (4.3) becomes

$$\eta \;=\; H(\xi) \;=\; \frac{\xi}{37}\cdot\frac{38\xi-1}{2\xi-1}\,, \tag{4.19}$$

and we have

$$\xi_\pm \;=\; \frac{1}{2} \pm \frac{3\sqrt{38}}{38}\,, \qquad \eta_\pm \;=\; \frac{1}{2} \pm \frac{3\sqrt{38}}{37}\,.$$

In the plane of the complex variable ξ we define the following four open regions:

$$\begin{aligned}
C_1 &= \{\operatorname{Im}\xi > 0,\;\; (\operatorname{Re}\xi - 1/2)^2 + (\operatorname{Im}\xi)^2 > 9/38\},\\
D_1 &= \{\operatorname{Im}\xi < 0,\;\; (\operatorname{Re}\xi - 1/2)^2 + (\operatorname{Im}\xi)^2 > 9/38\},\\
C_2 &= \{\operatorname{Im}\xi < 0,\;\; (\operatorname{Re}\xi - 1/2)^2 + (\operatorname{Im}\xi)^2 < 9/38\},\\
D_2 &= \{\operatorname{Im}\xi > 0,\;\; (\operatorname{Re}\xi - 1/2)^2 + (\operatorname{Im}\xi)^2 < 9/38\},
\end{aligned}$$

and in the plane of the complex variable η we define:

$$\begin{aligned}
C &= \{\operatorname{Im}\eta > 0\},\\
D &= \{\operatorname{Im}\eta < 0\}.
\end{aligned}$$

An easy inspection of the orientations of the borders of the four regions above in the ξ-plane, compared with the corresponding orientation of the real line in the η-plane, shows that the function (4.19) is a one-to-one mapping of both C_1 and C_2 onto C, and of both D_1 and D_2 onto D.

We now choose

$$Z = z = 9.$$

Then the stationary points of $f_z(\xi,\eta)$ are:

$$\begin{aligned}
(\xi_0,\eta_0) &= (0.4761\ldots,\;\; -4.6067\ldots),\\
(\xi_1,\eta_1) &= (0.5231\ldots + i\,0.1540\ldots,\;\; 0.6278\ldots - i\,0.6930\ldots),\\
&(\overline{\xi_1},\overline{\eta_1}),\\
(\xi_2,\eta_2) &= (23.0995\ldots,\;\; 12.1105\ldots),
\end{aligned} \tag{4.20}$$

and we have

$$
\begin{aligned}
\log|f_z(\xi_0,\eta_0)| &= -95.8085\ldots,\\
\log|f_z(\xi_1,\eta_1)| &= -95.8741\ldots,\\
\log|f_z(\xi_2,\eta_2)| &= 227.2982\ldots.
\end{aligned}
\tag{4.21}
$$

At each of the stationary points (4.20) the inequalities (4.8) hold. By (4.12) we have, for $z = 9$,

$$
\lim_{n\to\infty}\frac{1}{n}\log\left|\int_0^{-\infty}\left(\int_0^1 f_z(\xi,\eta)^n\,\frac{\mathrm{d}\xi}{\xi-\eta}\right)\frac{\mathrm{d}\eta}{\eta-z}\right| = \log|f_z(\xi_0,\eta_0)|. \tag{4.22}
$$

Owing to (4.15), (4.16) and (4.17), for the integrals $K_Z^{(0)}$, $J_Z^{(1)}$ and $J_Z^{(2)}$ we require the $\mathbb{C}^2$-saddle point method only in the case $Z = z > 1$, i.e., $Z = z = 9$ with our choice. The simplest integral to deal with is the double contour integral $J_z^{(2)}$ given by (4.7). In this case we take

$$
\begin{aligned}
&\mathit{\Delta} = C \cup D \cup (\eta_+, +\infty),\\
&(\xi_*,\eta_*) = (\xi_2,\eta_2),\\
&\mathit{\Xi} : \mathit{\Delta} \longrightarrow C_1 \cup D_1 \cup (\xi_+, +\infty),\\
&\mathit{\Gamma} = \{|\eta - z| = \eta_2 - z\}.
\end{aligned}
$$

Then (4.9) holds, with the maximum attained only at η_2. For every fixed $\eta \in \mathit{\Gamma}$, the function $f_z(\xi,\eta)$ vanishes at $\xi = 1$, and tends to infinity as $\xi \to \infty$ or $\xi \to \eta$. Since $\xi_+ < 1$, we have $1 \in \mathit{\Xi}(\mathit{\Delta})$. By the ordinary saddle point method in $\mathbb{C}$, there exists a contour δ_η enclosing η and passing through $\mathit{\Xi}(\eta)$, which satisfies (4.10) with the maximum on δ_η attained only at the saddle point $\xi = \mathit{\Xi}(\eta)$. By (4.11) we get, for $Z = z = 9$,

$$
\lim_{n\to\infty}\frac{1}{n}\log\left|\frac{1}{2\pi i}\oint_{|\eta-z|=\varrho_1}\left(\frac{1}{2\pi i}\oint_{|\xi-\eta|=\varrho_2} f_z(\xi,\eta)^n\,\frac{\mathrm{d}\xi}{\xi-\eta}\right)\frac{\mathrm{d}\eta}{\eta-z}\right| = \log|f_z(\xi_2,\eta_2)|.
$$

For the integral $K_z^{(0)}$ in (4.5) with $z = 9$ and with, e.g., $\operatorname{Im}\gamma_{0,1} < 0$, the relevant stationary point is (ξ_1,η_1). We take

$$
\begin{aligned}
&\mathit{\Delta} = D,\\
&(\xi_*,\eta_*) = (\xi_1,\eta_1),\\
&\mathit{\Xi} : \mathit{\Delta} \longrightarrow D_2.
\end{aligned}
$$

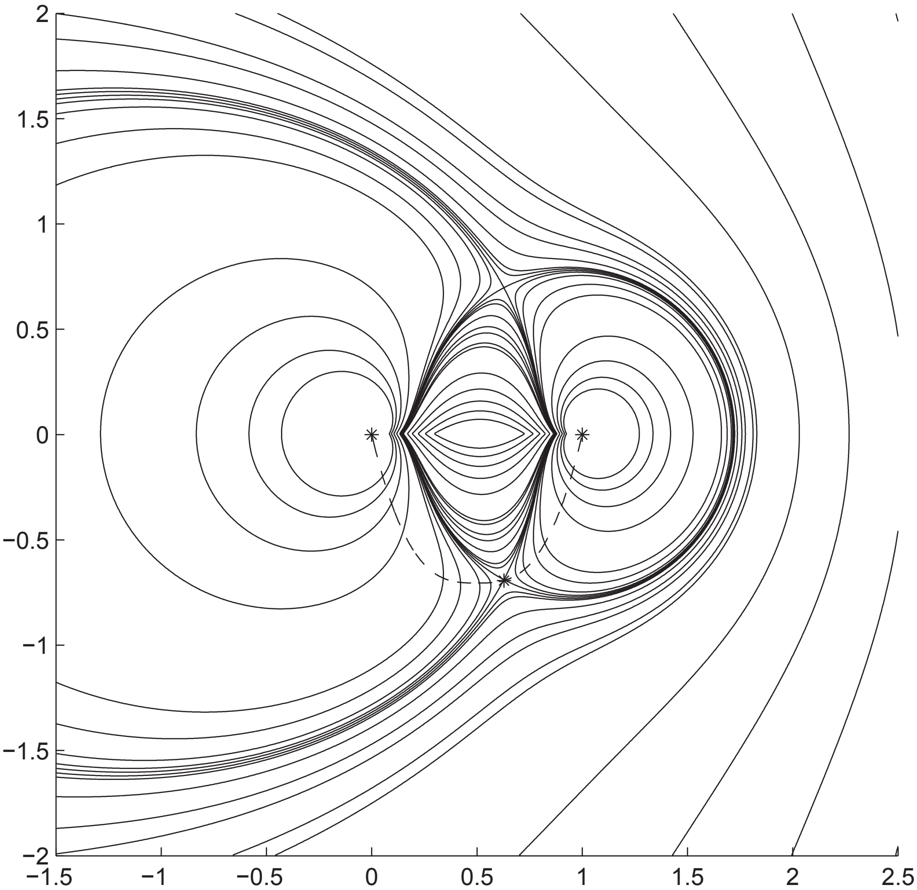

Figure 1: Level curves $|f_9(\Xi(\eta),\eta)| = \text{constant}$, and the path $\Gamma = \gamma_{0,1}$ for $K_9^{(0)}$.

For Γ we choose a path $\gamma_{0,1} \subset D$ going from 0 to 1 and passing through η_1, such that (4.9) holds with the maximum attained only at η_1. In the η-plane, the level curves $|f_z(\Xi(\eta),\eta)| = \text{constant}$, for several values of the constant around and including $|f_z(\xi_1,\eta_1)|$, are represented in Figure 1, where the integration path $\Gamma = \gamma_{0,1}$ is the dashed curve through the saddle point η_1 of the function $f_z(\Xi(\eta),\eta)$.

The image $\Xi(\gamma_{0,1})$ is an arc in D_2 passing through ξ_1 with endpoints 1/38 and 37/38. For every fixed $\eta \in \gamma_{0,1}$, $f_z(\xi,\eta)$ vanishes at $\xi = 0$ and $\xi = 1$, and tends to infinity as $\xi \to \infty$ or $\xi \to \eta$. By the saddle point method in $\mathbb{C}$, there exists a path δ_η from 0 to 1 in the upper half-plane $\operatorname{Im}\xi > 0$, passing through $\Xi(\eta)$ and satisfying (4.10) with the maximum attained only at $\xi = \Xi(\eta)$. By (4.11) we obtain, for $Z = z = 9$,

$$\lim_{n\to\infty} \frac{1}{n} \log \left| \int_{\gamma_{0,1}} \left(\int_0^1 f_z(\xi,\eta)^n \, \frac{\mathrm{d}\xi}{\xi-\eta} \right) \frac{\mathrm{d}\eta}{\eta - z} \right| = \log |f_z(\xi_1,\eta_1)|. \tag{4.23}$$

For the remaining integrals $J_z^{(1)}$ and $J_{1-z}^{(0)}$, the required asymptotic estimates are easily reduced to (4.22) and (4.23) by means of (2.15) and (2.16), with no further applications of the saddle point method. From (2.15) we get

$$\left|J_z^{(1)}\right| = \frac{1}{\pi}\left|\operatorname{Im} K_z^{(0)}\right| \leq \frac{1}{\pi}\left|K_z^{(0)}\right|,$$

whence

$$\limsup_{n\to\infty} \frac{1}{n}\log\left|J_z^{(1)}\right| \leq \lim_{n\to\infty}\frac{1}{n}\log\left|K_z^{(0)}\right|.$$

Therefore, by (4.23),

$$\limsup_{n\to\infty} \frac{1}{n}\log\left|\frac{1}{2\pi i}\oint\limits_{\Gamma_{0,1}}\left(\int\limits_0^1 f_z(\xi,\eta)^n\,\frac{\mathrm{d}\xi}{\xi-\eta}\right)\frac{\mathrm{d}\eta}{\eta-z}\right| \leq \log|f_z(\xi_1,\eta_1)|.$$

By (2.16),

$$\left|J_{1-z}^{(0)}\right| = \left|J_z^{(0)}\right|\left|1-\frac{K_z^{(0)}}{J_z^{(0)}}\right|. \tag{4.24}$$

From the values (4.21) we have $|f_z(\xi_1,\eta_1)| < |f_z(\xi_0,\eta_0)|$, whence, by (4.22) and (4.23),

$$\lim_{n\to\infty}\frac{K_z^{(0)}}{J_z^{(0)}} = 0.$$

Thus, by (4.24),

$$\lim_{n\to\infty}\frac{1}{n}\log\left|J_{1-z}^{(0)}\right| = \lim_{n\to\infty}\frac{1}{n}\log\left|J_z^{(0)}\right|.$$

Hence, for any ζ with $\operatorname{Im}\zeta \neq 0$, we get by (4.22),

$$\lim_{n\to\infty}\frac{1}{n}\log\left|\int\limits_0^{\zeta\infty}\left(\int\limits_0^1 f_{1-z}(\xi,\eta)^n\,\frac{\mathrm{d}\xi}{\xi-\eta}\right)\frac{\mathrm{d}\eta}{\eta-(1-z)}\right| = \log|f_z(\xi_0,\eta_0)|.$$

Altogether we have, for the double integrals (4.4), (4.5), (4.6) and (4.7) with the values (4.13) and with $z = 9$,

$$\lim_{n\to\infty} \frac{1}{n} \log \left|J_9^{(0)}\right| = \lim_{n\to\infty} \frac{1}{n} \log |J_9| = \lim_{n\to\infty} \frac{1}{n} \log \left|J_{-8}^{(0)}\right| = \lim_{n\to\infty} \frac{1}{n} \log |J_{-8}|$$
$$= \log |f_9(\xi_0, \eta_0)| = -95.8085\ldots,$$

$$\lim_{n\to\infty} \frac{1}{n} \log \left|K_9^{(0)}\right| = \lim_{n\to\infty} \frac{1}{n} \log \left|K_{-8}^{(0)}\right| = \log |f_9(\xi_1, \eta_1)| = -95.8741\ldots,$$

$$\limsup_{n\to\infty} \frac{1}{n} \log \left|J_9^{(1)}\right| = \limsup_{n\to\infty} \frac{1}{n} \log \left|J_{-8}^{(1)}\right| \le \log |f_9(\xi_1, \eta_1)| = -95.8741\ldots,$$

$$\lim_{n\to\infty} \frac{1}{n} \log \left|J_9^{(2)}\right| = \lim_{n\to\infty} \frac{1}{n} \log \left|J_{-8}^{(2)}\right| = \log |f_9(\xi_2, \eta_2)| = 227.2982\ldots$$

and, finally,

$$\limsup_{n\to\infty} \frac{1}{n} \log |K_9| \le -95.8741\ldots$$

on the basis of Lemma 2.6.

4.3

We summarise our discussion in Subsections 4.1 and 4.2 in the following form.

Proposition 4.1. *For $z > 1$ and for a fixed set of positive integers h, j, k, l, m, q such that $j + k - m$, $j + q - m$, $h + m - k$ and $h + q - k$ are also positive, and satisfying $h = j$ and $k + m = h + q$, i.e.,*

$$(h, j, k, l, m, q) = (j,\ h,\ j + q - m,\ l,\ h + q - k,\ q),$$

consider the four stationary points

$$\begin{aligned} &(\xi_0, \eta_0) \in \mathbb{R}^2, && 0 < \xi_0 < 1, \quad \eta_0 < 0, \\ &(\xi_1, \eta_1) \in \mathbb{C}^2, && 0 < \operatorname{Re} \xi_1 < 1, \quad 0 < \operatorname{Re} \eta_1 < 1, \\ &(\overline{\xi_1}, \overline{\eta_1}) \in \mathbb{C}^2, && \\ &(\xi_2, \eta_2) \in \mathbb{R}^2, && \xi_2 > 1, \quad \eta_2 > 0, \end{aligned}$$

solutions of the system (4.2)*. Define*

$$c_0 = -\log |f_z(\xi_0, \eta_0)|, \quad c_1 = -\log |f_z(\xi_1, \eta_1)| = -\log |f_z(\overline{\xi_1}, \overline{\eta_1})|$$
$$\text{and} \quad c_2 = \log |f_z(\xi_2, \eta_2)|,$$

where the function $f_z(\xi,\eta)$ is given by (4.1). If $c_0 < c_1$ we have the following asymptotic formulae:

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n}\log\left|J_z^{(0)}(hn,jn,kn,ln,mn,qn)\right| &= \\
\lim_{n\to\infty}\frac{1}{n}\log\left|J_{1-z}^{(0)}(hn,jn,kn,ln,mn,qn)\right| &= -c_0,
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n}\log\left|K_z^{(0)}(hn,jn,kn,ln,mn,qn)\right| &= \\
\lim_{n\to\infty}\frac{1}{n}\log\left|K_{1-z}^{(0)}(hn,jn,kn,ln,mn,qn)\right| &= -c_1,
\end{aligned}$$

$$\begin{aligned}
\limsup_{n\to\infty}\frac{1}{n}\log\left|J_z^{(1)}(hn,jn,kn,ln,mn,qn)\right| &= \\
\limsup_{n\to\infty}\frac{1}{n}\log\left|J_{1-z}^{(1)}(hn,jn,kn,ln,mn,qn)\right| &\le -c_1,
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n}\log\left|J_z^{(2)}(hn,jn,kn,ln,mn,qn)\right| &= \\
\lim_{n\to\infty}\frac{1}{n}\log\left|J_{1-z}^{(2)}(hn,jn,kn,ln,mn,qn)\right| &= \quad c_2.
\end{aligned}$$

5 Arithmetic ingredients

5.1

Lemma 5.1. *For a collection of real numbers $\gamma_1,\dots,\gamma_S$, assume that we have sequences of linear forms*

$$r_n^{(\mu)} = q_n\gamma_\mu - p_n^{(\mu)} \in \mathbb{Z}\gamma_\mu + \mathbb{Z}, \qquad \mu = 1,\dots,S,$$

satisfying the following conditions:

(i) *for some positive τ,*

$$\limsup_{n\to\infty}\frac{\log\left|r_n^{(S)}\right|}{n} = -\tau$$

and

$$\limsup_{n\to\infty}\frac{\log\left|r_n^{(\mu)}\right|}{n} < -\tau \qquad \text{for} \quad \mu = 1,\dots,S-1;$$

(ii) *the numbers $1,\gamma_1,\dots,\gamma_{S-1}$ are linearly independent over $\mathbb{Q}$.*

Then the numbers $1,\gamma_1,\dots,\gamma_{S-1}$ and γ_S are linearly independent over $\mathbb{Q}$.

Proof. First of all, we introduce an $\varepsilon > 0$ to quantify the second part of condition (i):

$$\limsup_{n\to\infty} \frac{\log \left|r_n^{(\mu)}\right|}{n} \le -\tau - \varepsilon \qquad \text{for} \quad \mu = 1, \dots, S-1.$$

Assume, on the contrary, that there is a linear form $\Lambda = a_0 + a_1\gamma_1 + \cdots + a_S\gamma_S$, with integer coefficients not all zero, such that $\Lambda = 0$. Write, for any n,

$$q_n\Lambda = \left(a_0q_n + a_1p_n^{(1)} + \cdots + a_Sp_n^{(S)}\right) + \left(a_1r_n^{(1)} + \cdots + a_Sr_n^{(S)}\right). \tag{5.1}$$

The left-hand side of this equality is 0 and the expression $a_1r_n^{(1)} + \cdots + a_Sr_n^{(S)}$ on the right-hand side tends to 0 as $n \to \infty$ by hypothesis (i). It follows that $a_0q_n + a_1p_n^{(1)} + \cdots + a_Sp_n^{(S)} = 0$ for all $n > n_1$, hence also

$$a_1r_n^{(1)} + \cdots + a_{S-1}r_n^{(S-1)} + a_Sr_n^{(S)} = 0$$

for all $n > n_1$ by (5.1). Dividing both sides of this equality by $r_n^{(S)}$ and computing the limit as $n \to \infty$ along the subsequence of indices n for which $\left|r_n^{(S)}\right| > \exp\big((-\tau - \varepsilon/2)n\big)$ we obtain $a_S = 0$, so that $\Lambda = 0$ translates into the relation

$$a_0 + a_1\gamma_1 + \cdots + a_{S-1}\gamma_{S-1} = 0.$$

By hypothesis (ii) this equality is only possible when $a_0 = a_1 = \cdots = a_{S-1} = 0$, implying now that all the coefficients of the linear form Λ are zero, a contradiction with our assumption on Λ. □

We briefly comment about how we use the lemma in the next section. Our choice will be $S = 3$,

$$\gamma_1 = \mathrm{Li}_1(1/z), \quad \gamma_2 = \tfrac{1}{2}\,\mathrm{Li}_1(1/z)^2 = -\,\mathrm{Li}_2(1/z) - \mathrm{Li}_2(1/(1-z)), \quad \gamma_3 = \mathrm{Li}_2(1/z),$$

so that condition (ii) is met. In addition, by Proposition 4.1, hypothesis (i) will be satisfied.

To derive estimates for the linear independence measure we will use the following extension of the arithmetical lemma established by Hata in [Ha2, Lemma 2.1] in the case $S = 2$. In his recent work [Ma], R. Marcovecchio extends this result to cover the following general S situation.

Lemma 5.2 ([Ma, Lemma 7.1])**.** *For a collection of real numbers $\gamma_1, \dots, \gamma_S$ which are linearly independent with 1 over $\mathbb{Q}$, assume that we have sequences of linear forms*

$$r_n^{(\mu)} = q_n\gamma_\mu - p_n^{(\mu)}, \qquad \mu = 1, \dots, S,$$

where $q_n,\, p_n^{(\mu)} \in \mathbb{Z} + i\sqrt{d}\,\mathbb{Z}$ *for a fixed integer* $d \ge 0$*, satisfying the following condition:*

For some positive σ *and* τ*,*

$$\lim_{n\to\infty} \frac{\log|q_n|}{n} = \sigma \quad \textit{and} \quad \limsup_{n\to\infty} \frac{\log\left|r_n^{(\mu)}\right|}{n} \le -\tau \quad \textit{for} \quad \mu = 1, \dots, S.$$

Then for any $\varepsilon > 0$ *and any* $(S+1)$*-tuple*

$$(a_0, a_1, \dots, a_S) \in \mathbb{Z}^{S+1}, \quad (a_0, a_1, \dots, a_S) \ne (0, 0, \dots, 0),$$

we have

$$|a_0 + a_1\gamma_1 + \dots + a_S\gamma_S| > C(\varepsilon)\, A^{-\sigma/\tau-\varepsilon},$$

where $A = \max\{|a_0|, \dots, |a_S|\}$ *and the constant* $C(\varepsilon) > 0$ *does not depend on the tuple.*

5.2

Marcovecchio's lemma suffices for our application, and the (short) discussion below is to set an alternative approach which may be applicable in a situation where only estimates for *upper limits* are known:

$$\limsup_{n\to\infty} \frac{\log|q_n|}{n} \le \sigma \quad \text{and} \quad \limsup_{n\to\infty} \frac{\log\left|r_n^{(\mu)}\right|}{n} \le -\tau \quad \text{for} \quad \mu = 1, \dots, S.$$

In order to make the standard Siegel-type argument work in this case, it suffices to have an $N \ge S+1$ such that, for all sufficiently large n, the rank of the $N \times (S+1)$ matrix

$$\big(\, q_{n+i},\, p_{n+i}^{(1)}, \dots, p_{n+i}^{(S)} \,\big)_{0 \le i \le N-1}$$

is full (that is, equal to $S+1$); then there is at least one row satisfying

$$p_{n+i} = a_0 q_{n+i} + a_1 p_{n+i}^{(1)} + \dots + a_S p_{n+i}^{(S)} \ne 0$$

for any given nontrivial form $\Lambda = a_0 + a_1\gamma_1 + \dots + a_S\gamma_S$. The fact that $p_n \ne 0$ for infinitely many indices n already follows from the linear independence of $1, \gamma_1, \dots, \gamma_S$; indeed, if $p_n = 0$ for all sufficiently large n, then taking the limit in (5.1) shows that $\Lambda = 0$, which is not possible. Verifying that $p_n \ne 0$ holds "sufficiently often" is subtler. To establish this we can use the fact that our construction guarantees that all the sequences involved are P-recursive (or holonomic) over $\mathbb{Q}$, that is, each sequence satisfies (its own) linear recurrence equation with polynomial coefficients from $\mathbb{Q}[n]$. In our settings, this follows from the Fundamental Corollary in [WZ, Section 2.1] that, given a fixed collection of positive rationals h, j, k, l, m, q such that $j+k-m>0$, $j+q-m>0$, and a rational $Z \ne 0$, the sequences of integrals

$$J_Z^{(\mu)}(hn, jn, kn, ln, mn, qn) \quad (\mu = 0, 1, 2), \qquad \text{where} \quad n = 1, 2, 3, \dots,$$

are P-recursive (or holonomic) over $\mathbb{Q}$. Moreover, the paper [WZ] provides one with an algorithm to compute the recurrence equations for each particular choice of the parameters, and even with an explicit estimate for the order of the recursions.

The following lemma is an immediate consequence of the fact that the set of holonomic sequences over a field $\mathbb{K}$ is a ring.

Lemma 5.3. *Given P-recursive sequences $p_n^{(1)}, \dots, p_n^{(r)}$ over $\mathbb{K}$, exactly one of the following options holds: either*

(i) *there exists a collection of numbers $a_1, \dots, a_r \in \mathbb{K}$ such that $a_1 p_n^{(1)} + \dots + a_r p_n^{(r)} = 0$ for all $n > n_0$, or*

(ii) *there is an integer $N \ge r$ such that the rank of the $N \times r$ matrix*

$$\big(p_{n+i}^{(1)}, \dots, p_{n+i}^{(r)}\big)_{0 \le i \le N-1}$$

is full for any $n > n_0$.

Proof. Consider the P-recursive sequence

$$p_n = \det\big(p_{n+i-1}^{(\mu)}\big)_{1 \le i, \mu \le r}$$

and denote by M the order of a linear recurrence equation it satisfies. If $p_n = 0$ for all sufficiently large n, we arrive at option (i). If this does not happen, then at least one of p_n, $p_{n+1}, \dots, p_{n+M}$ must be nonzero for all $n > n_0$; the latter implies option (ii) with the choice $N = M + r$. □

It is this lemma and the fact that option (i) is already excluded that allow us to deduce the fullness of the rank of the $N \times (S+1)$ matrix above.

Remark. In practice, we normally deal with a collection of sequences $\widetilde{p}_n^{(\mu)} = \lambda_n p_n^{(\mu)}$, where the sequences $p_n^{(\mu)}$ are holonomic ($\mu = 1, \dots, r$), while a *nonzero* sequence λ_n serves to 'correct' the arithmetic of the latter (for example, to have the sequences $\widetilde{p}_n^{(\mu)}$ to be integer-valued, rather than rational-valued). The statement of Lemma 5.3 remains valid for the modified sequences $\widetilde{p}_n^{(1)}, \dots, \widetilde{p}_n^{(r)}$, because the vanishing of $a_1 p_n^{(1)} + \dots + a_r p_n^{(r)}$ and of $a_1 \widetilde{p}_n^{(1)} + \dots + a_r \widetilde{p}_n^{(r)}$ are equivalent.

6 Linear independence

6.1

Let $h,\ j,\ k,\ l,\ m,\ q > 0$ be fixed integers satisfying $j+k-m$, $j+q-m$, $h+m-k$, $h+q-k > 0$, $h = j$ and $k+m = h+q$, and let H, H', α, β, δ be defined as in Proposition 3.1. We consider the integrals $J_Z^{(\mu)}$ ($\mu = 0,1,2$) and $K_Z^{(0)}$ (with $Z = z > 1$ or $Z = 1-z < 0$) where, as in Section 4, we take the parameters to be hn, jn, kn, ln, mn, qn, with the integer n varying from 1 to ∞. Following the strategy of [RV, Section 4], we use Proposition 3.1 to make the arithmetical correction of the corresponding polynomials $Q(z)$, $P_1(z)$, $P_2(z)$, $P(z)$, $R(z) \in \mathbb{Z}[z]$. Namely, taking

$$\Omega = \{\omega \in [0,1) : [(h+m-k)\omega] + [(j+k-m)\omega] < [h\omega] + [j\omega]\,\} \tag{6.1}$$

and applying the arguments in [RV, pp. 418–420] we conclude that

$$\varDelta_n^{-1}Q(z),\ \varDelta_n^{-1}P_1(z),\ \varDelta_n^{-1}P_2(z),\ \varDelta_n^{-1}P(z),\ \varDelta_n^{-1}R(z)\ \in\ \mathbb{Z}[z],$$

where

$$\varDelta_n = \prod_{\substack{p>\sqrt{Hn}\\ \{n/p\}\in\Omega}} p.$$

Note that all the above polynomials have degrees not exceeding

$$\delta n\ =\ (\alpha+\beta+h+j-l)n.$$

For a rational $z = s/r > 1$, let

$$\begin{aligned} c_3 &= H + H' + \alpha\,\log z + \beta\,\log(z-1) + \delta\,\log r - \int_\Omega \mathrm{d}\psi(x) \\ &= H + H' + \alpha\,\log s + \beta\,\log(s-r) + (h+j-l)\log r - \int_\Omega \mathrm{d}\psi(x), \end{aligned}$$

where $\psi(x)$ is the logarithmic derivative of the Euler gamma-function. Applying Propositions 3.1 and 4.1, Lemma 5.1 and Lemma 5.2 with $d = 0$ to the linear forms

$$\begin{aligned} r_n^{(1)} &= \varDelta_n^{-1} d_{Hn} d_{H'n} z^{\alpha n}(1-z)^{\beta n} r^{\delta n} J_z^{(1)} \in \mathbb{Z}\ \mathrm{Li}_1(1/z) + \mathbb{Z}, \\ r_n^{(2)} &= \varDelta_n^{-1} d_{Hn} d_{H'n} z^{\alpha n}(1-z)^{\beta n} r^{\delta n} K_z \in \mathbb{Z}\,\tfrac{1}{2}\,\mathrm{Li}_1(1/z)^2 + \mathbb{Z}, \\ r_n^{(3)} &= \varDelta_n^{-1} d_{Hn} d_{H'n} z^{\alpha n}(1-z)^{\beta n} r^{\delta n} J_z \in \mathbb{Z}\ \mathrm{Li}_2(1/z) + \mathbb{Z}, \end{aligned}$$

we arrive at the following general result.

Proposition 6.1. *For a fixed rational $z = s/r > 1$ and for a fixed set of positive integers h, j, k, l, m, q such that $j+k-m$, $j+q-m$, $h+m-k$ and $h+q-k$ are also positive, and satisfying $h = j$ and $k+m = h+q$, define c_0, c_1, c_2 as in Proposition 4.1 and c_3 as above. Then if $c_3 < c_0 < c_1$, the numbers*

$$1,\ \operatorname{Li}_1(1/z),\ \operatorname{Li}_2(1/z)\ \textit{and}\ \operatorname{Li}_2(1/(1-z))$$

are linearly independent over $\mathbb{Q}$. Furthermore, for any $\varepsilon > 0$ and any quadruple $(a_0, a_1, a_2, a_3) \in \mathbb{Z}^4 \setminus \{\mathbf{0}\}$ we have

$$|a_0 + a_1 \operatorname{Li}_1(1/z) + a_2 \operatorname{Li}_2(1/z) + a_3 \operatorname{Li}_2(1/(1-z))| > C(\varepsilon)\, A^{-(c_2+c_3)/(c_0-c_3)-\varepsilon},$$

where $A = \max\{|a_0|, |a_1|, |a_2|, |a_3|\}$ and the constant $C(\varepsilon) > 0$ does not depend on the quadruple.

6.2

Here is some arithmetical analysis of the particular case (4.13), with $r = 1$ and $z = s \geq 9$. We have $H = 60$, $H' = 38$, $\alpha = \beta = 0$, while the permutation $\boldsymbol{\varphi}$ allows us to eliminate the set of primes p for which the fractional parts of n/p belong to the set (6.1), i.e., to the following union:

$$\begin{aligned}
\Omega = {} & \left[\tfrac{1}{37}, \tfrac{1}{36}\right) \cup \left[\tfrac{2}{37}, \tfrac{1}{18}\right) \cup \left[\tfrac{3}{37}, \tfrac{1}{12}\right) \cup \left[\tfrac{4}{37}, \tfrac{1}{9}\right) \cup \left[\tfrac{5}{37}, \tfrac{5}{36}\right) \cup \left[\tfrac{6}{37}, \tfrac{1}{6}\right) \\
& \cup \left[\tfrac{7}{37}, \tfrac{7}{36}\right) \cup \left[\tfrac{8}{37}, \tfrac{2}{9}\right) \cup \left[\tfrac{9}{37}, \tfrac{1}{4}\right) \cup \left[\tfrac{10}{37}, \tfrac{5}{18}\right) \cup \left[\tfrac{11}{37}, \tfrac{11}{36}\right) \cup \left[\tfrac{12}{37}, \tfrac{1}{3}\right) \\
& \cup \left[\tfrac{13}{37}, \tfrac{13}{36}\right) \cup \left[\tfrac{14}{37}, \tfrac{7}{18}\right) \cup \left[\tfrac{15}{37}, \tfrac{5}{12}\right) \cup \left[\tfrac{16}{37}, \tfrac{4}{9}\right) \cup \left[\tfrac{17}{37}, \tfrac{17}{36}\right) \cup \left[\tfrac{18}{37}, \tfrac{1}{2}\right) \\
& \cup \left[\tfrac{19}{37}, \tfrac{10}{19}\right) \cup \left[\tfrac{20}{37}, \tfrac{21}{38}\right) \cup \left[\tfrac{21}{37}, \tfrac{11}{19}\right) \cup \left[\tfrac{22}{37}, \tfrac{23}{38}\right) \cup \left[\tfrac{23}{37}, \tfrac{12}{19}\right) \cup \left[\tfrac{24}{37}, \tfrac{25}{38}\right) \\
& \cup \left[\tfrac{25}{37}, \tfrac{13}{19}\right) \cup \left[\tfrac{26}{37}, \tfrac{27}{38}\right) \cup \left[\tfrac{27}{37}, \tfrac{14}{19}\right) \cup \left[\tfrac{28}{37}, \tfrac{29}{38}\right) \cup \left[\tfrac{29}{37}, \tfrac{15}{19}\right) \cup \left[\tfrac{30}{37}, \tfrac{31}{38}\right) \\
& \cup \left[\tfrac{31}{37}, \tfrac{16}{19}\right) \cup \left[\tfrac{32}{37}, \tfrac{33}{38}\right) \cup \left[\tfrac{33}{37}, \tfrac{17}{19}\right) \cup \left[\tfrac{34}{37}, \tfrac{35}{38}\right) \cup \left[\tfrac{35}{37}, \tfrac{18}{19}\right) \cup \left[\tfrac{36}{37}, \tfrac{37}{38}\right).
\end{aligned}$$

This yields

$$c_3 = 60 + 38 - 4.030167\ldots = 93.969832\ldots,$$

and on combining with our earlier computation of

$$c_0 = 95.808510\ldots, \quad c_1 = 95.874154\ldots \quad \text{and} \quad c_2 = 227.298288\ldots$$

for $z = 9$, we arrive at the linear independence over $\mathbb{Q}$ of 1, $\operatorname{Li}_1(1/9)$, $\operatorname{Li}_2(1/9)$ and $\operatorname{Li}_2(-1/8)$, with the corresponding measure bounded from above by

$$\frac{c_2 + c_3}{c_0 - c_3} = 174.727780\ldots.$$

z	(h,j,k,l,m,q)	c_0	c_1	c_2	c_3	$\frac{c_2+c_3}{c_0-c_3}$
9	$(37,37,22,14,23,8)$	95.80...	95.87...	227.29...	93.96...	174.72778...
10	$(39,39,23,15,24,8)$	102.82...	103.17...	245.76...	98.91...	88.227939...
11	$(26,26,15,10,16,5)$	69.18...	69.50...	167.58...	65.32...	60.443071...
12	$(40,40,23,16,24,7)$	108.79...	108.79...	261.95...	100.89...	45.937765...
13	$(42,42,24,17,25,7)$	115.77...	115.86...	279.89...	105.84...	38.833296...
14	$(32,32,18,13,19,5)$	88.92...	88.93...	216.65...	80.11...	33.699538...
15	$(27,27,15,11,16,4)$	75.61...	75.62...	185.52...	67.28...	30.350069...
16	$(41,41,23,17,24,6)$	116.73...	116.97...	285.50...	102.86...	27.993154...
17	$(36,36,20,15,21,5)$	103.29...	103.43...	253.75...	89.99...	25.853914...
18	$(45,45,25,19,26,6)$	130.65...	130.78...	320.61...	112.77...	24.236011...
19	$(31,31,17,13,18,4)$	90.25...	90.41...	223.55...	77.14...	22.947485...
20	$(49,49,27,21,28,6)$	144.72...	144.74...	356.17...	122.68...	21.725841...
21	$(58,58,32,25,33,7)$	172.85...	173.06...	425.73...	145.51...	20.893729...
22	$(35,35,19,15,20,4)$	104.39...	104.42...	259.37...	87.02...	19.940943...
23	$(44,44,24,19,25,5)$	132.52...	132.72...	328.90...	109.79...	19.298069...

Table 1: The last column reproduces the estimate from above for the linear independence measure over $\mathbb{Q}$ of 1, $\mathrm{Li}_1(1/z)$, $\mathrm{Li}_2(1/z)$ and $\mathrm{Li}_2(1/(1-z))$.

z	(h,j,k,l,m,q)	c_0	c_1	c_2	c_3	$\frac{c_2+c_3}{c_0-c_3}$
143/2	$(58,58,30,27,31,3)$	205.31...	205.37...	526.95...	205.20...	6940.3660...
742/3	$(89,89,45,43,46,2)$	368.47...	368.51...	968.26...	368.39...	16306.639...
2355/4	$(159,159,80,78,81,2)$	727.23...	727.26...	1930.87...	727.21...	122147.47...

Table 2: Linear independence over $\mathbb{Q}$ of 1, $\mathrm{Li}_1(1/z)$, $\mathrm{Li}_2(1/z)$ and $\mathrm{Li}_2(1/(1-z))$ when $z > 1$ is non-integral.

In Table 1 we summarise our numerical results for the integers z in the range $9 \le z \le 23$. Note that our choice of parameters there follows the pattern

$$(h, j, k, l, m, q) = (j,\ j,\ k,\ j - k - 1,\ k + 1,\ 2k - j + 1),$$

so that $H = j + k + 1$, $H' = j + 1$ and $\alpha = \beta = 0$.

For any integer $z \ge 24$ we take $(h, j, k, l, m, q) = (9, 9, 5, 4, 5, 1)$, so that $H = 14$, $H' = 9$, $\alpha = \beta = 0$ and $\Omega = \varnothing$, whence $c_3 = H + H' = 23$. Then $c_1 > c_0 > 27.625757$ and $c_2 < 67.623222$, and we get a linear independence measure bounded from above by

$$\frac{c_2 + c_3}{c_0 - c_3} < 19.6\,.$$

Table 2 contains some results for rational $z = s/r$, $1 < r < s$, where s is chosen to be least possible for each denominator $r = 2, 3, 4$ to obtain the linear independence result in Proposition 6.1.

6.3

The construction in this paper allows one to single out the approximations to $\mathrm{Li}_1(1/z)$ and $\mathrm{Li}_2(1/z)$, without considering $\mathrm{Li}_2(1/(1-z))$. Then it can be shown (also in a quantitative form) that, for integer $z \ge 7$ (hence $z \le -6$ as well), the numbers 1, $\mathrm{Li}_1(1/z)$ and $\mathrm{Li}_2(1/z)$ are linearly independent over $\mathbb{Q}$. These results are weaker than the ones given in [RV] and therefore are not discussed here.

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