The Brauer Group of A Finite Quantum Group

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Abstract

In this paper, we give a general method to compute the Brauer group of a finite quantum group, i.e., a faithfully projective coquasitriangular Hopf algebra over a commutative ring. Let (H, R) be a finite quantum group with an R-matrix R on $H \otimes H$. There exists a deformation $\mathcal{H}_R$ of H. This deformation $\mathcal{H}_R$ can be embedded into the Drinfel'd double $D(H)$ as a left coideal subalgebra. We construct a group $\mathrm{Gal}(\mathcal{H}_R)$ consisting of quantum commutative $\mathcal{H}_R^*$-bigalois objects. There is an exact sequence of group morphisms:

$$1 \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{BC}(k, H, R) \longrightarrow \mathrm{Gal}(\mathcal{H}_R)$$

where $\mathrm{Br}(k)$ is the usual Brauer group of k, $\mathrm{BC}(k, H, R)$ is the Brauer group of (H, R) with respect to the R-matrix R. The applications to the Brauer group of triangular Hopf algebras and the Brauer-Long group will be presented.

Contents:

Introduction

Let k be a commutative ring with identity, H a Hopf algebra over k with a bijective antipode. In [7], we introduced H-Azumaya algebras and the Brauer group $\mathrm{BQ}(k, H)$ classifying the H-Azumaya algebras. When H is a finite commutative and cocommutative Hopf algebra, the Brauer group $\mathrm{BQ}(k, H)$ turns out to be the Brauer-Long group introduced by F.W. Long in [17] which is the most proper generalization of the Brauer-Wall group $\mathrm{BW}(k)$ (see [34]).

In [8], we made further investigation on the basic properties of H-Azumaya algebras and studied in detail the split part of the Brauer group $\mathrm{BQ}(k,H)$ in order to find a non-abelian cohomological interpretation of some subgroups of $\mathrm{BQ}(k,H)$. This approach turns out to be difficulty as later we found that the Hopf automorphism group can be embedded into the Brauer group $\mathrm{BQ}(k,H)$, which showed that $\mathrm{BQ}(k,H)$ is not necessarily a torsion group (see [32] for detail). In a recent calculation of the Brauer group of the Sweedler 4-dimensional Hopf algebra cf.[33], we found that the Hopf automorphism group is not the only non-torsion part of $\mathrm{BQ}(k,H)$, the group of Galois objects plays the non-torsion role as well in the Brauer group $\mathrm{BQ}(k,H)$. Nevertheless, when H is commutative and cocommutative, the group of Galois objects and the group of Hopf automorphisms generate a subgroup which is isomorphic to a factor group of the group of bigalois objects cf.[5]. This fact strongly indicates that the group of bigalois objects plays the vital role in the computation of the full Brauer group of a finite quantum group. This indication was further strengthened by a beautiful exact sequence of the Brauer-Long group due to K-L. Ulbrich in [30] where the group of bigalois objects appears in the picture. However, when (H,R) is no longer commutative and cocommutative, the group of bigalois objects of H or H^* does not fit into an exact sequence of the Brauer group of H. The solution found in this paper is the deformation $\mathcal{H}_R$ of a finite quantum group (H,R), which is no longer a Hopf algebra, instead a left coideal subalgebra of the quantum double $D(H)$. The main idea of this paper is to embed the quotient group $\mathrm{BC}(k,H,R)/\mathrm{Br}(k)$ of a finite quantum group (H,R) into a suitable group of "bigalois objects" of the deformation $\mathcal{H}_R$, which is easier to compute (or to estimate). Since the full Brauer group $\mathrm{BQ}(k,H)$ of any finite Hopf algebra H is equal to $\mathrm{BC}(k,D(H)^*,R')$, where $(D(H)^*,R')$ is the dual of the Drinfel'd quantum double group, it is sufficient to consider the general case $\mathrm{BC}(k,H,R)$ for a finite quantum group (H,R).

In Section one, we recall the definition of the Brauer groups $\mathrm{BQ}(k,H)$ and $\mathrm{BC}(k,H,R)$ when (H,R) is a coquasitriangular Hopf algebra. In section 2, we define a new deformation $\mathcal{H}_R$ of a finite coquasitriangular Hopf algebra (H,R). $\mathcal{H}_R$ turns out to be a left coideal subalgebra of the quantum double $D(H)$ though it is not a Hopf subalgebra. The main result in this section is the existence of a covariant functor from the category of Yetter-Drinfel'd H-modules to the category of $\mathcal{H}_R$-bimodules cf.Prop.2.7.

In Section 3, we construct a group $\mathrm{Gal}(\mathcal{H}_R)$ consisting of $\mathcal{H}_R^*$-bigalois objects which are quantum commutative. the group $\mathrm{Gal}(\mathcal{H}_R)$ plays the significant role in the computation of the Brauer group $\mathrm{BC}(k,H,R)$.

In Section 4, we establish a group morphism $\widetilde{\pi}$ from the Brauer group $\mathrm{BC}(k,H,R)$ to the group $\mathrm{Gal}(\mathcal{H}_R)$. In order to define the morphism $\widetilde{\pi}$, we have to show that any element of $\mathrm{BC}(k,H,R)$ is represented by an H-Azumaya algebra which is an H^{op}-Galois extension of its coinvariants. Such an H-Azumaya algebra is called a Galois R-Azumaya algebra. The centralizer of the coinvariants of a Galois R-Azumaya algebra turns out to be an $\mathcal{H}_R^*$-bigalois object. The kernel of the morphism $\widetilde{\pi}$ is isomorphic to the usual Brauer group $\mathrm{Br}(k)$ of k. In section 5, we present two examples of the exact sequence (28). Section 6 is devoted to the special case when the finite quantum group (H,R) is triangular. Section 7 will focus on the Drinfel'd double case, that is, the full Brauer group of a finite Hopf algebra (not necessarily coquasitriangular). The last section goes to the application to the Brauer-Long group.

1 Preliminaries

Let k be a fixed commutative ring with unit. Throughout all algebras, unadorned $\otimes$, Hom are over k. A module (or an algebra) is said to be *finite* if it is faithfully projective (i.e. faithful, finitely generated and projective) as a k-module. A *finite quantum group* is a finite Hopf algebra over k which is co-quasitriangular (CQT), that is, there is an invertible element $R \in (H \otimes H)^*$, the convolution algebra of $H \otimes H$, subject to the following conditions:

CQT1. $R(h \otimes 1) = R(1 \otimes h) = \varepsilon(h)$,
CQT2. $R(x \otimes yz) = \sum R(x_{(1)} \otimes z)R(x_{(2)} \otimes y)$,
CQT3. $R(yz \otimes x) = \sum R(y \otimes x_{(1)})R(z \otimes x_{(2)})$,
CQT4. $R(x_{(1)} \otimes y_{(1)})x_{(2)}y_{(2)} = \sum R(x_{(2)} \otimes y_{(2)})y_{(1)}x_{(1)}$.

for all x, y and $z \in H$. Since H is faithfully projective, we may identify R with an invertible element $\sum R^1 \otimes R^2$ in $H^* \otimes H^*$. In this case, (H^*, R) is a quasitriangular Hopf algebra, namely, R as an element in the Hopf algebra $(H^*, \underline{\Delta}, \underline{\varepsilon})$ satisfies the conditions:

QT1. $\sum \underline{\varepsilon}(R^1)R^2 = \sum R^1\underline{\varepsilon}(R^2) = 1$,
QT2. $\sum \underline{\Delta}(R^1) \otimes R^2 = \sum R^1 \otimes r^1 \otimes R^2 r^2$,
QT3. $\sum R^1 \otimes \underline{\Delta}(R^2) = \sum R^1 r^1 \otimes r^2 \otimes R^2$,
QT4. $R\underline{\Delta}(p) = \underline{\Delta}^{op}(p)R$.

for all $p \in H^*$. Note that the conditions CQT2 + CQT3 are equivalent to the cocycle condition, i.e,

$$\sum R(x_{(1)} \otimes y_{(1)})R(x_{(2)}y_{(2)} \otimes z) = R(y_{(1)} \otimes z_{(1)})R(x \otimes y_{(2)}z_{(2)})$$

for all x, y and $z \in H$. On the other hand, R results in two linear Hopf algebra morphisms:

$$\begin{array}{lll} \Theta_l: & H^{cop} \longrightarrow H^*, & \Theta_l(h)(l) = R(h \otimes l), \\ \Theta_r: & H^{op} \longrightarrow H^*, & \Theta_r(h)(l) = R(l \otimes h). \end{array}$$

Thus the condition CQT4 can be interpreted as

$$\sum x_{(2)}y_{(2)}\Theta_l(x_{(1)})(y_{(1)}) = \sum y_{(1)}x_{(1)}\Theta_r(y_{(2)})(x_{(2)})$$

for all $x, y \in H$.

Let us now recall the definition of the Brauer group of a Hopf algebra with a bijective antipode and some related notions. Let H be a Hopf algebra with a bijective antipode (not necessarily finite). A Yetter-Drinfel'd H-module (simply, YD H-module) M is a crossed H-bimodule [35]. That is, M is a k-module which is at once a left H-module and a right H-comodule satisfying the following equivalent compatibility conditions [18, 5.1.1]:

1). $\sum h_{(1)} \cdot m_{(0)} \otimes h_{(2)}m_{(1)} = \sum (h_{(2)} \cdot m)_{(0)} \otimes (h_{(2)} \cdot m)_{(1)}h_{(1)}$,
2). $\sum (h \cdot m)_{(0)} \otimes (h \cdot m)_{(1)} = \sum h_{(2)} \cdot m_{(0)} \otimes h_{(3)}m_{(1)}S^{-1}(h_{(1)})$,

where the sigma notations for comodules and for comultiplications can be found in the standard reference book [28]. Denote by $\mathbf{Q}^H$ the category of YD H-modules. A YD H-module algebra is a YD H-module A such that A is a left H-module algebra and a right H^{op}-comodule algebra. For the details of H-(co)module algebras we refer to [28].

In [7] we defined the Brauer group of a Hopf algebra H by considering isomorphism classes of H-Azumaya algebras. A YD H-module algebra A is called *H-Azumaya* if it is faithfully projective as a k-module and if the following YD H-module algebra maps are isomorphisms:

$$\begin{array}{llll} F: & A\#\overline{A} & \longrightarrow & \mathrm{End}(A), & F(a\#\overline{b})(x) = \sum ax_{(0)}(x_{(1)}\cdot b), \\ G: & \overline{A}\#A & \longrightarrow & \mathrm{End}(A)^{op}, & G(\overline{a}\#b)(x) = \sum a_{(0)}(a_{(1)}\cdot x)b, \end{array}$$

where $\overline{A}$ is the H-opposite YD H-module algebra of A, that is, $\overline{A} = A$ as a YD H-module, but with the multiplication given by

$$\overline{a}\cdot\overline{b} = \overline{b_{(0)}(b_{(1)}\cdot a)}$$

for $\overline{a},\overline{b}\in\overline{A}$ (see [7] for the detail). For a faithfully projective YD H-module M, the endomorphism algebra $\mathrm{End}_k(M)$ is a YD H-module algebra with H-structures given by

$$\begin{array}{l} (h\cdot f)(m) = \sum h_{(1)}\cdot f(S(h_{(2)})\cdot m), \\ \sum f_{(0)}(m)\otimes f_{(1)} = \sum f(m_{(0)})_{(0)}\otimes S^{-1}(m_{(1)})f(m_{(0)})_{(1)}. \end{array}$$

The elementary H-Azumaya algebra $\mathrm{End}(M)^{op}$ has the different H-structures from those of $\mathrm{End}(M)$ (see [7] for detail).

Two H-Azumaya algebras A and B are Brauer equivalent (denoted by $A\sim B$) if there exist two faithfully projective YD H-modules M and N such that $A\#\mathrm{End}(M)\cong B\#\mathrm{End}(N)$. Note that $A\sim B$ if and only if A is H-Morita equivalent to B (cf.[7, Th.2.10]). The Brauer group of the Hopf algebra H is denoted by $\mathrm{BQ}(k,H)$. An element in $\mathrm{BQ}(k,H)$ represented by an H-Azumaya algebra A is indicated by $[A]$. The unit in $\mathrm{BQ}(k,H)$ is represented by $\mathrm{End}(M)$ for any faithfully projective YD H-module M.

If A is an H-Azumaya algebra, the left and right H-center defined by

$$\{a\in A\mid ax = \sum x_{(0)}(x_{(1)}\cdot a),\ \forall x\in A\},$$

resp.

$$\{a\in A\mid xa = \sum a_{(0)}(a_{(1)}\cdot x),\ \forall x\in A\},$$

are trivial.

Now let H be a CQT Hopf algebra with a CQT structure R. If M is a right H-(or H^{op})-comodule, The morphism Θ_l induces a left H-module structure on M as follows:

$$h\triangleright_1 m = \Theta_l(h)\cdot m = \sum m_{(0)}R(h\otimes m_{(1)}) \tag{1}$$

for $h\in H$ and $m\in M$. The H-action (1) together with the original H-coaction makes M a YD H-module cf.[8, 18]. Denote by $\mathbf{M}_R^H$ the category of YD H-modules with the left H-module structure (1) coming from the right H-comodule structure. It is obvious that $\mathbf{M}_R^H$ is a full braided monoidal subcategory of $\mathbf{Q}^H$.

When $A = M$ is a right H^{op}-comodule algebra, (1) makes A a left H-module algebra and hence a YDH-module algebra. In the sequel, a YD H-module algebra A is called an R-module algebra if the H-action on A comes from the H^{op}-coaction on A through R. An R-module algebra is said to be R-Azumaya if it is H-Azumaya. The subset of $\mathrm{BQ}(k,H)$ consisting of the elements represented by the R-Azumaya algebras turns out to be a subgroup of $\mathrm{BQ}(k,H)$, denoted by $\mathrm{BC}(k,H,R)$. It is obvious that $\mathrm{BC}(k,H,R)$ contains the Brauer group $\mathrm{Br}(k)$.

Dually, if H is a QT Hopf algebra with a QT structure R, then a left H-module algebra A is simultaneously a YD H-module algebra with the right H^{op}-comodule structure given by

$$A\longrightarrow A\otimes H^{op},\quad a\mapsto \sum R^2\cdot a\otimes R^1$$

for $a \in A$. The subset of $\mathrm{BQ}(k,H)$ consisting of the elements represented by the H-Azumaya algebras with right H^{op}-comodule structures stemming from left H-module structures in the above way, turns out to be a subgroup of $\mathrm{BQ}(k,H)$, denoted by $\mathrm{BM}(k,H,R)$. It is obvious that $\mathrm{BM}(k,H,R)$ contains the Brauer group $\mathrm{Br}(k)$.

The Brauer group $\mathrm{BQ}(k,H)$ is a special case of the Brauer group $\mathrm{Br}(\mathcal{C})$ of a braided monoidal category $\mathcal{C}$ as introduced in [32]. The fact that $\mathrm{BC}(k,H,R)$ is a subgroup of $\mathrm{BQ}(k,H)$ when (H,R) is a coquasitriangular Hopf algebra, can be explained in a categorical way. If $\mathcal{D}$ is a closed braided monoidal subcategory of a braided monoidal category $\mathcal{C}$, then the Brauer group $\mathrm{Br}(\mathcal{D})$ is a subgroup of $\mathrm{Br}(\mathcal{C})$. This fact allows us to consider various subgroups of the Brauer group $\mathrm{Br}(\mathcal{C})$ of a braided monoidal category $\mathcal{C}$ whenever $\mathcal{C}$ contains certain closed braided subcategories. For example, if (H,R) is a CQT Hopf algebra, then the category $\mathbf{M}^{H^{op}}$ of right H^{op}-comodules is a closed braided subcategory of the braided category of YD H^{op}-modules with the braiding φ given by:

$$M \otimes N \longrightarrow N \otimes M,\ m \otimes n \mapsto \sum n_{(0)} \otimes m_{(0)} R(n_{(1)} \otimes m_{(1)}),$$

where $m \in M$ and $n \in N$. The Brauer group $\mathrm{Br}(\mathbf{M}_R^H)$ of $\mathbf{M}_R^H$ is indeed $\mathrm{BC}(k,H,R)$. When H is a finite Hopf algebra, it is well-known that the category of YD H-modules is equivalent to the category of left $D(H)$-modules, where $D(H)$ is the Drinfel'd double of H. So we have that $\mathrm{BQ}(k,H) = \mathrm{BM}(k,D(H),R) = \mathrm{BC}(k,D(H)^*,R)$, where R is the canonical quasi-triangular structure on $D(H)$.

To end this section let us recall the notion of a Hopf Galois extensions by a Hopf algebra H. Let A be a right H-comodule algebra. A is said to be H-Galois if the canonical linear map

$$\beta:\ A \otimes_{A_0} A \longrightarrow A \otimes H$$

sending element $a \otimes b$ to the element $\sum ab_{(0)} \otimes b_{(1)}$, is an isomorphism, where

$$A_0 = \{x \in A \mid \sum x_{(0)} \otimes x_{(1)} = x \otimes 1\}$$

is the coinvariant subalgebra of A. For a general Hopf Galois theory one may refer to [21, 23, 24].

2 The deformation $\mathcal{H}_R$

In this section, we construct a deformation of a CQT Hopf algebra (H,R) which we will use through the whole paper. We start with the definition of the new product $\star$ on the vector space H:

$$h \star l = \sum l_{(2)} h_{(2)} R(S^{-1}(l_{(3)}) l_{(1)} \otimes h_{(1)}) = \sum h_{(2)} l_{(1)} R(l_{(2)} \otimes S(h_{(1)}) h_{(3)})) \qquad (2)$$

where h and l are in H. The second equation of (2) is evident by the definition of R. We show that $\star$ is indeed a multiplication on H.

Lemma 2.1. $(H,\star)$ is an algebra with unit 1.

Proof. It is sufficent to verify that $\star$ is associative. Given elements h, l and $m \in H$, we have

$$(h \star l) \star m \quad = \quad \sum (l_{(2)} h_{(2)}) \star m R(S^{-1}(l_{(3)}) l_{(1)} \otimes h_{(1)})$$

$$
\begin{aligned}
&= \sum m_{(2)}l_{(3)}h_{(3)}R(S^{-1}(m_{(3)})m_{(1)}\otimes l_{(2)}h_{(2)})R(S^{-1}(l_{(4)})l_{(1)}\otimes h_{(1)})\\
&= \sum m_{(3)}l_{(3)}h_{(4)}R(S^{-1}(m_{(5)})m_{(1)}\otimes h_{(3)})R(S^{-1}(m_{(4)})m_{(2)}\otimes l_{(2)})\\
&\quad R(S^{-1}(l_{(4)})\otimes h_{(1)})R(l_{(1)}\otimes h_{(2)}).
\end{aligned}
$$

Let $a = S^{-1}(m_{(4)})$, $b = l_{(1)}$ and $c = h_{(2)}$. By utilizing the formula (see [19, ?]):

$$
\sum R(a_{(1)}\otimes c_{(2)})R(a_{(2)}\otimes b_{(2)})R(b_{(1)}\otimes c_{(1)}) = \sum R(a_{(1)}\otimes b_{(1)})R(a_{(2)}\otimes c_{(1)})R(b_{(2)}\otimes c_{(2)})
$$

for all a, b and $c \in H$, we obtain

$$
\begin{aligned}
(h\star l)\star m &= \sum m_{(3)}l_{(3)}h_{(4)}R(S^{-1}(m_{(4)})m_{(2)}\otimes h_{(2)})R(S^{-1}(m_{(5)})m_{(1)}\otimes l_{(1)})\\
&\quad R(S^{-1}(l_{(4)})\otimes h_{(1)})R(l_{(2)}\otimes h_{(3)})\\
&= \sum m_{(3)}l_{(3)}h_{(2)}R(S^{-1}(m_{(4)}l_{(4)})m_{(2)}l_{(2)}\otimes h_{(1)})R(S^{-1}(m_{(5)}m_{(1)}\otimes l_{(1)})\\
&= \sum h\star(m_{(2)}l_{(2)})R(S^{-1}(m_{(3)})m_{(1)}\otimes l_{(1)})\\
&= h\star(l\star m).
\end{aligned}
$$

So $\star$ is indeed a multiplication of H. □

Denote by $\mathcal{H}_R$ the algebra $(H, \star)$. It is easy to see that the counit map ε of H is still an augmentation map from $\mathcal{H}_R$ to k.

In general, there is a double Hopf algebra for a CQT Hopf algebra (H, R) (not necessaily finite). This double Hopf algebra, denoted by $D[H]$ in this paper, is $H\otimes H$ as a coalgebra. The multiplication of $D[H]$ is defined by

$$
(h\otimes l)(h'\otimes l') = \sum hh'_{(2)}\otimes l_{(2)}l'R(h'_{(1)}\otimes l_{(1)})R(S(h'_{(3)})\otimes l_{(3)})
$$

for h, l, h' and $l' \in H$. The antipode of $D[H]$ is given by

$$
S(h\otimes l) = (1\otimes S(l))(S(h)\otimes 1)
$$

for all $h, l \in H$. The counit of $D[H]$ is $\varepsilon\otimes\varepsilon$.

For the detail of $D[H]$, we refer to [19]. In the sequel, we will write $h\bowtie l$ for an element in $D[H]$ and $H\bowtie H$ for $D[H]$ as well. Since H is finite in this paper, the canonical Hopf algbera morphism Θ_l induces an Hopf algebra morphism from $D[H]$ to $D(H)$, the Drinfel'd quantum double $H^{*op}\bowtie H$ (cf. [13]):

$$
\Phi:\ D[H]\longrightarrow D(H),\ \ \Phi(h\bowtie l) = \Theta_l(h)\bowtie l. \tag{3}
$$

When Θ_l is an isomorphism, we may identify $D[H]$ with $D(H)$. Thus any YD H-modules are automatically a left $D[H]$-module. Moreover, the following lemma says that $\mathcal{H}_R$ can be embedded into $D[H]$.

Lemma 2.2. The following linear map is an algebra monomorphism:

$$
\phi:\ \mathcal{H}_R\longrightarrow D[H],\ \ \phi(h) = \sum S^{-1}(h_{(2)})\bowtie h_{(1)}.
$$

Proof. Given $h, l \in \mathcal{H}_R$, we have

$$
\phi(h\star l) = \sum \phi(l_{(2)}h_{(2)})R(S^{-1}(l_{(3)})l_{(1)}\otimes h_{(1)})
$$

$$
\begin{aligned}
&= \sum S^{-1}(l_{(3)}h_{(3)}) \bowtie l_{(2)}h_{(2)}R(S^{-1}(l_{(4)})l_{(1)} \otimes h_{(1)}) \\
&= \sum S^{-1}(h_{(4)}S^{-1}(l_{(3)}) \bowtie l_{(2)}h_{(3)}R(S^{-1}(l_{(4)}) \otimes h_{(1)})R(l_{(1)} \otimes h_{(2)}) \\
&= \sum S^{-1}(h_{(4)}S^{-1}(l_{(3)}) \bowtie h_{(2)}l_{(1)}R(S^{-1}(l_{(4)}) \otimes h_{(1)})R(l_{(2)} \otimes h_{(3)}) \\
&= \sum (S^{-1}(h_{(2)}) \bowtie h_{(1)})(S^{-1}(l_{(2)}) \bowtie l_{(1)}) \\
&= \phi(h)\phi(l).
\end{aligned}
$$

It is obvious that ϕ is injective. □

Proposition 2.3. $\mathcal{H}_R$ is a left $D[H]$-comodule algebra.

Proof. Define a linear map from $\mathcal{H}_R$ to $D[H] \otimes \mathcal{H}_R$ as follows:

$$\chi: \mathcal{H}_R \longrightarrow D[H] \otimes \mathcal{H}_R,\ \chi(h) = \sum (S^{-1}(h_{(3)}) \bowtie h_{(1)}) \otimes h_{(2)}.$$

It is evident that χ is a left $D[H]$-comodule map. We have to show that χ is an algebra map. Indeed, if $h, l \in \mathcal{H}_R$, then

$$
\begin{aligned}
\chi(h \star l) &= \sum \chi(h_{(2)}l_{(1)})R(l_{(2)} \otimes S(h_{(1)})h_{(3)}) \\
&= \sum S^{-1}(h_{(4)}l_{(3)}) \bowtie h_{(2)}l_{(1)} \otimes h_{(3)}l_{(2)}R(l_{(4)} \otimes S(h_{(1)})h_{(5)}) \\
&= \sum S^{-1}(h_{(4)}l_{(3)}) \bowtie h_{(2)}l_{(1)} \otimes h_{(3)}l_{(2)}R(l_{(4)} \otimes h_{(5)}) \\
&\quad R(S^{-1}(l_{(5)}) \otimes h_{(1)}) \\
&= \sum S^{-1}(l_{(4)}h_{(5)}) \bowtie h_{(2)}l_{(1)} \otimes h_{(3)}l_{(2)}R(l_{(3)} \otimes h_{(4)}) \\
&\quad R(S^{-1}(l_{(5)}) \otimes h_{(1)}) \\
&= \sum S^{-1}(l_{(4)}h_{(5)}) \bowtie h_{(2)}l_{(1)} \otimes l_{(3)}h_{(4)}R(l_{(2)} \otimes h_{(3)}) \\
&\quad R(S^{-1}(l_{(5)}) \otimes h_{(1)}) \\
&= \sum S^{-1}(l_{(6)}h_{(6)}) \bowtie h_{(2)}l_{(1)} \otimes l_{(3)}h_{(5)}R(S^{-1}(l_{(4)})l_{(2)} \otimes h_{(4)}) \\
&\quad R(S^{-1}(l_{(7)}) \otimes h_{(1)})R(l_{(5)} \otimes h_{(3)}) \\
&= \sum (S^{-1}(h_{(4)}) \bowtie h_{(1)})(S^{-1}(l_{(5)}) \bowtie l_{(1)}) \otimes l_{(3)}h_{(3)} \\
&\quad R(S^{-1}(l_{(4)})l_{(2)} \otimes h_{(2)}) \\
&= \sum (S^{-1}(h_{(3)}) \bowtie h_{(1)})(S^{-1}(l_{(3)}) \bowtie l_{(1)}) \otimes h_{(2)} \star l_{(2)} \\
&= \chi(h)\chi(l).
\end{aligned}
$$

□

Lemma 2.2 and Proposition 2.3 say that $\mathcal{H}_R$ can be embedded into $D[H]$ as left coideal subalgebra. In fact, $\mathcal{H}_R$ can be further embedded into $D(H)$ as a left coideal subalgebra.

Corollary 2.4. The composite algebra map:

$$\mathcal{H}_R \xrightarrow{\ \phi\ } D[H] \xrightarrow{\ \Phi\ } D(H)$$

is injective, and $\mathcal{H}_R$ is isomorphic to a left coideal subalgebra of $D(H)$.

Proof. Since H is faithfully flat over k, we have that the kernel of Φ is $\mathrm{Ker}(\Theta_l) \bowtie H$. If $\Phi\Theta_l(h) = 0$ for some $h \in \mathcal{H}_R$, then

$$\phi(h) = \sum S^{-1}(h_{(2)}) \bowtie h_{(1)} \in \mathrm{Ker}(\Theta_l) \bowtie H.$$

It follows that

$$h = \sum \varepsilon(S^{-1}(h_{(2)}))h_{(1)} = (\varepsilon \otimes \iota)\phi(h) = 0.$$

Since Φ is a Hopf algebra map, $\Phi\phi(\mathcal{H}_R)$ is a left coideal subalgebra of $D(H)$. $\square$.

Let us now consider Yetter-Dinfel'd H-modules and $\mathcal{H}_R$-bimodules. Let M be a Yetter-Drinfel'd module over H, or a left $D(H)$-module. The following composite map:

$$\mathcal{H}_R \otimes M \xrightarrow{\phi\otimes\iota} D[H] \otimes M \xrightarrow{\Phi\otimes\iota} D(H) \otimes M$$

makes M a left $\mathcal{H}_R$-module. If we write $-\triangleright$ for the above left action, then we have the explicit formula:

$$h -\triangleright m = \sum (h_{(2)} \cdot m_{(0)})R(S^{-1}(h_{(4)}) \otimes h_{(3)}m_{(1)}S^{-1}(h_{(1)})) \tag{4}$$

for $h \in \mathcal{H}_R$ and $m \in M$.

Since there is an augmentation map ε on $\mathcal{H}_R$, we may define the $\mathcal{H}_R$-invariant of a left $\mathcal{H}_R$-module M which is

$$M^{\mathcal{H}_R} = \{m \in M \mid h -\triangleright m = \varepsilon(h)m, \forall h \in \mathcal{H}_R\}.$$

When a left $\mathcal{H}_R$-module comes from a YD H-module we have

Lemma 2.5. Let M be a YD H-module. Then

$$M^{\mathcal{H}_R} = \{m \in M \mid h \cdot m = h \triangleright_1 m = \sum m_{(0)}R(h \otimes m_{(1)}), \forall h \in H\}.$$

Proof. By definition of the action of $\mathcal{H}_R$ on M, we have $h -\triangleright m = \sum S^{-1}(h_{(2)}) \triangleright_1 (h_{(1)} \cdot m)$ for any $h \in \mathcal{H}_R$ and $m \in M$. It follows that the latter set is contained in $M^{\mathcal{H}_R}$.

Conversely, if $m \in M^{\mathcal{H}_R}$, then we have

$$\begin{aligned} h \cdot m &= \sum h_{(3)} \triangleright_1 (S^{-1}(h_{(2)}) \triangleright_1 (h_{(1)} \cdot m)) \\ &= \sum h_{(2)} \triangleright_1 (h_{(1)} -\triangleright m) \\ &= \sum h_{(2)} \triangleright_1 (\varepsilon(h_{(1)})m) \\ &= h \triangleright_1 m \end{aligned}$$

for any $h \in \mathcal{H}_R$ and $m \in M$. $\square$

Following Lemma 2.5, we obtain that the invariant submodule $M^{\mathcal{H}_R}$ is the maximal submodule sitting in the subcategory $\mathbf{M}_R^H$, where the left H-action is the induced one $\triangleright_1$. Thus we get an idempotent radical κ:

$$\kappa : \ \mathbf{Q}^H \longrightarrow \mathbf{M}_R^H, \ \kappa(M) = M^{\mathcal{H}_R}.$$

It follows that *a YD H-module is sitting in $\mathbf{M}_R^H$ if and only if $\mathcal{H}_R$ acts trivally on M, i.e, $M = M^{\mathcal{H}_R}$*

Now we will define a right $\mathcal{H}_R$-module structure on a YD H-module M. Observe that the right H-comodule structure of M induces two left H-module structures. The first one is (1), and the second one is given by

$$h \triangleright_2 m = \sum m_{(0)}R(S(m_{(1)}) \otimes h) \tag{5}$$

for $h \in H$ and $m \in M$. With this second left H-action M, M becomes a right $D[H]$-module.

Lemma 2.6. Let M be a Yetter-Drinfel'd H-module. Then M is a right $D[H]$-module defined by

$$m \leftharpoondown (h \bowtie l) = S(l) \triangleright_2 (S(h) \cdot m) \tag{6}$$

for $m \in M$ and $h, l \in H$. Moreover, if A is a YD H-module algebra, then (6) makes A into a right $D[H]^{cop}$-module algebra.

Proof. It is sufficient to show that

$$m \leftharpoondown [(1 \bowtie l)(h \bowtie 1)] = S(h) \cdot (S(l) \triangleright_2 m)$$

for $h, l \in H$ and $m \in M$. Indeed, we have

$$\begin{aligned}
& m \leftharpoondown [(1 \bowtie l)(h \bowtie 1)] \\
= & \sum m \leftharpoondown (h_{(2)} \bowtie l_{(2)}) R(h_{(1)} \otimes l_{(1)}) R(S(h_{(3)} \otimes l_{(3)}) \\
= & \sum S(l_{(2)}) \triangleright_2 (S(h_{(2)}) \cdot m) R(h_{(1)} \otimes l_{(1)}) R(S(h_{(3)} \otimes l_{(3)}) \\
= & \sum S(h_{(3)}) \cdot (S(l_{(3)}) \triangleright_2 m) R(h_{(4)} \otimes l_{(4)}) R(S(h_{(2)}) \otimes l_{(2)}) \\
& R(h_{(1)} \otimes l_{(1)}) R(S(h_{(5)}) \otimes l_{(5)}) \\
= & S(h) \cdot (S(l) \triangleright_2 m)
\end{aligned}$$

for $h, l \in H$ and $m \in M$. The second statement is obvious. $\square$

Unfortunately, the right $D[H]$-module structure (6) does not match the canonical left $D[H]$-module structure induced by Hopf algebra map Φ to give a $D[H]$-bimodule structure on M. However the right $\mathcal{H}_R$-module structure on M given by

$$M \otimes \mathcal{H}_R \xrightarrow{\iota \otimes \phi} M \otimes D[H] \longrightarrow M,$$

precisely,

$$m \triangleleft\!\!- h = \sum (h_{(3)} \cdot m_{(0)}) R(h_{(4)} m_{(1)} S^{-1}(h_{(2)}) \otimes h_{(1)}) \tag{7}$$

for $m \in M$ and $h \in \mathcal{H}_R$, together with the left $\mathcal{H}_R$-module structure (4) does give an $\mathcal{H}_R$-bimodule structure on M.

Proposition 2.7. Let M be a YD H-module. Then M is an $\mathcal{H}_R$-bimodule via (4) and (7).

Proof. Given $h, l \in \mathcal{H}_R$ and $m \in M$, we have to prove

$$(l -\!\!\triangleright m) \triangleleft\!\!- h = l -\!\!\triangleright (m \triangleleft\!\!- h)$$

i.e.,

$$\sum S(h_{(1)}) \triangleright_2 (h_{(2)} \cdot (S^{-1}(l_{(2)}) \triangleright_1 (l_{(1)} \cdot m))) = \sum S^{-1}(l_{(2)}) \triangleright_1 (l_{(1)} \cdot (S(h_{(1)}) \triangleright_2 (h_{(2)} \cdot m))).$$

Indeed, we have

$$\begin{aligned}
& \sum S(h_{(1)}) \triangleright_2 (h_{(2)} \cdot (S^{-1}(l_{(2)}) \triangleright_1 (l_{(1)} \cdot m))) \\
= & \sum S^{-1}(l_{(4)}) \triangleright_1 (S(h_{(2)}) \triangleright_2 (l_{(2)} h_{(4)} \cdot m)) R(l_{(1)} \otimes h_{(3)}) R(l_{(3)} \otimes S(h_{(1)})) \\
= & \sum S(h_{(3)}) \triangleright_2 (S^{-1}(l_{(5)}) \triangleright_1 (l_{(2)} h_{(6)} \cdot m)) R(l_{(1)} \otimes h_{(5)}) \\
& R(l_{(3)} \otimes S(h_{(1)})) R(l_{(6)} \otimes S(h_{(4)}) R(S^{-1}(l_{(4)}) \otimes S(h_{(2)})) \\
= & \sum S(h_{(1)}) \triangleright_2 (S^{-1}(l_{(3)}) \triangleright_1 (l_{(2)} h_{(4)} \cdot m)) R(l_{(1)} \otimes h_{(3)}) R(l_{(4)} \otimes S(h_{(2)})) \\
= & \sum S(h_{(1)}) \triangleright_2 (S^{-1}(l_{(3)}) \triangleright_1 (h_{(3)} l_{(1)} \cdot m)) R(l_{(2)} \otimes h_{(4)}) R(l_{(4)} \otimes S(h_{(2)})) \\
= & \sum S(h_{(1)}) \triangleright_2 (h_{(2)} \cdot (S^{-1}(l_{(2)}) \triangleright_1 (l_{(1)} \cdot m))).
\end{aligned}$$

So M is an $\mathcal{H}_R$-bimodule. $\square$

Remark that the right $\mathcal{H}_R$-invariant of a YD H-module is different from the left one described in Lemma 2.5. One may apply the same argument in Lemma 2.5 to obtain the right invariant set of a YD H-module M is

$$\{m \in M \mid h \cdot m = h \triangleright_2 m, \forall h \in H.\}$$

Like the left invariants, the set of right invariants is the maximal right-right YD H-submodule of M. Combining Lemma 2.3, 2.6 and Proposition 2.7, we obtain the following:

Corollary 2.8. If A is a YD H-module algebra, then A is an $\mathcal{H}_R$-bimodule algebra in the sense that

$$\begin{aligned} h \multimap (ab) &= \sum (h_{(-1)} \cdot a)(h_{(0)} \multimap b) \\ (ab) \multimapinv h &= \sum (a \cdot h_{(0)})(b \multimapinv h_{(-1)}) \end{aligned}$$

for $a, b \in A$ and $h \in \mathcal{H}_R$, where $\chi(h) = \sum h_{(-1)} \otimes h_{(0)} \in D[H] \otimes \mathcal{H}_R$.

Now we are looking at the dual coalgebra H_R^* which is a quotient coalgebra of the dual Hopf algebra $D[H]^*$ by Lemma 2.2. Let us view R as an invertible element $\sum R^1 \otimes R^2$ in $H^* \otimes H^*$. Then the comultiplication of the Hopf algebra $D[H]^*$ reads as follows

$$\begin{aligned} \Delta(p \otimes q) &= \sum p_{(1)} \otimes R(q_{(1)} \otimes p_{(2)})R^{-1} \otimes q_{(2)} \\ &= \sum (p_{(1)} \otimes R^1 q_{(1)} S(r^1)) \otimes (R^2 p_{(2)} r^2 \otimes q_{(2)}) \end{aligned}$$

where $p, q \in H^*$ and we use the same S to denote the antipode of H^* (one may easily tell the difference from the context). Note that as an algebra $D[H]^*$ is the tensor product $H^* \otimes H^*$.

Lemma 2.9. H_R^* is a left $D[H]^*$-module coalgebra with the comultiplication given by

$$\begin{aligned} \Delta_R(p) &= \sum R^2 r^2 p_{(2)} \otimes r^1 p_{(1)} S^{-1}(R^1) \\ &= R^2 p_{(1)} r^2 \otimes p_{(2)} r^1 S^{-1}(R^1) \end{aligned}$$

where $R = r$, $p \in H_R^*$. The counit ε of H is now a group-like element of H_R^*. $\square$

Actually H_R^* is a quotient coalgebra in the sense that the following coalgebra map is sujective:

$$\phi^* : \ D[H]^* \longrightarrow H_R^*, \ p \bowtie q \mapsto qS^{-1}(p). \tag{8}$$

So the action of $D[H]^*$ on H_R^* reads as follows:

$$p \bowtie q \rightharpoondown x = qxS^{-1}(p) \tag{9}$$

for $p \bowtie q \in D[H]^*$ and $x \in H_R^*$. In particular, we have

$$p \bowtie q \rightharpoondown \varepsilon = \phi^*(p \bowtie q) = qS^{-1}(p)$$

for any $p \bowtie q \in D[H]^*$.

In the sequel, we will write $\Delta_R(p) = \sum p^{(1)} \otimes p^{(2)}$, $p \in H_R^*$ in order to distinguish $\Delta_R(p)$ from $\Delta(p) = \sum p_{(1)} \otimes p_{(2)}$, the usual comultiplication of the dual Hopf algebra H^*. Let us figure out the relation between $\mathcal{H}_R^*$-comodule structure and the H^*-comodule structures of a YD

H-module. Let M be a YD H-module. We use the following summation notation for the dual H^*-comodule structure of M

$$M \longrightarrow M \otimes H^*, \ m \mapsto \sum m_{[0]} \otimes m_{[1]},$$

and the usual Sweedler notation $\sum m_{(0)} \otimes m_{(1)}$ for the H-comodule stucture of M. The right and left $\mathcal{H}_R^*$-comodule structure will be indicated by the Sweedler "upcase" notations.

It is not difficult to check that the right $D[H]^*$-comodule structure of M reads as follows:

$$M \longrightarrow M \otimes D[H]^*, \ m \mapsto \sum m_{[0](0)} \otimes m_{[0](1)} \bowtie m_{[1]}.$$

Similarly, one may get the dual left $D[H]^*$-comodule structure of (6). It follows from (8) that we obtain the corresponding dual right $\mathcal{H}_R^*$-comodule structure of (4) and the dual left $\mathcal{H}_R^*$-comodule structure of (7) respectively:

$$\begin{array}{ll} M \longrightarrow M \otimes \mathcal{H}_R^*, & \sum m^{(0)} \otimes m^{(1)} = \sum m_{[0](0)} \otimes m_{[1]} S^{-1}(\Theta_r(m_{[0](1)})), \\ M \longrightarrow \mathcal{H}_R^* \otimes M, & \sum m^{(-1)} \otimes m^{(0)} = \sum \Theta_l(m_{[0](1)}) m_{[1]} \otimes m_{[0](0)}. \end{array} \tag{10}$$

Now let A be a YD H-module algebra. Then A is both a left and right $D[H]$-module algebra, and therefore an $\mathcal{H}_R$-bimodule algebra in the sense of Corollary 2.8. We present the dual comodule structures of the formulas in Coro.2.8 because they will be frequently used in the sequel. Let us use the following upcase Sweedler notations for the dual left and right $D[H]^*$-comodule structures of a YD H-module algebra A:

$$\sum a^{[-1]} \otimes a^{[0]}, \ \ \sum a^{[0]} \otimes a^{[1]}$$

for an element $a \in A$. Then the dual comodule versions of the formulas in Corollary 2.8 read as follows:

$$\begin{array}{l} \sum (ab)^{(0)} \otimes (ab)^{(1)} = \sum a^{[0]} b^{(0)} \otimes a^{[1]} \rightharpoondown b^{(1)}, \\ \sum (ab)^{(-1)} \otimes (ab)^{(0)} = \sum a^{(0)} b^{[0]} \otimes b^{[-1]} \rightharpoondown a^{(-1)} \end{array} \tag{11}$$

for $a, b \in A$. We will call A a *right (or left) $\mathcal{H}_R^*$-comodule algebra* in the sense of (11).

Finally, for a YD H-module M, we will write $M_\diamond$ (or ${}_\diamond M$) for the right (or left) $\mathcal{H}_R^*$-coinvariants. For instance,

$$M_\diamond = \{m \in M \mid \sum m^{(0)} \otimes m^{(1)} = m \otimes \varepsilon.\}$$

It is obvious that $M_0 = M^{\mathcal{H}_R}$ cf.Lemma 2.5.

3 The group $\mathrm{Gal}(\mathcal{H}_R)$

In this section, we construct a group $\mathrm{Gal}(\mathcal{H}_R)$ of "Galois" objects for the deformation $\mathcal{H}_R$. The group $\mathrm{Gal}(\mathcal{H}_R)$ plays the vital part in this paper.

Definition 3.1. Let A be a right $D[H]^*$-comodule algebra. $A/A_\diamond$ is said to be a right $\mathcal{H}_R^*$-Galois extension if the linear map

$$\beta^r : \ A \otimes_{A_\diamond} A \longrightarrow A \otimes \mathcal{H}_R^*, \ \ \beta^r(a \otimes b) = \sum a^{(0)} b \otimes a^{(1)}$$

is an isomorphism. Similarly, if A is a left $D[H]^*$-comodule algebra, then $A/_0A$ is said to be left Galois if the linear map

$$\beta^l:\ A\otimes_{\diamond A}A\longrightarrow\mathcal{H}_R^*\otimes A,\quad \beta^l(a\otimes b)=\sum b^{(-1)}\otimes ab^{(0)}$$

is an isomorphism.

Note that the reason why we use the linear maps β^r and β^l, instead of the canonical linear maps defining the Hopf-Galois extensions (see [23, 3.7]), will be seen in the next section where our derived objects are Galois extensions of the type in Definition 3.1. However, when $\mathcal{H}_R$ is a Hopf algebra (although it is not the case in general), the definitions in 3.1 are equivalent to the classical ones. The following are characterizations of a right $\mathcal{H}_R^*$-Galois object, i.e, a right $\mathcal{H}_R^*$-Galois extension of k cf.[23, 3.7] and [16].

Lemma 3.2. Let A be a right $D[H]^*$-comodule algebra such that $A_\diamond=k$. The following statements are equivalent:

1). A/k is $\mathcal{H}_R^*$-Galois.
2). $A\#\mathcal{H}_R$ is canonically isomorphic to $\mathrm{End}(A)$, and A is finitely projective k-module.

A right $D[H]^*$-comodule algebra satisfying Lemma 3.2 is called *a right $\mathcal{H}_R^*$-Galois object.* Similarly, one may have the symmetric version of Lemma 3.2 for a left $\mathcal{H}_R^*$-Galois object. Note that a $D[H]$-linear algebra map between two Galois objects is an isomorphism. What we are interested in, in this section, are those *$\mathcal{H}_R^*$-bigalois objects which are both left and right $\mathcal{H}_R^*$-Galois such that the left and right $\mathcal{H}_R^*$-coactions commutes.*

Denote by $\mathcal{E}(\mathcal{H}_R)$ the category of YD H-module algebras A such that A are $\mathcal{H}_R^*$-bigalois objects. The morphisms in $\mathcal{E}(\mathcal{H}_R)$ are YD H-module algebra isomorphisms. We are going to define a product in the category $\mathcal{E}(\mathcal{H}_R)$.

Given two objects X and Y in $\mathcal{E}(\mathcal{H}_R)$, we define $X\wedge Y$ as a subset of $X\#_\theta Y$, the latter is the braided product algebra in $\mathbf{M}_R^H$:

$$X\wedge Y=\{\sum x_i\#y_i\in X\#_\theta Y\mid \sum x_i\triangleleft\!\!- h\#y_i=\sum x_i\#h-\!\!\triangleright y_i,\forall h\in\mathcal{H}_R.\}\tag{12}$$

Lemma 3.3. Let X,Y be two objects of $\mathcal{E}(\mathcal{H}_R)$. Then

$$X\wedge Y=\{\sum x_i\#y_i\in X\#_\theta Y\mid \sum h_{(1)}\cdot x_i\#h_{(2)}\triangleright_1 y_i=\sum h_{(1)}\triangleright_2 x_i\#h_{(2)}\cdot y_i,\forall h\in H\}.\tag{13}$$

Proof. Let T be the set

$$\{\sum x_i\#y_i\in X\#_\theta Y\mid \sum h_{(1)}\cdot x_i\#h_{(2)}\triangleright_1 y_i=\sum h_{(1)}\triangleright_2 x_i\#h_{(2)}\cdot y_i,\forall h\in H\}.$$

In order to simplify the computation, we will write $x\#y$ for an element $\sum x_i\#y_i$ in $X\#Y$ in the sequel. Given $x\#y\in X\wedge Y$ and $h\in H$, we have

$$\begin{aligned}
&\sum h_{(1)}\cdot x\#h_{(2)}\triangleright_1 y\\
=&\sum h_{(1)}\triangleright_2(S(h_{(2)}\triangleright_2(h_{(3)}\cdot x))\#h_{(4)}\triangleright_1 y\\
=&\sum h_{(1)}\triangleright_2(x\triangleleft\!\!- h_{(2)})\#h_{(3)}\triangleright_1 y\\
=&\sum h_{(1)}\triangleright_2 x\#h_{(3)}\triangleright_1(h_{(2)}-\!\!\triangleright y)\\
=&\sum h_{(1)}\triangleright_2 x\#h_{(4)}\triangleright_1(S^{-1}(h_{(3)})\triangleright_1(h_{(2)}\cdot y))\\
=&\sum h_{(1)}\triangleright_2 x\#h_{(2)}\cdot y.
\end{aligned}$$

So $x\#y$ belongs to T and we have that $X \wedge Y \subseteq T$.

Conversely, if $x\#y \in T$, and $h \in \mathcal{H}_R$, we have

$$\begin{aligned}
x \multimapdotinv h\#y &= \sum S(h_{(1)}) \triangleright_2 (h_{(2)} \cdot x)\#y \\
&= \sum S(h_{(1)}) \triangleright_2 (h_{(2)} \cdot x)\#S^{-1}(h_{(4)})h_{(3)} \triangleright_1 y \\
&= \sum S(h_{(1)}) \triangleright_2 (h_{(2)} \triangleright_2 x)\#S^{-1}(h_{(4)}) \triangleright_1 (h_{(3)} \cdot y) \\
&= \sum x\#S^{-1}(h_{(2)}) \triangleright_1 (h_{(1)} \cdot y) \\
&= x\#h \multimap\!\triangleright y.
\end{aligned}$$

It follows that $T \subseteq X \wedge Y$. □

The above alternative definition (13) of $X \wedge Y$ is more practicable when we check if an element is in $X \wedge Y$. Moreover, it hints that we might have an left H-action on $X \wedge Y$ given by

$$h \cdot \sum (x_i\#y_i) = \sum h_{(1)} \cdot x_i\#h_{(2)} \triangleright_1 y_i = \sum h_{(1)} \triangleright_2 x_i\#h_{(2)} \cdot y_i \tag{14}$$

whenever $\sum x_i\#y_i \in X \wedge Y$ and $h \in H$. (14) is indeed a left H-module strucuture on $X \wedge Y$. To show this, one may easily check that the element $h \cdot (x\#y)$ is sitting in $X \wedge Y$ by Lemma 3.3 if $x\#y \in X \wedge Y$.

Lemma 3.4. $X \wedge Y$ with the H-action (14) and the H-coaction inheriting from $X\#Y$ is a YD H-module.

Proof. Let us first check that $X \wedge Y$ is an H-subcomodule of $X\#_\theta Y$. Given $x\#y \in X \wedge Y$, $h \in H$ and $p \in H^*$, we have

$$\begin{aligned}
& \sum h_{(1)} \cdot x_{(0)}\#h_{(2)} \triangleright_1 y_{(0)} < p, y_{(1)}x_{(1)} > \\
= & \sum h_{(1)} \cdot (p_{(2)} \cdot x)\#h_{(2)} \triangleright_1 (p_{(1)} \cdot y) \\
= & \sum p_{(3)} \cdot (h_{(2)} \cdot x)\#p_{(2)} \cdot (h_{(3)} \triangleright_1 y) < p_{(1)}, h_{(4)} >< p_{(4)}, S^{-1}(h_{(1)}) > \\
= & \sum p_{(3)} \cdot (h_{(2)} \triangleright_2 x)\#p_{(2)} \cdot (h_{(3)} \cdot y) < p_{(1)}, h_{(4)} >< p_{(4)}, S^{-1}(h_{(1)}) > \\
= & \sum h_{(1)} \triangleright_2 (p_{(2)} \cdot x)\#h_{(2)} \cdot (p_{(1)} \cdot y) \\
= & \sum h_{(1)} \triangleright_2 x_{(0)}\#h_{(2)} \cdot y_{(0)} < p, y_{(1)}x_{(1)} > .
\end{aligned}$$

It follows that $X \wedge Y$ is a right H-subcomodule of $X\#_\theta Y$.

Now we show that the right H-comodule $X \wedge Y$ together with the left H-module structure (14) is a YD H-module. Indeed, for $h \in H$ and $x\#y \in X \wedge Y$, we have

$$\begin{aligned}
\rho(h \cdot (x\#y)) &= \sum \rho(h_{(1)} \cdot x\#h_{(2)} \triangleright_1 y) \\
&= \sum h_{(2)} \cdot x_{(0)}\#h_{(5)} \triangleright_1 y_{(0)} \otimes h_{(6)}y_{(1)}S^{-1}(h_{(4)})h_{(3)}x_{(1)}S^{-1}(h_{(1)}) \\
&= \sum h_{(2)} \cdot (x_{(0)}\#y_{(0)}) \otimes h_{(3)}y_{(1)}x_{(1)}S^{-1}(h_{(1)})
\end{aligned}$$

for any $h \in H$ and $x\#y \in X \wedge Y$. □

Proposition 3.5. If X, Y are two objects of $\mathcal{E}(\mathcal{H}_R)$, then $X \wedge Y$ is an object of $\mathcal{E}(\mathcal{H}_R)$.

Proof. First of all, we have to show that $X\wedge Y$ is a subalgebra of $X\#_\theta Y$. Let $x\#y$ and $x'\#y'$ be two elements of $X\wedge Y$. For any $h\in H$, we have

$$\begin{aligned}
&\sum h_{(1)}\cdot(xx'_{(0)})\#h_{(2)}\triangleright_1(y_{(0)}y')R(x'_{(1)}\otimes y_{(1)})\\
=\ &\sum(h_{(1)}\cdot x)(h_{(2)}\cdot x'_{(0)})\#(h_{(3)}\triangleright_1 y_{(0)})(h_{(4)}\triangleright_1 y')R(x'_{(1)}\otimes y_{(1)})\\
=\ &\sum(h_{(1)}\cdot x)(h_{(2)}\cdot x'_{(0)})\#y_{(0)}y'_{(0)}R(h_{(3)}\otimes y_{(1)})R(h_{(4)}\otimes y'_{(1)})R(x'_{(1)}\otimes y_{(2)})\\
=\ &\sum(h_{(1)}\cdot x)(h_{(4)}\cdot x'_{(0)})\#y_{(0)}y'_{(0)}R(h_{(5)}x'_{(1)}S^{-1}(h_{(3)})\otimes y_{(1)})\\
&R(h_{(2)}\otimes y_{(2)})R(h_{(6)}\otimes y'_{(1)})\\
=\ &\sum(h_{(1)}\cdot x\#h_{(2)}\triangleright_1 y)(h_{(3)}\cdot x'\#h_{(4)}\triangleright_1 y).
\end{aligned}$$

Similarly, one may obtain

$$\begin{aligned}
&\sum h_{(1)}\triangleright_2(xx'_{(0)})\#h_{(2)}\cdot(y_{(0)}y')R(x'_{(1)}\otimes y_{(1)})\\
=\ &\sum(h_{(1)}\triangleright_2 x\#h_{(2)}\cdot y)(h_{(3)}\triangleright_2 x'\#h_{(4)}\cdot y).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\sum h_{(1)}\cdot(xx'_{(0)})\#h_{(2)}\triangleright_1(y_{(0)}y')R(x'_{(1)}\otimes y_{(1)})\\
=\ &\sum(h_{(1)}\cdot x\#h_{(2)}\triangleright_1 y)(h_{(3)}\cdot x'\#h_{(4)}\triangleright_1 y)\\
=\ &\sum(h_{(1)}\triangleright_2 x\#h_{(2)}\cdot y)(h_{(3)}\triangleright_2 x'\#h_{(4)}\cdot y)\\
=\ &\sum h_{(1)}\triangleright_2(xx'_{(0)})\#h_{(2)}\cdot(y_{(0)}y')R(x'_{(1)}\otimes y_{(1)}).
\end{aligned}$$

By Lemma 3.3, we have shown that $X\wedge Y$ is a subalgebra of $X\#_\theta Y$s and hence a H^{op}-comodule subalgebra of $X\#_\theta Y$. Moreover, the previous computations actually showed that $X\wedge Y$ is a left H-module algebra with respect to the H-action (14). It follows from Lemma 3.4 that $X\wedge Y$ is a YD H-module algebra.

It remains to show that $X\wedge Y$ is an $\mathcal{H}_R^*$-bigalois object. Observe that the $\mathcal{H}_R$-bimodule structure on $X\wedge Y$ induced by the YD H-module structure defined above is as follows:

$$h\multimap(\sum x_i\#y_i)=\sum h\multimap x_i\#y_i,\quad(\sum x_i\#y_i)\lhd\!\!-\,h=\sum x_i\#y_i\lhd\!\!-\,h \tag{15}$$

whenever $h\in\mathcal{H}_R$ and $\sum x_i\#y_i\in X\wedge Y$. We verify the first formula and leave the second one to the reader. Indeed,

$$\begin{aligned}
h\multimap(x_\# y) &= \sum S^{-1}(h_{(2)})\triangleright_1(h_{(1)}\cdot(x\#y))\\
&= \sum S^{-1}(h_{(3)})\triangleright_1(h_{(1)}\cdot x\#h_{(2)}\triangleright_1 y)\\
&= \sum S^{-1}(h_{(2)})\triangleright_1(h_{(1)}\cdot x)\#y\\
&= h\multimap x\#y,
\end{aligned}$$

for any $h\in\mathcal{H}_R$ and $x\#y\in X\wedge Y$. Thus the right (or left) $\mathcal{H}_R^*$-comodule structure of $X\wedge Y$ only comes from the right one of X (or the left one of Y).

Applying (15), we may easily obtain that both the left and right coinvariant subalgebras of $X\wedge Y$ are trivial. To show that $X\wedge Y$ is a right $\mathcal{H}_R^*$-Galois object, we need a new object $(X\otimes X)\wedge Y$ which is not in $\mathcal{E}(\mathcal{H}_R)$, but which makes sense with respect to the product $\wedge$. Consider $X\otimes X$ as a product YD H-module algebra in the category $\mathbf{Q}^H$. By Coro.2.8,

$X\otimes X$ is an $\mathcal{H}_R$-biomodule algebra. So the product defined in (12) makes sense even though $X\otimes X$ is not $\mathcal{H}_R^*$-Galois. For instance, $\mathcal{H}_R$ acts on $X\otimes X$ from the right hand side in the following way:

$$(x\otimes x') \multimap h = \sum x \multimap h_{(0)} \otimes x' \leftharpoondown h_{(-1)}$$

for $h \in \mathcal{H}_R$ and $x\otimes x' \in X\otimes X$.

Now one may take a while to check that the following diagram is commutative:

$$\begin{tikzcd}
(X\wedge Y)\otimes(X\wedge Y) \arrow[rr, "\beta^r_{X\wedge Y}"] \arrow[dr, "\tau"'] & & (X\wedge Y)\otimes \mathcal{H}_R^* \\
& (X\otimes X)\wedge Y \arrow[ur, "\delta"'] &
\end{tikzcd}$$

where τ is the linear map given by

$$\tau((x\#y)\otimes(x'\#y')) = \sum (x\otimes x_{(0)})\#(x_{(1)} \triangleright_1 y)y'$$

and the linear map δ is defined by

$$\delta((x'\otimes x)\#y) = \sum x^{(0)}x'\#y\otimes x^{(1)}.$$

However, we have to show that τ and δ are well-defined. We leave the easier verification of τ to the reader, and show that δ is well-defined.

Observe that the multiplication map of X: $X\otimes X \longrightarrow X$ is an $\mathcal{H}_R$-bimodule map because X is an $\mathcal{H}_R$-bimodule algebra in the sense of Coro.2.8. The multiplication map then induces an $\mathcal{H}_R$-bimodule map:

$$\mu:\ (X\otimes X)\wedge Y \longrightarrow X\wedge Y.$$

If we can prove that the map η:

$$(X\otimes X)\wedge Y \longrightarrow (X\otimes X)\wedge Y\otimes \mathcal{H}_R^*$$

given by $\eta(x\otimes x')\#y = \sum (x^{(0)}\otimes x')\#y\otimes x^{(1)}$, is well-defined, then δ is well-defined because δ actually is the composite map of η with μ:

$$\delta:\ (X\otimes X)\wedge Y \xrightarrow{\eta} (X\otimes X)\wedge Y\otimes \mathcal{H}_R^* \xrightarrow{\mu} X\wedge Y\otimes \mathcal{H}_R^*.$$

Indeed, to show that η is well-defined, it is equivalent to show that for any $l\in\mathcal{H}_R$, and $(x\otimes x')\#y \in (X\otimes X)\wedge Y$, the element

$$(l \multimapdotinv x\otimes x')\#y$$

is still in $(X\otimes X)\wedge Y$. This is the case since

$$\begin{aligned}
(l \multimapdotinv x\otimes x') \multimap h\#y &= \sum ((l \multimapdotinv x) \multimap h_{(0)} \otimes x' \leftharpoondown h_{(-1)})\#y \\
&= \sum (l \multimapdotinv (x \multimap h_{(0)}) \otimes x' \leftharpoondown h_{(-1)})\#y \\
&= \sum (l \multimapdotinv x\otimes x')\#h \multimapdotinv y
\end{aligned}$$

for any $h, l \in \mathcal{H}_R$ and $(x\otimes x')\#y \in (X\otimes X)\#Y$.

Since both τ and δ are obviously isomorphisms, we obtain that $\beta_{X\wedge Y}$ is an isomorphism. It follows that $X \wedge Y$ is a right $\mathcal{H}_R^*$-Galois object. Similarly, one may show that $X \wedge Y$ is a left $\mathcal{H}_R^*$-Galois object, and $X \wedge Y$ is an object in $\mathcal{E}(\mathcal{H}_R)$. $\square$

Now let H^* be the covolution algebra of H. There is a canonical YD H-module stucture on H^* such that H^* is a YD H-module algebra:

$$\begin{array}{ll} h \cdot p = \sum p_{(1)} < p_{(2)}, h >, & \mathrm{H-action} \\ h^* \cdot p = \sum h^*_{(2)} p S^{-1}(h^*_{(1)}), & \mathrm{H-coaction} \end{array} \tag{16}$$

for $h^*, p \in H^*$ and $h \in H$.

Lemma 3.6. H^* is an object in $\mathcal{E}(\mathcal{H}_R)$.

Proof. It is sufficient to show that the induced $\mathcal{H}_R$-bimodule structure (4) and (7) on H^* is the same as the dual $\mathcal{H}_R$-bimodule sructure stemming from the comultiplication Δ_R (see Lemma 2.9) of $\mathcal{H}_R^*$. Indeed, given $p \in H^*$, $h \in \mathcal{H}_R$, we have

$$\begin{aligned} h \multimap p &= \sum S^{-1}(h_{(2)}) \triangleright_1 (h_{(1)} \cdot p) \\ &= \sum \Theta_l(S^{-1}(h_{(3)}))(h_{(1)} \cdot p) S^{-1}(\Theta_l(S^{-1}(h_{(2)})) \\ &= \sum \Theta_l(S^{-1}(h_{(3)})) p_{(1)} \Theta_l(h_{(2)}) < p_{(2)}, h_{(1)} > \\ &= \sum R^2 p_{(1)} r^2 < p_{(2)}, h_{(1)} >< r^1, h_{(2)} >< S^{-1}(R^1), h_{(3)} > \\ &= \sum R^2 p_{(1)} r^2 < p_{(2)} r^1 S^{-1}(R^1), h > \\ &= \sum p^{(1)} < p^{(2)}, h > . \end{aligned}$$

Similarly, one have $p \multimapinv h = \sum p^{(2)} < p^{(1)}, h >$ for any $h \in \mathcal{H}_R$ and $p \in H^*$. Since $\mathcal{H}_R^*$ is a quotient coalgebra of $D[H]^*$, we have that H^* with the $\mathcal{H}_R^*$-bicomodule stucture Δ_R is an $\mathcal{H}_R^*$-bigalois object. $\square$

Denote by I the object H^* described in Lemma 3.6. I is infact a unit in the category of $\mathcal{E}(\mathcal{H}_R)$.

Proposition 3.7. The category $\mathcal{E}(\mathcal{H}_R)$ is a monoidal category with product $\wedge$ and the unit I.

Proof. It is sufficient to show that $I \wedge X \cong X \cong X \wedge I$ for any $X \in \mathcal{E}(\mathcal{H}_R)$. We show that $I \wedge X \cong X$. The proof of $X \wedge I \cong X$ is similar. Let ρ^+ be the composite map of the flip map with the right $\mathcal{H}_R^*$-comodule structure of X. We show that

$$\rho^+ : X \longrightarrow I \wedge X, \ \rho^+(x) = \sum x^{(1)} \# x^{(0)}$$

is the desired isomorphism in $\mathcal{E}(\mathcal{H}_R)$. By Lemma 3.6, we have

$$\begin{aligned} \sum x^{(1)} \multimapinv h \# x^{(0)} &= \sum x^{(2)} < x^{(1)}, h > \# x^{(0)} \\ &= \sum x^{(1)} \# h \multimap x^{(0)} \end{aligned}$$

for $h \in \mathcal{H}_R$ and $x \in X$. So ρ^+ is a well-defined map. Second, we verify that ρ^+ is an algebra map. Let $x, y \in X$ and $h \in \mathcal{H}_R$. we compute $\rho^+(x)\rho^+(y)$.

$$\rho^+(x)\rho^+(y) = \sum (x^{(1)} \# x^{(0)})(y^{(1)} \# y^{(0)})$$

$$
\begin{aligned}
&= \sum x^{(1)} y^{(1)}_{(0)} \# x^{(0)}_{(0)} R(y^{(1)}_{(1)} \otimes x^{(1)}_{(1)}) \\
&= \sum x^{(1)} (\Theta_l(x^{(0)}_{(1)}) \cdot y^{(1)}) \# x^{(0)}_{(0)} y^{(0)} \\
&= \sum x_{[1]} \Theta_r(S(x_{[0](3)})) \Theta_r(x_{[0](2)}) y_{[1]} \Theta_r(S(y_{[0](1)})) \Theta_r(S(x_{[0](1)})) \# x_{[0](0)} y_{[0](0)} \\
&= \sum x_{[1]} y_{[1]} \Theta_r(S(y_{[0](1)} x_{[0](1)})) \# x_{[0](0)} y_{[0](0)} \\
&= \sum (xy)^{(1)} \otimes (xy)^{(0)} \\
&= \sum \rho^+(xy).
\end{aligned}
$$

It is obvious that ρ^+ is a YD H-module morphism, hence a morphism in $\mathcal{E}(\mathcal{H}_R)$. $\square$

Denote by $E(\mathcal{H}_R)$ the set of the isomorphism classes of objects in $\mathcal{E}(\mathcal{H}_R)$. Proposition 3.7 says that $E(\mathcal{H}_R)$ is a semigroup. In the rest of this section we show that $E(\mathcal{H}_R)$ contains a subgroup.

Let X be an object in $\mathcal{E}(\mathcal{H}_R)$. Let $\overline{X}$ be the opposite algebra in $\mathbf{M}^H_R$. That is, $\overline{X} = X$ as a right H^{op}-comodule, but with the multiplication given by

$$
\overline{x} \circ \overline{y} = \sum \overline{y_{(0)} x_{(0)}} R(y_{(1)} \otimes x_{(1)})
$$

when $\overline{x}, \overline{y} \in \overline{X}$. Since the H-action on X does not give an H-module algebra structure on $\overline{X}$, we have to define a new H-action on $\overline{X}$ such that $\overline{X}$ together with the inherited H^{op}-comodule structure is a YD H-module algebra. Let H act on $\overline{X}$ as follows

$$
\begin{aligned}
h \rightharpoonup \overline{x} &= \sum \overline{S(h_{(4)}) \cdot x_{(0)}} R(h_{(6)} \otimes x_{(2)}) R(x_{(1)} \otimes S(h_{(3)})) R(h_{(5)} \otimes S(h_{(2)})) u^{-1}(h_{(1)}) \\
&= \sum \overline{h^u_{(3)} \cdot (h_{(2)} \triangleright_2 (h_{(5)} \triangleright_1 x))} R(S(h_{(4)}) \otimes h_{(1)})
\end{aligned} \tag{17}
$$

where $h \in \mathcal{H}_R$, $\overline{x} \in \overline{X}$, $h^u = \sum S(h_{(2)}) u^{-1}(h_{(1)})$ and $u = \sum S(R^2) R^1 \in H^*$ is the Casimir element of H^*. Since the square of the antipode of H^* is an inner automorphism induced by the Casimir element u, we have the formula (see [19]):

$$
\begin{aligned}
&\sum u(h_{(1)}) h_{(2)} = \sum S^2(h_{(1)}) u(h_{(2)}), \\
&\sum u^{-1}(h_{(1)}) S^{-1}(h_{(2)}) = \sum S(h_{(1)}) u^{-1}(h_{(2)})
\end{aligned} \tag{18}
$$

for any $h \in H$. We will use the formulas (18) quite often in the sequel. For instance, one may change the order of the actions $\cdot, \triangleright_1$ and $\triangleright_2$ in the formula (17) in order to have an alternative formula:

$$
h \rightharpoonup \overline{x} = \sum \overline{h_{(1)} \triangleright_2 (h_{(5)} \triangleright_1 (S^{-1}(h_{(4)}) \cdot x))} R(S^2(h_{(6)}) \otimes h_{(3)}) \tag{19}
$$

for $h \in \mathcal{H}_R$ and $x \in \overline{X}$.

Lemma 3.8. Let X be an object in $\mathcal{E}(\mathcal{H}_R)$. Then $\overline{X}$ together with the H-action (17) is a YD H-module algebra.

Proof. We show that $\overline{X}$ together with (17) is a YD H-module, and leave to the reader the tedious check that $\overline{X}$ is left H-module algebra.

First we show that (17) is a left H-module structure on $\overline{X}$. Given $h, l \in H$ and $\overline{x} \in \overline{X}$, we have

$$
\begin{aligned}
& l \rightharpoonup (h \rightharpoonup \overline{x}) \\
= & \sum l \rightharpoonup \overline{S(h_{(4)}) \cdot x_{(0)}} R(h_{(6)} \otimes x_{(2)}) R(x_{(1)} \otimes S(h_{(3)})) R(h_{(5)} \otimes S(h_{(2)})) u^{-1}(h_{(1)})
\end{aligned}
$$

$$
\begin{aligned}
&= \sum \overline{S(l_{(4)})\cdot(S(h_{(4)})\cdot x_{(0)})}R(l_{(6)}\otimes S(h_{(4)})x_{(2)}h_{(8)})R(S(h_{(5)})x_{(1)}h_{(7)}\otimes S(l_{(3)}))\\
&\quad R(l_{(5)}\otimes S(l_{(2)}))u^{-1}(l_{(1)})R(h_{(10)}\otimes x_{(4)})R(x_{(3)}\otimes S(h_{(3)}))R(h_{(9)}\otimes S(h_{(2)}))u^{-1}(h_{(1)})\\
&= \sum \overline{S(h_{(6)}l_{(6)})\cdot x_{(0)}}R(l_{(8)}\otimes h_{(8)})R(l_{(9)}\otimes S(h_{(3)}))R(l_{(10)}\otimes x_{(3)})\\
&\quad R(h_{(5)}\otimes l_{(5)})R(x_{(1)}\otimes S(h_{(4)}))R(h_{(7)}\otimes S(l_{(3)}))R(l_{(7)}\otimes S(l_{(2)}))\\
&\quad R(h_{(10)}\otimes x_{(4)})R(x_{(2)}\otimes S(h_{(4)}))R(h_{(9)}\otimes S(h_{(2)}))u^{-1}(h_{(1)})u^{-1}(l_{(1)})\\
&= \sum \overline{S(l_{(4)}h_{(4)})\cdot x_{(0)}}R(h_{(5)}\otimes l_{(5)})R(l_{(8)}\otimes h_{(8)})R(l_{(9)}h_{(9)}\otimes S(h_{(2)}))\\
&\quad R(h_{(7)}l_{(7)}\otimes S(l_{(2)}))R(x_{(1)}\otimes S(l_{(4)}h_{(4)}))R(l_{(10)}h_{(10)}\otimes x_{(2)})u^{-1}(h_{(1)})u^{-1}(l_{(1)})\\
&= \sum \overline{S^{-1}(l_{(3)}h_{(3)})\cdot x_{(0)}}((u^{-1}\otimes u^{-1})R_{21}R)(l_{(5)}\otimes h_{(5)})\\
&\quad R(S^{-1}(l_{(4)}h_{(4)})\otimes l_{(1)}h_{(1)})R(l_{(6)}h_{(6)}\otimes x_{(2)})R(x_{(1)}\otimes S^{-1}(l_{(2)}h_{(2)}))\\
&= \sum \overline{S^{-1}(l_{(3)}h_{(3)})\cdot x_{(0)}}u^{-1}(l_{(5)}h_{(5)})R(S^{-1}(l_{(4)}h_{(4)})\otimes l_{(1)}h_{(1)})\\
&\quad R(l_{(6)}h_{(6)}\otimes x_{(2)})R(x_{(1)}\otimes S^{-1}(l_{(2)}h_{(2)}))\\
&= \sum \overline{S(l_{(4)}h_{(4)})\cdot x_{(0)}}R(l_{(5)}h_{(5)}\otimes S(l_{(2)}h_{(2)}))R(l_{(6)}h_{(6)}\otimes x_{(2)})\\
&\quad R(x_{(1)}\otimes S(l_{(3)}h_{(3)}))u^{-1}(l_{(1)}h_{(1)})\\
&= lh \rightharpoonup \overline{x},
\end{aligned}
$$

where we used the identity $(u^{-1}\otimes u^{-1})R_{21}R = \Delta(u^{-1})$ (see [19] for the proof), and $R_{21} = \sum R^2\otimes R^1$.

Second we show that $\overline{X}$ is a YD H-module. Given $h\in H$ and $\overline{x}\in\overline{X}$, we compute $\rho(h\rightharpoonup\overline{x})$.

$$
\begin{aligned}
\rho(h\rightharpoonup\overline{x}) &= \sum \overline{(S(h_{(4)})\cdot x_{(0)})_{(0)}}\otimes (S(h_{(4)})\cdot x_{(0)})_{(1)}R(h_{(6)}\otimes x_{(2)})\\
&\quad R(x_{(1)}\otimes S(h_{(3)}))R(h_{(5)}\otimes S(h_{(2)}))u^{-1}(h_{(1)})\\
&= \sum \overline{S(h_{(5)})\cdot x_{(0)}}\otimes S(h_{(4)})x_{(1)}h_{(6)}R(x_{(2)}\otimes S(h_{(3)}))\\
&\quad R(h_{(8)}\otimes x_{(3)})R(h_{(7)}\otimes S(h_{(2)}))u^{-1}(h_{(1)})\\
&= \sum \overline{S(h_{(5)})\cdot x_{(0)}}\otimes x_{(2)}S(h_{(3)})h_{(6)}u^{-1}(h_{(1)})\\
&\quad R(h_{(8)}\otimes x_{(3)})R(x_{(1)}\otimes S(h_{(4)})R(h_{(6)}\otimes S(h_{(3)}))\\
&= \sum \overline{S(h_{(5)})\cdot x_{(0)}}\otimes x_{(2)}h_{(7)}S^{-1}(h_{(1)})u^{-1}(h_{(2)})\\
&\quad R(h_{(8)}\otimes x_{(3)})R(x_{(1)}\otimes S(h_{(4)}))R(h_{(7)}\otimes S(h_{(2)}))\\
&= \sum \overline{S(h_{(5)})\cdot x_{(0)}}\otimes h_{(8)}x_{(3)}S^{-1}(h_{(1)})u^{-1}(h_{(2)})\\
&\quad R(h_{(7)}\otimes x_{(2)})R(x_{(1)}\otimes S(h_{(4)}))R(h_{(6)}\otimes S(h_{(3)}))\\
&= \sum h_{(2)}\rightharpoonup\overline{x_{(0)}}\otimes h_{(3)}x_{(1)}S^{-1}(h_{(1)}).
\end{aligned}
$$

So $\overline{X}$ is a YD H-module algebra. □

Now we are looking at the $\mathcal{H}_R$-bimodule structure of $\overline{X}$ steming from the new YD H-module structure of $\overline{X}$.

Lemma 3.9. The $\mathcal{H}_R$-bimodule structure on $\overline{X}$ is given by

$$
\begin{aligned}
h \multimap \overline{x} &= \sum\overline{S^{-1}(h_{(2)})\cdot(h_{(1)}\triangleright_2 x)} \equiv \overline{h\twoheadrightarrow x},\\
\overline{x} \multimapinv h &= \sum\overline{S(h_{(1)})\cdot(h_{(2)}\triangleright_1 x)} \equiv \overline{x\twoheadleftarrow h}.
\end{aligned}
\tag{20}
$$

Proof. We verify the right $\mathcal{H}_R$-action, and leave the check of the left $\mathcal{H}_R$-action to the reader.

Indeed, if $h \in \mathcal{H}_R$ and $\overline{x} \in \overline{X}$, we have

$$\begin{aligned}
\overline{x} \triangleleft\!- h &= \sum S(h_{(1)}) \triangleright_2 (h_{(2)} \rightharpoonup \overline{x}) \\
&= \sum \overline{h_{(4)} \triangleright_1 (S^{-1}(h_{(3)}) \cdot x)} u(h_{(1)}) R(S(h_{(5)}) \otimes S^{-1}(h_{(3)})) \text{ by using (19)} \\
&= \sum \overline{S^{-1}(h_{(4)}) \cdot (h_{(7)} \triangleright_1 x)} R^{-1}(h_{(6)} \otimes S^{-1}(h_{(5)})) \\
&\quad R(h_{(8)} \otimes S^{-1}(h_{(3)})) R(S(h_{(9)} \otimes S^{-1}(h_{(2)})) u(h_{(1)}) \\
&= \sum \overline{S^{-1}(h_{(2)}) \cdot (h_{(5)} \triangleright_1 x)} u^{-1}(h_{(3)}) u(h_{(1)}) \\
&= \sum \overline{S(h_{(3)}) \cdot (h_{(4)} \triangleright_1 x)} u^{-1}(h_{(2)}) u(h_{(1)}) \\
&= \sum \overline{S(h_{(1)}) \cdot (h_{(2)} \rightharpoonup x)}.
\end{aligned}$$

□

From (20), one immidiately obtain that the left and right $\mathcal{H}_R$-invariant subalgebra are trivial. Moreover, the $\mathcal{H}_R$-bimodule structure (20) on $\overline{X}$ can be translated into the $\mathcal{H}_R^*$-bicomodule structure on $\overline{X}$ in the following way:

$$\begin{array}{l}
\sum \overline{x}^{(0)} \otimes \overline{x}^{(1)} = \sum \overline{x_{(0)[0]}} \otimes \Theta_l(S(x_{(1)})) S^{-1}(x_{(0)[1]}), \\
\sum \overline{x}^{(-1)} \otimes \overline{x}^{(0)} = \sum S(x_{(0)[1]}) \Theta_r(x_{(1)}) \otimes \overline{x_{(0)[0]}}.
\end{array} \tag{21}$$

Now Let us recall that a YD H-module algebra A is said to be *quantum commutative* if

$$ab = \sum b_{(0)}(b_{(1)} \cdot a) \tag{22}$$

for any $a, b \in A$. That is, A is a commutative algebra in $\mathbf{Q}^H$.

Lemma 3.10. Let A be quantum commutative YD H-module algebra. Let $\overline{A}$ be the opposite algebra in $\mathbf{M}_R^H$. Then the multiplication of $\overline{A}$ reads as follows:

$$\begin{aligned}
\overline{a} \circ \overline{b} &= \sum \overline{a_{(0)}(S(a_{(1)}) \twoheadrightarrow b)} \\
&= \sum \overline{(a \twoheadleftarrow b_{(1)}) b_{(0)}}
\end{aligned} \tag{23}$$

for $a, b \in A$, where the actions $\twoheadleftarrow$ and $\twoheadrightarrow$ are defined in (20).

Proof. Let $\overline{a}$ and $\overline{b}$ be two elements in $\overline{A}$. By definition, we have

$$\overline{a} \circ \overline{b} = \sum \overline{b_{(0)}(b_{(1)} \triangleright_1 a)}.$$

Since A is also quantum commutative in $\mathbf{Q}^H$, we have

$$\begin{aligned}
\overline{a} \circ \overline{b} &= \sum \overline{b_{(0)}(b_{(1)} \triangleright_1 a)} \\
&= \sum \overline{S(b_{(1)}) \cdot (b_{(2)} \triangleright_1 a) b_{(0)}} \\
&= \overline{(a \twoheadleftarrow b_{(1)}) b_{(0)}}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\overline{a} \circ \overline{b} &= \sum \overline{(S(a_{(1)}) \triangleright_2 b) a_{(0)}} \\
&= \sum \overline{a_{(0)}(a_{(1)} \cdot (S(a_{(2)}) \triangleright_2 b))} \\
&= \overline{a_{(0)}(S(a_{(1)}) \twoheadrightarrow b)}
\end{aligned}$$

for any $\overline{a}, \overline{b} \in \overline{A}$.

Proposition 3.11. Let X be an object in $\mathcal{E}(\mathcal{H}_R)$ such that X is quantum commutative in $\mathbf{Q}^H$. Then the opposite algebra $\overline{X}$ in $\mathbf{M}_R^H$ is an object in $\mathcal{E}(\mathcal{H}_R)$.

Proof. We have to show that $\overline{X}$ is an $\mathcal{H}_R^*$-bigalois objects. Let f be the following linear map:

$$f:\ X \otimes X \longrightarrow X \otimes X$$

given by $f(x \otimes y) = \sum x_{(0)} \otimes S(x_{(1)}) \twoheadrightarrow y$. It is easy to see that f is a linear isomorphism. Let us compute the canonical Galois linear map $\beta^l_{\overline{X}}$ from $\overline{X} \otimes \overline{X}$ to $\mathcal{H}_R^* \otimes \overline{X}$. By applying the formulas (21) and (23), we obtain

$$\begin{aligned}
\beta^l_{\overline{X}}(\overline{y} \otimes \overline{x}) &= \sum \overline{x}^{(-1)} \otimes \overline{y} \circ \overline{x}^{(0)} \\
&= \sum S(x_{(0)[1]})\Theta_r(x_{(1)}) \otimes \overline{y} \circ \overline{x_{(0)[0]}} \\
&= \sum S(x_{(0)[1]})\Theta_r(x_{(1)}) \otimes \overline{y} \circ \overline{x_{(0)[0]}} \\
&= \sum S(x_{(0)[1]})\Theta_r(x_{(1)}) \otimes \overline{y_{(0)}(y_{(1)} \twoheadrightarrow x_{(0)[0]})} \\
&= \sum S(x_{(0)[1]})\Theta_r(x_{(1)}) \otimes \overline{y_{(0)}(y_{(1)} \twoheadrightarrow x)_{(0)[0]}}
\end{aligned}$$

for $\overline{x}, \overline{y} \in \overline{X}$, where the last equation holds because $\overline{X}$ is an $\mathcal{H}_R^*$-bicomodule. Since X is right $\mathcal{H}_R^*$-Galois, we have the Galois isomorphism β^r_X:

$$\beta^r_X(x \otimes y) = \sum x_{[0](0)}y \otimes x_{[1]}S^{-1}(\Theta_r(x_{[0](1)}))$$

for any $x, y \in X$. β^r_X in turn induces an isomorphism

$$\gamma^r : X \otimes X \longrightarrow X \otimes \mathcal{H}_R^*$$

given by $\gamma^r(y \otimes x) = \sum yx_{(0)[0]} \otimes S(x_{(0)[1]})\Theta_r(x_{(1)})$. It follows that $\beta^l_{\overline{X}}$ is the following composite linear isomorphism if we identify the vector space X with $\overline{X}$:

$$X \otimes X \xrightarrow{\ f\ } X \otimes X \xrightarrow{\ \gamma^r\ } X \otimes \mathcal{H}_R^* \xrightarrow{\ \tau\ } \mathcal{H}_R^* \otimes X.$$

where τ is the flip map.

Similarly, one may verify that $\beta^r_{\overline{X}}$ is the composite isomorphism:

$$X \otimes X \xrightarrow{\ g\ } X \otimes X \xrightarrow{\ \gamma^l\ } \mathcal{H}_R^* \otimes X \xrightarrow{\ \tau\ } X \otimes \mathcal{H}_R^*,$$

where g and γ^l are given by

$$\begin{aligned}
&g(x \otimes y) = \sum x \twoheadleftarrow y_{(1)} \otimes y_{(0)}, \\
&\gamma^l(x \otimes y) = \sum S^{-1}(x_{(0)[1]}x_{[1]}) \otimes x_{(0)[0]}y
\end{aligned}$$

for $x, y \in X$. So $\overline{X}$ is indeed an $\mathcal{H}_R^*$-bigalois object, and hence an object in $\mathcal{E}(\mathcal{H}_R)$. $\square$

Now we are able to state our main theorem in this section. Denote by $\mathrm{Gal}(\mathcal{H}_R)$ the subset of $E(\mathcal{H}_R)$ consisting of the isomorphism classes of objects in $\mathcal{E}(\mathcal{H}_R)$ such that the objects are quantum commutative in $\mathbf{Q}^H$.

Theorem 3.12. The set $\mathrm{Gal}(\mathcal{H}_R)$ is a group.

Proof. First of all we show that $\mathrm{Gal}(\mathcal{H}_R)$ is a sub-semigroup of $E(\mathcal{H}_R)$. It is obvious that I is a quantum commutative algebra and $[I] \in \mathrm{Gal}(\mathcal{H}_R)$. Suppose that $[X]$ and $[Y]$ are two elements of $\mathrm{Gal}(\mathcal{H}_R)$. We have to verify that $X \wedge Y$ is quantum commutative. For simplicity, we will write $x\#y$ for an element $\sum x_i\#y_i$ of $X \wedge Y$. Given $x\#y, a\#b \in X \wedge Y$, we compute $\sum(x_{(0)}\#y_{(0)})(y_{(1)}x_{(1)} \rightharpoonup (a\#b))$.

$$\begin{aligned}
&\sum(x_{(0)}\#y_{(0)})(y_{(1)}x_{(1)} \rightharpoonup (a\#b) \\
= &\sum(x_{(0)}\#y_{(0)})(y_{(1)}x_{(1)} \triangleright_2 a\#y_{(2)}x_{(2)} \cdot b) \\
= &\sum x_{(0)}(S(y_{(1)})y_{(2)}x_{(1)} \triangleright_2 a)\#y_{(0)}(y_{(3)}x_{(2)} \cdot b) \\
= &\sum x_{(0)}(x_{(1)} \triangleright_2 a)\#y_{(0)}(y_{(1)} \cdot (x_{(2)} \cdot b)) \\
= &\sum x_{(0)}(x_{(1)} \cdot a)\#y_{(0)}(y_{(1)} \cdot (x_{(2)} \triangleright_1 b)) \\
= &\sum ax_{(0)}\#(x_{(1)} \triangleright_1 b)y \\
= &(a\#b)(x\#y).
\end{aligned}$$

This means that $X \wedge Y$ is quantum commutative as well.

Second we show that $\overline{X}$ is quantum commutative if X is quantum commutative. Indeed, given $\overline{x}, \overline{y} \in \overline{X}$, we have

$$\begin{aligned}
\sum \overline{y_{(0)}} \circ (y_{(1)} \rightharpoonup \overline{x}) &= \sum \overline{y_{(0)}} \circ \overline{S(y_{(4)}) \cdot x_{(0)}} R(y_{(6)} \otimes x_{(2)}) R(x_{(1)} \otimes S(y_{(3)})) \\
&\quad R(y_{(5)} \otimes S(y_{(2)})) u^{-1}(y_{(1)}) \\
&= \sum \overline{S(y_{(6)}) \cdot x_{(0)}) y_{(0)}} R(S(y_{(5)}) x_{(1)} y_{(7)} \otimes y_{(1)}) \\
&\quad R(y_{(9)} \otimes x_{(3)}) R(x_{(2)} \otimes S(y_{(4)})) R(y_{(8)} \otimes S(y_{(3)})) u^{-1}(y_{(2)}) \\
&= \sum \overline{S(y_{(5)}) \cdot x_{(0)}) y_{(0)}} R(S(y_{(4)}) x_{(1)} y_{(7)} \otimes y_{(1)}) R(y_{(8)} \otimes x_{(3)}) \\
&\quad R(x_{(2)} y_{(7)} \otimes S(y_{(3)})) u^{-1}(y_{(2)}) \\
&= \sum \overline{S(y_{(5)}) \cdot x_{(0)}) y_{(0)}} R(S(y_{(4)}) x_{(1)} y_{(7)\otimes y_{(1)}} \otimes y_{(1)}) R(y_{(8)} \otimes x_{(3)}) \\
&\quad R(S(x_{(2)} y_{(7)}) \otimes y_{(2)})) u^{-1}(y_{(3)}) \\
&= \sum \overline{S(y_{(4)}) \cdot x_{(0)}) y_{(0)}} R(S(y_{(3)}) \otimes y_{(1)}) R(y_{(5)} \otimes x_{(1)}) u^{-1}(y_{(2)}) \\
&= \sum \overline{S(y_{(4)}) \cdot x_{(0)}) y_{(0)}} u^{-1}(y_{(3)}) R(S^{-1}(y_{(2)}) \otimes y_{(1)}) R(y_{(5)} \otimes x_{(1)}) \\
&= \sum \overline{S(y_{(1)}) \cdot x_{(0)}) y_{(0)}} R(y_{(2)} \otimes x_{(1)}) \\
&= \sum \overline{y_{(0)} x_{(0)}} R(y_{(1)} \otimes x_{(1)}) \\
&= \overline{x} \circ \overline{y}.
\end{aligned}$$

Thus we have proved that $[\overline{X}]$ is an element of $\mathrm{Gal}(\mathcal{H}_R)$ if $[X] \in \mathrm{Gal}(\mathcal{H}_R)$. Finally we show that $[\overline{X}]$ is the inverse of $[X]$ in $\mathrm{Gal}(\mathcal{H}_R)$.

In order to show that $X \wedge \overline{X} \cong I$ in $\mathcal{E}(\mathcal{H}_R)$, it is suffcient to construct a non-zero YD H-module algebra morphism from I to $X \wedge \overline{X}$. Since X is a right $\mathcal{H}_R^*$-Galois object we have the isomorphism:

$$\beta^r : X \otimes X \longrightarrow X \otimes \mathcal{H}_R^*.$$

Write $\sum U_i^p \otimes V_i^p$ for the inverse image ${\beta^r}^{-1}(1 \otimes p)$ of an element $p \in \mathcal{H}_R^*$. We claim that $\sum U_i^p \otimes \overline{V_i^p}$ is in $X \wedge \overline{X}$ for any $p \in \mathcal{H}_R^*$. Indeed, if $p \in \mathcal{H}_R^*$, we have the formulas

$$\begin{aligned}
\beta^r(\sum U_i^p \otimes V_i^p) &= \sum U_i^{p(0)} V_i^p \otimes U_i^{p(1)} = 1 \otimes p, \\
\gamma^r(\sum U_i^p \otimes \overline{V_i^p}) &= \sum U_i^p \overline{V_i^p}^{(0)} \otimes \overline{V_i^p}^{(-1)} = 1 \otimes p
\end{aligned}$$

where $\sum U_i^p \overline{V_i^p}^{(0)} \otimes \overline{V_i^p}^{(-1)}$ should be understood as

$$\gamma^r(\sum U_i^p \otimes \overline{V_i^p}) = \sum U_i^p V^p_{i(0)[0]} \otimes S(V^p_{i(0)[1]})\Theta_r(V^p_{i(1)}).$$

Write $\sum X_j^p \otimes Y_j^p$ for the element $\beta^{l^{-1}}(p\otimes 1)$ whenever $p \in \mathcal{H}_R^*$. Then we have

$$\sum X_j^{\overline{a}^{(1)}} \otimes Y_j^{\overline{a}^{(1)}} \overline{a}^{(0)} = \overline{a} \otimes 1 \tag{24}$$

for any $\overline{a} \in \overline{X}$. Applying formula (24), we obatin

$$\begin{aligned}\sum(\iota\otimes\gamma^r)(1\otimes U_i^p \otimes \overline{V_i^p}) &= 1\otimes 1\otimes p\\ &= \sum U_i^p \overline{V_i^p}^{(0)} \otimes 1 \otimes \overline{V_i^p}^{-1}\\ &= \sum U_i^p X_j^q \otimes Y_j^q \overline{V_i^p}^{(0)} \otimes \overline{V_i^p}^{-1}\\ &= \sum(\iota\otimes\gamma^r)(U_i^p X_j^q \otimes Y_j^q \otimes \overline{V_i^p}^{(0)})\end{aligned}$$

where $q = \overline{V_i^p}^{(1)}$. Since $\iota\otimes\gamma^r$ is an isomorphism, we obtain

$$\sum 1\otimes U_i^p \otimes \overline{V_i^p} = \sum U_i^p X_j^q \otimes Y_j^q \otimes \overline{V_i^p}^{(0)} \tag{25}$$

where again $q = \overline{V_i^p}^{(1)}$. Now let $\beta^l \otimes \iota$ act on (25), we get

$$\sum U_i^{p(-1)} \otimes U_i^{p(0)} \otimes \overline{V_i^p} = \sum \overline{V_i^p}^{(1)} \otimes U_i^p \otimes \overline{V_i^p}^{(0)}.$$

It follows that

$$\sum U_i^p \triangleleft\!- h\# V_i^p = \sum U_i^p \# h -\!\triangleright V_i^p,$$

for any $h \in \mathcal{H}_R$.

Next we show that the well-defined map

$$\Psi : I \longrightarrow X \wedge \overline{X},\ \Psi(p) = \beta^{r^{-1}}(1\otimes p)$$

is an algebra map. In order to simplify the notations, we write $a\#\overline{b}$ and $c\#\overline{d}$ for $\Psi(q)$ and $\Psi(p)$ respectively if q and p are two elements of I. Since

$$\sum a^{(0)} b \otimes b^{(1)} = \sum a_{[0](0)} b \otimes a_{[1]} S^{-1}(\Theta_r(a_{[0](1)}) = 1 \otimes p$$

is equivalent to

$$\sum a_{[0]} b_{(0)} \otimes a_{[1]} \Theta_r(b_{(1)}) = 1\otimes p,$$

It is suficient to show that

$$\sum x_{[0]} y_{(0)} \otimes x_{[1]} \Theta_r(y_{(1)}) = 1 \otimes qp,$$

where

$$x\#\overline{y} = (a\#\overline{b})(c\#\overline{d}) = \sum ac_{(0)} \# \overline{d_{(0)} b_{(0)}} R(d_{(1)} c_{(1)} \otimes b_{(1)}).$$

Indeed, we have

$$\begin{aligned}&\sum x_{[0]} y_{(0)} \otimes x_{[1]} \Theta_r(y_{(1)})\\ =\ &\sum a_{[0]} c_{(0)[0]} d_{(0)} b_{(0)} \otimes a_{[1]} c_{(0)[1]} \Theta_r(d_{(1)}) \Theta_r(b_{(1)}) R(d_{(2)} c_{(1)} \otimes b_{(2)}).\end{aligned}$$

Applying the equations:

$$\begin{aligned}
&\sum c_{(0)[0]}d_{(0)} \otimes c_{(0)[1]}\Theta_r(d_{(1)}) \otimes S^{-1}(\Theta_l(d_{(2)}c_{(1)}))\\
&= \sum c_{[0](0)}d_{(0)} \otimes R^{-1}(c_{[1]}\Theta_r(d_{(2)}) \otimes S^{-1}(\Theta_l(c_{[0](1)}d_{(1)}))R\\
&= 1 \otimes (\iota \otimes S)(R^{-1}(p \otimes 1)R)\\
&= \sum 1 \otimes S(R^1)pr^1 \otimes S(R^2r^2)
\end{aligned}$$

where $R = r$, we obtain

$$\begin{aligned}
&\sum x_{[0]}y_{(0)} \otimes x_{[1]}\Theta_r(y_{(1)})\\
= &\sum a_{[0]}b_{(0)} \otimes a_{[1]}S(R^1)pr^1\Theta_r(b_{(1)}) < S(R^2r^2), b_{(2)} >\\
= &\sum a_{[0]}b_{(0)} \otimes a_{[1]}(S(R^1) \cdot p)\Theta_r(b_{(1)}) < S(R^2, b_{(2)} >\\
= &\sum a_{[0]}b_{(0)} \otimes a_{[1]}p_{(0)}\Theta_r(b_{(1)})R(S(p_{(1)}) \otimes S(b_{(2)}))\\
= &\sum a_{[0]}b_{(0)} \otimes a_{[1]}p_{(0)}(p_{(1)} \cdot \Theta_r(b_{(1)}))\\
= &\sum a_{[0]}b_{(0)} \otimes a_{[1]}\Theta_r(b_{(1)})p\\
= &\ 1 \otimes qp.
\end{aligned}$$

So Ψ is inded an algebra map.

Finally, we show that Ψ is a YD H-module map so that it is an $\mathcal{H}_R$-bimodule map (or a $\mathcal{H}_R^*$-bicomodule map) as well. Given $p \in I$ and $h \in \mathcal{H}_R$, we have

$$\sum U^p_{i[0]}V^p_{i(0)} \otimes U^p_{i[1]}\Theta_r(V^p_{i(1)}) = 1 \otimes p.$$

It implies that

$$\begin{aligned}
h \cdot \Psi(p) &= \sum h_{(1)} \cdot U_i^p \# h_{(2)} \triangleright_1 \overline{V_i^p}\\
&= \sum U^p_{i[0]} \# \overline{V^p_{i(0)}} < h, U^p_{i[1]}\Theta_r(V^p_{i(1)}) >\\
&= \sum U_i^{p_{(1)}} \# \overline{V_i^{p_{(1)}}} < h, p_{(2)}) >\\
&= \Psi(h \cdot p).
\end{aligned}$$

To show that Ψ is H^{op}-colinear, we verify that Ψ is left H^*-linear. Indeed, if $p \in H^*$ and $q \in I$, we have

$$\begin{aligned}
\Psi(p \cdot q) &= \sum \Psi(p_{(2)}qS^{-1}(p_{(1)}))\\
&= \sum \Psi(p_{(2)})\Psi(q)\Psi(S^{-1}(p_{(1)}))\\
&= \sum \Psi(q)_{(0)}(\Psi(q)_{(1)} \cdot \Psi(p_{(2)}))\Psi(S^{-1}(p_{(1)}))\\
&= \sum \Psi(q)_{(0)}\Psi(p_{(2)}))\Psi(S^{-1}(p_{(1)})) < \Psi(q)_{(1)}, p_{(3)} >\\
&= \sum \Psi(q)_{(0)} < \Psi(q)_{(1)}, p >,
\end{aligned}$$

where we used the facts that $X \wedge \overline{X}$ is quantum commutative, that Ψ is an algebra map and that Ψ is H-linear. So Ψ is indeed a YD H-module algebra map, and $I \cong X \wedge \overline{X}$ in $\mathcal{E}(\mathcal{H}_R)$. This proved that $[\overline{X}]$ is a right inverse of $[X]$ in $Gal(\mathcal{H}_R)$. Since any element of $\mathrm{Gal}(\mathcal{H}_R)$ has a right inverse element in $\mathrm{Gal}(\mathcal{H}_R)$, $\mathrm{Gal}(\mathcal{H}_R)$ is a group. $\square$

4 The exact sequence

In this section, we investigate the R-Azumaya algebras which are Galois extensions of the coinvariants, and establish a group morphism from $\mathrm{BC}(k, H, R)$ to the group $\mathrm{Gal}(\mathcal{H}_R)$ constructed in the previous section. The main result will be the exact sequence (28). In the sequel, we will write

$$M_0 = \{m \in M \mid \sum m_{(0)} \otimes m_{(1)} = m \otimes 1\}$$

for the coinvariant k-submodule of a right H-comodule M, in order to make a difference between $\mathcal{H}_R^*$-coinvariants and H-coinvariants. We start with a special elementary R-Azumaya algebra.

Lemma 4.1. Let $M = H^{op}$ be the right H^{op}-comodule, and let A be the induced R-Azumaya algebra $\mathrm{End}(M)$. Then $A \cong H^{*op} \# H^{op}$, where the left H^{op}-action on H^{*op} is given by $h \cdot p = \sum p_{(1)} < p_{(2)}, S^{-1}(h) >= S^{-1}(h) \rightharpoonup p$, whenever $h \in H^{op}$ and $p \in H^{*op}$.

Proof. Let π be the representation map of the regular left H^{op}-module H^{op}:

$$\pi : \ H^{op} \longrightarrow \mathrm{End}(M).$$

We claim that π is a right H^{op}-colinear algebra map. Indeed, by definition, the right H^{op}-comodule structure of $\pi(h)$ for $h \in H^{op}$ is given by

$$\begin{aligned}
\rho(\pi(h))(x \otimes 1) &= \sum (\pi(h)(x_{(0)}))_{(0)} \otimes S^{-1}(x_{(1)})(\pi(h)(x_{(0)}))_{(1)} \\
&= \sum (x_{(1)}h)_{(1)} \otimes S^{-1}(x_{(2)})(x_{(1)}h)_{(2)} \\
&= \sum x_{(1)}h_{(1)} \otimes S^{-1}(x_{(3)})x_{(2)}h_{(2)} \\
&= \sum \pi(h_{(1)})(x) \otimes h_{(2)}
\end{aligned}$$

for any $x \in M$. It follows that $\rho(\pi(h)) = \sum \pi(h_{(1)}) \otimes h_{(2)}$. So π is a right H^{op}-colinear algebra map. This fact implies that the right H^{op}-comodule algebra A is a smash product $A_0 \# H^{op}$, where A_0 is the coinvariant subalgebra of A.

Now we show that A_0 is isomorphic to H^{*op}. It is obvious that

$$A_0 = \mathrm{End}^H(M)$$

the subalgebra of all H^{op}-colinear endomorphisms of M. Since M is H^{op} by our choice, we know that the linear map

$$\lambda : \ H^{*op} \longrightarrow \mathrm{End}^H(H^{op}), \ \lambda(p)(x) = \sum p(x_{(1)})x_{(2)}$$

for $p \in H^{*op}$ and $x \in H^{op}$, is an algebra isomorphism.

Finally, if $p \in H^{*op}$, $h \in H^{op}$ and $x \in M$, then

$$\begin{aligned}
\pi(h)(\lambda(p)(x)) &= \sum x_{(2)}h < p, x_{(2)} > \\
&= \sum (S^{-1}(h_{(1)}) \rightharpoonup p)(\pi(h_{(2)})(x)).
\end{aligned}$$

It follows that the action of H^{op} on H^{*op} is

$$h \cdot p = S^{-1}(h) \rightharpoonup p = \sum p_{(1)} < p_{(2)}, S^{-1}(h) >$$

for $h \in H^{op}$ and $p \in H^{*op}$. □

Corollary 4.2. Any element of $\mathrm{BC}(k,H,R)$ can be represented by an R-Azumaya algebra which is a smash product.

Proof. Let $[A]$ be an element of $\mathrm{BC}(k,H,R)$. Since $\mathrm{End}(H^{op})$ represents the unit of $\mathrm{BC}(k,H,R)$, we have $[A\#\mathrm{End}(H^{op})]=[A]$. Now the composite algebra map

$$H^{op}\xrightarrow{\lambda}\mathrm{End}(H^{op})\hookrightarrow A\#\mathrm{End}(H^{op})$$

is still H^{op}-colinear. It follows that $A\#\mathrm{End}(H^{op})$ is a smash product algebra $B\#H^{op}$ where $B=(A\#\mathrm{End}(H^{op}))_0$. □

Since any smash product algebra is a Galois extension of its coinvariants, we have that *any element of* $\mathrm{BC}(k,H,R)$ *can be represented by an* R*-Azumaya algebra which is a* H^{op}*-Galois extension of its coinvariants.*

Lemma 4.3. Let A be an R-Azumaya algebra. If A is an H^{op}-Galois extension of A_0, then $\overline{A}$ is a H^{op}-Galois extension of A_0^{op}.

Proof. Since A/A_0 is H^{op}-Galois, we have the canonical isomorphism:

$$\beta_A':\ A\otimes_{A_0}A\longrightarrow A\otimes H^{op},\ a\otimes b\mapsto\sum a_{(0)}b\otimes b_{(1)}.$$

Since the flip map τ is an linear isomorphism from $A\otimes_{A_0}A$ to $\overline{A}\otimes_{A_0^{op}}\overline{A}$, β_A' gives an isomorphism

$$\eta:\ \overline{A}\otimes_{A_0^{op}}\overline{A}\longrightarrow\overline{A}\otimes H^{op},\ \eta(\overline{a}\otimes\overline{b})=\sum\overline{a_{(0)}b}\otimes a_{(1)}.$$

Define a linear map:

$$\xi:\ \overline{A}\otimes H^{op}\longrightarrow A\otimes H^{op}$$

sending an element $a\otimes h$ to $\sum\overline{a_{(0)}}\otimes h_{(3)}R(h_{(2)}\otimes a_{(1)}S(h_{(1)}))$. We show that ξ is a linear isomorphism. As remarked in the previous section, we may view R as an element $\sum R^1\otimes R^2$ in $H^*\otimes H^*$ which is a QT structure of H^*. Then the element $u=\sum R^2S^{-1}(R^1)$ is the Casimir element of H^* which is invertible. Thus we may rewrite the linear map ξ as the following composite map:

$$A\otimes H^{op}\xrightarrow{\iota\otimes\widetilde{u}}A\otimes H^{op}\xrightarrow{\sigma}A\otimes H^{op}$$

where $\widetilde{u}$ is defined by $\widetilde{u}(h)=h\leftharpoonup u=\sum h_{(2)}u(h_{(1)})$, and σ is defined by

$$\sigma(a\otimes h)=\sum h_{(1)}\cdot a\otimes h_{(2)}=\sum a_{(0)}\otimes h_{(2)}R(h_{(1)}\otimes a_{(1)})$$

for all $a\in A$ and $h\in H^{op}$. Since $\widetilde{u}$ and σ are linear isomorphisms, we have that ξ is an isomorphism. Now we show that the canonical map $\beta_{\overline{A}}$ is a composite map of isomorphisms. We have

$$\begin{aligned}\xi\circ\beta_A'\circ\tau(\overline{a}\otimes\overline{b}) &= \sum\xi(\sum\overline{b_{(0)}a}\otimes b_{(1)})\\ &= \sum\overline{b_{(0)}a_{(0)}}\otimes b_{(4)}R(b_{(3)}\otimes a_{(1)}b_{(1)}S(b_{(2)}))\\ &= \sum\overline{b_{(0)}a_{(0)}}\otimes b_{(2)}R(b_{(1)}\otimes a_{(1)})\\ &= \sum\overline{a}\overline{b_{(0)}}\otimes b_{(1)}\\ &= \beta_{\overline{A}}(\overline{a}\otimes\overline{b})\end{aligned}$$

for any $\overline{a}\in\overline{A}$ and $\overline{b}\in\overline{B}$. So $\beta_{\overline{A}}$ is an isomorphism, and $\overline{A}/A_0^{op}$ is an H^{op}-Galois extension.□

In the sequel, an R-Azumaya algebra A is said to be *Galois* if it is a right H^{op}-Galois extension of its coinvariant subalgebra A_0. Let A be a Galois R-Azumaya algebra. Denote by $\pi(A)$ the centralizer subalgebra $C_A(A_0)$ of A_0 in A. A_0 is clearly an R-subalgebra of A. The Miyashita-Ulbrich action of H on $\pi(A)$ is given by

$$h \rightharpoonup a = \sum X_i^h a Y_i^h, \tag{26}$$

where $\sum X_i^h \otimes Y_i^h = \beta^{-1}(1 \otimes h)$, for $h \in H$. It is well-known (e.g. see [29, 7]) that $\pi(A)$ together with the action (26) is a new YD H-module algebra. Moreover, $\pi(A)$ is quantum commutative in the sense of (22). In fact, in (22) the element b can be any element in A. By Corollary 2.8, $\pi(A)$ is an $\mathcal{H}_R$-bimodule algebra, or A is an $\mathcal{H}_R^*$-bicomodule algebra.

Lemma 4.4. Let A be a Galois R-Azumaya algebra. Then $\pi(A)/k$ is an $\mathcal{H}_R^*$-biextension.

Proof. Given $a \in \pi(A)_\diamond$, then by Lemma 2.5, we have

$$h \rightharpoonup a = h \triangleright_1 a$$

for any $h \in H$. Then for any element $b \in A$, we have

$$\begin{aligned} ab &= \sum b_{(0)}(b_{(1)} \rightharpoonup a) \\ &= \sum b_{(0)}(b_{(1)} \triangleright_1 a). \end{aligned}$$

This means that a is an element in the left H-center of A which is trivial. So $\pi(A)_\diamond = k$. Similarly, if a is in ${}_\diamond\pi(A)$, then we have

$$\begin{aligned} ab &= \sum b_{(0)}(b_{(1)} \rightharpoonup a) \\ &= \sum b_{(0)}(b_{(1)} \triangleright_2 a) \\ &= \sum b_{(0)} a_{(0)} R^{-1}(a_{(1)} \otimes b_{(1)}), \end{aligned}$$

for any $b \in A$. This implies that

$$\sum a_{(0)} b_{(0)} R(a_{(1)} \otimes b_{(1)}) = ba$$

for any $b \in A$. So a is in the right H-center of A which is trivial as well. It follows that $\pi(A)$ is an $\mathcal{H}_R^*$-biextension. $\square$

Recall from [29] that when a Galois H^{op}-comodule algebra A is an Azumaya algebra, the centralizer $\pi(A)$ is a right H^*-Galois extension of k with respect to the Miyashita-Ulbrich action (26). This is no loger the case when A is an R-Azumaya algebra. However, we will see that A would be an $\mathcal{H}_R^*$-Galois object, instead of an H^*-Galois object.

Proposition 4.5. Let A be a Galois R-Azumaya algebra, and $\pi(A)$ be as above. Then $\pi(A)$ is an object in $\mathrm{Gal}(\mathcal{H}_R)$.

Proof. Let F be the canonical H-linear algebra isomorphism:

$$A\#\overline{A} \longrightarrow \mathrm{End}(A).$$

It is easy to see that F induces an algebra isomorphism:

$$\pi(A)\#\overline{\pi(A)} \longrightarrow \mathrm{End}_{A_0-A_0}(A)$$

where the latter is the subalgebra of all A_0-biendomorphisms of A. Since A/A_0 is H^{op}-Galois, we have a linear isomorphism:

$$\delta:\ \mathrm{Hom}(H,\pi(A)) \longrightarrow \mathrm{End}_{A_0-A_0}(A)$$

given by $\delta(f)(a)=\sum a_{(0)}f(a_{(1)})$ (see [12, 3.2]). Define a linear map α as follows:

$$\alpha:\ \pi(A)\otimes\pi(A) \longrightarrow \pi(A)\otimes H^*$$

by sending an element $a\otimes b$ to $\sum a_{[0]}b_{(0)}\otimes a_{[1]}\Theta_r(b_{(1)})$. One may take a while to check that α and δ fit the following commutative diagram:

$$\begin{tikzcd}
\pi(A)\otimes\pi(A) \arrow[r, "F"] \arrow[d, "\alpha"'] & \mathrm{End}_{A_0-A_0}(A) \\
\pi(A)\otimes H^* \arrow[r, "\cong"] & \mathrm{Hom}(H,\pi(A)). \arrow[u, "\delta"']
\end{tikzcd}$$

Note that here we view F as a linear isomorphism from $\pi(A)\otimes\pi(A)$ to $\mathrm{End}_{A_0-A_0}(A)$. It follows from the above commutative diagram that α is a linear isomorphism. It is evident that the canonical Galois linear map β^r is now the composite isomorphism:

$$\pi(A)\otimes\pi(A)\xrightarrow{\alpha}\pi(A)\otimes H^*\xrightarrow{\eta}\pi(A)\otimes H^*$$

where η is given by $\eta(a\otimes p)=\sum a_{(0)}\otimes pS^{-1}(\Theta_r(a_{(1)}))$ whenever $a\in\pi(A)$ and $p\in H^*$. So we obtain that $\pi(A)$ is a right $\mathcal{H}_R^*$-Galois object.

Similarly, one has the commutative diagram

$$\begin{tikzcd}
\pi(A)\otimes\pi(A) \arrow[r, "G"] \arrow[d, "\alpha'"'] & \mathrm{End}_{A_0-A_0}(A)^{op} \\
\pi(A)\otimes H^* \arrow[r, "\sim"] & \mathrm{Hom}(H,\pi(A)) \arrow[u, "\delta"']
\end{tikzcd}$$

where α' is given by

$$\alpha'(a\otimes b)=\sum a_{(0)[0]}b\otimes a_{(0)[1]}\Theta_l(a_{(1)})$$

for $a,b\in\pi(A)$. Let ζ and η' be the linear automorphisms of $\pi(A)\otimes H^*$ given by

$$\zeta(a\otimes p)=\sum a_{[0]}\otimes S(a_{[1]})p$$
$$\eta'(a\otimes p)=\sum a_{(0)}\otimes pS(\Theta_l(a_{(1)}))$$

for any $a\in A$ and $p\in H^*$. We have

$$\eta'\zeta\alpha'(a\otimes b)=\sum ab_{[0](0)}\otimes S(\Theta_l(b_{[0](1)})b_{[1]})$$

for any $a,b\in\pi(A)$. It follows that the Galois linear map β^l is the composite isomorphism $(S^{-1}\otimes\iota)\tau\eta'\zeta\alpha'$, where τ is the flip map. So $\pi(A)$ is a left $\mathcal{H}_R^*$-Galois object. This completes the proof. □

Now we are ready to show that π induces a group morphism from the Brauer group $\mathrm{BC}(k,H,R)$ to the group $\mathrm{Gal}(\mathcal{H}_R)$.

Proposition 4.6. Let A and B be two Galois R-Azumaya algebras. Then we have $\pi(A\#B) = \pi(A)\wedge\pi(B)$.

Proof. It is obvious that $\pi(A\#B)\subseteq\pi(A)\#\pi(B)$ because $A_0\otimes B_0 = A_0\#B_0\subseteq(A\#B)_0$. For an element $h\in H^{op}$, we let

$$\begin{aligned}\sum\beta_A^{-1}(1\otimes h) = X_j^h\otimes Y_j^h\in A\otimes_{A_0}A\\ \sum\beta_B^{-1}(1\otimes h) = U_i^h\otimes V_i^h\in B\otimes_{B_0}B\end{aligned}$$

Then we have

$$\sum(X_j^h\#1)\otimes(Y_j^h\#1) = \beta_{A\#B}^{-1}(1\otimes h) = \sum(1\#U_i^h)\otimes(1\#V_i^h).$$

This implies that the Miyashita-Ulbirich action of H on $\pi(A\#B)$ can be written in two folds:

$$\begin{aligned}h\rightharpoonup a\#b &= \sum(X_j^h\#1)(a\#b)(Y_j^h\#1)\\ &= \sum(1\#U_i^h)(a\#b)(1\#V_i^h)\end{aligned}$$

where $a\#b$ should be read as a sum of elements in $A\#B$. Precisely, we have for $a\#b\in\pi(A\#B)$,

$$\begin{aligned}h\rightharpoonup(a\#b) &= \sum(X_j^h\#1)(a\#b)(Y_j^h\#1)\\ &= \sum X_j^h aY_{j(0)}^h\#Y_{j(1)}^h\triangleright_1 b\\ &= \sum h_{(1)}\rightharpoonup a\#h_{(2)}\triangleright_1 b.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}h\rightharpoonup(a\#b) &= \sum(1\#U_i^h)(a\#b)(1\#V_i^h)\\ &= \sum a_{(0)}\#(a_{(1)}\triangleright_1 U_i^h)bV_i^h\\ &= \sum a_{(0)}\#U_{i(0)}^h bV_i^h R(a_{(1)}\otimes U_{i(1)}^h)\\ &= \sum a_{(0)}R(a_{(1)}\otimes S^{-1}(h_{(1)}))\#h_{(2)}\rightharpoonup b\\ &= \sum h_{(1)}\triangleright_2 a\#h_{(2)}\rightharpoonup b.\end{aligned}$$

This means that $a\#b$ is sitting in $\pi(A)\wedge\pi(B)$ by Lemma 3.3. It follows that $\pi(A\#B)\subseteq\pi(A)\wedge\pi(B)$.

Conversely, if $a\#b$ is an element of $\pi(A)\wedge\pi(B)$, we show that $a\#b\in\pi(A\#B)$. Indeed, given $x\#y\in(A\#B)_0$, we have

$$\sum x_{(0)}\#y_0\otimes y_{(1)}x_{(1)} = x\otimes y\otimes 1,$$

or

$$\sum x_{(0)}\#y\otimes x_{(1)} = \sum x\#y_{(0)}\otimes S(y_{(1)}).$$

These two formulas lead to the equations:

$$(a\#b)(u\#v) = \sum au_{(0)}\#(x_{(1)}\triangleright_1 b)v$$

$$
\begin{aligned}
&= \sum x_{(0)}(x_{(1)} \rightharpoonup a)\#(x_{(2)} \triangleright_1 b)v \\
&= \sum x_{(0)}(x_{(1)} \triangleright_2 a)\#(x_{(2)} \rightharpoonup b)y \\
&= \sum x_{(0)}(x_{(1)} \triangleright_2 a)\#y_{(0)}(y_{(1)}x_{(2)} \rightharpoonup b) \\
&= \sum x_{(0)}(x_{(1)} \triangleright_2 a)\#yb \\
&= \sum x(S(y_{(1)}) \triangleright_2 a)\#y_{(0)}b \\
&= \sum xa_{(0)}\#(a_{(1)} \triangleright_1 y)b \\
&= (x\#y)(a\#b),
\end{aligned}
$$

where we used the quantum commutativity (22). This implies that $a\#b \in \pi(A\#B)$. So $\pi(A) \wedge \pi(B) \subseteq \pi(A\#B)$, and hence they are equal. □

Lemma 4.7. Let M be a finite right H^{op}-comodule, and $A = \mathrm{End}(M)$ be the elementary R-Azumaya algebra. If A is a Galois R-Azumaya algebra, then $\pi(A) \cong I$.

Proof. Since M is a right H^{op}-comodule, we may view M as a left H^*-module. The representation map

$$\lambda : H^* \longrightarrow A,\ \lambda(p)(m) = p \cdot m = \sum m_{(0)}p(m_{(1)})$$

sends H^* into the subalgebra $\pi(A)$ because $A_0 = \mathrm{End}_{H^*}(A)$. Thus λ gives an algebra morphism from I (see Lemma 3.6) to $\pi(A)$. If we can prove that λ is a YD H-module map, then λ becomes an $\mathcal{H}_R$-bimodule map, and hence an isomorphism between the two Galois objects in $\mathrm{Gal}(\mathcal{H}_R)$. By definition (16), λ is right H^{op}-colinear. We show that λ is left H-linear as well.

To show that $\lambda(h \cdot p) = h \rightharpoonup \lambda(p)$ for $h \in H$ and $p \in I$, it is sufficient (or equivalent) to show that

$$\lambda(p)f = \sum f_{(0)}\lambda(f_{(1)} \cdot p)$$

for any $f \in A$. Given $m \in M$ and $f \in A$, we have

$$\sum f_{(0)}\lambda(f_{(1)} \cdot p)(m) = f_{(0)}(m_{(0)}) < p, m_{(1)}f_{(1)} > .$$

Since

$$\sum f_{(0)}(m) \otimes f_{(1)} = \sum f(m_{(0)})_{(0)} \otimes S(m_{(1)})f(m_{(0)})_{(1)},$$

we have

$$
\begin{aligned}
\sum f_{(0)}\lambda(f_{(1)} \cdot p)(m) &= \sum f_{(0)}(m_{(0)}) < p, m_{(1)}f_{(1)} > \\
&= \sum f(m)_{(0)} < p, f(m)_{(1)} > \\
&= p \cdot f(m) \\
&= \lambda(p)f(m).
\end{aligned}
$$

This proves that λ is a YD H-module algebra map, and hence an isomorphism because I and $\pi(A)$ are $\mathcal{H}_R^*$-bigalois objects. □

Lemma 4.8. π induces a group morphism $\widetilde{\pi}$ from $\mathrm{BC}(k, H, R)$ to $\mathrm{Gal}(\mathcal{H}_R)$, where $\widetilde{\pi}([A]) = [\pi(A)]$ and A is a Galois R-Azumaya algebra representing the class $[A]$ in $\mathrm{BC}(k, H, R)$.

Proof. First we show that $\widetilde{\pi}$ is well-defined. Suppose that A abd B are two Galois R-Azumaya algebras representing the same class in $\mathrm{BC}(k, H, R)$. Then there are two finite right H^{op}-modules M, N such that

$$A\#\mathrm{End}(M) \cong B\#\mathrm{End}(N).$$

Let H^{op} be the regular right H^{op}-comodule. By Lemma 4.1, $\mathrm{End}(H^{op})$ is a Galois R-Azumaya algebra. Since $\mathrm{End}(M)\#\mathrm{End}(H^{op}) \cong \mathrm{End}(M \otimes H^{op})$ is still a Galois R-Azumaya algebra, by Lemma 4.7 we have

$$\pi(\mathrm{End}(M)\#\mathrm{End}(H^{op})) \cong \pi(\mathrm{End}(M \otimes H^{op})) \cong I.$$

This implies that

$$\begin{aligned} \pi(A) &\cong \pi(A) \wedge I \\ &\cong \pi(A\#\mathrm{End}(M)\#\mathrm{End}(H^{op})) \\ &\cong \pi(B\#\mathrm{End}(N)\#\mathrm{End}(H^{op})) \\ &\cong \pi(B) \wedge I \\ &\cong \pi(B). \end{aligned}$$

So we obtain that $\widetilde{\pi}([A]) = \widetilde{\pi}([B])$, and $\widetilde{\pi}$ is well-defined. $\square$

In order to figure out the kernel of $\widetilde{\pi}$, we need two more preparations. Recall from [2] that an action of a Hopf algebra H on an algebra A is called an *inner action* if there is an invertible element u in the convolution algebra $\mathrm{Hom}(H, A)$ such that

$$h \cdot a = \sum u(h_{(1)}) a u^{-1}(h_{(2)})$$

for any $a \in A$ and $h \in H$. If in addition, u is an algebra map, then the action of H is called a *strongly inner action*.

Lemma 4.9. Let A be a Galois R-Azumaya algebra such that $\pi(A) \cong I$. Then the action of H^{*cop} (or the coaction of H^{op}) on A is strongly inner.

Proof. By assumption, there is a YD H-module algebra isomorphism

$$\psi : \ I \longrightarrow \pi(A).$$

Thus the action and the coaction of H on $\pi(A)$ are determined by the corresponding action and coactions of H on I through ψ. Namely, we have:

$$\begin{array}{l} h \rightharpoonup \psi(p) = \sum \psi(p_{(1)}) < p_{(2)}, h >, \\ h^* \cdot \psi(p) = \sum \psi(h^*_{(2)}) \psi(p) \psi(S^{-1}(h^*_{(1)}) \end{array} \tag{27}$$

where $p \in I, h^* \in H^{*op}$ and $h \in H$. In particular, the H^{op}-coaction on $\pi(A)$ is strongly inner. We show that this inner action extends to the inner action on A. Indeed, given $a \in A$ and $h^* \in H^{*op}$, we have

$$\begin{aligned} h^* \cdot a &= \sum a_{(0)} < h^*, a_{(1)} > \\ &= \sum \psi(h^*_{(3)} S^{-1}(h^*_{(2)})) a_{(0)} < h^*_{(1)}, a_{(1)} > \\ &= \sum \psi(h^*_{(3)}) a_{(0)} (a_{(1)} \rightharpoonup \psi(S^{-1}(h^*_{(2)}))) < h^*_{(1)}, a_{(2)} > \\ &= \sum \psi(h^*_{(f)}) a_{(0)} \psi(S^{-1}(h^*_{(3)})) < S^{-1}(h^*_{(2)}), a_{(1)} >< h^*_{(1)}, a_{(2)} > \\ &= \sum \psi(h^*_{(2)}) a \psi(S^{-1}(h^*_{(1)}). \end{aligned}$$

This means that the algebra map $\psi : H^{*op} \longrightarrow \pi(A) \hookrightarrow A$ induces a strongly inner action of H^{*op} on A. $\square$

Lemma 4.10. Let A be a Galois R-Azumaya algebra such that $\pi(A) \cong I$, and B any R-Azumaya algebra. Then as algebras
(1). $A \otimes B \cong A\#B$,
(2). $\overline{A} \cong A^{op}$,
(3). A is an Azumaya algebra.

Proof. (1). Let $\psi : I \longrightarrow \pi(A)$ be an isomorphism of the two Galois objects in $\mathcal{E}(\mathcal{H}_R)$. We define a linear map ξ as follows:

$$\xi : A \otimes B \longrightarrow A\#B,\ \xi(a \otimes b) = \sum a\psi(\Theta_r(S(b_{(1)})))\#b_{(0)}.$$

It is easy to see that ξ is an isomorphism. We verify that ξ is an algebra map as well. Indeed, If $a, c \in A$ and $b, d \in B$, then

$$\begin{aligned}
\xi((a \otimes b)(c \otimes d)) &= \sum ac\psi(\Theta_r(S(d_{(1)}b_{(1)})))\#b_{(0)}d_{(0)} \\
&= \sum a\psi(\Theta_r(S(b_{(3)})))\psi(\Theta_r(b_{(2)}))c\psi(\Theta_r(S(d_{(1)}))) \\
&\quad \psi(\Theta_r(S(b_{(1)})))\#b_{(0)}d_{(0)} \\
&= \sum a\psi(\Theta_r(S(b_{(2)})))[\Theta_r(b_{(1)}) \cdot (c\psi(\Theta_r(d_{(1)})))]\#b_{(0)}d_{(0)} \\
&= \sum (a\psi(\Theta_r(S(b_{(1)}))\#b_{(0)})(c\psi(\Theta_r(d_{(1)})))\#d_{(0)}) \\
&= \xi(a \otimes b)\xi(c \otimes d).
\end{aligned}$$

(2) and (3). The proof of Part (1) shows that there is a linear map $\nu : \overline{A} \longrightarrow A^{op}$ given by

$$\nu(\overline{a}) = \sum \psi(\Theta_r(S(a_{(1)})))a_{(0)}.$$

We show that ν is the desired algebra isomorphism.

First ν is an algebra map. Given $\overline{a}, \overline{b} \in \overline{A}$, we have

$$\begin{aligned}
\nu(\overline{a}\overline{b}) &= \sum \nu(\overline{b_{(0)}a_{(0)}})R(b_{(1)} \otimes a_{(1)}) \\
&= \sum \psi(\Theta_r(S(a_{(1)}b_{(1)})))b_{(0)}a_{(0)}R(b_{(2)} \otimes a_{(2)}) \\
&= \sum \psi(\Theta_r(S(b_{(2)}a_{(2)})))b_{(0)}a_{(0)}R(b_{(1)} \otimes a_{(1)}) \\
&= \sum \psi(\Theta_r(S(b_{(1)}a_{(2)})))(\Theta_r(a_{(1)}) \cdot b)a_{(0)} \\
&= \sum \psi(\Theta_r(S(b_{(1)}a_{(3)})))\psi(\Theta_r(a_{(2)}))b\psi(\Theta_r(S(a_{(1)})))a_{(0)} \\
&= \sum \psi(\Theta_r(S(b_{(1)})))b_{(0)}\psi(\Theta_r(S(a_{(1)})))a_{(0)} \\
&= \nu(\overline{a}) \circ \nu(\overline{b}).
\end{aligned}$$

Now one may easily check that the following diagram commutes:

$$\begin{tikzcd}
A \otimes \overline{A} \arrow[r, "\xi"] \arrow[dr, "\iota \otimes \nu"'] & A\#\overline{A} \arrow[r, "F"] & \mathrm{End}(A) \\
& A \otimes A^{op} \arrow[ur, "\mathrm{can}"'] &
\end{tikzcd}$$

Since $F \circ \xi$ is an isomorphism, we obtain that $\iota \otimes \nu$ is injective and *can* is surjective. Since all the algebras involved are finite, ν and *can* are isomorphisms. So A is an Azumaya algebra and $\overline{A} \cong A^{op}$ as algebras. $\square$

Theorem 4.11. We have an exact sequence of group morphisms:

$$1 \longrightarrow \mathrm{Br}(k) \xrightarrow{\iota} \mathrm{BC}(k, H, R) \xrightarrow{\widetilde{\pi}} \mathrm{Gal}(\mathcal{H}_R). \tag{28}$$

Proof. Suppose that A, B are two Galois R-Azumaya algebras such that $[A], [B] \in \mathrm{Ker}(\widetilde{\pi})$. By Lemma 4.11, A and B are Azumaya algebras, and $A\#B \cong A \otimes B$. This implies that there is a gropup morphism

$$\zeta : \mathrm{Ker}(\widetilde{\pi}) \longrightarrow \mathrm{Br}(k), \zeta([A]) = [A]$$

by forgetting the H-structures on A, where $[A] \in \mathrm{Ker}(\widetilde{\pi})$ is represented by a Galois R-Azumaya algebra. It is evident that $\zeta \circ \iota = id$, the identity map on $\mathrm{Br}(k)$. If we can show that ζ is also injective, then $\mathrm{Ker}(\widetilde{\pi}) \cong \mathrm{Br}(k)$. Indeed, if $\zeta([A]) = 1 \in \mathrm{Br}(k)$, then there is a finite k-module M such that $A \cong \mathrm{End}(M)$ as an algebra. By Lemma 4.9, the coaction of H^{*op} on A is strongly inner. So there is an H^{*op}-coaction on M such that M is a right H^{*op}-comodule and $A \cong \mathrm{End}(M)$ as H^{*op}-comodule algebra. This implies that $[A] = [\mathrm{End}(M)] = 1$ in $\mathrm{BC}(k, H, R)$. It follows that the sequence (28) is exact. $\square$

Note that the exact sequence (28) indicates that the factor group $\mathrm{BC}(k, H, R)/\mathrm{Br}(k)$ is completely determined by the $\mathcal{H}_R^*$-bigalois objects. In particular, when k is an algebraic field, $\mathrm{BC}(k, H, R)$ is a subgroup of $\mathrm{Gal}(\mathcal{H}_R)$.

5 Examples

In this section, we let k be a field. Let H_4 be the Sweedler 4-dimensional Hopf algebra over k. That is, H_4 is generated by two elements g and h satisfying

$$g^2 = 1, h^2 = 0, gh + hg = 0.$$

The comultiplication, the counit and the antipode are give as follows:

$$\begin{array}{ll} \Delta(g) = g \otimes g, & \Delta(h) = 1 \otimes h + h \otimes g, \\ \varepsilon(g) = 1, & \varepsilon(h) = 0, \\ S(g) = g, & S(h) = gh. \end{array}$$

There is a family of CQT structures R_t on H_4 parameterized by $t \in k$ as follows:

R_t	1	g	h	gh
1	1	1	0	0
g	1	-1	0	0
h	0	0	t	-t
gh	0	0	t	t

It is not hard to check that the Hopf algebra morphisms Θ_l and Θ_r induced by R_t are as follows:

$$\begin{array}{ll}\Theta_l: & H_4^{cop} \longrightarrow H_4^*, \quad \Theta_l(g) =, \overline{1} - \overline{g}, \quad \Theta_l(h) = t(\overline{h} - \overline{gh}),\\ \Theta_r: & H_4^{op} \longrightarrow H_4^*, \quad \Theta_r(g) = \overline{1} - \overline{g}, \quad \Theta_r(h) = t(\overline{h} + \overline{gh}).\end{array}$$

When t is non-zero, Θ_l and Θ_r are isomorphisms, so that H_4 is a self-dual Hopf algebra.

Then H_{R_0} is nothing else but the H_4 itself. When t is not zero, we have that H_{R_t} is a 4-dimensional algebra generated by two elements u and v such that u and v satisfy the relations:

$$u^2 = 1, uv + vu = 0, v^2 = t(1-u).$$

The double algebra $D[H_4]$ with respect to R_t is generated by four elements, g_1, g_2, h_1 and h_2 such that

$$\begin{array}{ll} g_i^2 = 1, \ h_i^2 = 0, & g_ih_j + h_jg_i = 0,\\ g_1g_2 = g_2g_1, & h_1h_2 + h_2h_1 = t(1 - g_1g_2).\end{array}$$

The comultiplication of $D[H_4]$ is easy because the Hopf subalgebras generated by g_i, h_i, $i = 1, 2$, are isomorphic to H_4. Thus the algebra embedding ϕ reads as follows:

$$H_{R_t} \longrightarrow D[H_4], \ \phi(u) = g_1g_2, \phi(v) = g_1(h_1 + h_2).$$

Let us first consider the triangular case (H_4, R_0). In this case, We have an exact and split sequence cf.[33]:

$$1 \longrightarrow (k,+) \longrightarrow \mathrm{BC}(k, H, R_0) \longrightarrow \mathrm{BW}(k, H) \longrightarrow 1. \tag{29}$$

where $(k,+)$ is the addition group which is isomorphic to the group of H_4-bi-Galois objects [22].

On the other hand, the Brauer-Wall group $\mathrm{BW}(k)$ is $\mathrm{BC}(k, k\mathbb{Z}_2, R')$, where $k\mathbb{Z}_2$ is the sub-Hopf algebra of H_4 generated by the group-like element $g \in H_4$, and R' is the restriction of R_0 to $k\mathbb{Z}_2$. The following exact squence is well-known:

$$1 \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{BC}(k, k\mathbb{Z}_2) \longrightarrow Q_2(k) \longrightarrow 1,$$

where $Q_2(k)$ is the group of graded quadratic extensions of k, which is nothing but $\mathrm{Gal}(\mathcal{Z}_{R'})$ where $\mathcal{Z}_{R'} \cong k\mathbb{Z}_2$ is the deformation of $(k\mathbb{Z}_2, R')$. It follows that

$$\mathrm{Gal}(\mathcal{H}_{R_0}) = (k,+) \times Q_2(k)$$

and we have an exact sequence:

$$1 \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{BC}(H_4, R_0) \longrightarrow (k,+) \times Q_2(k) \longrightarrow 1.$$

to be continued $\cdots$.

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