On the Completions of the Spaces of Metrics on an Open Manifold

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Abstract

We construct a complete metric d_0 on the space of continuous complete Riemannian metrics on a smooth manifold X of dimension n. Using that metric, we are able to show, that the space ${}^{b,m}\mathcal{M}(X)$ defined in [Eich] is complete when suppplied with the uniform structure defined in the same paper.

1 A Metric on the Space of C^0 Riemannian Metrics.

Lemma 1.1 *For $a > 0$, $b > 0$, $c > 0$ the inequality*

$$\frac{|a-b|}{a+b} \leq \frac{|a-c|}{a+c} + \frac{|c-b|}{c+b} \tag{1}$$

holds.

Proof: The inequality is symmetric in a and b, and we may thus assume $a \geq b$ without loss of generality. For $c \leq b \leq a$, we estimate directly

$$\frac{|a-b|}{a+b} \leq \frac{|a-c|+|c-b|}{a+b} \leq \frac{|a-c|}{a+c} + \frac{|c-b|}{c+b}$$

This proves (1) in this case. In the case $b < c \leq a$ we calculate

$$\frac{|a-b|}{a+b} - \frac{|a-c|}{a+c} - \frac{|c-b|}{c+b} = \frac{a-b}{a+b} - \frac{a-c}{a+c} - \frac{c-b}{c+b}$$

$$= \frac{(a-b)(a+c)(c+b) - (a-c)(a+b)(c+b) - (c-b)(a+b)(a+c)}{(a+b)(a+c)(c+b)}$$

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The function above the line is a 2'nd degree polynomial in c with roots a and b. The leading coefficient is given by $(a-b)+(a+b)-(a+b)=a-b>0$. It follows, that the polynomial is ≥ 0 for $c>a$ and for $c<b$, and ≤ 0 for $b\leq c\leq a$. This proves (1) in this case. Finally we consider the case $b\leq a\leq c$. In this case

$$\frac{|a-b|}{a+b}-\frac{|a-c|}{a+c}-\frac{|c-b|}{c+b}=\frac{a-b}{a+b}-\frac{c-a}{a+c}-\frac{c-b}{c+b}$$
$$=\frac{-(a+3b)(c-a)(c+\frac{(3a+b)b}{a+3b})}{(a+b)(a+c)(c+b)}$$

Again, the position of the roots and the sign of the leading coefficient of the polynomial above the line gives, that this is non-positive for $b\leq a\leq c$. This proves the lemma. □

Lemma 1.2 *Let X be a smooth manifold and let $C^0\mathcal{M}=C^0\mathcal{M}(X)$ denote the space of complete C^0 metrics on X. I.e. the space*

$$C^0\mathcal{M}=\{g\in C^0(T^*X\otimes T^*X)\mid g \text{ is strictly positive definite in every } x\in X \text{ and } (X,g) \text{ is complete.}\}$$

Then the function $d_0: C^0\mathcal{M}\times C^0\mathcal{M}\mapsto\mathbb{R}$ given by

$$d_0(g,g')=\sup_{x\in X}\sup_{Y\in(T_xX\setminus\{0\})}\frac{|g_x(Y,Y)-g'_x(Y,Y)|}{g_x(Y,Y)+g'_x(Y,Y)} \tag{2}$$

is a metric on $C^0\mathcal{M}$ and $(C^0\mathcal{M},d_0)$ is a complete metric space.

Proof: The function d_0 is defined for any pair of metrics because of the inequality

$$0\leq\frac{|g_x(Y,Y)-g'_x(Y,Y)|}{g_x(Y,Y)+g'_x(Y,Y)}\leq 1$$

for any $x\in X$ and $Y\in T_xX$. Moreover, it is easy to see, that $d_0(g,g')=0$ implies $g=g'$ and that $d_0(g,g')=d_0(g',g)$. For $g,g',g''\in C^0\mathcal{M}$, the triangle inequality is the inequality

$$\sup_{x\in X}\sup_{Y\in(T_xX\setminus\{0\})}\frac{|g_x(Y,Y)-g'_x(Y,Y)|}{g_x(Y,Y)+g'_x(Y,Y)}\leq\sup_{x\in X}\sup_{Y\in(T_xX\setminus\{0\})}\frac{|g_x(Y,Y)-g''_x(Y,Y)|}{g_x(Y,Y)+g''_x(Y,Y)}$$
$$+\sup_{x\in X}\sup_{Y\in(T_xX\setminus\{0\})}\frac{|g''_x(Y,Y)-g'_x(Y,Y)|}{g''_x(Y,Y)+g'_x(Y,Y)} \tag{3}$$

which immediately follows from Lemma 1.1.

It remains to prove that $(C^0\mathcal{M},d_0)$ is complete. Let $\{g_k\}_{k\in\mathbf{N}}$ be a Cauchy sequence in $(C^0\mathcal{M},d_0)$. Let $K\subseteq X$ be a simply connected compact subset of X and let V be a continuous non-vanishing vector-field on K. Then

$$\sup_{x\in K}\frac{|g_{k,x}(V,V)-g_{p,x}(V,V)|}{g_{k,x}(V,V)+g_{p,x}(V,V)}\leq\delta \tag{4}$$

If we for a moment let k be fixed, we notice, that $g_{k,x}(V,V)$ is bounded from above, and see, that if

$$\{g_{p,x}(V,V)\}_{p>N,x\in K} \tag{5}$$

is not bounded from above in p and x, then (4) is impossible for $\delta < 1$. Further, since $g_{k,x}(V,V)$ is a continuous nonvanishing function on the compact set K, it is bounded from below by some $\varepsilon > 0$. Thus if (5) is not bounded from below by some $\varepsilon' > 0$ in p and x, (4) would be impossible for $\delta < 1$. It follows, that the sequence of functions $\{g_{k,x}(V,V)\}$ is a Cauchy sequence in $C^0(K)$ with a lower bound. Since $C^0(K)$ is a Banach-space and convergence in $C^0(K)$ implies pointwise convergence, we see, that the limit is a non-vanishing positive continuous function. By the polarization identity, for V_1, V_2 continuous nonvanishing vector-fields on K, $g_{k,x}(V_1,V_2) = \frac{1}{2T}(g_{k,x}(V_1 + TV_2, V_1 + TV_2) - g_{k,x}(V_1 - TV_2, V_1 - TV_2))$, where T is choosen sufficiently big, so that $V_1 \pm TV_2$ is zero-free. It follows, that the limit

$$g_x(V_1,V_2) = \lim_{k\to\infty} g_{k,x}(V_1,V_2)$$

exists and gives a continuous function with values in the symmetric bilinear maps $TX \otimes TX \mapsto \mathbb{R}$. Moreover, g_x is strictly positive definite for each x. Finally, it is easy to establish the identity

$$\sup_{x\in K} \frac{|g_{k,x}(V,V) - g_x(V,V)|}{g_{k,x}(V,V) + g_x(V,V)} \leq \delta$$

This gives, that the contributions over compact subsets may be glued together to form a C^0 metric g on X, and that $g_k \to g$ with respect to the metric d_0.

We have to show, that g is complete if each g_k is complete. This will follow from the fact, that if g, g' are metrics such that g makes X complete, but g' does not, then $d_0(g,g') = 1$. All Riemannian metrics induce the same topology on X, and thus have the same convergent sequences. It therefore follows, that if g is complete but g' is not, there exists a sequence of points $\{x_k\}_{k\in\mathbb{N}} \subset X$ such that $\{x_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence with respect to g', but not with respect to g. Choosing smooth curves which almost realizes the minimal g'-distances between the x_k's and using the formula for curve length, we find, that there is a sequence of points $\{y_p\}_{p\in\mathbb{N}}$ and vectors $\{Y_p\}_{p\in\mathbb{N}}$ such that $Y_p \in T_{y_p}X$ and

$$\frac{|g_{y_p}(Y_p,Y_p)|}{|g'_{y_p}(Y_p,Y_p)|} \to \infty$$

From that it easily follows, that $d_0(g,g') = 1$. □

2 The Uniform Structure on ${}^{b}_{m}\mathcal{M}$.

In [Eich] a sequence of uniform structures is defined on the space $\mathcal{M} = \mathcal{M}(X)$ of complete smooth metrics on X. We will not repeat the construction from [Eich], but only notice, that for each $m \in \mathbb{N}$ a basis for the uniform structure is given by the sets

$$V_\delta = \{(g,g') \in \mathcal{M} \times \mathcal{M} \mid \sup_{x\in X} |g_x - g'_x|_{g'} < \infty \text{ and}$$

$$ {}^{b,m}|g-g'|_g = \sup_{x\in X}|g_x - g'_x|_g + \sum_{j=0}^{m-1}\sup_{x\in X}\left|(\nabla^g)^j(\nabla^g - \nabla^{g'})_x\right|_g < \delta\} $$

for $\delta > 0$. Let in the following ${}^b_m\mathcal{M}$ denote the topological space $(\mathcal{M}, \tau_m)$, where τ_m is the topology induced on $\mathcal{M}$ by the m'th uniform structure.

We notice, that the neigbourhood basis has a countable subbasis. Thus any convergent net has a convergent subsequence, and any Cauchy net has a Cauchy subsequence. Consequently we will be able to work with sequences rather than nets. The Cauchy sequences of ${}^b_m\mathcal{M}$ can be described by

$$ \{g_\nu\}_{\nu\in\mathbf{N}} \text{ is a Cauchy sequence iff} $$

$$ \forall\delta > 0 : \exists\nu_0 > 0 : \{x_\nu\}_{\nu>\nu_0} \times \{x_\nu\}_{\nu>\nu_0} \subseteq V_\delta $$

The space $C^0\mathcal{M}(X)$ will be considered as a complete metric space with the metric d_0 defined in the previous section. We are ready to prove the main result of this paper:

Theorem 2.1 *The natural inclusion ${}^b_m\mathcal{M} \mapsto C^0\mathcal{M}$ is continuous and extends to a continuous map ${}^b_m\overline{\mathcal{M}} \mapsto C^0\overline{\mathcal{M}}$, where ${}^b_m\overline{\mathcal{M}}$ is the completion of ${}^b_m\mathcal{M}$.*

Proof: It is enough to prove, that the inclusion sends Cauchy sequences into Cauchy sequences. But this follows immediately, since $(g, g') \in V_\delta$ implies that $|g_x - g'_x|_g < \delta$ for all $x \in X$. This again implies that

$$ \sup_{x\in X}\sup_{Y\in(T_xX\setminus\{0\})}\frac{|g_x(Y,Y) - g'_x(Y,Y)|}{g_x(Y,Y) + g'_x(Y,Y)} < \delta $$

□

Corollary 2.2 *The space ${}^b_m\overline{\mathcal{M}}$ consists of strictly positive definite metrics.*

Proof: Convergence in ${}^b_m\overline{\mathcal{M}}$ as well as in $C^0\mathcal{M}$ implies pointwise convergence. Thus the extension of the inclusion to ${}^b_m\overline{\mathcal{M}}$ is also just the inclusion. The corollary follows since elements of $C^0\mathcal{M}(X)$ are strictly positive definite. □

Corollary 2.3 *The space ${}^{b,m}\mathcal{M}$ defined in [Eich] is complete.*

Proof: By definition

$$ {}^{b,m}\mathcal{M} = {}^b_m\overline{\mathcal{M}} \cap C^m\mathcal{M} $$

where $C^m\mathcal{M}$ is the space of strictly positive definite C^m metrics on X. By Corollary 2.2 and since all elements of ${}^b_m\overline{\mathcal{M}}$ are m times continuously differentiable it follows, that

$$ {}^{b,m}\mathcal{M} = {}^b_m\overline{\mathcal{M}} $$

This proves the corollary. □

The results above can also be applied to the spaces $\mathcal{M}^{p,r}$ defined in [Eich, p. 268]. Let $\mathcal{M}(I, B_k)$ be the space of smooth complete metrics g on X, such that the injectivity radius of (X,g) has a lower bound, and such that $|(\nabla^g)^j R^g|$ is bounded for $j \le k$. Let for $g, g' \in \mathcal{M}(I, B_k)$, $p \in [1,\infty)$ and $r \in \mathbb{N}$:

$$|g - g'|_{g,p,r} = \left(\int_X (|g - g'|^p_{g,x} + \sum_{i=0}^{r-1} |(\nabla^g)^i (\nabla^{g'} - \nabla^g)|^p_{g,x}) d\mathrm{vol}_x(g) \right)^{\frac{1}{p}}$$

For $k \ge r > \frac{n}{p} + 1$, a metrizable uniform structure on $\mathcal{M}(I, B_k)$ is given by the following neigbourhood basis

$V_\delta = \{(g, g') \in \mathcal{M}(I, B_k) \times \mathcal{M}(I, B_k) \mid g$ and g' are quasi isometric and $|g - g'|_{g,p,r} < \delta\}$

Let $\mathcal{M}^p_r(I, B_k)$ be the space $\mathcal{M}(I, B_k)$ supplied with the uniform structure given above, and let $\overline{\mathcal{M}^p_r}(I, B_k)$ be the completion of $\mathcal{M}^p_r(I, B_k)$. Then by definition [Eich, p. 268],

$$\mathcal{M}^{p,r}(I, B_k) = \overline{\mathcal{M}^p_r}(I, B_k) \cap C^s\mathcal{M}$$

where $C^s\mathcal{M}$ is the space of positive definite C^s-metrics and $s < r - \frac{n}{p}$.

Lemma 2.4 *Let $p \in [1,\infty)$ and assume $k \ge r > \frac{n}{p} + 1$. Then*

$$\mathcal{M}^{p,r}(I, B_k) = \overline{\mathcal{M}^p_r}(I, B_k)$$

In particular, $\mathcal{M}^{p,r}(I, B_k)$ is complete.

Proof: By the Sobolev imbedding theorm for manifolds with bounded geometry, there is a continuous imbedding

$$\mathcal{M}^p_r(I, B_k) \hookrightarrow^b_0 \mathcal{M}$$

which extends to a continuous imbedding

$$\overline{\mathcal{M}^p_r}(I, B_k) \hookrightarrow^b_0 \overline{\mathcal{M}}$$

Since both sides are pointwise defined as tensors, and the right side consists of strictly positive metrics, we conclude, that also the left side consists of strictly positive metrics. This proves the lemma. □

References

[Eich] J. Eichhorn: "Spaces of Riemannian Metrics on Open Manifolds." Results in Mathematics **27** (1995)