Reduction theory decreasing dimensions

for $Sp_n(\mathbb{Z})$ and some $O_n(f,\mathbb{Z})$

by

Alexander Scheutzow

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Introduction: In order to study homological properties of an arithmetic group Γ , it is helpful to know a contractible locally finite (regular) cell complex C on which it acts with finite quotient. Then, for instance, $C/\tilde{\Gamma}$ ($\tilde{\Gamma}$ a torsionfree subgroup of finite index in Γ) is a finite Eilenberg-MacLane complex for $\tilde{\Gamma}$, whose homologies may thus be computed combinatorially (cf. [Br], Ch.I, 4). Starting with the (contractible) symmetric space of maximal compact subgroups of the corresponding real group G , such a complex is provided by Borel-Serre compactification ([BS]). However, for more explicit calculations it would be advantageous to find deformation retracts as small as possible with all the desired properties. For GL and SL groups, such subspaces have been described in [Se2], [So], [Me], [SV], and [As]. This paper, which is basically a shortened translation of [Sc], intends to define similar spaces for $Sp_n(\mathbb{Z})$ and for the automorphism groups of indefinite unimodular quadratic $\mathbb{Z}$-lattices (i.e. "inner product $\mathbb{Z}$-spaces" in the sense of [MH]; their discriminant is ± 1). We shall decrease dimensions manifestly by only 1 , but it will be shown that further savings can be made once the complex has been computed. The dimension eventually achieved cannot be less than the virtual cohomological dimension of Γ , which means in our cases (by [BS], 11.4.3) that at most r dimensions can be disposed of, where r is the Witt index (resp. n/2). - It is quite likely that the main results remain valid without the unimodularity assumption, but in this case some of the proofs would become much longer (cf. also [Gr1] and [Gr2]). Generalisations from $\mathbb{Z}$ to other rings and the introduction of certain "sets of weights" (on classes like the ones defined in 2.5 below, cf. [As]) may be possible as well. -

Among the criteria for the formulation of this text are brevity and concreteness; the reader is asked to accept occasional fluctuation in language (which is mainly geometrical like in [As]), and the onus of some straightforward verifications. Since the proofs are logically interwoven to a high degree, variants of known facts are sometimes included without mention, and the distinction "lemma" - "corollary" etc. loses most of its meaning.

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§ 1 The real groups G

1.0 Notations and definitions: Let $M_*(*)$ denote the set of matrices, $*^t$ the transpose of a matrix. I_* is the unit matrix, and $I_{a,b} := \begin{pmatrix} I_b \\ 0 \end{pmatrix} \in M_{a,b}(\mathbb{Z})$. $\mathbb{R}^m$ and $\mathbb{Z}^m$ mean sets of columns. If S is an additive subgroup of $\mathbb{R}^m$, and M a matrix out of $M_m(\mathbb{R})$, then S endowed with the bilinear form $v,w \mapsto v^tMw$ will be called (S,M), so that $\mathrm{Aut}(S,M) := \{ g \in GL(S) \mid g^tMg = M \}$. The columns of a matrix A are indexed A_i, and by abuse of notation no difference will be made between A and the set $\{A_i\}$ (in given order). E.g., saying " m is a basis of $m\cdot\mathbb{Z}^n$ " is to mean that $\{m_i\}$ is a $\mathbb{Z}$-basis of that module. $\mathcal{U} = \mathcal{U}_m \subset M_m(\mathbb{R})$ shall contain the upper triangular matrices, $\mathcal{U}_m^+$ those with positive diagonal entries. $\mathcal{L}$ and $\mathcal{L}^+$ are the analogous notations for lower triangular matrices. -

By the Witt decomposition theorem, the real groups to be considered may be fixed as

$$G := \mathrm{Aut}(\mathbb{R}^n, J) \text{ , where } J := \begin{pmatrix} & & I_r \\ & I_s & \\ \epsilon I_r & & \end{pmatrix} ,$$

$r \geq 1$ (expressing that the form be indefinite),

$\epsilon = \pm 1$ (to determine whether the form is to be symmetric or symplectic),

$n = s + 2r$, and

$[\ s = 0 \text{ if } \epsilon = -1\ ]$.

The data r , s , and ϵ are assumed fixed throughout this paper, thus J , G etc. are fixed as well. - An unspecified trisection of the index set $1\ldots n$ in a matrix means $1\ldots r$, $r+1\ldots r+s$, and $r+s+1\ldots n$. As an example to this notation, note that $(a,b,c) \in G$ is equivalent to

$$0 = a_i^tJa_j = c_i^tJc_j = a_i^tJb_k = b_k^tJc_i \text{ and } \tag{1}$$
$$b_k^tJb_l = \delta_{lm} , \quad a_i^tJc_j = \delta_{ij} \quad \text{for } i,j=1\ldots r , \quad k,l=1\ldots s .$$

Finally, $K := G \cap O_n(\mathbb{R})$, which is a maximal compact subgroup of G .

1.1 Let Φ consist of those $g \in G$ which have the form $\begin{pmatrix} \ell & * & * \\ 0 & I_s & * \\ 0 & 0 & * \end{pmatrix}$ with $\ell \in \mathfrak{U}_r^+$. Calculation shows that in fact Φ is the set of all

$$g = \begin{pmatrix} \ell & -\ell p^t & -\ell x \\ 0 & I_s & p \\ 0 & 0 & \ell^{-t} \end{pmatrix} \in GL_n(\mathbb{R}) \qquad (1)$$

with $\ell \in \mathfrak{U}^+$ and

$$x^t + \epsilon x - p^t p = 0 . \qquad (2)$$

1.2 Remarks: Define $M^{\approx} := \{v \in \mathbb{R}^n | m^t J v = 0 \ \ \forall\, m \in M\}$ and $M^{\perp} := \{v \in \mathbb{R}^n | m^t v = 0 \ \ \forall\, m \in M\}$ for the remainder of this text. One may convince oneself of the following facts by geometric considerations or by straightforward matrix computation:

(i): Let b be a basis of a totally isotropic subspace S of $(\mathbb{R}^n, J)$, in other words, $b \in M_{n,\rho}(\mathbb{R})$ has rank $\rho \le r$, and $b^t J b = 0$. Then we have, orthogonally for both bilinear forms,

$$\mathbb{R}^n = S \perp (S^{\approx} \cap S^{\perp}) \perp S^{\approx} . \qquad (1)$$

Also there is a $u \in K$ which respects this decomposition, i.e. its columns distribute on it in given order. $u \cdot b = \ell \cdot I_{n,\rho}$, $\ell \in \mathfrak{U}_\rho^+$, can be forced.

(ii): Let S be as in (i), then $S^{\approx}$ is positive semidefinite (and $S^{\approx} \cap S^{\perp}$ positive definite) iff $\rho = r$.

(iii): ("Iwasawa decomposition"): For $g \in G$ there is exactly one $u \in K$ such that $u \cdot g \in \Phi$. -

(iv): As an open set of coset representatives, Φ is homeomorphic to ${}_K\backslash^G$. Furthermore, the data ℓ, x, and p from 1.1 define a homeomorphism from ϕ to some $\mathbb{R}_+^{\mu} \times \mathbb{R}^{\nu}$ (take 1.1(3) into account).

<u>1.3 Lemma:</u> For this technical lemma (which, however, also has a geometrical meaning), we consider a space $(\mathbb{R}^n, J_\lambda)$ with an $\lambda \in GL_s(\mathbb{R})$ and $J_\lambda := \begin{pmatrix} & & I_r \\ & \lambda^t\lambda & \\ I_r & & \end{pmatrix}$. Let $B := \left(\begin{array}{c|c} \begin{matrix} a \\ o \end{matrix} & * \\ \hline 0 & * \\ \hline 0 & * \end{array}\right)$ be a basis of a totally isotropic subspace in it with $a \in M_i(\mathbb{R})$ (this implies of course $a \in GL_i(\mathbb{R})$; $i=0$ is allowed). Then we have $B = \left(\begin{array}{c|c} \begin{matrix} a \\ o \end{matrix} & * \\ \hline 0 & * \\ \hline 0 & \begin{matrix} o \\ b \end{matrix} \end{array}\right)$,

$b \in M_{r-i,j}$ $(j \le r-i)$; and if in addition $B\cdot\mathbb{R}^r \cap I_{n,r}\cdot\mathbb{R}^r = I_{n,i}\cdot a\cdot\mathbb{R}^i$, then $rk(b) = j$.

<u>Proof:</u> To prove the last assertion (the others are straightforward), assume $b\cdot v = 0$, $v \in \mathbb{R}^{r-i}$. Then $B\cdot\begin{pmatrix}0\\v\end{pmatrix}$ is an isotropic vector of the form $\begin{pmatrix}*\\*\\0\end{pmatrix}$, by $(\mathbb{R}^n, J_\omega)$-isotropy even $\begin{pmatrix}*\\0\\0\end{pmatrix}$, so the assumption forces it to be in $I_{n,i}\cdot a\cdot\mathbb{R}^i$, thus $B\cdot\begin{pmatrix}0\\v\end{pmatrix} = I_{n,i}\cdot a\cdot w = B\cdot\begin{pmatrix}w\\0\end{pmatrix}$, $w \in \mathbb{R}^i$. As B is a basis, $v = 0$ q.e.d..

<u>§ 2 The arithmetic groups Γ</u>

<u>2.0</u> A "lattice" $(\mathbb{Z}^n, M)$ with a symmetric $M \in GL_n(\mathbb{Z})$ is called of type II if the quadratic form induced by M asumes only even values on $\mathbb{Z}^n$. Otherwise it is said to be of type I. Obviously the type is invariant under isomorphisms.

2.1 Fact: For a triple r , s , ϵ , there is only one isomorphism class of lattices $(\mathbb{Z}^n, M)$ such that $(\mathbb{R}^n, M)$ is isomorphic to $(\mathbb{R}^n, J)$, except if $\epsilon = 1$ and $s \equiv 0 \bmod 8$, where two classes exist, one of each type.- Proofs can be found e.g. in [MH] I, 3.5 (case $\epsilon=-1$) resp. [Sel] V, Th. 4 to 6 and §3 ($\epsilon=1$).

2.2 In these exceptional cases we take the liberty to deal with the class of type II only, to avoid further complications. This (inessential) restriction can be justified by the fact that the automorphism groups of the two lattice classes are commensurable up to isomorphism, as shown in 2.3 below. Thus it suffices to regard but a single group Γ (r , s , and ϵ always assumed fixed). Fix a matrix $k_{(1)} \in SL_8(\mathbb{Q})$ such that ${}^t k_{(1)} k_{(1)} \in SL_8(\mathbb{Z})$ and $(\mathbb{Z}^8, {}^t k_{(1)} k_{(1)})$ is of type II (cf. [Sel] V, 1.4), and define $k_{(i)} \in SL_{8i}(\mathbb{Q})$ to be the direct matrix product of i samples of $k_{(1)}$. Now we define permanently:

$$\kappa := \begin{cases} I_s & s \text{ not divisible by } 8 \text{ (or } s=0\text{)} \\ k_{(i)} & s = 8i\ ,\ i \in \mathbb{N} \end{cases} ,$$

$$J_\kappa := \begin{pmatrix} & & I_r \\ & {}^t\kappa\kappa & \\ \epsilon I_r & & \end{pmatrix} \in SL_n(\mathbb{Z}) ,$$

$$\underline{\kappa} := \begin{pmatrix} I_r & & \\ & \kappa & \\ & & I_r \end{pmatrix} \in SL_n(\mathbb{Q}) ,$$

$$\Gamma := \mathrm{Aut}(\mathbb{Z}^n, J_\kappa) = \{ \gamma \in GL_n(\mathbb{Z}) \mid {}^t\gamma J_\kappa \gamma = J_\kappa \} .$$

Since $J = \underline{\kappa}^{-t} J_\kappa \underline{\kappa}^{-1}$, obviously $(\mathbb{R}^n, J_\kappa)$ is isomorphic to $(\mathbb{R}^n, J)$. Also, $(\mathbb{Z}^n, J_\kappa)$ is of type II if $\epsilon=1$ and $s \equiv 0 \bmod 8$, as desired.

<u>2.3</u> In the latter case, a representative of type I is $(\mathbb{Z}^n,J)$ (if $s\neq 0$) resp. $(\mathbb{Z}^n,M)$ with $M := \begin{pmatrix} I_r & \\ & -I_r \end{pmatrix}$ (if $s=0$, whence $J = J_\kappa$). We have $\Gamma = GL_n(\mathbb{Z}) \cap \kappa^{-1}G\kappa \cong \kappa\, GL_n(\mathbb{Z})\, \kappa^{-1} \cap G$, the latter group has a subgroup of mutually finite index in common with $\mathrm{Aut}(\mathbb{Z}^n,J) = GL_n(\mathbb{Z}) \cap G$, as can be seen by clearing denominators. The same argument works for $s=0$, where $\begin{pmatrix} 1/2\ I_r & -1/2\ I_r \\ I_r & I_r \end{pmatrix}$ takes the role of κ . So the missing groups are commensurable up to isomorphism to our Γ's .

<u>2.4</u> In proofs we will also have to consider $(\mathbb{Z}^s,M)$ with a *positive definite* $M \in SL_s(\mathbb{Z})$. We collect some facts:

<u>(i):</u> For a given s there is only a finite number of isomorphy classes of such objects; if $s \leq 7$, just a single one.

<u>(ii):</u> The spaces $(\mathbb{Q}^s,M)$, s fixed, are pairwise isomorphic, in particular to $(\mathbb{Q}^s,I_s)$ (follows e.g. from [Sel], Ch.V, 1.3.6, 2.11 Th.2, Ch.IV Th.7 & 9).

<u>(iii):</u> Type II again only occurs if $8|s$, e.g. with $(\mathbb{Z}^s,k^t_{(s/8)}k_{(s/8)})$ ([Sel], Ch..V §2 Th.2 Cor.1). -

<u>2.5</u> Thanks to these results, we may permanently fix a finite nonempty set $\Omega \subset SL_s(\mathbb{Q})$ (!), such that $\omega^t\omega \in SL_s(\mathbb{Z})$ $\forall\ \omega\in\Omega$, and that $\{(\mathbb{Z}^s,\omega^t\omega) \mid \omega \in \Omega \}$ is a set of representatives for those isomorphy classes from 2.4(i), except that in the case $s \equiv 0 \bmod 8$ only those of type II are admitted. We may assume $\kappa \in \Omega$. Kneser's algorithm from [Kn] can be used to calculate Ω , which he actually accomplished for

$s \leq 16$.

2.6 Define for each $\omega \in \Omega$

$$J_\omega := \begin{pmatrix} & & I_r \\ & \omega^t\omega & \\ \epsilon I_r & & \end{pmatrix} \in SL_n(\mathbb{Z}) ,$$

then $(\mathbb{R}^n, J_\omega)$ is isomorphic to $(\mathbb{R}^n, J_\kappa)$, and $(\mathbb{Z}^n, J_\omega)$ is of type II if $8|s$, so $(\mathbb{Z}^n, J_\omega)$ is isomorphic to $(\mathbb{Z}^n, J_\kappa)$ by 2.1. This means that we can fix permanently matrices $B_\omega \in GL_n(\mathbb{Z})$ (which may be viewed as bases of $\mathbb{Z}^n$), such that

$$B_\omega^t J_\kappa B_\omega = J_\omega . \tag{1}$$

It is convenient to take $B_\kappa = I_n$.

2.7 The lattice $(\mathbb{Z}^n, J_\kappa)$ will now be examined from a geometric point of view, imbedded into $(\mathbb{R}^n, J_\kappa)$. We begin with some more definitions: For an $S \subset \mathbb{R}^n$ let

$S^o := \{ z \in \mathbb{Z}^n \mid z^t J_\kappa \sigma = 0 \;\; \forall \sigma \in S \}$,

$\langle S \rangle$: be the $\mathbb{R}$-span of S .

A subgroup U of $\mathbb{Z}^n$ will be called *sublattice* only if

$$U \supset \langle U \rangle \cap \mathbb{Z}^n \quad (\text{"}\subset\text{" always holds}) . \tag{1}$$

Note that for two sublattices U and V one has

$$U \subset V \Rightarrow [\, U = V \iff \operatorname{rk} U = \operatorname{rk} V \,] . \tag{2}$$

A set which is a basis of a sublattice is called primitive.

$\mathcal{N}$: will denote the set of maximal totally isotropic sublattices.

2.8 Remarks: (i) (Gauss): If $P \subset Q$ are two sublattices, any basis

(p_i) of P can be completed to one (p_i, q_j) of Q. In particular, if $P \neq Q$, one finds an element of Q that is primitive with (p_i) (cf. 2.7(2)).

<u>(ii):</u> As sublattices correspond one-to-one to their $\mathbb{Q}$-spans in $\mathbb{Q}^n$, they can be handled like vector spaces. E.g., together with P and Q, P^o and $P \cap Q$ are sublattices; $\mathrm{rk}\, P^o = n - \mathrm{rk}\, P$.

<u>(iii):</u> Every element of $\mathcal{N}$ has rank r. To see this, take bases (p_i), (p_i, q_j), and $(p_i, q_j, t_k) =: B$ of P, P^o, and $\mathbb{Z}^n$ respectively according to (i). Then $B^t J_\kappa B$ has the form $\begin{pmatrix} o & o & * \\ o & * & * \\ * & * & * \end{pmatrix}$ and determinant 1. Thus the lattice spanned by (q_j) is of discriminant 1 and anisotropic (because P was maximal totally isotropic), so it must be definite (cf. [Sel],Ch.V, Th.3), i.e. $\langle (q_j) \rangle$ anisotropic. From $\langle P^o \rangle = \langle P \rangle \oplus \langle (q_j) \rangle$, it follows that $\langle P \rangle$ is maximal totally isotropic, hence the assertion. - Note that this statement is generally false if one admits a *non-unimodular* matrix (symmetric of Witt index r) instead of J_κ. -

<u>2.9 Theorem:</u> Given a matrix ('basis') b such that $b \cdot \mathbb{Z}^r \in \mathcal{N}$, one can find a $\gamma \in \Gamma$ and a unique $\omega \in \Omega$ such that $b = \gamma \cdot B_\omega \cdot I_{n,r}$.

<u>Proof:</u> Once a completion $B = (b,c,d) \in GL_n(\mathbb{Z})$ satisfying

$$B^t J_\kappa B = J_\omega , \tag{1}$$

has been constructed, $\gamma := B \cdot B_\omega^{-1}$ is in Γ (because of (1) and 2.6(1)) and fits the theorem.

To achieve this, we start with a completion like the one from 2.8(iii), so we have $B^tJ_\kappa B = \begin{bmatrix} 0 & 0 & * \\ 0 & * & * \\ * & * & * \end{bmatrix}$ and

$$\det(b^tJ_\kappa d) = 1 = \det(c^tJ_\kappa c) \ . \qquad (2)$$

This basis will be modified step by step to fulfill the remaining requirements of (1). Firstly, assume by induction that $d_{(i)} := \{d_j | j \le i\}$ already satisfies the following ones:

$$b \cup c \cup d_{(i)} \text{ is primitive,} \qquad (3)$$

$$b_k^tJ_\kappa d_j = \delta_{k,j} \quad (k=1\ldots r,\ j\le i), \text{ and} \qquad (4)$$

$$d_j^tJ_\kappa d_m = 0 \quad (j,m\le i) \ . \qquad (5)$$

Then the $\mathbb{Z}$-span $S_{(i)}$ of (3) and $\{b_j | j>i\}^o$ are two sublattices both of rank $r+s+i$ (cf. 2.8(ii)), the latter containing the former, therefore they are equal. Now change d_{i+1} to complement (3) to form a basis of $\{b_j | j>i+1\}^o$ (cf. 2.8(i,ii)). Without losing any achievements (always assume those d_k with $k>i+1$ adjustable to restore the basis property automatically), we may still alter d_{i+1} by adding elements of $S_{(i)}^o$ to it. Taking $d_{i+1} - \sum_{j\le i} (b_j^tJ_\kappa d_j)\cdot d_j$ as our new d_{i+1}, we have $b_j^tJ_\kappa d_{i+1} = 0$ for $j\ne i+1$. In fact, abbreviating $\eta := b_{i+1}^tJ_\kappa d_{i+1}$, the matrix $b^tJ_\kappa d = \left[\begin{array}{c|cc} I_{r-1} & o & * \\ \hline 0 & \eta & * \\ & o & * \end{array}\right]$ has determinant ± 1 by (2), thus $\eta = 1$ can be obtained by adjusting the sign of d_{i+1}. Next we require $d_{i+1}^tJ_\kappa d_{i+1}$ to be even. If s is a multiple of 8, this must be the case already, by definition of J_κ (which is hereby motivated). Otherwise $c\mathbb{R}^s$ is anisotropic and positive semidefinite (since $b\mathbb{R}^r \oplus c\mathbb{R}^s = b^{\approx}$, cf. 2.8(iii) and 1.2(ii)), i.e. positive definite, also its discriminant is 1 (cf. (2)), therefore, by 2.4, $c\cdot\mathbb{Z}^n$ contains an element v with $v^tJ_\kappa v$ odd. In the same vein an above, we may still add elements of b^o

to d_{i+1} , replacing the latter by $d_{i+1}+v$, if necessary, and then by $d_{i+1} - 1/2\ (d_{i+1}^t J_\kappa d_{i+1})\cdot b_{i+1} - \sum_{j\le i} (d_j^t J_\kappa d_{i+1})$. This continues our induction hypotheses, so we end up with $b^t J_\kappa b = 0$, $d^t J_\kappa d = 0$, and $b^t J_\kappa d = I_r$. - Now the c_i can be doctored on by elements of $b\cdot\mathbb{Z}^r$ at liberty: Taking $c_i - \sum_{j=1}^{r} (c_i^t J_\kappa d_j)\cdot b_j$ for c_i yields $c_i^t J_\kappa d_j = 0$ and conserves $c_i^t J_\kappa b_j = 0$ for all i and j . As above one sees that the matrix $c^t J_\kappa c$ defines a positive definite quadratic form $\mathbb{Z}^n \to \mathbb{Z}$ which in the case $8|s$ assumes even values only. By definition of Ω , there must be a ω out of it and a $m \in GL_s(\mathbb{Z})$ such that $m^t c^t J_\kappa cm = \omega^t\omega$. The replacement $c\cdot m$ for c fulfills with $c^t J_\kappa c = \omega^t\omega$ the only remaining condition for (1). -

To prove the uniqueness assertion, assume $b^o = b\cdot\mathbb{Z}^r \oplus c\cdot\mathbb{Z}^r = b\cdot\mathbb{Z}^r \oplus \bar{c}\cdot\mathbb{Z}^r$ to be two decompositions of the above kind, and π_b , π_c the projections induced by the first one. Then for $v,w \in \bar{c}\cdot\mathbb{Z}^s$ we have $(\pi_c(v))^t J_\kappa(\pi_c(w)) = (\pi_b(v)+\pi_c(v))^t J_\kappa(\pi_b(w)+\pi_c(w)) = v^t J_\kappa w$; thus $\pi_c|_{\bar{c}\cdot\mathbb{Z}^s}$ is a homomorphism of positive definite lattices, bijective by a symmetry argument. The corresponding matrices (versions of $\omega^t\omega$) therefore are congruent; by definition of Ω they must be equal, whence the claim.

§ 3 Reduction theory for Γ

3.0 The real group belonging to Γ is $\underline{\kappa}^{-1}G\underline{\kappa}$. However, it is more comfortable (in view of 1.2(i)) and topologically equivalent to let Γ act on the space

$$\mathcal{G} := G\cdot\underline{\kappa}$$

by multiplication from the right, since $K\cdot\mathcal{G}\cdot\Gamma = \mathcal{G}$ holds. The corresponding homogeneous space is

$$X := {}_K\backslash^{\mathcal{G}} .$$

In the sequel it is understood that most constructions on $\mathcal{G}$ descend to X . - The following reduction theory of that action is centered about the notion of $\mathcal{N}$, using the results of § 2 :

3.1 Proposition: Given $\tilde{h} \in \mathcal{G}$ and a basis b of $b\cdot\mathbb{Z}^r \in \mathcal{N}$, one can find $\gamma \in \Gamma$, $u \in K$, and a unique $\omega \in \Omega$ such that $h := u\tilde{h}\gamma \in \mathcal{G}$ has:

$$h = \begin{pmatrix} \ell & -\ell p^t\omega & -\ell x \\ & \omega & p \\ & & \ell^{-t} \end{pmatrix} \cdot B_\omega^{-1} \quad , \text{ and} \tag{1}$$

$$\gamma\cdot B_\omega \cdot I_{n,r} = b \quad . \tag{2}$$

ℓ can be required to be in $\mathcal{U}^+$ or to be in $\mathcal{L}^+$; the data ℓ , x , and p may now be regarded as functions $\ell(h) = \ell(h,\omega)$ etc. of h and ω satisfying

$$x^t + \epsilon x - p^t p = 0 \quad . \tag{3}$$

Let us define permanently one more function of that kind:

$$y = y(h) = y(h,\omega) := \ell^{-1}\ell^{-t} = (\ell^{-t})^t(\ell^{-t}) \quad .$$

Then one may switch between the $\mathcal{U}^+$ and the $\mathcal{L}^+$ option without changing z , x , and p .

Proof: Take γ and ω from 2.9 to satisfy (2). Now $\tilde{h}\cdot\gamma \in \mathcal{G}$, whence calculation with 2.3(1) and 2.6(1) yields $\tilde{h}\cdot\gamma\cdot B_\omega\cdot\begin{pmatrix} I_r & \omega^{-1} \\ & I_r\end{pmatrix} \in G$. For this matrix we can therefore find a u from 1.2(i) (suitably adjusted in case of the $\mathcal{L}^+$ option, whence the last claim), so (1) and (3) = 1.1(2) follow. -

3.2 Define the volume of a matrix m as

$$\mathrm{vol}\ m := (\det({}^t m\, m))^{1/2} \geq 0 \quad .$$

We collect some facts about this definition for reference:

(i): $\mathrm{vol}(u\cdot m) = \mathrm{vol}\ m$ if $u \in O_*(\mathbb{R})$,

(ii): $\mathrm{vol}(m\cdot\sigma) = \mathrm{vol}\ m$ if $\sigma \in \pm SL_*(\mathbb{R})$; therefore

(a) we can define

$$\mathrm{vol}\ M := \mathrm{vol}\ m \text{ if } m \text{ is a } \mathbb{Z}\text{-basis of the lattice } M = m\cdot\mathbb{Z}^\mu \ ;$$

(this agrees with the notion of ordinary volume of a fundamental parallelotope. The reader may find geometric formulations for all these remarks.),

(b) $\mathrm{vol}\begin{pmatrix}\alpha & * \\ 0 & *\end{pmatrix} = \mathrm{vol}\begin{pmatrix}\alpha & 0 \\ 0 & *\end{pmatrix}$ by column operations, if α is regular.

(iii): $\mathrm{vol}(m\cdot\sigma) \geq \mathrm{vol}\ m$ if $\sigma \in M_*(\mathbb{Z}) \cap GL_*(\mathbb{R})$,

(iv): If m is a square matrix, $\mathrm{vol}\ m = |\det m|$,

(v): $\mathrm{vol}\begin{pmatrix} a \\ b \\ \vdots \end{pmatrix} = \det^{1/2}({}^t a\, a + {}^t b\, b + \ldots)$, e.g.

(vi): $\mathrm{vol}\begin{pmatrix} a & 0 \\ 0 & b \\ & 0 \end{pmatrix} = \mathrm{vol}\ a \cdot \mathrm{vol}\ b$, (vii): vol $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq \det^{1/2}(a^t a + c^t c) = \text{vol} \begin{pmatrix} a \\ c \end{pmatrix}$ ("empty" matrices allowed).

(viii): Given two subgroups A and B of $\mathbb{Z}^n$ with bases a resp. b, such that $\langle A\rangle \cap \langle B\rangle = \{0\}$, and some $g \in GL_n(\mathbb{R})$. Then

$$\text{vol}\ (g\cdot(\ \langle A\cup B\rangle \cap \mathbb{Z}^n\ )) \leq \text{vol}\ (g\cdot a)\cdot \text{vol}\ (g\cdot b) \quad .$$

To prove this, (i) allows us to calculate with respect to a ONB (basis out of $O_n(\mathbb{R})$) whose columns span first $g\cdot\langle A\rangle$, then $g\cdot(\langle A\rangle\oplus\langle B\rangle)$. I.e., $g\cdot(a,b)$ transforms to $\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \\ & 0 \end{pmatrix}$ with $\det \alpha \neq 0$. Now $A+B \subset \langle A\cup B\rangle \cap \mathbb{Z}^n$;

$$\begin{aligned} \text{vol}(g(\langle A\cup B\rangle\cap\mathbb{Z}^n)) &\leq \text{vol}(g(A+B)) && \text{(by (iii))} \\ &= \text{vol}(g\cdot(a,b)) = \text{vol}\ \alpha\cdot\text{vol}\ \gamma && \text{(by (ii,b), (vi))} \\ &\leq \text{vol}\ \alpha\cdot\text{vol}\begin{pmatrix}\beta\\ \gamma\end{pmatrix} && \text{(by (vii))} \\ &= \text{vol}(g\cdot a)\cdot\text{vol}(g\cdot b) \quad \text{q.e.d..} \ - \end{aligned}$$

(ix): Define $\lambda_j : GL_i(\mathbb{R}) \to \mathbb{R}^+$, $j\leq i$ such that $\lambda_j(m)$ denotes the product of the square roots of the j smallest eigenvalues of $m^t m$. This is a continuous function, and given $b \in M_{i,j}(\mathbb{R})$, $m \in GL_i(\mathbb{R})$, one has

$$\text{vol}(m\cdot b) \geq \lambda_j(m)\cdot\text{vol}\ b \quad .$$

(Written with respect to an appropriate ONB, this amounts to the known "minimax principle"). -

Two auxiliary reduction concepts have to be recalled:

3.3 We shall encounter the following sets:

$$L_\omega := \{\ z \in \omega\cdot\mathbb{Z}^s \mid z^t z \in 2\mathbb{Z}\ \} \quad , \ \omega \in \Omega \ .$$

They are in fact groups (lattices), and $L_\omega = \mathbb{R}^s$. The "Dirichlet domains" (sometimes called "Voronoi cells")

$$\Delta_\omega := \{\ p \in \mathbb{R}^s \mid p^t p \le (p+z)^t (p+z) \quad \forall\ z \in L_\omega\ \}\ ,\ \omega \in \Omega\ ,$$

are compact rectilinear polyhedra with finitely many faces, and for any $p \in \mathbb{R}^s$ there is a $\lambda \in \Lambda_\omega$ with $p + \lambda \in \Delta_\omega$.

3.4 A real i-by-j matrix b of rank j shall be called M'-reduced (cf. [Wa], 2, §3), if [the form defined by] $b^t b$ is Minkowski-reduced (M-reduced for short) in the more familiar sense (ibd. or [Bo], §2). Thus for any basis b there is a $\sigma \in GL_j(\mathbb{Z})$ such that $b\cdot\sigma$ ("change of $\mathbb{Z}$-basis") is M'-reduced. Together with b , $u\cdot b$ $(u \in O_i(\mathbb{R}))$ is also M'-reduced.

3.5 Lemma: For a number c and a matrix $m \in GL_i(\mathbb{R})$, there are only finitely many subgroups B of $\mathbb{Z}^i$ with $\mathrm{vol}(m\cdot B) \le c$.

Proof: For such a B of rank, say, ρ let b be a M'-reduced basis of $m\cdot B$, and $\mu(m) := \min\{v^t v \mid v \in m\mathbb{Z}^i - \{0\}\}\ \ 0$. The "Minkowski inequality" (cf. [Wa], 2, §7) yields

$$\prod_{\nu=1}^{i} b_\nu^t b_\nu \le k\cdot\det(b^t b) \le k\cdot\mathrm{vol}^2 b \le k\cdot c^2\ ,$$

where k depends but on ρ . This implies

$$b_\nu^t b_\nu \le kc^2\cdot\Big(\prod_{\lambda\ne\nu} b_\lambda^t b_\lambda\Big)^{-1} \le k\cdot c^2\cdot\mu(m)^{1-i}\ .$$

As m is fixed and $b_\nu \in m\cdot\mathbb{Z}^i$, all b_ν and thus B must be from a finite set q.e.d..

3.6 Corollary: In a nonempty family of subgroups of $\mathbb{Z}^\mu$, those elements B with minimal $\mathrm{vol}(m\cdot B)$ for a fixed $m \in GL_\mu(\mathbb{R})$ form a finite nonempty subset.

3.7 We are now ready to apply these results to the situation of 3.1. For $h \in \mathcal{G}$ define

$$\mathcal{X}(h) := \{ N \in \mathcal{N} \mid \mathrm{vol}\, hN \le \mathrm{vol}\, hM \quad \forall\, M \in \mathcal{N} \} \quad .$$

According to 3.6, this is always a finite nonempty set. Note that $\langle h\cdot N\rangle$ (N integral) is maximal totally isotropic in $(\mathbb{R}^n,J)$ iff $N \in \mathcal{N}$. ny $\gamma \in \Gamma$ (which by left application causes a permutation of $\mathcal{N}$) and $u \in K$ gives

$$N \in \mathcal{X}(h) \quad \Rightarrow \quad \mathrm{vol}(uh\gamma\cdot\gamma^{-1}N) = \mathrm{vol}\, hN \le \mathrm{vol}(h\cdot\gamma M) = \mathrm{vol}\, uh\gamma M$$

for all $M \in \mathcal{N}$ (cf. 3.2). This translates to

$$\mathcal{X}(u\cdot h\cdot\gamma) = \gamma^{-1}\mathcal{X}(h) \quad . \tag{1}$$

In particular $\mathcal{X}$ descends to X . lso, let

$$k(h) := \mathrm{vol}\, hN \quad \text{for a} \quad N \in \mathcal{X}(h) \ ;$$

then (1) implies

$$k(u\cdot h\cdot\gamma) = k(h) \quad . \tag{2}$$

3.8 Our prospective fundamental domain is the following set:

$$F := \bigcup_{\omega\in\Omega} F_\omega \quad ,$$

where F_ω consists of all $h \in \mathcal{G}$ with the following properties:

(a) h has the form $h = \begin{pmatrix} \ell & -\ell p^t\omega & -\ell x \\ & \omega & p \\ & & \ell^{-t} \end{pmatrix} \cdot B_\omega^{-1}$ from 3.1(1), implying 3.1(2) ,

(b) $\mathrm{vol}\, hN \ge \mathrm{vol}\, hB_\omega I_{n,r} \quad [= \mathrm{vol}(I_{n,r}\cdot\ell) = \det \ell\ ] \quad \forall\, M \in \mathcal{N}$, or equivalently: $B_\omega I_{n,r}\mathbb{Z}^r \in \mathcal{X}(h)$, or likewise: $|\det \ell| = k(h)$,

(c) ℓ^{-t} is M′-reduced, i.e. y is M-reduced ,

(d) $\ell \in \mathcal{L}^+$ (say) ,

(e) if $\epsilon = -1$, $|x_{ij}| \le 1/2$ for all $1 \le i,j \le r$,

(f) if $\epsilon = 1$, we have

(f_1) $p_i \in \Delta_\omega$ (cf. 3.3) for all $1 \le i \le r$,

(f_2) p_1 is in a fundamental domain of the finite group $\mathrm{Aut}(\omega^{-t}\mathbb{Z}^s, I_s) \subset O_s(\mathbb{Q})$; this domain may be defined by demanding p_1 to have minimal euclidean distance in its orbit from a chosen non-fixed-point in $\mathbb{R}^n$;

(f_3) $|x_{ij} - x_{ji}| \le 1$ for all $1 \le i,j \le r$.

<u>3.9 Theorem:</u> Given any $h \in \mathcal{G}$ and $N \in \mathcal{X}(h)$, there are $\omega \in \Omega$, $\gamma \in \Gamma$ and $u \in K$ such that $\gamma \cdot N = B_\omega I_{n,r}\mathbb{Z}^r$ and $u \cdot h \cdot \gamma \in F_\omega$. Thus

$K \cdot F \cdot \Gamma = \mathcal{G}$ or, slovenly, ${}_U\backslash^{(U\cdot F)} \cdot \Gamma = X$.

<u>Proof:</u> Much like in 2.9, imagine that h is being changed to some $u \cdot h \cdot \gamma$ until $h \in F_\omega$. 3.1 yields conditions (a) and (b) of 3.8 already (keep in mind 3.7(1)). With a suitable $B_\omega \begin{pmatrix} c^{-t} & & \\ & I_s & \\ & & c \end{pmatrix} B_\omega^{-1} \in \Gamma$

($\Leftrightarrow c \in GL_r(\mathbb{Z})$ by 2.6(1)) , (c) is obtained. Achieve (d) as in 3.1, then we are still free to apply any $\gamma \in \Gamma$ of the form

$B_\omega \begin{pmatrix} I_r & * & * \\ & I_s & * \\ & & I_r \end{pmatrix} B_\omega^{-1}$. With 2.6 again, we see this to be equivalent to:

$$B_\omega^{-1} \cdot \gamma \cdot B_\omega = \begin{pmatrix} I_r & -Q^t\omega^t\omega & -Z \\ & I_s & Q \\ & & I_r \end{pmatrix} \in SL_n(\mathbb{Z}) \quad \text{with} \tag{1}$$

$$Z^t + \epsilon Z = Q^t \omega^t \omega Q \quad . \tag{2}$$

The effect of this application is

$$\ell \mapsto \ell \ , \tag{3}$$

$$p \mapsto p + \omega Q \ , \tag{4}$$

$$x \mapsto x + Z + p^t \omega Q \quad . \tag{5}$$

Let us first consider the case $\epsilon = -1$. We can find numbers Z_{ij} for $i \leq j$ such that $|Z_{ij} + x_{ij}| \leq 1/2$. This is true for $i > j$ as well if we set $Z_{ij} := -Z_{ji}$, as x is also symmetric [3.1(3)]. This Z makes a valid γ with (2), and produces (e) by (5) (all ω , p , Q etc. are "empty", thus $Q^t\omega^t\omega Q = 0$ etc.). -

Now let $\epsilon = 1$. Find columns $q_i \in L_\omega$ from 3.3 such that $p_i + q_i \in \Delta_\omega$ and put $Q_i := \omega^{-1} q_i \in \mathbb{Z}^s$ (!) . Another integral matrix is then given by $Z_{ij} := \begin{cases} 0 & i<j \\ 1/2\ Q_i^t \omega^t \omega Q_i & i=j \\ Q_i^t \omega^t \omega Q_j & i>j \end{cases}$ (this explains the use of L_ω).

These data satisfy (1) and (2), yielding (f_1) by (4). - Now a $\gamma = B_\omega \begin{pmatrix} I_r & m \\ & I_r \end{pmatrix} B_\omega^{-1} \in \Gamma$ (i.e. $m \in \mathrm{Aut}(\mathbb{Z}^s, \omega^t\omega)$) can be admitted; if (a) is restored by means of $u = \begin{pmatrix} I_r & \omega m^{-1} \omega^{-1} \\ & I_r \end{pmatrix} \in K$, nothing is lost: In fact the effect on h is $p \mapsto \omega m^{-1} \omega^{-1} p = \omega^{-t} m^t \omega^t p$, $\ell \mapsto \ell$, $x \mapsto x$; as $\omega^{-t} m^t \omega^t$ is an arbitrary element of $ut(\omega\mathbb{Z}^s, I_r)$, which group conserves L_ω and Δ_ω , (f_2) can be assumed additionally. -

At last, (1) can still be activated with $Q = 0$. Choosing Z_{ij} for $i < j$ to satisfy $|Z_{ij} - 1/2\ (x_{ji} - x_{ij})| \leq 1/2$, we find again that the same holds for $i > j$ if we stick to (2). Let us rewrite (5) as

$\tilde{\tilde{x}} = x + Z$, so we have $|\tilde{\tilde{x}}_{ij}-\tilde{\tilde{x}}_{ji}| = |x_{ij}+Z_{ij}-x_{ji}-Z_{ji}| =$
$2\cdot|1/2\ (x_{ij}-x_{ji}) - Z_{ji}| \le 1$, which settles (f_3). -

<u>3.10 Corollary:</u> <u>(i):</u> If $\epsilon = 1$ and δ_ω is the maximum of $(v^t v)^{1/2}$ for $v \in \Delta_\omega$ ("radius"; $\delta_\omega := 0$ if $s=0$), one obtains:

$$0 \le x_{ii} \le 1/2\ \delta_\omega^2 \quad \text{, and} \tag{1}$$

$$|x_{ij}| \le 1/2\ (\delta_\omega + 1) \quad . \tag{2}$$

<u>(ii):</u> p and x are bounded on F_ω (thus on F).

<u>Proof:</u> (i): From 3.1(3) we find $x_{ii} = 1/2\ p_i^t p_i \ge 0$, whence (1), and $|x_{ij}+x_{ji}| = |p_i^t p_j| \le \delta_\omega^2$ (Schwartz inequality), thus $|x_{ij}| =$
$1/2\ |(x_{ij}-x_{ji}) + (x_{ij}+x_{ji})| \le 1/2\ (\delta_\omega+1)$ by 3.6(f_3).
(ii) can be collected from 3.6(e,f_1) and (i).

<u>§ 4 Reduction constants and finiteness properties</u>

<u>4.0</u> In the case $\epsilon=-1$, $s=0$, the above amounts to the well-known Lagrange or Minkowski theory on $SL_2(\mathbb{Z})$. To extend some classical results about that to our general case, we make use of the traditional notion of Siegel sets (cf. [Bo],2.7). Also we shall define a fund of constants c_i , i.e. positive real numbers which depend only on r , s , and ϵ . All this, and the use of the letters x and y , is intended as a reverence for C.L. Siegel, to whom the results of §§ 3 and 4 are due in the case $\epsilon=-1$ ([Si], cf. 9.1 below) in a different formulation. The reader who prefers a "modern" language will not find it hard to translate this. -

From the proof of our "main reduction theorem" we forestall the principal step for future reuse. Note that

$$h \in F_\omega \Rightarrow k(h) = \det \ell = \det^{-1/2} y \quad , \text{ and} \tag{1}$$

$$\ell \in \mathcal{L}^+ \Rightarrow \ell_{11} = y_{11}^{-1/2} \quad . \tag{2}$$

<u>4.1 Proposition:</u> Let h be as in 3.1 with $\ell \in \mathcal{L}^+$. Assume that

$$\mathrm{vol}(h \cdot B_\omega \cdot I_{n,r}) \leq \mathrm{vol}\ h \cdot E \tag{1}$$

for all those $E \in M_{n,r}(\mathbb{Z})$ such that $E \cdot \mathbb{Z}^r \in \mathcal{N}$ and $B_\omega^{-1} E$ looks like $I_{n,r}$ but for the first column. Then

<u>(i):</u> in case $\epsilon = 1$, $\ell_{11}^2 \leq c_1$,

<u>(ii):</u> if however $\epsilon = -1$, $\ell_{11}^2 \leq (1 - x_{11}^2)^{-1/2}$ provided that $|x_{11}| \quad 1$.

<u>Proof:</u> We shall concoct a particular E of that kind. For an onset, write

$$B_\omega^{-1} E = \left(\begin{array}{c|c} \frac{\mu}{z} v^t v & 0 \\ 0 & I_{r-1} \\ \hline 2\frac{\mu}{z} \omega^{-1} \beta v & 0 \\ \hline -2\frac{\mu}{z} \beta^2 & 0 \\ 0 & \end{array} \right) ,$$

where $\mu = \mu(\omega)$ is the common denominator of the components of $2 \cdot \omega^{-1}$. Once $\beta \in \mathbb{Z}$ and $v \in \mathbb{Z}^s$ have been chosen arbitrarily, there is a $z \in \mathbb{N}$ so that the matrix remains integral and becomes primitive. Computing the isotropy property as usually, we find that E satisfies all our requirements.

Now let us choose $\beta \neq 0$ and v for the case $s \neq 0$ as follows: Abbreviating

$$\lambda := (\beta p_1 - v)^t (\beta p_1 - v) \quad ,$$

denoting the volume of the s-dimensional unit sphere by $d(s)$ and fixing some $0 < e < 1$, we demand

$$\lambda \leq \frac{e}{\mu} \quad , \text{ and} \tag{2}$$

$$|\beta| \le \frac{2^s \ \mu^{s/2}}{d(s)\ e^{s/2}} \quad . \tag{3}$$

This is not too immodest, because (2) and (3), read as conditions on *real* β and v, define a "skew cylinder" in $\mathbb{R}^{s+1}$ of Lebesgue volume 2^{s+1} ; the Minkowski lattice point theorem gives us integral such data, not both zero. In fact $\beta \neq 0$, otherwise (2) would say $v^t v \le \frac{e}{\mu} < 1$, i.e. $v = 0$. With this E, let us compute (1):

$$|\det \ell| = \operatorname{vol} hB_\omega I_{n,r} \le \operatorname{vol} hE = \operatorname{vol}\left(\begin{array}{c|c} \frac{\mu}{z}\ell_{11}\lambda & \begin{matrix}\ell_{22}\cdot & 0 \\ 0 & \cdot\ell_{rr}\end{matrix} \\ * & * \\ \hline 2\frac{\mu}{z}\beta(v-\beta p_1) & 0 \\ \hline -2\frac{\mu}{z}\ell_{11}^{-1}\beta^2 & 0 \\ 0 & \end{array}\right) \ldots$$

[the 1,1-entry may be deduced from $(\mathbb{R}^n, J)$-isotropy; now by 3.2(iii) this expression cannot get smaller if we replace $\frac{\mu}{z}$ by $\mu := \max(\mu(\omega))$, also 3.2(ii)(b) and 3.2(v) allow us to put 0 for $*$ and continue:]

$$\begin{aligned} &\le \mu\ (\ell_{11}^2\lambda^2 + 4\beta^2\lambda + 4\beta^4\ell_{11}^{-2})^{1/2}\cdot|\ell_{22}\cdot\ldots\cdot\ell_{rr}| \\ &= |\det \ell|\cdot\mu\cdot(\lambda + 2\beta^2\ell_{11}^{-2}) \quad . \end{aligned}$$

Thus $\mu(\lambda+2\beta^2\ell_{11}^{-2}) \ge 1$; $\ell_{11}^{-2} \ge \frac{1-\mu\lambda}{2\mu\beta^2} \ge \dfrac{1-e}{2\mu\dfrac{2^{2s}\mu^s}{d(s)^2 e^s}} > 0$ by (2) and (3), or

$$\ell_{11}^2 \le \frac{2^{2s+1}\mu^{s+1}}{d(s)^2(e^s - e^{s+1})} \quad .$$

The best choice of e would have been $\frac{s}{s+1}$, yielding

$$c_1 = \frac{2^{2s+1}\mu^{s+1}(s+1)^{s+1}}{d(s)^2 s^s} \quad .$$

The case $s=0$ is left to be dealt with. Or rather assume only $s\le 7$ (whence ω and B_ω are just I_* bz 2.4*i)) to obtain alternative constants lateron. Take $E = \begin{pmatrix} 0 & \begin{matrix} & \circ \\ I_{r-1} & \end{matrix} \\ \hline 0 & 0 \\ \hline \begin{matrix} 1 \\ 0 \end{matrix} & 0 \end{pmatrix}$. Then (1) reads:

$$|\det \ell| \le \operatorname{vol} \begin{pmatrix} \begin{matrix} -\ell_{11}x_{11} \\ * \end{matrix} & \begin{matrix} \ell_{22} \cdot & \circ \\ * & \cdot \ell_{rr} \end{matrix} \\ \hline p_1 & 0 \\ \hline \begin{matrix} \ell_{11}^{-1} \\ 0 \end{matrix} & 0 \end{pmatrix} = |\det \ell| \, (x_{11}^2+\ell_{11}^{-2}p_1^t p_1+\ell_{11}^{-4})^{1/2}$$

(derived the same way as above), which yields with 3.1(3):

$$\ell_{11}^2 \le (1-x_{11})^{-1} \quad \text{if} \quad \epsilon = 1 \quad \text{and} \quad |x_{11}| \quad 1 \quad , \tag{4}$$

in particular

$$\ell_{11}^2 \le 1 \quad \text{if} \quad \epsilon = 1 \quad \text{and} \quad s = 0 \quad ,$$

and also part (ii) of the proposition. -

<u>4.2 Theorem:</u> For $h \in F_\omega$, $y_{11} \ge c_2 \quad 0$ holds.

<u>Proof:</u> The conditions for proposition 4.1(i) are fulfilled by 3.8(a,b,d), thus

$$y_{11} \ge 1/c_1 \quad \text{(remember (4.0(2)) if} \quad \epsilon=1 \ .$$

Furthermore, 3.8(a,b,e) and 4.1(ii) result in

$$y_{11} \ge (1-x_{11}^2)^{1/2} \ge (3/4)^{1/2} \quad \text{for} \quad \epsilon=-1 \ , \ \text{q.e.d.} \ -$$

To obtain the promised alternative constants in the cases $1 \le s \le 7$ we use the fact that $\delta_{I_s} = \max\{ 1 \ , \ s^{1/2}/2 \}$ (proof by induction; L_{I_n} is known as "The D_n-Lattice"). So 3.10(1), 4.0(2), and 4.1(4) combine to

$$y_{11} \ge 1-x_{11} \ge 1 - 1/2 \ \delta_{I_s}^2 = \min\{1/2, \ 1-s/8\} > 0 \quad ,$$

since $s \le 7$ was presumed ("coincidence").

4.3 Corollary: $\kappa(h) \le c_3$ holds for all $h \in \mathcal{G}$.

Proof: Because of 3.7(2), it suffices to consider $h \in F_\omega$ for any ω . Thus y is M-reduced, the "Hermite inequality" runs as $y_{11} \le c_4 \det y$ (cf. [Sel], 2.2), whence by 4.0(2) and 4.2 :

$$\kappa(h) = (\det y)^{-1/2} \le (c_4 y_{11})^{-1/2} \le (c_4/c_2)^{1/2} =: c_3 \quad .$$

4.4 Corollary: There is a $g_\omega \in GL_n(\mathbb{Q})$ for each $\omega \in \Omega$, such that $F_\omega \cdot g_\omega$ is contained in a Siegel domain for $GL_n(\mathbb{R})$.

Proof: Set $\iota := \begin{pmatrix} & & 1 \\ & \cdot & \\ 1 & & \end{pmatrix} \in O_r(\mathbb{Z})$, $\underline{\iota} := \begin{pmatrix} \iota & & \\ & I_s & \\ & & I_r \end{pmatrix}$, and define

$g_\omega := B_\omega \begin{pmatrix} \iota & & \\ & \omega^{-1} & \\ & & I_r \end{pmatrix} \in GL_n(\mathbb{Q})$. Then for any $h \in F_\omega$ we obtain

$$h' := \underline{\iota} \cdot h \cdot g_\omega = \begin{pmatrix} \iota \ell \iota & -\iota \ell p^t & -\iota \ell x \\ & I_s & p \\ & & \ell^{-t} \end{pmatrix} \in O_n(\mathbb{R}) \cdot F_\omega \cdot g_\omega \cap \mathfrak{A}_n^+ \ .$$

We know already from 3.8(c) and [Bo], 2.7, that

$\ell^{-t} =: \operatorname{diag}(\ell_{11}^{-1}, \ldots, \ell_{rr}^{-1}) \cdot \rho$ lies in a fixed Siegel domain, i.e. $\rho \in \mathfrak{A}_r^+$ comes out of a fixed compact set and

$$0 < \ell_{ii}^{-1} \le c_5 \ell_{i+1,i+1}^{-1} \quad \forall\ 1 \le i \quad r \ . \tag{1}$$

The analogous decomposition of h' is

$$h' = \operatorname{diag}(\ell_{rr} \ldots \ell_{11}, 1 \ldots 1, \ell_{11}^{-1} \ldots \ell_{rr}^{-1}) \cdot \begin{pmatrix} \iota \rho^{-t} \iota & -\iota \rho^{-t} p^t & -\iota \rho^{-t} x \\ & I_s & p \\ & & \rho \end{pmatrix} \ . \tag{2}$$

Like ρ , the last matrix is bounded owing to 3.10(ii). The analogon of (1) holds by (1) itself, plus (this is the crucial point) 4.2, which contributes $\ell_{11}/1 = 1/\ell_{11}^{-1} = y_{11}^{-1/2} \le c_2^{-1/2}$. Since Siegel domains are invariant under left multiplication with $O_n(\mathbb{R})$, the proof is finished.

<u>4.5 Corollary</u> ("Siegel property"): The set $\{\gamma \in \Gamma | \ K\cdot F\cdot\gamma \cap K\cdot F \neq \phi \}$ ("neighbours") is finite.

<u>Proof:</u> Let Σ be a Siegel domain comprising all those (fininely many) Siegel domains mentioned in 4.3, so $KF_\omega g_\omega \subset O_n(\mathbb{R})\cdot\Sigma = \Sigma$, $g_\omega \in GL_n(\mathbb{Q})$. If $KF\gamma \cap KF \neq \phi$, there are $\omega, \psi \in \Omega$ with $KF_\omega\gamma \cap KF_\psi \neq \phi$, thus $\Sigma g_\omega^{-1}\gamma \cap \Sigma g_\psi^{-1} \neq \phi$. The "Siegel theorem" (cf. [Bo], Th. 4.6) tells us that this holds for only finitely many $\gamma \in GL_n(\mathbb{Z})$ even.

<u>4.6 Corollary:</u> If $h \in F_\omega$, $\mathfrak{X}(h)$ it is taken out of a finite stock.

<u>Proof:</u> 3.9 applied to $M \in \mathfrak{X}(h)$ gives us some $u\cdot h\cdot\gamma \in F_\psi$, thus $h \in F_\omega \cap KF_\psi\gamma$, which means that γ is of the sort 4.5. In 3.9 we had $M = \gamma B_\omega \mathbb{Z}^r$, proving the claim.

<u>4.7 Corollary:</u> $\#\mathfrak{X}(h)$ is bounded on $\mathcal{G}$ (resp. on X).

<u>Proof:</u> This is a direct consequence of 4.6, since by 3.7(1), it suffices to consider $h \in F_\omega$.

<u>4.8(i) Lemma:</u> The conditions 3.8(b) follow from finitely many of them.

<u>Proof:</u> Such a condition is indispensable with a particular $M \in \mathcal{N}$ iff there is a $h \in \mathcal{G}$ satisfying all 3.8, except that $\mathrm{vol}\, hM < \mathrm{vol}\, hB_\omega I_{n,r}$ violates 3.8(b). Then $h^{(\tau)} := \begin{pmatrix} \tau^{-1}I_r & & \\ & I_s & \\ & & \tau I_r \end{pmatrix}\cdot h \quad (\tau\geq 1)$ preserves 3.8(a), (c) through (f), and in fact 3.8(b) for $N \neq M$, since

$$\mathrm{vol}(h^{(\tau)}B_\omega I_{n,r}) = \mathrm{vol}\begin{pmatrix}\tau^{-1}\ell\\ 0\\ 0\end{pmatrix} = \tau^{-r}\ \mathrm{vol}\ hB_\omega I_r \quad , \tag{2}$$

whereas, if $\begin{pmatrix}A\\ B\\ C\end{pmatrix}$ is a $\mathbb{Z}$-basis of hN ,

$$\mathrm{vol}(h^{(\tau)}N) = \tau^{-r}\ \det{}^{1/2}(A^tA+\tau^2B^tB+\tau^4C^tC) \geq \tau^{-r}\ \mathrm{vol}\ hN \quad . \tag{3}$$

Now choose a basis m of M whose first i columns span the sublattice $M \cap B_\omega I_{n,r}\mathbb{Z}^r$ (of course, $i < r$). Then $h\cdot m = \left(\begin{array}{c|c} a & * \\ 0 & * \\ \hline & * \\ \hline & 0 \\ & b \end{array}\right)$

with $\det a \neq 0 \neq \det b$ by 1.3, hence (use 3.2(ii,b) etc.)

$$\mathrm{vol}(h^{(\tau)}M) \geq \tau^{-i}|\det a|\cdot\tau^{r-i}|\det c| \quad 0 \quad . \tag{4}$$

From (2), (4) and the fact that everything is continuous in τ , one deduces that there must be some $\tau > 1$ such that $\mathrm{vol}(h^{(\tau)}M) = \mathrm{vol}(h^{(\tau)}B_\omega I_{n.r})$, which means $h^{(\tau)} \in F_\omega$, $M \in \mathfrak{X}(h^{(\tau)})$. 4.6 shows that only a finite number of M can occur this way. -

(ii) The reader who feels like some more exercise may convince himself by these means that each F_ω is nonempty and in fact noncompact, and (more important) that all hypersurfaces of $\mathcal{G}$ or X defined by $\mathrm{vol}\ M - \mathrm{vol}\ N = 0$ ($M, N \in \mathcal{N}$) are nonsingular.

4.9 Corollary: Each F_ω , and thus F , is closed.

Proof: All the inequalities of 3.8 describe closed sets. It remains to show that finitely many of them suffice to define F . This is known for 3.8(c) (cf. [Wa],1,§8 for a proof analogous to the above one), and 4.8 settles 3.8(b).

4.10 In 4.6 one may have noticed that the "reason" for F to be noncompact is that $k(h)$ can become arbitrarily small (see also 5.2 below). Indeed, we have a

Corollary: Given any $k > 0$, $F(k) := \{h \in F \mid k(h) \geq k\}$ is compact.

Proof: This set is closed as we have seen in 4.9, so we are done once we know that each $(F_\omega \cap F(k))\cdot g_\omega$ lies inside some compact set. In fact it is contained in a Siegel set by 4.4, thus all that is left to do is to find a positve lower bound for ℓ_{rr} in 4.4(2) (which then supplies an upper bound for ℓ_{rr}^{-1} free of charge). From 4.4(1) we compute:

$$\begin{aligned}\ell_{rr} &= \ell_{11}^{-1}\cdot\ldots\cdot\ell_{r-1,r-1}^{-1}\cdot\det \ell \geq c_5^{-(r-1)(r-2)/2}\ell_{11}^{-(r-1)}\det \ell \\ &= c_5' y_{11}^{(r-1)/2}(\det y)^{-1/2} \geq c_5' c_2^{(r-1)/2}(\det y)^{-1/2} \qquad \text{(by 4.2!)} \\ &\geq c_5' c_2' k \quad \text{(by 4.0(1), 3.8(b)[third version], and hypothesis).}\end{aligned}$$

§ 5 The retraction

5.0 To understand the deformation retraction (of $\mathcal{G}$ or X) defined below, it might be helpful to imagine the orbit space $X/_\Gamma$ as a set of "rotating lattices" $K\cdot h\cdot\mathbb{Z}^n$. Roughly speaking, the process then amounts for such a lattice to finding the intersection of the "smallest totally isotropic sublattices" and inflating its $\mathbb{R}$-span while compressing some perpendicular $\mathbb{R}$-space, until a further such "smallest sublattice" exists. This procedure (the principle of which is similar to the one used in [So] and [As] and in fact dates back to Procrustes) is repeated as many times as possible. - We start with some lemmas on $\mathcal{N}(*)$, of which 5.4 is most important:

5.1 Lemma: For any (relatively) compact set $C \subset \mathcal{G}$ the set $\bigcup_{h\in C} \mathcal{N}(h)$ is finite.

Proof: If $h \in C$, $N \in \mathcal{N}(h)$, 4.3 and 3.2(ix) yield: $c_3 \geq k(h) = \mathrm{vol}\ hN \geq \lambda_r(h)\ \mathrm{vol}\ N \geq \lambda(C)\ \mathrm{vol}\ N$ (with some $\lambda(C) > 0$); 3.5 grants the lemma.

5.2 Corollary: $k(*)$ is continuous on $\mathcal{G}$ (or X).

Proof: Choose a relatively compact neighbourhood C of $h \in \mathcal{G}$. Thereon $k(h')$ is the minimum of $\mathrm{vol}\ h'N_i$ (N_i provided by 5.1), i.e. of finitely many continuous functions.

5.3 Lemma: Given $M \subset \mathcal{N}$, $H(M) := \{h \in \mathcal{G} \mid M \subset \mathcal{N}(h)\}$ is closed.

Proof: Straightforward from the definition.

5.4 Lemma: For $h \in \mathcal{G}$ there is a neighbourhood U such that $\mathcal{N}(h') \subset \mathcal{N}(h)$ for all $h' \in U$.

Proof: Take C and the N_i from 5.2, $H(*)$ from 5.3, then $U = C - \bigcup_{\{i \mid N_i \notin \mathcal{N}(h)\}} H(\{N_i\})$ fulfills the requirements.

5.5 Our deformation will run along geodesics of $\mathcal{G}$. This notion presumes a totally isotropic subspace S of $(\mathbb{R}^n, J)$, which will later be obtained as indicated in 5.0.

For any subspace $V \subset \mathbb{R}^n$ (with ONB $\{v_i\}$), let

$$\pi_V : \mathbb{R}^n \to V \quad [\ w \mapsto \sum_i (v_i^t w) v_i\ ] \tag{1}$$

denote the Euclidean orthogonal projection onto V. We can now define

$$\rho_S(t) \in M_n(\mathbb{R}) \quad \text{for } t \quad 0$$

by postulating that the corresponding linear application be (remember 1.2(1))

$$w \mapsto t \cdot \pi_S(w) + \pi_{(S^{\approx} \cap S^{\perp})}(w) + t^{-1} \cdot \pi_{S^{\approx\perp}}(w) \quad . \tag{2}$$

5.6 Remarks:

(i): With a $u \in K$ from 1.2(i), we have

$$\rho_S(t) = u \cdot \left(\begin{array}{cc|c|cc} tI_d & & & & \\ & I_{r-d} & & & \\ \hline & & I_s & & \\ \hline & & & t^{-1}I_d & \\ & & & & I_{r-d} \end{array} \right) \cdot u^{-1} \quad , \quad \text{hence} \quad \rho_S(t) \in G \quad .$$

(ii): $\rho_S(1) = I_n$.

5.7 Define:

$R(h) := \bigcap_{N_i \in \mathcal{N}(h)} N_i$ (which is a sublattice, possibly $\{0\}$),

$\mathcal{G}_i := \{h \in \mathcal{G} \mid \operatorname{rk} R(h) \le i\}$.

Note that

$$\mathcal{N}(h') \subset \mathcal{N}(h) \Rightarrow R(h') \supset R(h) \quad . \tag{1}$$

By 3.7(1), these and all the following constructions descend to X (thus X_i makes sense), furthermore, Γ acts on $\mathcal{G}_i$. As soon as we have seen that each $\mathcal{G}_i$ deforms to its subspace $\mathcal{G}_{i-1}$, we know that $\mathcal{G}_0$ is a deformation retract of $\mathcal{G}_r = \mathcal{G}$. So let us regard some $i \ge 1$ as fixed. For an R of rank i put

$$\mathscr{G}_R := \{h \in \mathscr{G} \mid R(h) = R\} \subset \mathscr{G}_i - \mathscr{G}_{i-1} \quad .$$

These form a disjoint union

$$\mathscr{G}_i = \mathscr{G}_{i-1} \;\dot{\cup}\; \dot{\bigcup_R} \mathscr{G}_R \quad .$$

The $\mathscr{G}_R$ are open relative to $\mathscr{G}_i$ (5.4; thus $\mathscr{G}_{i-1}$ is closed), and in fact the relative boundaries $\overline{\mathscr{G}_R} - \mathscr{G}_R$ are all contained in $\mathscr{G}_{i-1}$:

$$R(h) \subsetneq R \quad \text{for} \quad h \in \overline{\mathscr{G}_R} - \mathscr{G}_R \tag{2}$$

(use 5.4 and (1)). Therefore it suffices to deform each $\overline{\mathscr{G}_R}$ onto this boundary. Assume from now on R fixed as well. Let us abbreviate

$$h_{(\tau)} := \rho_{\langle h \cdot R\rangle}(\tau)\cdot h \quad \in \mathscr{G} \quad (\text{for} \quad h \in \mathscr{G} \ , \ \tau > 0) \quad .$$

$$h_{(1)} = h \quad (5.6(\text{ii})) \ . \tag{3}$$

<u>5.8 Lemma:</u> $h_{(\tau)}$ is continuous in h and τ .

<u>Proof:</u> Show that the u in 5.6(i) for $S = \langle hR\rangle$ can be chosen to depend continuously on h .

<u>5.9</u> In order to study the behaviour of our volumes under this first approach of a deformation, we define:

$$\varphi(h,\tau,M) := \frac{\operatorname{vol} h_{(\tau)}M}{\tau^i \cdot k(h)} \quad (h \in \mathscr{G} \ , \ \tau > 0 \ , \ M \in \mathscr{N}\ ) \tag{1}$$

(recall that $i = \operatorname{rk} R$ is fixed). We know from 5.8 and 5.2 that this function is continuous in h and τ , also by definition

$$\varphi(h,1,M) \geq 1 \quad \forall\, h \ , \ M \quad , \tag{2}$$

$$\varphi(h,1,M) = 1 \iff M \in \mathscr{M}(h) \quad . \tag{3}$$

The following lemma forms the most important cog wheel of this machinery:

5.10 Lemma: Let $h \in \mathcal{G}$, $M \in \mathcal{N}$.

(a) If $R \subset M$, $\varphi(h,\tau,M)$ is constant in τ ;

(b) Otherwise $\varphi(h,\tau,M)$ is strictly decreasing in τ .

Proof: Write $P := \langle h\cdot R\rangle$, $L := \langle h\cdot M\rangle$, m some basis of M . Choose an $u \in O_n(\mathbb{R})$ that respects the Euclidean-orthogonal decomposition

$$\mathbb{R}^n = (L \cap P) \perp (L \cap P^{\approx} \cap P^{\perp}) \perp (L \cap P^{\approx\perp}) \perp ((L \cap P)^{\perp} \cap P) \\ \perp ((L \cap P^{\approx} \cap P^{\perp})^{\perp} \cap P^{\approx} \cap P^{\perp}) \perp ((L \cap P^{\approx\perp}) \cap P^{\approx\perp}) \qquad (1)$$

with dimensions, say,

$$\alpha,\ \beta,\ \gamma,\ i\text{-}\alpha,\ \omega, \text{ and } i\text{-}\gamma . \qquad (2)$$

Also choose some $b \in SL_r(\mathbb{R})$ such that the columns of $h\cdot m\cdot b$ respect

$$L = hmb\mathbb{R}^r = (L \cap P) \perp (L \cap P^{\approx} \cap P^{\perp}) \perp (L \cap P^{\approx\perp}) \perp (L \cap Q^{\perp})$$

(Q simply denoting the span of the previous three spaces). Thus

$$\rho_{\langle hR\rangle}(\tau) = u\cdot\begin{pmatrix} \tau I_\alpha &&&&& \\ & I_\beta &&&& \\ && \tau^{-1}I_\gamma &&& \\ &&& \tau I_{i-\alpha} && \\ &&&& I_\omega & \\ &&&&& \tau^{-1}I_{i-\gamma}\end{pmatrix}\cdot u^{-1} , \quad hmb = u\cdot\begin{pmatrix} A && \\ & B & \\ && C \\ &&& D \\ &&& D' \\ &&& D''\end{pmatrix} .$$

The D^* have $\delta := r\text{-}\alpha\text{-}\beta\text{-}\gamma$ columns. We compute, using 3.5(i) etc.:

$$\tau^{-i}\text{vol } h_{(\tau)}M = \tau^{-i}\text{vol}(u^{-1}\rho_{\langle hR\rangle}(\tau)u\cdot u^{-1}hmb)$$
$$= \tau^{-i}|\det \tau A|\,|\det B|\,|\det \tau^{-1}C|\cdot\det{}^{1/2}(\tau^2 D^tD + D'^tD' + \tau^{-2}D''^tD'')$$
$$= \tau^{-i+\alpha-\gamma+\delta}|\det A \det B \det C|\cdot\det{}^{1/2}(D^tD + \tau^{-2}D'^tD' + \tau^{-4}D''^tD''). \qquad (4)$$

Ad (a): We have $P \subset L \subset P^{\approx}$ by total istropy, (3) and (1) thus yield $\gamma = 0 = \delta$ and $i\text{-}\alpha = 0$, hence (4) reads: $t^{-i}\text{vol } h_{(\tau)}M = |\det A \det B|$

which settles this case.

Ad (b): The dimension formula contributes $\beta+\gamma = \dim(L\cap P^{\perp}) \geq n-(n-r+i) = r-i$, or $-i+\alpha-\gamma+\delta = -i+r-\beta+2\gamma \leq -\gamma \leq 0$, thus $\tau^{-i+\alpha-\gamma+\delta}$ is decreasing, even strictly so if $\gamma \neq 0$. Also, if one rewrites out of (4): $D^tD+\tau^{-2}D'^tD'+\tau^{-4}D''^tD'' = D^tD+\tau^{-2}\begin{pmatrix}D'\\D''\end{pmatrix}^t\begin{pmatrix}D'\\D''\end{pmatrix}+(\tau^{-4}-\tau^{-2})D''^tD''$, one can see that making τ smaller amounts to adding a positive semidefinite matrix, which is definite if $\mathrm{rk}\begin{pmatrix}D'\\D''\end{pmatrix} = \delta > 0$ .. Therefore we are through if we either show the latter condition or $\gamma \neq 0$. Now $P \not\subseteq L \not\subseteq \tilde{\tilde{P}}$ (L being maximal totally isotropic); by (3) $\delta = 0$ implies $\gamma \neq 0$. Thus only the case $\delta \neq 0$ is left: Assume $z \in \mathbb{R}^{\delta}$ with $\begin{pmatrix}D'\\D''\end{pmatrix}\cdot z = 0$, then $hmb\cdot\begin{pmatrix}0\\z\end{pmatrix}$ lies in $u\cdot\begin{pmatrix}0\\D\\0\end{pmatrix}\cdot\mathbb{R}^{\delta} \subset (L \cap P)^{\perp} \cap P$ (cf. (1)), but also in $hmb\mathbb{R}^r = L$, hence it is 0 , and $z = 0$ since hmb is a basis. So the rank claim is proved as well. -

<u>5.11 Corollary:</u> For $h \in \mathcal{G}_R$ and $N \in \mathcal{X}(h)$, 5.9(3) and 5.10(b) yield

$$\varphi(h,\tau,N) = 1 \ , \text{ i.e. } \mathrm{vol}\ h_{(\tau)}N = \tau^{i}k(h) \quad \forall\ \tau > 0 \ , \tag{1}$$

providing a more suggestive version of 5.9(1):

$$\varphi(h,\tau,M) = \frac{\mathrm{vol}\ h_{(\tau)}M}{\mathrm{vol}\ h_{(\tau)}N} \quad \forall\ M \in \mathcal{N} \ . \tag{2}$$

<u>5.12</u> The value of τ at which the deformation will be stopped is

$$\tau_o(h) := \inf\{\tau \geq 1 \mid \exists\ M : \ R \not\subseteq M \ , \ \varphi(h,\tau,M) = 1\} \quad (h\in\overline{\mathcal{G}_R})$$

(admit ∞ for the moment).

5.13 Lemmas:

(i): From 5.7(2) and 5.9(3) deduce

$$\tau_o(h) = 1 \quad \text{if} \quad h \in \overline{\mathscr{G}_R} - \mathscr{G}_R \quad .$$

(ii): Now assume $h \in \mathscr{G}_R$. 5.9 then says that $\varphi(h,1,M)$ 1 if $M \notin \mathfrak{M}(h)$, in particular if $R \not\subseteq M$. φ being contionuous, we see that $\tau_o(h)$ is actually the infimum of all $\tau \geq 1$ such that

$$\varphi(h,\tau,M) \leq 1 \quad \forall\, R \not\subseteq M \quad (\text{or} \quad \forall\, M \notin \mathfrak{M}(h) \ , \text{ using } 5.10(a)),$$

or equivalently (by 5.11(2) and 5.10(a)) such that

$$\mathfrak{M}(h_{(\tau)}) \neq \mathfrak{M}(h) \quad . \tag{1}$$

5.11(2) then offers

$$\boldsymbol{k}(h_{(\tau)}) = \tau^{i}\, \boldsymbol{k}(h) \quad \forall\, 1 \leq \tau < \tau_o(h) \quad ,$$

thus the hard-earned boundedness assertion 4.3 yields

$$\tau_o(h) < \infty$$

and we can write:

$$h_o := h_{(\tau_o(h))} \in \mathscr{G} \quad \text{for all} \quad h \in \overline{\mathscr{G}_R} \ .$$

(iii): Furthermore, one can replace "inf" by "min" . Indeed, for $h \in \mathscr{G}_R$ (otherwise (i) hits the point), the opposite assumption $\mathfrak{M}(h_o) = \mathfrak{M}(h)$ (cf. (1)) would imply $\mathfrak{M}(h_{(\tau_o(h)+\eta)}) \subset \mathfrak{M}(h)$ $(0 \leq \eta \leq \text{some } \eta_o > 0)$ by 5.4 and continuity, nay even "=" using 5.10(a). That would mean $\varphi(h,\tau_o(h)+\eta,M) > 1 \ \ \forall\, M \notin \mathfrak{M}(h),\ 0 \leq \eta \leq \eta_o$ (cf. 5.11(2)) contradicting the definition.

5.14 Lemma: Let us point out:

(i) $h \in \mathcal{G}_R \Rightarrow h_{(\tau)} \in \mathcal{G}_R \quad \forall\, 1 \le \tau < \tau_o(h)$;

(ii) $h \in \overline{\mathcal{G}_R} \Rightarrow h_o \in \overline{\mathcal{G}_R} - \mathcal{G}_R$.

Proof: (i) is immediate from 5.13(1). By continuity this and 5.13(i) imply $h_o \in \overline{\mathcal{G}_R}$. But an $N \in \mathcal{X}(h_o)$ responsible for (1) cannot have $R \subset N$ (5.10(a)!), which means $h_o \notin \mathcal{G}_R$.

5.15 Now we can define our deformation

$$\zeta_R : \overline{\mathcal{G}_R} \times [0,1] \to \overline{\mathcal{G}_R} ,$$
$$(h,\omega) \mapsto h_{(\tau_o(h)^{\omega})} .$$

We have seen that we are indeed moving inside $\overline{\mathcal{G}_R}$ and ending up with the untouched subspace $\overline{\mathcal{G}_R} - \mathcal{G}_R$. The only (and most crucial) assertion left to prove is

5.16 Lemma: $\tau_o(h)$ is continuous in $h \in \overline{\mathcal{G}_R}$.

Proof: We shall find some relatively open subset U of $\overline{\mathcal{G}_R}$ with $h \in U$ (h regarded as fixed), on which τ_o is continuous. Denote

$$\{N_j \mid j \in I \text{ (finite nonempty index set)}\} := \{N \in \mathcal{X}(h_o) \mid R \not\subseteq N\} .$$

We find $\varphi(h,\tau_o(h),N_j) = 1$ (5.11(4) resp. 5.9(3)) and $\varphi(h,\tau_o(h)+1987,N_j) < 1$ (5.10(b)) for all j . By continuity there is some U' , relatively open in $\mathcal{G}_R$, where $\varphi(h',\tau_o(h)+1987,N_j) < 1$ $\forall\, h' \in U'$, $j \in I$ (I is finite). But we had $\varphi(h',1,N_j) \ge 1$, so strict monotony (5.10(b)) gives us unique $\tau_j(h') \in [1, \tau_o(h)+1987]$, such that $\varphi(h',\tau_j(h'),N_j) = 1$. Each of these is a bounded function having a closed graph (by uniqueness and continuity, behold the pivot point!), thus it is continuous. Therefore $\sigma_j(h') := h'_{\tau_j(h')}$ is another continuous function $U' \to \mathcal{G}$. Now let V be a neighbourhood of h_o in $\mathcal{G}$ where $\mathfrak{X}(h'') \subset \mathfrak{X}(h_o)$ $\forall\, h'' \in V$ according to 5.4. Then

$$U := \bigcap_{j\in I} \sigma_j^{-1}(V)$$

is open with respect to U' , thus to $\overline{\mathcal{G}_R}$ as well, and contains h , because $\sigma_j(h) = h_{(\tau_j(h))} = h_{(\tau_o(h))} \in V \quad \forall j$. A last continuous function is given by $\tau'(h') := \min\{\tau_j(h') | j \in I\}$. The claim

$$\tau_o(h') = \tau'(h') \quad \forall\, h \in U$$

would finish the proof. Now "$\leq$" can be assembled from the definitions. The assumption "$<$" on the other hand implies that there is some $h' \in U$ and M with $\varphi(h',\tau_o(h'),M) = 1$ and $\varphi(h',\tau'(h'),M) < 1$, so no element of $\mathfrak{X}(h_o)$ is in $\mathfrak{X}(h'_{(\tau'(h'))})$ since we had $\varphi(h',\tau'(h),N_j) \geq 1 \quad \forall j$.(always use 5.10(b)). But fixing a $j(h') \in I$ for $h' \in U$ such that $\tau'(h') = \tau_{j(h')}(h')$ holds, yields $h'_{(\tau'(h'))} = \sigma_{j(h')}(h')$; also the definitions of U and V demand $\mathfrak{X}(\sigma_{j(h')}(h')) \subset \mathfrak{X}(h_o)$, whence the desired contradiction.

<u>5.17 Theorem:</u> $X_0 := {}_K\backslash^{\mathcal{G}}0 = \{Kh \in X | \bigcap_{N\in\mathfrak{X}(h)} N = \{0\}\}$ has trivial homotopy type, and Γ operates on it (discontinuously and properly).

<u>Proof:</u> X was seen in 1.2(iv) to be contractible, so the above homotopy equivalence, which also makes sense on X (even on $X/_\Gamma$), transfers it to X_0 . The other assertions are also inherited from X (well known)

and will result once more from § 7. -

§ 6 Compactness

6.0 We want to see by means of 4.10 that $\mathcal{G}_0 \cap F$ is compact. Put

$$\mathcal{G}_0' := \{h \in \mathcal{G} \mid \exists\, N_1, N_2 \in \mathcal{X}(h) : N_1 \cap N_2 = \{0\}\} \quad \text{and}$$

$$\mathcal{G}_0'' := \{h \in \mathcal{G}_0 \mid N_i \cap N_j \neq \{0\} \quad \forall\, N_i, N_j \in \mathcal{X}(h)\} \quad ,$$

thus

$$\mathcal{G}_0 = \mathcal{G}_0' \,\dot{\cup}\, \mathcal{G}_0'' \quad . \tag{1}$$

6.1 Theorem: $h \in \mathcal{G}_0' \Rightarrow k(h) \geq 1$.

Proof: By 3.7(2) and 3.9 we may assume $h \in F_\omega \cap \mathcal{G}_0'$ and with some pair N_1, N_2 from the definition: $B_\omega I_{n,r}\mathbb{Z}^r = N_1$; $B_\omega^{-1}N_2 =: \begin{pmatrix} A \\ B \\ C \end{pmatrix} \cdot \mathbb{Z}^r$, $\begin{pmatrix} A \\ B \\ C \end{pmatrix} \in M_{n,r}(\mathbb{Z})$. Actually, $\det C \neq 0$, because 1.3 applies. Hence

$$k(h) = \operatorname{vol} hN_2 = \operatorname{vol} \begin{pmatrix} * \\ * \\ \ell^{-t}C \end{pmatrix}$$

$$\geq |\det \ell^{-t}C| = \det^{-1/2}(\ell^t\ell)\cdot|\det C| = k(h)^{-1}|\det C| \geq k(h)^{-1},$$

as claimed.

6.2 To show the corresponding property for $\mathcal{G}_0''$, we need an important

Lemma: There is a constant $c_6 > 0$ such that for any $h \in \mathcal{G}$ and any totaly isotropic sublattice T of $(\mathbb{Z}^n, J_\kappa)$ there is a $T' \in \mathcal{N}$ with $T \subset T'$ and $\operatorname{vol} hT' \leq c_6 \cdot \operatorname{vol} h$.

<u>Proof:</u> From 3.2(i,ii) we see that this claim only concerns the "rotating lattice" $Kh\Gamma$ and therefore need only be proved for representatives. Also put $\operatorname{rk} T =: \tau < r$ (otherwise the lemma is trivial). 3.6 guarantees a $T' \in \mathcal{N}$ with $T \subset T'$ and

$$\operatorname{vol} hT' \le \operatorname{vol} hN \quad \text{for all} \quad N \in \mathcal{N} \quad \text{containing} \quad T \ . \tag{1}$$

Entering into 3.1 with an anterior completion of some basis of T to one of T', we may assume besides 3.1(1) (with $\ell \in \mathfrak{L}^+$):

$$T' = B_\omega \cdot I_{n,r} \cdot \mathbb{Z}^r \quad , \qquad T = B_\omega \cdot \begin{pmatrix} 0 \\ I_\tau \\ \hline 0 \\ \hline 0 \end{pmatrix} \cdot \mathbb{Z}^\tau \quad . \tag{2}$$

Write $\ell^{-t} = \begin{pmatrix} \lambda & * \\ 0 & * \end{pmatrix}$ with $\lambda \in GL_\rho(\mathbb{R})$, $\rho := r - \tau$. Now let us proceed similarly as in 3.9: Apply $\gamma := B_\omega \cdot \left(\begin{array}{c|c|c} \begin{matrix} m & \\ & I_\tau \end{matrix} & & \\ \hline & I_s & \\ \hline & & \begin{matrix} m^{-t} & \\ & I_\tau \end{matrix} \end{array}\right) \cdot B_\omega^{-1}$ to make λ M'-reduced, renew $\ell^{-t} \in \mathfrak{A}^+$, and in case $\epsilon = -1$ achieve $|x_{11}| \le 1/2$ by means of some $\gamma = \begin{pmatrix} I_r & -Z \\ & I_r \end{pmatrix}$. - Now any $E \cdot \mathbb{Z}^r \in \mathcal{N}$ with

$B_\omega^{-1} E = \begin{pmatrix} * & 0 \\ & I_{r-1} \\ \hline * & 0 \\ \hline * & 0 \end{pmatrix}$ contains T by (2), so (1) implies $\operatorname{vol} hT' \le \operatorname{vol} hE$ for all such E . But these are precisely the conditions of 4.1, which yields $\ell_{11} \le c_1^{1/2}$ resp. $\ell_{11} \le 2^{1/2} \cdot 3^{-1/4}$. If η_ρ denotes the familiar Hermite constant, we also have

$\ell_{11}^{-\rho} = \lambda_{11}^{\rho} \le \eta_\rho^{\rho/2} \det \lambda = \eta_\rho^{\rho/2} \prod_{j=1}^{\rho} \ell_{jj}^{-1}$. All this composes to

$$\operatorname{vol} hT = \det \begin{pmatrix} \ell_{\rho+1,\rho+1} & & \\ & \ddots & \\ * & & \ell_{rr} \end{pmatrix} = \det \ell \cdot \prod_{j=1}^{\rho} \ell_{jj}^{-1}$$

$$\geq \ell_{11}^{-\rho}\eta_\rho^{-\rho/2}\det \ell \geq \psi_\rho \det \ell = \psi_\rho \text{ vol } hT' \quad (\text{some } \psi_\rho > 0)$$

by (2). One may (or may not for tighter calculation) take the smallest ψ_ρ to obtain the lemma.

<u>6.3 Theorem:</u> $k(h) \geq c_7$ for all $h \in \mathcal{G}_0^n$.

<u>Proof:</u> Fix $h \in \mathcal{G}_0^n$ and an arrangement $\mathcal{X}(h) =: \{N_j \mid j=1,\ldots,\#\mathcal{X}(h)\}$.

$$S_i := \bigcap_{j=1}^{i} N_j \quad ;$$

and let σ be the largest i with $S_i \neq \{0\}$. The definition of $\mathcal{G}_0^n$ tells us $2 \leq \sigma < \#\mathcal{X}(h)$. In order to show inductively the existence of constants k_i such that

$$\text{vol } hS_i \leq k_i \cdot k(h) \quad , \tag{1}$$

assume $i < \sigma$ fixed as well. Similarly to 6.2, take $h' :=: uh\gamma$ from 3.1 of the form 3.1(1) with $\ell := \ell(h') \in \mathcal{U}^+$ (!) and

$$\gamma B_\omega I_{n,r}\mathbb{Z}^r = N_{i+1} \quad , \quad \gamma B_\omega I_{n,\alpha}\mathbb{Z}^\alpha = S_{i+1} \quad ; \tag{2}$$

As $S_{i+1} = S_i \cap N_{i+1}$ (and $B_\omega^{-1}\gamma^{-1}S_i$ is totally isotropic in $(\mathbb{R}^n, J_\omega)$), 1.3 is applicable: $S_i = \gamma B_\omega \left(\begin{array}{cc} I_\alpha & * \\ o & \\ \hline 0 & * \\ \hline 0 & \begin{array}{c} o \\ b \end{array} \end{array}\right) \in M_{n,\alpha+\beta}(\mathbb{Z})$, where $\text{rk } b = \beta$. Writing $\ell =: \begin{pmatrix} \mu & * \\ 0 & \nu \end{pmatrix}$ ($\nu \in GL_\beta(\mathbb{R})$), and using η_* as in 6.2, we thus have (employ 3.2 as usually):

$$\text{vol } hS_i = \text{vol } h'\gamma^{-1}S_i = \text{vol} \left(\begin{array}{cc} \mu & * \\ o & \\ \hline 0 & * \\ \hline & o \\ 0 & \nu^{-t}b \end{array}\right) \geq \det \mu \det{}^{1/2}(b^t\nu^{-1}\nu^{-t}b)$$

$$\geq \text{vol } h'\gamma^{-1}S_{i+1} \cdot \eta_\beta^{\beta/2} \min{}^{\beta/2}\{(\nu^{-t}bz)^t(\nu^{-t}bz) \mid z \in \mathbb{Z}^\beta - \{0\}\}$$

$$\geq \text{vol } hS_{i+1} \cdot \eta_\beta^{\beta/2} \min{}^{\beta/2}\{(\ell^{-t}z)^t(\ell^{-t}z) \mid z \in \mathbb{Z}^r - \{0\}\}$$

$$\geq \text{vol } hS_{i+1} \cdot \eta_\beta^{\beta/2} \cdot c_2^{\beta/2} \tag{3}$$

(to obtain the last assertion from 4.2 and 3.8(c), observe that the respective minimum is not changed by the process of 3.9 if we start it with h' and $I_{n,r}\mathbb{Z}^r$). This settles (1). In fact we can take

$$k_2 = 1 \tag{4}$$

since for $i=1$ (3) reads: $k(h) = \text{vol } hS_1 \geq \det \mu \det^{-1}\nu \, |\det e|$ $\geq \det^2\mu \det^{-1}\ell \cdot 1 = \text{vol}^2 hS_2 \, (k(h))^{-1}$. Now, σ being bounded (4.7),

$$\text{vol } hS_\sigma \leq c_8 \, k(h) \tag{5}$$

holds. If we set

$S := N_1 \cap N_{\sigma+1}$ ($\neq \{0\}$ by definition of $\mathcal{G}_0^n$), and

$T := \langle S \cup S_\sigma \rangle \cap \mathbb{Z}^n \subset N_1$,

we have (from (1) and (4), renumerating the N_j)

$$\text{vol } hS \leq k_2 \cdot k(h) = k(h) \quad . \tag{6}$$

Since we postulated $S \cap S_\sigma = S_{\sigma+1} = \{0\}$, 3.2(viii) is tailored to measure:

$$\text{vol } hT \leq \text{vol } hS \text{ vol } hS_\sigma \leq c_8(k(h))^2 \quad . \tag{7}$$

At last let $T' \in \mathcal{N}$ be obtained from 6.2 for our T , then

$k(h) \leq \text{vol } hT' \leq c_6 \text{ vol } hT \leq c_6c_8(k(h))^2$ or

$k(h) \geq (c_6c_8)^{-1} =: c_7$. -

<u>6.4 Remarks:</u> <u>(i)</u> The reader may have noticed that some of those estimates can be tightened by known methods. Here however just one improvement will be given that provides further insight as well: Assume the order of the N_j chosen such that σ is minimal, which

implies $M_j := \bigcap_{\substack{i\neq j \\ 1\leq i\leq\sigma+1}} N_i \neq (0) \quad \forall\ 1\leq j\leq\sigma+1$. The spaces $\langle M_j\rangle$ are linearly independent $[\ \langle M_j\rangle \cap \langle \bigcup_{i\neq j} M_i\rangle \subset \langle M_j\cap N_j\rangle = (0)\ ]$, and together they span a totally isotropic subspace M of $(\mathbb{R}^n, J_\kappa)$ $[\ m_i\in M_i\ ,\ m_j\in M_j \Rightarrow m_i, m_j \in N_k\ ]$, which means $3 \leq \sigma+1 \leq \dim M \leq r$ or

$$2 < \sigma < r \quad ; \tag{1}$$

$$\mathscr{G}_0'' = \phi \quad \text{if} \quad r = 1 \text{ or } 2 \quad .$$

Also, with suitable permutations of $\{N_j \mid j\leq\sigma\}$ we can enter into the proof at 6.3(5) to obtain $\operatorname{vol} hM_j \leq c_8 k(h)$ $\forall j$; the role of T is played by $\bar{T} := M \cap \mathbb{Z}^n$ which allows σ-fold application of 3.5(viii):

$$\operatorname{vol} h\bar{T} \leq \prod_j \operatorname{vol} hM_j \leq c_8^{\sigma+1} k(h)^{\sigma+1} \quad . \tag{2}$$

Although this does not look like a general improvement of 6.3(7), for $r = 3$ (where we had $\sigma = 2$ and $c_8 = 1$) we get $3 \geq \dim M = \operatorname{rk} \bar{T} \geq 2+1$, thus $\bar{T} \in \mathscr{N}$, and (2) specifies: $k(h) \leq \operatorname{vol} h\bar{T} \leq k(h)^3$;

$$c_7 = 1 \quad \text{(defined in 6.3)} \quad \text{if} \quad r = 3 \quad .$$

Other intersection lattices may offer special advantages as well. -

(ii) The bisection 6.0(1) is not completely natural. Indeed, 6.1 (and the part of 3.9 referred to therein) can be viewed as "limiting case" of 6.3 (resp. 6.2). -

6.5 Theorem: Given any subgroup $\tilde{\Gamma} \leq \Gamma$ of finite index, $X_0/_{\tilde{\Gamma}} = {}_K\backslash \mathscr{G}_0/_{\tilde{\Gamma}}$ is compact.

Proof: This space is the projection image of $\bigcup_\gamma (\mathscr{G}_0 \cap F)\cdot\gamma$ (γ representing left cosets), which is compact by 4.10 and the above.

6.6 Remark: Despisers of Siegel sets may use the results of this section to show that $\mathcal{G}_0$ is contained in some "Stuhler-Grayson domain of semistability" as defined in [Gr1], and take Grayson's compactness result (ibd., Th. 7.18) instead of 4.10. (Notice that those domains have the same dimension as $\mathcal{G}$.)

§ 7 Triangulability

7.0 In this chapter we deviate from our general concept by providing a kind of cross-country existence proof, which in return might be applied to similar cases without much modification. Thus substantial simplifications can be anticipated in actual calculations. [By the way, skilful modelling may start with taking 3.8e) and f) not too literally.] Recall the following

7.1 Definition: A subset of $\mathbb{R}^m$ is called *semialgebraic set* ("sas"), if it is a (finite!) Boolean combination of some $\{x \in \mathbb{R}^n \mid P(x) \underset{\leq}{=} 0\}$, $P(*)$ polynomials. A function $\mathbb{R}^k \supset \quad \longrightarrow \mathbb{R}^m$ whose graph is a sas will be refered to as *semialgebraic application* ("saa") (cf. [Co], 1.1, 2.9).

7.2 Remarks and quotations:

(i) G , $\mathcal{G}$, $GL_n(\mathbb{R})$ etc. (but not Γ) are canonically sas's.

(ii) If f , g , and h are saa's, so are $f(g(*))$ and h^{-1} if (and where) defined.

(iii) Any algebraic function (e.g. division and square root) is a saa on its domain of definition.

(iv) The closure of a sas and the image of a sas under a saa are again sas's ([Co], 2.7, 2.10)

(v) (Lojasiewicz, cf. [Co], Theorem 4.10): If a compact set S is the disjoint union of finitely many sas's S_i , then there is a homeomorphic saa f onto a simplicial complex $C = \dot{\cup}\, \sigma_i$ (σ_i affine simplices in some real space), such that each S_i is a union of some $f^{-1}(\sigma_j)$.

(vi) Such a $f^{-1}(\sigma_j)$ is a sas by (ii) and (iii).

7.3: Now let us draw forth that old Φ from 1.1, regarded as a sas in its real hull. It is homeomomorphic to X , and with the $u(*)$ from 1.2 the projection writes

$$\psi : \mathcal{G} \to \Phi , \quad h \mapsto u(h\underline{\kappa}^{-1})\, h\, \underline{\kappa}^{-1} \quad .$$

$u(*)$ is a saa by 7.2(iii) ("orthogonalisation process"), thus ψ as well. The action of a $\gamma \in \Gamma$ descends to the saa

$$\varphi \mapsto \varphi \circ \gamma := u(\varphi\underline{\kappa}\gamma\underline{\kappa}^{-1})\cdot\varphi\, \underline{\kappa}\, \gamma\, \underline{\kappa}^{-1} \quad (\varphi \in \Phi) \quad .$$

Writing $\bar{F}$ for the sas $\psi(F)$, we have

$$\bar{F} \circ \Gamma = \Phi \quad .$$

7.4 Actually we need a sas $F' \subset \bar{F}$ of Γ-orbit representatives. Firstly, a corresponding set for $GL_n(\mathbb{Z})$ acting on $\mathcal{P}_n^+$ can (e.g.) be constructed as follows: Switch to the equivalent model of positive definite matrices (always check saa and sas properties) acted on *linearly* by $g \in GL_n(\mathbb{Z})$: $f \mapsto {}^t g f g$. Voronoi's reduction theory ([Vo]) divides this space into convex polyhedral cones edged by some semidefinite matrices. $GL_n(\mathbb{Z})$ permutes the latter as well (not just projectively!), thus also their barycentres. A system of representatives of the induced projective barycentric subdivision (freed from its indefinite parts) consists of finitely many simplicial cones and serves our purpose. - Now let M be the version of this set in a "tilted $\mathcal{U}_n^+$" that contains Φ, and γ_i be Γ-coset representatives of the finite (by Siegel property) set $\{g \in GL_n(\mathbb{Z}) \mid F \cap Mg \neq \phi\}$. The somewhat brutal definition $F' := \dot{\bigcup_i} (F \cap M\gamma_i)$ has the desired features (in practice a hand-made sample is to be prefered). - $F_o := F' \cap \psi(\mathcal{G}_0)$ is another sas by 4.6, thus $\overline{F_o}$ as well (barring always denotes closure).

<u>7.5 Lemma:</u> We can fix a triangulation $\overline{F_o} =: \dot{\bigcup}_{i=1}^{\lambda} \sigma_i$ such that each σ_i is a sas, $F_o = \dot{\bigcup}_{i=1}^{\mu} \sigma_i$ and

$$[\ \sigma_i, \sigma_j < \sigma_k\ ,\ \sigma_i \circ \gamma \cap \sigma_j \neq \phi\ ] \ \Rightarrow\ \gamma = I_n \quad . \tag{1}$$

<u>Proof:</u> Apply 7.2(v) to $\overline{F_o} = F_o \cup (\overline{F_o} - F_o)$. As F_o is a set of Γ-representatives in $\psi(\mathcal{G}_0)$, (1) can be obtained from this triangulation by barycentric subdivision wherever necessary (performed affinely in the affine model first to conserve the sas features).

<u>7.6</u> To force respect for the Γ-action (whose saa property will chiefly provide the necessary authority), we refine one $\overline{\sigma_i}$ after the other ($i \le \mu$). For the moment we content ourselves with a regular cell

structure. Assume $S_k := \bigcup_{i=1}^{k-1} (\overline{\sigma_i} \circ \Gamma)$ already carries a Γ-invariant such structure $\{\tau_m\}$ with sas's for cells, all σ_i $(i \le k-1)$ and $\overline{\sigma_j} \cap S_k$ $(j \ge k)$ being unions of τ_m's . Now $\overline{\sigma_k}$ [$\subset \psi(F)$] can have nonempty intersection with only finitely many $\overline{\sigma_\lambda} \circ \gamma$ or τ_m [$\subset$ some $\sigma_* \circ \gamma'$] by Siegel property 4.5. These induce a disjoint decomposition of $\overline{\sigma_k}$; 7.2(v) applies once more. Since all interfering τ_m are *contained* in $\overline{\sigma_k}$, they form a subcomplex that is being refined. Transport of that triangulation to $\overline{\sigma_k} \circ \Gamma$ makes no problem because of 7.5(4). The result is a new complex S_{k+1} recovering all induction hypotheses, thus eventually $F_o \circ \Gamma = \psi(\mathcal{G}_0)$ is made such a complex. Now it is easy to refine this to a topological triangulation (e.g. of $\psi(\mathcal{G}_0)/_\Gamma$) by centrally subdividing all closed cells (they are closed simplices with their faces refined!) in an order of increasing dimension, respecting each time the triangulation of the boundary accomplished before.

<u>7.7 Theorem:</u> $X_0/_{\tilde{\Gamma}}$ is triangulable for any subgroup $\tilde{\Gamma}$ of Γ .
<u>Proof:</u> The structure of 7.6 obviously descends to $\psi(\mathcal{G}_0)/_{\tilde{\Gamma}}$, which is homeomorphic to $X_0/_{\tilde{\Gamma}}$.

§ 8 Dimension

8.0 We shall see that out of the r steps in § 5 (of which but the last achieved compactness relative to Γ), only the first one reduces the dimension by 1 .

Abbreviate U for $I_{n,r}\mathbb{Z}^r$, and $\mathfrak{X}_o$ for $\{U, J\cdot U\}$.

8.1 Lemma: $\mathfrak{X}(\underline{\kappa})$ consists of those $M\cdot\mathbb{Z}^n$ with $M = \begin{pmatrix} A & 0 \\ 0 & 0 \\ 0 & B \end{pmatrix}$ such that (A,B) is an r-by-r permutation matrix. (Hence $\underline{\kappa}M = M$.)

Proof: As $\underline{\kappa}^t\underline{\kappa} \in SL_n(\mathbb{Z})$ and $\mathrm{vol}(\underline{\kappa}I_{n,r}) = 1$, $k(\underline{\kappa})$ must be 1 . Thus $N\mathbb{Z}^r \in \mathfrak{X}(\underline{\kappa})$, $N =: \begin{Bmatrix} a \\ b \\ c \end{Bmatrix}$, means in case $\epsilon = 1$:

$$a^tc + b^t\kappa^t\kappa b + c^ta = 0 \quad , \text{ and} \tag{1}$$

$$\det(N^t\underline{\kappa}^t\underline{\kappa}N) = 1 \quad . \tag{2}$$

From (1) we deduce

$$(a-c)^t(a-c) = a^ta + b^t\kappa^t\kappa b + c^tc = N^t\underline{\kappa}^t\underline{\kappa}N \quad , \tag{3}$$

thus $\det(a-c) = \pm 1$, and replacing N by $N\cdot(a-c)^{-1}$ ("change of basis"), (3) reads

$$N^t\underline{\kappa}^t\underline{\kappa}\, N = I_r \quad . \tag{5}$$

For $\epsilon = -1$, $(N\mathbb{Z}^r, I_{2r})$ is a lattice of discriminant 1 in $(\mathbb{Z}^{2r}, I_{2r})$, hence it must respect the "irreducible decomposition" $\perp_{i=1}^{2r} (\mathbb{Z},1)$ of the latter (cf. [MH], I.3.1 and II.6.4), i.e., (5) holds after change of basis. - From (5) and the isotropy condition, $N\mathbb{Z}^r$ can easily be seen to be one of the $M\mathbb{Z}^n$ of the claim; on the other hand these obviously *are* in $\mathfrak{X}(\underline{\kappa})$.

8.2 Lemma: In every neighbourhood V of $\underline{\kappa}$ there is a $h \in \mathscr{G}_0$ with $\mathfrak{X}(h) = \mathfrak{X}_o$.

Proof: If $r = 1$, $\underline{\kappa}$ itself does the job. So let $r \geq 2$. Define $(\delta(\xi,\zeta))_{ij} := \begin{cases} \xi & i=j \\ \zeta & i\neq j \end{cases} \in M_r(\mathbb{R})$ and $\eta(\xi,\zeta) \in \mathfrak{T}_r^+$ such that $\eta^t\eta = \delta$ wherever δ is positive definite. For a fixed $\xi > 1$, one can see by computation that $\eta(\xi,\zeta)_{ii}$ is strictly decreasing in $\zeta \geq 0$ (unless $i=1$ where it is constant) as well as in i ("collapsing r-pod"). Therefore a continuous function $\zeta(\xi) \geq 0$ is defined by postulating

$$\det \eta(\xi,\zeta(\xi)) = 1 \quad , \tag{1}$$

and for $\rho \leq r-1$, $\xi > 1$ we have

$$\theta_\rho := \det \eta(\xi,\zeta(\xi))_{i,j\leq\rho} > 1 \tag{2}$$

Let us take

$$h := \begin{bmatrix} \eta(\xi,\zeta(\xi))^{-t} & & \\ & \kappa & \\ & & \eta(\xi,\zeta(\xi)) \end{bmatrix} \in V$$

with small enough a $\xi > 1$ to have $\mathfrak{X}(h) \subset \mathfrak{X}(\underline{\kappa})$ (it exists by continuity, 5.4, and $\zeta(1) = 0$). Now consider some M with its A and B from 8.1, such that $0 \neq \mathrm{rk}(B) =: \beta \neq r$. Observe $(B,A)^{-t}\delta^\sigma(B,A)^{-1} = \delta^\sigma$, $(\sigma = \pm 1)$. Therefore

$$\begin{aligned} \mathrm{vol}\, hM &= \mathrm{vol}\, \eta^{-t}A \ \mathrm{vol}\, \eta B = \mathrm{vol}\, \eta^{-t}\begin{bmatrix} 0 \\ I_{r-\beta} \end{bmatrix} \cdot \mathrm{vol}\, \eta I_{r,\beta} \\ &= (\theta_r^{-1}/\theta_\beta^{-1})\cdot\theta_\beta = \theta_\beta^2 > 1 \quad \text{(cf. (1) and (2))}. \end{aligned}$$

On the other hand $\operatorname{vol} hU = \operatorname{vol} hJU = 1$ by (1), which amounts to the assertion.

8.3 Theorem: The dimension of X_0 (as a cell complex, say,) is one less than that of X .

Proof: The two dimensions are not equal: In Φ , $\psi(\mathcal{G}_0)$ (even $\psi(\mathcal{G}_{r-1})$) is recruited from the smooth algebraic hypersurfaces defined by $\operatorname{vol} \varphi\underline{\kappa}M = \operatorname{vol} \varphi\underline{\kappa}N$ ($M, N \in \mathfrak{M}(\varphi\underline{\kappa})$, cf. 4.8(ii)), actually from locally finitely many of them, as 5.4 shows. - Conversly, if one intersects the ψ-image of a 5.4-neighbourhood of the h found in 8.2 with one of those manifolds, the result is contained in $\psi(\mathcal{G}_0)$, proving that the latter space cannot have a smaller dimension than claimed.

8.4 Trying to make that finite (simplicial) cell complex $X_0/_\Gamma$ smaller still by hand is a tempting idea. Namely, if a cell is a face of only one other cell, both cells may be removed ("pressed in with the thumb"). However, remember that in order to cater for those important torsion-free subgroups, we are availed only by manipulations *that lift to deformations of* X_0 . We shall see that this can indeed be effected whenever $r \geq 2$.

8.5 Assume from now on $r \geq 2$, and that $\mathfrak{M}(*)$ is constant on each simplex of X_0 (we can achieve this by additionaly accommodating the finite number of interfering sas's $\{ \varphi \in \Phi \mid \mathfrak{M}(\varphi\underline{\kappa}) = \mathfrak{M} \}$ in the process of triangulating $\overline{F_o}$ in 7.5). We know that there is a highest dimensional simplex adjacent to $K\cdot\underline{\kappa}$, whereon $\mathfrak{X}(*) = \mathfrak{X}_o$. Applying some $\left(M\,\middle|\,\begin{matrix} o \\ I_s \\ o \end{matrix}\,\middle|\,JM\right)$ (M as in 8.1), which is in Γ and stabilizes $\underline{\kappa}$, we find that there is another such cell having $\{M\mathbb{Z}^r, JM\mathbb{Z}^r\}$ for $\mathfrak{X}$. So there must be a 1-codimensional simplex τ near $K\cdot\underline{\kappa}$ which is face of exactly one of those simplices with $\mathfrak{X} = \mathfrak{X}_o$; call the latter σ. On τ we must have $\mathfrak{X} \supset \mathfrak{X}_o \dot\cup \{N\}$ with some $N \in \mathfrak{X}(\underline{\kappa})$. In lemma 8.6 we are going to see that actually "=" holds, so that near τ outside $\bar\sigma$ $\mathfrak{X}(*)$ must be either $\{U,N\}$ or $\{JU,N\}$. But this means that we are leaving X_0 ; τ and σ (and their Γ-translates) can indeed be removed from X_0 without changing the homotopy type.

<u>8.6 Lemma:</u> In fact $\mathfrak{X} = \mathfrak{X}_o \dot\cup \{N\}$ on τ.

<u>Proof:</u> Suppose $\mathfrak{X}$ contains four lattices U, JU, N, and L. Then at least one of the following two cases applies:

a) Some canonical unit vector is in $U \cap L \cap N$, some other in $U \cap L \cap JN$, and a third one in $U \cap JL \cap JN$, or the roles of U and JU or those of L and N are interchanged.

b) Such a unit vector is found in $U \cap L \cap JN$ and another one in $U \cap JL \cap N$. -

Now regard the three dimensional submanyfold of Φ consisting of those matrices that differ from I_n only in the corresponding entries, namely

$$\begin{pmatrix} a & & & & & \\ & b & & & & \\ & & c & & & \\ & & & \ddots & & \\ & & & & a^{-1} & & \\ & & & & & b^{-1} & \\ & & & & & & c^{-1} \end{pmatrix} \text{ (case a)) resp. } \begin{pmatrix} a & & & & \epsilon ac \\ & b & & -bc & \\ & & \ddots & & \\ & & & a^{-1} & \\ & & & & b^{-1} \end{pmatrix} \text{ (case b)). The}$$

equations $\mathrm{vol}\ \varphi U = \mathrm{vol}\ \varphi JU = \mathrm{vol}\ \varphi L = \mathrm{vol}\ \varphi N$, transformed into polynomials, have no solutions in that subspace but I_n , *even if complex entries* a , b , c *are admitted.* This contradicts the fact that the codimension in X of τ is 2 . (Note that arguments like this one can be used to prove 8.2 and 8.3 likewise.) -

8.7 Such procedures may obviously be performed as long as boundary cells like τ are available. The key lies in those sets $\mathfrak{X}$ whose members have a common intersection if any one of them is removed. This implies $\#\mathfrak{X} \le r$ (cf. 6.4), which gives an intuitive illustration of the fact that the codimension of the final deformation retract in X cannot exceed r . -

§ 9 Special cases

9.1 Siegel's version of our §§ 3 and 4 in the case $\epsilon = -1$ ([Si]) translates as follows: His space H of symmetric matrices $z = x + iy \in M_r(\mathbb{C})$ (x,y real, y positive definite) is our ϕ via

$$\iota : \begin{pmatrix} \ell & -\ell x \\ & \ell^{-t} \end{pmatrix} \mapsto x + i\ell^{-1}\ell^{-t}$$

(cf. [Si], VI Lemma 6). Also, his *left* Γ-action

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \hat{\circ}\, z := (Az+B)\ (Cz+D)^{-1}$$

is anti-isomorphic to our right hand one by

$$\iota(\varphi\circ\gamma) = \gamma^{-1} \hat{\circ}\, \iota(\varphi) \quad .$$

As Γ acts transitively on $\mathcal{N}$ (cf. 3.1(2)), 3.8(b) can be written

$$|\det \ell(\varphi)| \le |\det \ell(\varphi\circ\gamma)| \quad \forall\ \gamma \in \Gamma \quad .$$

This is precisely Siegel's reduction condition that $\det y^{-1}$ $(= \det^2 \ell)$ be minimal in the Γ-orbit. Thus the two fundamental domains correspond to one another except that he wants y^{-1} to be M-reduced, not y.

9.2 For the case $\epsilon = -1$, $r = 2$, Gottschling computed the bounding hypersurfaces of that fundamental domain ([Go]). His "Assoziationsklassen" of matrix pairs (C,D) are the elements $\begin{pmatrix} C^t \\ D^t \end{pmatrix} \cdot \mathbb{Z}^r$ of $\mathcal{N}$. Thus $\mathcal{G}_0 \cap F$ corresponds to the union of those faces that belong to a pair (C,D) such that $\begin{pmatrix} C^t \\ D^t \end{pmatrix} \cdot \mathbb{Z}^r \cap I_{n,r}\mathbb{Z}^r = \{0\}$ (cf. also [Si], VI, §30). These are exactly the ones given in [Go],(10), namely $(C,D) = (S,I_r)$ with

$$S \in \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix}, \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}, \begin{pmatrix} e & 0 \\ 0 & -e \end{pmatrix}, \begin{pmatrix} 0 & e \\ e & 0 \end{pmatrix}, \begin{pmatrix} e & e \\ e & 0 \end{pmatrix}, \begin{pmatrix} 0 & e \\ e & e \end{pmatrix} \;\middle|\; e = \pm 1 \right\} .$$

The incidence structure of these manifolds has not yet been computed.

9.3 If (ϵ,r,s) is $(-1,1,0)$, $(1,1,1)$, or $(1,1,2)$, G acts on the "upper half plane" resp. the "upper half space" in a well-known manner, and in fact our deformation retract could have been obtained by tthe methods of [Sel] resp. [Me] (making the obvious adjustments owing to the fact that under the "Weil isomorphisms" Γ does not remain integral). -

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