ALEXANDER MODULES OF IRREDUCIBLE C-GROUPS

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ABSTRACT. A complete description of the Alexander modules of knotted n-manifolds in the sphere S^{n+2}, $n \geq 2$, and of irreducible Hurwitz curves is given. This description is applied to calculation of the first homology groups of cyclic coverings of the sphere S^{n+2} and of the projective complex plane $\mathbb{CP}^2$ branched respectively alone knotted n-manifolds and along irreducible Hurwitz (in particular, algebraic) curves.

INTRODUCTION

A class $\mathcal{C}$ of C-groups and its subclass $\mathcal{H}$ of Hurwitz C-groups (see definitions below) play very important role in geometry of codimension two submanifolds. For example, it is well known that the knot and link groups (given by Wirtinger presentations) are C-groups and any C-group G can be realized as the group of a linked n-manifold if $n \geq 2$, that is, as the fundamental group $\pi_1(S^{n+2} \setminus V)$ of the complement of a closed oriented manifold V without boundary, $\dim_{\mathbb{R}} V = n$, in the $(n+2)$-dimensional sphere S^{n+2} (see [8]) and viceversa. Note also that a C-group G is isomorphic to $\pi_1(S^{n+2} \setminus S^n)$, $n \geq 3$, for some linked n-dimensional spheres S^n if and only if $H_2 G = 0$ ([5]). Some other results related to description of groups $\pi_1(S^{n+2} \setminus S^n)$ can be found in [15] and [4].

If $H \subset \mathbb{CP}^2$ is an algebraic or, more generally, Hurwitz[1] (resp., pseudo-holomorphic) curve of degree m, then the Zariski – van Kampen presentation of $\pi_1 = \pi_1(\mathbb{CP}^2 \setminus (H \cup L))$ defines on π_1 a structure of a Hurwitz C-group of degree m, where L is a line at "infinity" (that is, L is a fiber of linear projection $\mathrm{pr} : \mathbb{CP}^2 \to \mathbb{CP}^1$ and it is in general position with respect to H; if H is a pseudo-holomorphic curve, then pr is given by a pencil of pseudo-holomorphic lines). In [9], it was proved that any Hurwitz C-group G of degree m can be realized as the fundamental group $\pi_1(\mathbb{CP}^2 \setminus (H \cup L))$ for some Hurwitz (resp. pseudo-holomorphic) curve H, $\deg H = 2^n m$, with singularities of the form $w^m - z^m = 0$, where n depends on the Hurwitz C-presentation of G. So the class $\mathcal{H}$ coincides with the class $\{ \pi_1(\mathbb{CP}^2 \setminus (H \cup L)) \}$ of the fundamental

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[1]The definition of Hurwitz curves can be found in [6] or in [2].

groups of the complements of "affine" Hurwitz (resp., of "affine" pseudo-holomorphic) curves and it contains the subclass of the fundamental groups of complements of affine plane algebraic curves.

By definition, a *C-group* is a group together with a finite presentation

$$G_W = \langle x_1, \dots, x_m \mid x_i = w_{i,j,k}^{-1} x_j w_{i,j,k},\ w_{i,j,k} \in W \rangle, \qquad (1)$$

where $W = \{w_{i,j,k} \in \mathbb{F}_m \mid 1 \le i, j \le m,\ 1 \le k \le h(i,j)\}$ is a collection consisting of elements of the free group $\mathbb{F}_m$ generated by free generators $x_1, \dots, x_m$ (it is possible that $w_{i_1,j_1,k_1} = w_{i_2,j_2,k_2}$ for $(i_1, j_1, k_1) \neq (i_2, j_2, k_2)$), and $h : \{1, \dots, m\}^2 \to \mathbb{Z}$ is some function. Such a presentation is called a *C-presentation* (C, since all relations are conjugations). Let $\varphi_W : \mathbb{F}_m \to G_W$ be the canonical epimorphism. The elements $\varphi_W(x_i) \in G$, $1 \le i \le m$, and the elements conjugated to them are called the *C-generators* of G. Let $f : G_1 \to G_2$ be a homomorphism of C-groups. It is called a *C-homomorphism* if the images of the C-generators of G_1 under f are C-generators of the C-group G_2. C-groups are considered up to C-isomorphisms. Properties of C-groups were investigated in [7], [9], [12],[11].

A C-presentation (1) is called a *Hurwitz C-presentation of degree* m if for each $i = 1, \dots, m$ the word $w_{i,i,1}$ coincides with the product $x_1 \dots x_m$, and a C-group G is called a *Hurwitz C-group* (*of degree* m) if for some $m \in \mathbb{N}$ it possesses a Hurwitz C-presentation of degree m. In other words, a C-group G is a Hurwitz C-group of degree m if there are C-generators $x_1, \dots, x_m$ generating G such that the product $x_1 \dots x_m$ belongs to the center of G. Note that the degree of a Hurwitz C-group G is not defined canonically and depends on the Hurwitz C-presentation of G. Denote by $\mathcal{H}$ the class of all Hurwitz C-groups.

It is easy to show that G/G' is a finitely generated free abelian group for any C-group G, where $G' = [G, G]$ is the commutator subgroup of G. A C-group G is called *irreducible* if $G/G' \simeq \mathbb{Z}$ and we say that G consists of k *irreducible components* if $G/G' \simeq \mathbb{Z}^k$. If a Hurwitz C-group G is realized as the fundamental group $\pi_1(\mathbb{CP}^2 \setminus (H \cup L))$ of the complement of some Hurwitz curve H, then the number of irreducible components of G is equal to the number of irreducible components of H. Similarly, if a C-group G consisting of k irreducible components is realized as the group of a linked n-manifold V, $G = \pi_1(S^{n+2} \setminus V)$, then the number of connected components of V is equal to k.

A free group $\mathbb{F}_n$ with fixed free generators is a C-group and for any C-group G the canonical C-epimorphism $\nu : G \to \mathbb{F}_1$, sending the C-generators of G to the C-generator of $\mathbb{F}_1$, is well defined. Denote by N its kernel. Note that if G is an irreducible C-group, then N coincides with G'. In what follows we consider only the irreducible case.

Let G be an irreducible C-group. The C-epimorphism ν induces the following exact sequence of groups

$$1 \to G'/G'' \to G/G'' \xrightarrow{\nu_*} \mathbb{F}_1 \to 1,$$

where $G'' = [G', G']$. The C-generator of $\mathbb{F}_1$ acts on G'/G'' by conjugation $\widetilde{x}^{-1} g \widetilde{x}$, where $g \in G'$ and $\widetilde{x}$ is one of the C-generators of G. Denote by t this action. The group $A_0(G) = G'/G''$ is an abelian group and the action t defines on $A_0(G)$ a structure of Λ-module, where $\Lambda = \mathbb{Z}[t, t^{-1}]$ is the ring of Laurent polynomials with integer coefficients. The Λ-module $A_0(G)$ is called the *Alexander module* of the C-group G. The action t induces an action $h_{\mathbb{C}}$ on $A_{\mathbb{C}} = A_0(G) \otimes \mathbb{C}$ and it is easy to see that its characteristic polynomial $h_{\mathbb{C}} \in \mathbb{Q}[t]$. The polynomial $\Delta(t) = a \det(h_{\mathbb{C}} - t\mathrm{Id})$, where $a \in \mathbb{N}$ is the smallest number such that $a \det(h_{\mathbb{C}} - t\mathrm{Id}) \in \mathbb{Z}[t]$, is called the *Alexander polynomial* of the C-group G. If H is either an algebraic, or Hurwitz, or pseudo-holomorphic irreducible curve in $\mathbb{CP}^2$ (resp., $V \subset S^{n+2}$ is a knotted (that is, connected smooth oriented without boundary) n-manifold, $n \geq 1$) and $G = \pi_1(\mathbb{CP}^2 \setminus (H \cup L))$ (resp., $G = \pi_1(S^{n+2} \setminus V)$), then the Alexander module $A_0(G)$ of the group G and its Alexander polynomial $\Delta(t)$ are called the *Alexander module and Alexander polynomial* of the curve H (resp., of the knotted manifold V). Note that the Alexander module $A_0(H)$ and the Alexander polynomial $\Delta(t)$ of a curve H do not depend on the choice of the generic (pseudo)-line L. Results related to the Alexander modules of knotted spheres are stated in [16], [17].

In [2] and [10], properties of the Alexander polynomials of Hurwitz curves were investigated. In particular, it was proved that if H is an irreducible Hurwitz curve of degree d, then its Alexander polynomial $\Delta(t)$ has the following properties

(*i*) $\Delta(t) \in \mathbb{Z}[t]$, $\deg \Delta(t)$ is an even number;
(*ii*) $\Delta(0) = \Delta(1) = 1$;
(*iii*) $\Delta(t)$ is a divisor of the polynomial $(t^d - 1)^{d-2}$,

and, moreover, a polynomial $P(t) \in \mathbb{Z}[t]$ is the Alexander polynomial of an irreducible Hurwitz curve if and only if the roots of $P(t)$ are roots of unity and $P(1) = 1$.

Let $G = \pi_1(\mathbb{CP}^2 \setminus (H \cup L))$ be the fundamental group of the complement of an irreducible affine Hurwitz curve (resp., $G = \pi_1(S^{n+2} \setminus V)$ is the group of a knotted n-manifold, $n \geq 1$). The homomorphism $\nu : G \to \mathbb{F}_1$ defines an infinite unramified cyclic covering $f_\infty : X_\infty \to \mathbb{CP}^2 \setminus (H \cup L)$ (resp., $f_\infty : X_\infty \to S^{n+2} \setminus V$). We have $H_1(X_\infty, \mathbb{Z}) = G'/G''$ and the action of t on $H_1(X_\infty, \mathbb{Z})$ coincides with the action of a generator h of the covering transformation group of the covering f_∞.

For any $k \in \mathbb{N}$ denote by $\mathrm{mod}_k : \mathbb{F}_1 \to \mu_k = \mathbb{F}_1/\{t^k\}$ the natural epimorphism to the cyclic group μ_k of degree k. The covering f_∞ can be factorized through the cyclic covering $f'_k : X'_k \to \mathbb{CP}^2 \setminus (H \cup L)$ (resp., $f'_k : X'_k \to S^{n+2} \setminus V$) associated with the epimorphism $\mathrm{mod}_k \circ \nu$, $f_\infty = f'_k \circ g_k$. Since a Hurwitz curve H has only analytic singularities, the covering f'_k can be extended (see [2]) to a map $\widetilde{f}_k : \widetilde{X}_k \to X$ branched along H and, maybe, along L. Here $\widetilde{X}_k$ is a closed four dimensional variety locally isomorphic over a singular point of H to a complex analytic singularity given by an equation $w^k = F(u,v)$, where $F(u,v)$ is a local equation of H at its singular point. In addition, $\widetilde{X}_k$ is locally isomorphic over a neighbourhood of an intersection point of H and L to the singularity locally given by $w^k = vu^d$, where d is the smallest non-negative integer for which $m+d$ is divisible by k. The variety $\widetilde{X}_k$, if $\widetilde{f}_k^{-1}(L) \subset \mathrm{Sing}\, \widetilde{X}_k$, can be normalized (as in the algebraic case) and we obtain a covering $\widetilde{f}_{k,\,\mathrm{norm}} : \widetilde{X}_{k,\,\mathrm{norm}} \to \mathbb{CP}^2$ in which $\widetilde{X}_{k,\,\mathrm{norm}}$ is a singular analytic variety at its finitely many singular points. The map $\widetilde{f}_{k,\,\mathrm{norm}}$ is branched along H and, maybe, along the line "at infinity" L (if k is not a divisor of $\deg H$, then $\widetilde{f}_{k,\,\mathrm{norm}}$ is branched along L). One can resolve the singularities of $\widetilde{X}_{k,\,\mathrm{norm}}$ and obtain a smooth manifold $\overline{X}_k$, $\dim_{\mathbb{R}} \overline{X}_k = 4$. Let $\sigma : \overline{X}_k \to \widetilde{X}_{k,\,\mathrm{norm}}$ be a resolution of the singularities, $E = \sigma^{-1}(\mathrm{Sing}\, \widetilde{X}_{k,\,\mathrm{norm}})$ the preimage of the set of singular points of $\widetilde{X}_{k,\,\mathrm{norm}}$, and $\overline{f}_k = \widetilde{f}_{k,\,\mathrm{norm}} \circ \sigma$. The action h induces an action $\overline{h}_k$ on $\overline{X}_k$ and an action t on $H_1(\overline{X}_k, \mathbb{Z})$.

Similarly, the covering $f'_k : X'_k \to S^{n+2} \setminus V$ can be extended to a smooth map $f_k : X_k \to S^{n+2}$ branched along V, where X_k is a smooth compact $(n+2)$-manifold, and the action t induces actions h_k on X_k and h_{k*} on $H_1(X_k, \mathbb{Z})$. The action h_{k*} defines on $H_1(X_k, \mathbb{Z})$ a structure of Λ-module.

In [2], it was shown that for any Hurwitz curve H, a covering space $\overline{X}_k$ can be embedded as a symplectic submanifold to a complex projective rational 3-fold on which the symplectic structure is given by an integer Kähler form, and it was proved that the first Betti number $b_1(\overline{X}_k) = \dim_{\mathbb{C}} H_1(\overline{X}_k, \mathbb{C})$ of $\overline{X}_k$ is equal to $r_{k,\neq 1}$, where $r_{k,\neq 1}$ is the number of roots of the Alexander polynomial $\Delta(t)$ of the curve $\bar{H}$ which are k-th roots of unity not equal to 1.

Let M be a Noetherian Λ-module. We say that M is $(t-1)$-*invertible* if the multiplication by $t-1$ is an automorphism of M. A Λ-module M is called t-*unipotent* if for some $n \in \mathbb{N}$ the multiplication by t^n is the identity automorphism of M. The smallest $k \in \mathbb{N}$ such that

$$t^k - 1 \in \mathrm{Ann}(M) = \{f(t) \in \Lambda \mid f(t)v = 0 \text{ for } \forall v \in M\}$$

is called the *unipotence index* of t-unipotent module M.

Let M be a Noetherian $(t-1)$-invertible Λ-module. A t-invertible Λ-modules $A_n(M) = M/(t^k-1)M$ is called the *k-th derived Alexander module* of M and if M is the Alexander module of a C-group G (resp., of a knotted n-manifold V, resp., of a Hurwitz curve H), then $A_k(M)$ is called the *k-th derived Alexander module* of G (resp., of V, resp., of H) and it will be denoted by $A_k(G)$ (resp., $A_k(V)$, resp., $A_k(H)$)

The main results of the article are the following statements.

Theorem 0.1. *A Λ-module M is the Alexander module of a knotted n-manifold, $n \geq 2$, if and only if it is a Noetherian $(t-1)$-invertible Λ-module.*

Theorem 0.2. *Let V be a knotted n-manifold, $n \geq 1$, and $f_k : X_k \to S^{n+2}$ the cyclic covering branched along V. Then $H_1(X_k, \mathbb{Z})$ is isomorphic to the k-th Alexander module $A_k(V)$ of V as a Λ-module.*

Similar statements hold in the case of algebraic and, more generally, of Hurwitz (resp., pseudo-holomorphic) curves.

Theorem 0.3. *A Λ-module M is the Alexander module of an irreducible Hurwitz (resp., pseudo-holomorphic) curve if and only if it is a Noetherian $(t-1)$-invertible t-unipotent Λ-module. In particular, the Alexander module of an irreducible algebraic plane curve is a Noetherian $(t-1)$-invertible t-unipotent Λ-module.*

The unipotence index of the Alexander module $A_0(H)$ of an irreducible plane algebraic (resp., Hurwitz or pseudo-holomorphic) curve H is a divisor of $\deg H$.

Corollary 0.4. *The Alexander module $A_0(H)$ of an irreducible plane algebraic (resp., Hurwitz or pseudo-holomorphic) curve H is finitely generated over $\mathbb{Z}$, that is, $A_0(H)$ is a finitely generated abelian group.*

A finitely generated abelian group G is the Alexander module $A_0(H)$ of some irreducible Hurwitz or pseudo-holomorphic curve H if and only if there are an integer m and an automorphism $h \in Aut(G)$ such that $h^m = Id$ and $h - Id$ is also an automorphism of G.

Theorem 0.5. *Let H be an algebraic (resp., Hurwitz or pseudo-holomorphic) irreducible curve in $\mathbb{CP}^2$, $\deg H = m$, and $\overline{f}_k : \overline{X}_k \to \mathbb{CP}^2$ be a resolution of singularities of the cyclic covering of degree $\deg \overline{f}_k = k$ branched along H and, maybe, alone the line "at infinity" L. Then*

$$H_1(\overline{X}_k \setminus E, \mathbb{Z}) \simeq A_k(H),$$
$$H_1(\overline{X}_k, \mathbb{Q}) \simeq A_k(H) \otimes \mathbb{Q},$$

where $A_k(H)$ is the k-th Alexander module of H and $E = \sigma^{-1}(\mathrm{Sing}\, \widetilde{X}_{k,\mathrm{norm}})$.

It should be noticed that in general case the homomorphism $H_1(\overline{X}_k \setminus E, \mathbb{Z}) \simeq A_k(H) \to H_1(\overline{X}_k, \mathbb{Z})$, induced by the embedding $\overline{X}_k \setminus E \hookrightarrow \overline{X}_k$, is an epimorphism and it is not necessary to be an isomorphism (see Example 4.6).

Corollary 0.6. *Let H be an algebraic (resp, Hurwitz or pseudo-holomorphic) irreducible curve in $\mathbb{CP}^2$, $\deg H = m$, and $\overline{f}_k : \overline{X}_k \to \mathbb{CP}^2$ be a resolution of singularities of the cyclic covering of degree $\deg f_k = k$ branched along H and, maybe, alone the line "at infinity". Then*

(i) *the first Betti number $b_1(\overline{X}_k)$ of $\overline{X}_k$ is an even number;*
(ii) *if $k = p^r$, where p is prime, then $H_1(\overline{X}_k, \mathbb{Q}) = 0$;*
(iii) *if k and m are coprime, then $H_1(\overline{X}_k, \mathbb{Z}) = 0$;*
(iv) *$H_1(\overline{X}_2, \mathbb{Z})$ is a finite abelian group of odd order.*

Note also that any C-group G can be realized (see [9]) as $\pi_1(\Delta^2 \setminus (C \cap \Delta^2))$, where $\Delta^2 = \{|z| < 1\} \times \{|w| < 1\} \subset \mathbb{C}^2$ is a bi-disc and $C \subset \mathbb{C}^2$ is a non-singular algebraic curve such that the restriction of $\mathrm{pr}_1 : \Delta^2 \to \{|z| < 1\}$ to $C \cap \Delta^2$ is a proper map. Therefore the analogue of Theorems 0.1 and 0.2 and corollaries of them hold also in this case.

The proof of Theorems 0.1 and 0.3 is given in section 3. In section 1, properties of Noetherian $(t-1)$-invertible Λ-modules are described and section 2 is devoted to Noetherian t-unipotent Λ-modules. In section 4, Theorems 0.2 and 0.5 are proved and some other corollaries of them are stated.

1. $(t-1)$-invertible Λ-modules

1.1. **Criteria of $(t-1)$-invertibility.** Before to describe $(t-1)$-invertible Λ-modules, let us recall that the ring $\Lambda = \mathbb{Z}[t, t^{-1}]$ is Noetherian. Each element $f \in \Lambda$ can be written in the form

$$f = \sum_{n_- \leq i \leq n_+} a_i t^i \in \mathbb{Z}[t, t^{-1}],$$

where $n_-, n_+, i, a_i \in \mathbb{Z}$. If $n_- \geq 0$ for $f \in \Lambda$, then $f \in \mathbb{Z}[t]$ and it will be called a *polynomial.*

For any $n \in \mathbb{Z}$, $n \neq 0$, a $\mathbb{Z}$-homomorphism

$$f(t) = \sum a_i t^i \mapsto f(n) = \sum a_i n^i$$

is well defined. The image $f(n)$ of $f(t)$ is called the *value* of $f(t)$ at n. If $f(t)$ is a polynomial, then its value $f(0) = a_0$ is also well defined.

We begin with the following lemma.

Lemma 1.1. *A Noetherian Λ-module M is $(t-1)$-invertible if and only if the multiplication by $t-1$ is a surjective endomorphism of M.*

Proof. Lemma follows from some more general statement. Namely, any surjective Λ-endomorphism $f : M \to M$ of a Noetherian Λ-module M is an isomorphism. Indeed, if $\ker f \neq 0$, then the chain of submodules

$$\ker f \subset \ker f^2 \subset \dots \subset \ker f^n \subset \dots$$

is strictly increasing, since f is an epimorphism. This contradicts the Noetherian property of the module M. □

Let M be a Noetherian $(t-1)$-invertible Λ-module. Consider an element $v \in M$ and denote by $M_v =< v >$ a principal submodule of M generated by v. Since M is Noetherian, any principle submodule of M is contained in a maximal principle submodule of M.

Lemma 1.2. *Any maximal principal submodule M_v of $(t-1)$-invertible module M is $(t-1)$-invertible.*

Proof. Since M is $(t-1)$-invertible module, there is an element $v_1 \in M$ such that $v = (t-1)v_1$. Therefore $M_v \subset M_{v_1}$. Since M_v is a maximal principle submodule of M, we have $M_v = M_{v_1}$. Therefore $v_1 \in M_v$ and the multiplication by $t-1$ defines a surjective endomorphism of M_v. To complete the proof we apply Lemma 1.1. □

A principal submodule $M_v \subset M$ is isomorphic to Λ/Ann_v, where $\mathrm{Ann}_v = \{f \in \Lambda | fv = 0\}$ is the *annihilator* of v. The annihilator Ann_v of an element $v \in M$ is an ideal of Λ. Denote by

$$\mathrm{Ann}(M) = \bigcap_{v\in M} \mathrm{Ann}_v = \{g(t) \in \Lambda \mid g(t)v = 0 \text{ for } \forall v \in M\}$$

the *annihilator* of M.

Lemma 1.3. *A principal Λ-module $M = \Lambda/I$ is a $(t-1)$-invertible if and only if the ideal I contains a polynomial $f(t)$ such that $f(1) = 1$.*

Proof. Let M is generated by an element $v \in M$.

If a polynomial $f(t)$ such that $f(1) = 1$ is contained in $I = \mathrm{Ann}_v$, then $f(t)$ can be expressed in the form

$$f(t) = (t-1)g(t) + 1 \tag{2}$$

for some polynomial $g(t)$. Therefore $v = (t-1)v_1$, where $v_1 = -g(t)v$. Thus, the multiplication by $t-1$ is a surjective automorphism of M and hence, by Lemma 1.1, the multiplication by $t-1$ is an isomorphism of M.

Conversely, if M is $(t-1)$-invertible, then there is an element $v_1 \in M$ such that $v = (t-1)v_1$. Let $v_1 = h(t)v$ for some $h(t) \in \Lambda$. We have $(1-(t-1)h(t))v = 0$. Therefore $1-(t-1)h(t) \in \mathrm{Ann}_v = I$. There is an integer k such that $f(t) = t^k(1-(t-1)h(t)) \in I \cap \mathbb{Z}[t]$. It is easy to see that $f(1)=1$. □

As a consequence of Lemma 1.3 we obtain the following Lemma.

Lemma 1.4. *Any principal submodule of a principal $(t-1)$-invertible module M is $(t-1)$-invertible.*

Proof. Indeed, let M be generated by an element $v \in M$ and its submodule M_1 be generated by $v_1 = h(t)v$. Then $\mathrm{Ann}_v \subset \mathrm{Ann}_{v_1}$.

Since M is $(t-1)$-invertible, by Lemma 1.3, there is a polynomial $f(t) \in \mathrm{Ann}_v$ such that $f(1)=1$. Applying again Lemma 1.3, we have that M_1 is $(t-1)$-invertible, since $f(t) \in \mathrm{Ann}_{v_1}$. □

Proposition 1.5. *Any submodule of a Noetherian $(t-1)$-invertible Λ-module M is $(t-1)$-invertible.*

Proof. Let N is a submodule of M. Since M is a Noetherian Λ-module, the submodule N is generated by a finite set of elements, say $v_1,\dots,v_n$. By Lemma 1.4, each principal submodule $M_{v_i} \subset N \subset M$ is $(t-1)$-invertible. Therefore the multiplication by $t-1$ is a surjective endomorphism of N, since it is surjective on each $M_{v_i} \subset N$ and the elements $v_1,\dots,v_n$ generate the module N. To complete the proof, we apply Lemma 1.1. □

Proposition 1.6. *Any factor module of a Noetherian $(t-1)$-invertible Λ-module M is $(t-1)$-invertible.*

Proof. It follows from Lemma 1.1. □

Lemma 1.7. *Let $M_1,\dots,M_k$ be Noetherian $(t-1)$-invertible Λ-modules. Then the direct sum $M=\bigoplus_{i=1}^k M_i$ is a Noetherian $(t-1)$-invertible Λ-module.*

Proof. Obvious.

Corollary 1.8. *Any Noetherian $(t-1)$-invertible Λ-module M is a the factor module of a direct sum $\bigoplus_{j=1}^n \Lambda/I_j$ of principle $(t-1)$-invertible Λ-modules Λ/I_j.*

Proof. Since M is a Noetherian Λ-module, it is generated by a finite set of elements, say $v_1,\dots,v_n$. By Proposition 1.5, each principal submodule $M_{v_i} \subset M$ is $(t-1)$-invertible and, obviously, there is an epimorphism $\bigoplus_{j=1}^n M_{v_i} \mapsto M$. □

Remark 1.9. *An abelian group G admits a structure of $(t-1)$-invertible Λ-module if and only if it has an automorphism t such that $t-1$ is also an automorphism. If G is finitely generated and $t \in Aut\,G$ is chosen, then G is a Noetherian Λ-module.*

Note that in general case an abelian group admits many structures of $(t-1)$-invertible Λ-modules. For example, the group $\mathbb{Z}/9\mathbb{Z}$ admits 3 such structures: either $tv = 2v$, or $tv = 5v$, or $tv = 8v$, where v is a generator of $\mathbb{Z}/9\mathbb{Z}$.

Theorem 1.10. *A Noetherian Λ-module M is $(t-1)$-invertible if and only if there is a polynomial $f(t) \in Ann(M)$ such that $f(1) = 1$.*

Proof. If M is $(t-1)$-invertible, then, by Proposition 1.5, its each principal submodule M_v is also $(t-1)$-invertible. Therefore, by Lemma 1.3, the annihilator Ann_v of $v \in M$ contains a polynomial $f_v(t)$ such that $f_v(1) = 1$. If M is generated by $v_1, \dots, v_n$, then the polynomial $f(t) = f_{v_1}(t) \dots f_{v_n}(t)$ is a desired one.

Let us show that if there is a polynomial $f(t) \in \mathrm{Ann}(M)$ such that $f(1) = 1$, then M is a $(t-1)$-invertible module. Indeed, in this case by Lemma 1.3, each principle submodule M_v of M is $(t-1)$-invertible. Therefore the multiplication by $t-1$ is an isomorphism of M, since it is an isomorphism of each principle submodule M_v of M. □

As a consequence of Theorem 1.10 we obtain that *any Noetherian $(t-1)$-invertible module M is a torsion Λ-module* and, consequently,

$$\dim_{\mathbb{Q}} M \otimes \mathbb{Q} < \infty.$$

The following proposition will be used in the proof of Theorems 0.1 and 0.3.

Proposition 1.11. *Any Noetherian $(t-1)$-invertible Λ-module M is isomorphic to a factor module Λ^n/M_1 of a free Λ-module Λ^n, where the submodule M_1 is generated by elements $w_1, \dots, w_n, \dots, w_{n+k}$ of Λ^n such that*

(*i*) *for $i = 1, \dots, n$ the vector $w_i = (0, \dots, 0, f_i(t), 0, \dots, 0)$, where a polynomial $f_i(t)$ stands on the i-th place and it is such that $f_i(1) = 1$,*
(*ii*) $w_{n+j} = (t-1)\overline{w}_{n+j} = ((t-1)g_{j,1}(t), \dots, (t-1)g_{j,n}(t))$ *for $j = 1, \dots, k$, where $g_{j,l}(t)$ are polynomials,*
(*iii*) *if for some $m \in \mathbb{N}$ the polynomial $t^m - 1 \in Ann(M)$, then for $i = 1, \dots, n$ the vector $w_{n+i} = (0, \dots, 0, t^m - 1, 0, \dots, 0)$, where the polynomial $t^m - 1$ stands on the i-th place.*

Proof. Let us choose generators $v_1, \dots, v_n$ of the Noetherian Λ-module M. Then, by Theorem 1.10, there are polynomials $f_i(t) \in \mathrm{Ann}_{v_i}$ such that $f_i(1) = 1$. Obviously, there is an epimorphism

$$h_1 : \bigoplus_{i=1}^{n} \Lambda/(f_i(t)) \to M$$

of Λ-modules such that $h(u_i)=v_i$ for $u_i=(0,\dots,0,1,0\dots,0)$ where 1 stands on the i-th place. The kernel $N=\ker h$ is a Noetherian Λ-module. Let it be generated by

$$u_{n+1}=(g_{1,1}(t),\dots,g_{1,n}(t)),\ \dots,\ u_{n+k}=(g_{k,1}(t),\dots,g_{k,n}(t)).$$

Without loss of generality, we can assume that all $g_{i,j}(t)$ are polynomials.

By Theorem 1.10, the Λ-module $\bigoplus_{i=1}^n \Lambda/(f_i(t))$ is $(t-1)$-invertible and by Proposition 1.5, N is also $(t-1)$-invertible Λ-module. Therefore the elements $(t-1)u_{n+1},\dots,(t-1)u_{n+k}$ are also generate N.

If for some $m\in\mathbb{N}$ the polynomial $t^m-1\in \operatorname{Ann}(M)$, then the elements $(0,\dots,0,t^m-1,0,\dots,0)\in N$, where the polynomial t^m-1 stands on the i-th place. Therefore we can add the elements $(0,\dots,0,t^m-1,0,\dots,0)$ to the set $(t-1)u_{n+1},\dots,(t-1)u_{n+k}$ and renumber the elements $\overline{u}_{n+1},\dots,\overline{u}_{n+k}$ (here we put $k:=n+k$) of the obtained set generating N so that $\overline{u}_{n+j}=(0,\dots,0,t^n-1,0,\dots,0)\in N$ for $j=1,\dots,n$, where the polynomial t^m-1 is stands on the j-th place.

Now, to complete the proof, notice that the kernel M_1 of the composite map $h\circ\nu:\Lambda^n\to M$ of h and the natural epimorphism $\nu:\Lambda^n\to\bigoplus_{i=1}^n\Lambda/(f_i(t))$ is generated by the elements

$$w_i=(0,\dots,0,f_i(t),0,\dots,0),\qquad i=1,\dots,n,$$

where the polynomial $f_i(t)$ stands on the i-th place, and the elements

$$w_{n+i}=(f_{i,1}(t),\dots,f_{i,n}(t))\in\Lambda^n,\qquad i=1,\dots,k,$$

where the coordinates $f_{i,j}(t)$ of each w_{n+i} coincide with the coordinates $\overline{g}_{i,j}(t)$ of $\overline{u}_{n+i}=(\overline{g}_{i,j}(t),\dots,\overline{g}_{i,j}(t))$. $\square$

1.2. **$\mathbb{Z}$-torsion submodules of $(t-1)$-invertible Λ-modules.** An element v of a Λ-module M is said to be of a *finite order* if there is $m\in\mathbb{Z}\setminus\{0\}$ such that $mv=0$. A Λ-module M is called *$\mathbb{Z}$-torsion* if all its elements are of finite order. For any Λ-module M denote by M_{fin} a subset of M consisting of all elements of finite order. It is easy to see that M_{fin} is a $\mathbb{Z}$-torsion Λ-module. If M is a Noetherian $(t-1)$-invertible Λ-module, then M_{fin} is also a Noetherian $(t-1)$-invertible Λ-module, and it follows from Propositions 1.5 and 1.6 that there is an exact sequence of Λ-modules

$$0\to M_{fin}\to M\to M_1\to 0$$

in which M_1 is a Noetherian $(t-1)$-invertible Λ-module free from elements of finite order.

Let $M=M_{fin}$ be a Noetherian $(t-1)$-invertible Λ-module. Since M is finitely generated over Λ, there is an integer $d\in\mathbb{N}$ such that $dv=0$ for all $v\in M$ (such d will be called an *exponent* for M). Let $d=p_1^{r_1}\dots p_n^{r_n}$ be its prime factorization. Denote by $M(p_i)$ the subset of M consisting of all elements $v \in M$ such that $p_i^r v = 0$ for some $r \in \mathbb{N}$. It is easy to see that $M(p)$ is a Λ-submodule of M and we call it the *p-submodule of M*.

Theorem 1.12. *Let $M = M_{fin}$ be a Noetherian $(t-1)$-invertible Λ-module and $d = p_1^{r_1} \dots p_n^{r_n}$ its exponent. Then M is the direct sum*

$$M = \bigoplus_{i=1}^{n} M(p_i)$$

of its p-submodules.

Proof. It coincides with the proof of similar Theorem for abelian groups (see, for example, Theorem 8.1 in [14]). □

Since the ring $\Lambda = \mathbb{Z}[t, t^{-1}]$ is Noetherian, any its ideal I is finitely generated. Denote by $I_{\rm pol} = I \cap \mathbb{Z}[t]$ the ideal of the ring $\mathbb{Z}[t]$. It is well known that $I = \Lambda I_{\rm pol}$, that is, any ideal I of Λ is generated by polynomials.

Recall that $\mathbb{Z}[t]$ is a factorial ring. Its units are precisely the units of $\mathbb{Z}$, and its prime elements are either primes of $\mathbb{Z}$ or polynomials $q(t) = \sum a_i t^i$ which are irreducible in $\mathbb{Q}[t]$ and have content 1 (that is, the greatest common divisors of the coefficients a_i of $q(t)$ are equal to 1). It follows from Euclidean algorithm that for any two polynomials $q_1(t), q_2(t) \in \mathbb{Z}[t]$ there are polynomials $h_1(t), h_2(t), r(t) \in \mathbb{Z}[t]$ and a constant $d \in \mathbb{Z}$, $d \neq 0$, such that

$$h_1(t)q_1(t) + h_2(t)q_2(t) = dr(t), \tag{3}$$

where $r(t)$ is the greatest common divisor of the polynomials $q_1(t)$ and $q_2(t)$.

Lemma 1.13. *Let M be a Noetherian $(t-1)$-invertible Λ-module and let $t^n - 1 \in Ann(M)$ for some $n = p^r$, where p is prime. Then M is $\mathbb{Z}$-torsion.*

Proof. If $t^n - 1 = (t-1)(t^{n-1} + \dots + t + 1)$ belongs to $\mathrm{Ann}(M)$, then the polynomial $g_n(t) = t^{n-1} + \dots + t + 1 \in \mathrm{Ann}(M)$, since M is $(t-1)$-invertible. For $n = p^r$ in the factorization

$$g_{p^r}(t) = \prod_{i=1}^{r} \Phi_{p^i}(t) = \prod_{i=1}^{r} \sum_{j=0}^{p-1} t^{jp^{i-1}}$$

each factor is an irreducible element of Λ.

By Theorem 1.10, there is a polynomial $f(t) \in \mathrm{Ann}(M)$ such that $f(1) = 1$ and if $n = p^r$ for some prime p, then $f(t)$ and $g_{p^r}(t)$ have not common irreducible divisors. Indeed, if $g(t)$ is a divisor of $f(t)$, then we should have $g(1) = \pm 1$, since $f(1) = 1$, but $\Phi_{p^i}(1) = p$ for each i. Therefore, there are polynomials $h_1(t)$, $h_2(t)$, and a constant $d \in \mathbb{N}$ such that $h_1(t)f(t) + h_2(t)g_{p^r}(t) = d$ and hence if $g_{p^r}(t) \in \mathrm{Ann}(M)$, then $d \in \mathrm{Ann}(M)$, that is, M is $\mathbb{Z}$-torsion. □

1.3. **Principle $(t-1)$-invertible Λ-modules.** Let I be a non-zero ideal of the ring Λ. Denote by I_m the subset of I_{pol} consisting of all polynomials $f(t)$ having the smallest degree (let m be this smallest degree). Note that if $f(t) \in I_m \setminus \{0\}$, then $f(0) \neq 0$.

Consider any two polynomials $f_1(t), f_2(t) \in I_m$ and write them in the form $f_i(t) = d_i q_i(t)$, where $d_i \in \mathbb{Z}$ and the polynomials $q_i(t)$ have content 1. We have $q_1(t) = q_2(t)$. Indeed, for their common greatest divisor $r(t)$ we have $\deg r(t) \leq m$ and, moreover, $\deg r(t) = m$ if and only if $q_1(t) = q_2(t)$. On the other hand, it follows from (3) that $d_2 h_1(t) f_1(t) + d_1 h_2(t) f_2(t) = d_1 d_2 d r(t)$ for some polynomials $h_1(t), h_2(t)$. Therefore $d_1 d_2 d r(t) \in I_{\mathrm{pol}}$ and we should have $\deg r(t) = m$.

Applying again Euclidean algorithm for integers, we obtain that if two polynomials $f_i(t) = d_i q(t)$ belong to I_m, then $d_0 q(t)$ belongs also to I_m, where d_0 is the greatest common divisor of d_1 and d_2. Thus there is a polynomial $f_m(t) = d_m q(t) \in I_m$ such that any polynomial $f(t) \in I_m$ is divided by $f_m(t)$. The polynomial $f_m(t)$ is defined uniquely up to multiplication by ± 1 and it will be called a *leading generator* of I.

Let I be a non-zero ideal of Λ and $f(t) = d_m q(t)$ be its leading generator. Then any polynomial $h(t) \in I$ should be divisible by $q(t)$. Indeed, as above it is easy to show that if $r(t)$ is the greatest common divisor of $f(t)$ and $h(t)$, then there is a constant d such that $dr(t) \in I$ and since $\deg q(t)$ is minimal for polynomials belonging to I, we should have the equality $r(t) = q(t)$.

The above considerations give rise to the following proposition.

Proposition 1.14. *Let $M = M_v$ be a principle $(t-1)$-invertible Λ-module generated by an element v. Then the annihilator Ann_v is generated by a finite set of polynomials $f_1(t), \dots, f_k(t)$, where $f_i(t) = d_i q_i(t)$, $d_i \in \mathbb{Z}$, $d_i \neq 0$, and $q_i(t)$ have content 1 for all i, such that $f_1(t), \dots, f_k(t)$ satisfy the following properties:*

- (i) $\deg f_1 < \deg f_2 \leq \cdots \leq \deg f_k$,
- (ii) $f_i(0) \neq 0$ *for all* i,
- (iii) $q_1(1) = 1$,
- (iv) $q_1(t) \mid q_i(t)$ *for* $i = 2, \dots, k$,
- (v) $|d_i| > 1$ *for* $i = 1, \dots, k-1$, $d_k = 1$, *and* $q_k(1) = 1$.

A set of generators of Ann_v is said to be *good* if it satisfies properties (i) – (v) from Proposition 1.14. We will distinguish the principal $(t-1)$-invertible Λ-modules $M = M_v$ as follows. We say that M_v is of *finite type* if in a good system $f_1(t), \dots, f_k(t)$ of generators of Ann_v the leading generator $f_1(t) \equiv d_1$ is a constant (that is, $q_1(t) \equiv 1$). A principle Λ-module M_v is said to be of *mixed type* if in a good system $f_1(t), \dots, f_k(t)$ of generators of Ann_v the degree of the leading generator $f_1 = d_1 q_1(t)$ is greater than one and $\mid d_1 \mid \geq 2$. It follows from the above considerations that if a principle $(t-1)$-invertible Λ-module $M = M_v$ is not of finite or mixed types, then for the leading generator $f_1(t) = q_1(t)$ of a good system of generators of Ann_v we should have $q_1(1) = 1$ and therefore Ann_v is a principle ideal generated by $q_1(t)$, since any polynomial $h(t) \in \mathrm{Ann}_v$ is divisible by $q_1(t)$. Such principle $(t-1)$-invertible Λ-modules will be called *bi-principle*.

It is easy to see that if $M = M_v$ is a principle Λ-module of finite type and $d_1 \in \mathbb{Z}$ is the leading generator of Ann_v, then all elements of M have order d_1, that is, a principle Λ-module M_v is of finite type if and only if it is $\mathbb{Z}$-torsion.

If $M = M_v$ is a bi-principle Λ-module, then M has not non-zero elements of finite order. Indeed, let $q(t)$ be a generator of Ann_v. If an element $v_1 = h(t)v$ has order m, then $mh(t) \in \mathrm{Ann}_v$, that is, $mh(t)$ is divisible by $q(t)$. Since t is a unite of Λ, we can assume that $h(t)$ is a polynomial, and since $q(1) = 1$, the polynomial $h(t)$ should be divisible by $q(t)$, that is, $v_1 = 0$.

If $M = M_v$ is a Λ-module of mixed type, then there is an exact sequence of Λ-modules

$$0 \to M_1 \to M \to M_2 \to 0$$

in which M_1 is a principle Λ-module of finite type and M_2 is a bi-principle Λ-module. Indeed, let $d_1q_1(t)$ be the leading generator of Ann_v. Put $v_1 = q_1(t)v$. Then it is easy to see that the Λ-module $M_1 = M_{v_1} \subset M$, generated by v_1, is of finite type and the Λ-module $M_2 = M/M_1 \simeq \Lambda/(q_1)$ is bi-principle.

1.4. **Finitely $\mathbb{Z}$-generated $(t-1)$-invertible Λ-modules.** Each Λ-module M can be considered as a $\mathbb{Z}$-module, that is as an abelian group.

Proposition 1.15. *A Noetherian $(t-1)$-invertible Λ-module M is finitely generated over $\mathbb{Z}$ if and only if there is a polynomial*

$$q(t) = \sum_{i=0}^{n} a_i t^i \in Ann(M)$$

such that $a_n = a_0 = 1$.

Proof. In the beginning, we prove Proposition 1.15 in the case when $M = M_v$ is a principal Λ-module.

It is easy to see that if there is a polynomial $q(t) = \sum_{i=0}^{n} a_i t^i \in \mathrm{Ann}_v$ such that $a_n = a_0 = 1$, then M is generated over $\mathbb{Z}$ by the elements $v, tv, \dots, t^{n-1}v$.

Let a Λ-module $M = M_v$ be finitely generated over $\mathbb{Z}$ and $h_1(t)v, \dots, h_m(t)v$ its generators. Since the multiplication by t is an isomorphism of M, we can assume that $h_i(t)$, $i = 1, \dots, m$, are polynomials such that $h_i(0) = 0$. Put $n - 1 = \max(\deg h_1(t), \dots, \deg h_m(t))$. Since $h_1(t)v, \dots, h_m(t)v$ generate M over $\mathbb{Z}$, there are integers $b_1, \dots, b_m$ and $c_1, \dots, c_m$ such that

$$v = \sum b_i h_i(t)v \quad \text{and} \quad t^n v = \sum c_i h_i(t)v.$$

Therefore the polynomials $1-\sum b_ih_i(t)$ and $t^n-\sum c_ih_i(t)$ belong to Ann_v. Then the polynomial $t^n+1-\sum(b_i+c_i)h_i(t)$ is a desired one.

In general case, a Noetherian $(t-1)$-invertible Λ-module M is generated by a finite set of elements $v_1,\dots,v_m$, and M is finitely generated over $\mathbb{Z}$ if and only if for all v_i the principal submodules $M_{v_i}\subset M$ are finitely generated over $\mathbb{Z}$.

If $g(t)\in \mathrm{Ann}(M)$, then $g(t)\in \mathrm{Ann}_{v_i}$ for $i=1,\dots,m$. In particular, if there is $q(t)=\sum_{i=0}^n a_it^i\in \mathrm{Ann}(M)$ such that $a_n=a_0=1$, then all M_{v_i} (and consequently, M) are finitely generated over $\mathbb{Z}$.

If for all i the principal submodules $M_{v_i}\subset M$ are finitely generated over $\mathbb{Z}$, then there are polynomials $q_i(t)=\sum_{j=0}^{n_i}a_{i,j}t^j\in \mathrm{Ann}_{v_i}$ such that $a_{i,n_i}=a_{i,0}=1$. Put $n=\sum n_i$. Then the polynomial

$$q(t)=q_1(t)\dots q_n(t)=t^n+1+\sum_{j=1}^{n-1}a_jt^j\in \mathrm{Ann}(M),$$

since $q(t)\in \mathrm{Ann}_{v_i}$ for all v_i. □

It follows from Proposition 1.15 that there are a lot of $(t-1)$-invertible bi-principle modules $M=\Lambda/I$ which are not finitely generated over $\mathbb{Z}$. More precisely, it is easy to see that *a bi-principle $(t-1)$-invertible module $M=\Lambda/I$ is finitely generated over $\mathbb{Z}$ if and only if the ideal $I=\langle q(t)\rangle$ is generated by a polynomial $q(t)=\sum_{i=0}^n a_it^i$ such that $q(1)=1$ and its coefficients a_0 and a_n are equal to ±1.*

For example, for each $m\in\mathbb{N}$ a $(t-1)$-invertible bi-principle module

$$M_m=\Lambda/\langle (m+1)t-m\rangle$$

is not finitely generated over $\mathbb{Z}$.

Theorem 1.16. *Let M be a Noetherian $\mathbb{Z}$-torsion $(t-1)$-invertible module. Then M is finitely generated over $\mathbb{Z}$.*

Proof. By Theorem 1.12, M is isomorphic the direct sum $\bigoplus M(p_i)$ of a finite number of its p-submodules. Therefore it suffices to prove Theorem in the case when M has exponent p^r, where p is a prime number. Next, by Corollary 1.8, M is a factor module of the direct sum $\bigoplus_{j=1}^n \Lambda/I_j$ of principle $(t-1)$-invertible Λ-modules Λ/I_j and in our case we can assume without loss of generality that each ideal I_j contains p^{r_j} for some r_j. Thus it suffices to prove Theorem in the case when $M=M_v$ is a principle $(t-1)$-invertible Λ-module of exponent p^r, that is, $I=\mathrm{Ann}_v$ contains a number p^r and a polynomial $g(t)$ such that $g(1)=1$.

Let $r=1$ and $g(t)=\sum a_it^i$. Denote by $g_1(t)=\sum_{p|a_i}a_it^i$ and put $\overline{g}(t)=g(t)-g_1(t)$. Then $\overline{g}(t)\in\mathrm{Ann}_v$, since $g(t),g_1(t)\in\mathrm{Ann}_v$. It is easy to see that $\overline{g}_1(1)$ and p are coprime, since $g(1) = 1$ and $g_1(1) \equiv 0 \mod p$. Moreover, by construction, each coefficient of the polynomial $\overline{g}(t)$ and p are coprime. Multiplying by t^{-k}, we can assume that $\overline{g}(0) \neq 0$. Let $\overline{g}(t) = \sum_{i=0}^{m} \overline{a}_i t^i$. Since $\overline{a}_m$ and p are coprime, one can find integers b_1 and c_1 such that $b_1\overline{a}_m + c_1 p = 1$. Similarly, there are integers b_2 and c_2 such that $b_2\overline{a}_0 + c_2 p = 1$. Therefore the polynomial $(b_1 t + b_2)\overline{g}(t) + p(c_1 t^{m+1} + c_2) \in I$ and it is equal to $h(t) = t^{m+1} + 1 + \sum_{i=1}^{m}(b_1\overline{a}_{i-1} + b_2\overline{a}_i)t^i$. Therefore, by Proposition 1.15, M_v is finitely generated over $\mathbb{Z}$.

Now consider general case of a principle $(t-1)$-invertible Λ-module of exponent p^r. Assume that for any principle $(t-1)$-invertible Λ-module M' of exponent p^{r_1}, where $r_1 < r$, M' is finitely generated over $\mathbb{Z}$. Let $M = M_v$ is a principle $(t-1)$-invertible Λ-module M of exponent p^r. Then the submodule M_{v_1} of M generated by $v_1 = p^{r-1}v$ is of exponent p and the factor module $M_{\overline{v}} = M_v/M_{v_1}$ is of exponent p^{r-1}. Now, the proof follows from the exact sequence

$$0 \to M_{v_1} \to M \to M/Mv_1 \to 0. \quad \square$$

Corollary 1.17. *Any Noetherian $\mathbb{Z}$-torsion $(t-1)$-invertible module is finite, that is, it is a finite abelian group.*

Lemma 1.18. *A group $G = \bigoplus_{i=1}^{n}(\mathbb{Z}/2^{r_i}\mathbb{Z})^{m_i}$ does not admit a structure of $(t-1)$-invertible Λ-module if $r_i \neq r_j$ for $i \neq j$ and one of $m_i = 1$.*

Proof. Assume that G has a structure of $(t-1)$-invertible Λ-module. Then for any r the subgroup 2^rG of G is its Λ-submodule and, by Propositions 1.5 and 1.6, 2^rG and $G/2^rG$ are $(t-1)$-invertible Λ-modules. Therefore, without loss of generality, we can assume that

$$G = (\mathbb{Z}/2\mathbb{Z}) \oplus (\bigoplus_{i=1}^{n}(\mathbb{Z}/2^{r_i}\mathbb{Z})^{m_i}),$$

where all $r_i \geq 2$ and $m_i \geq 2$. Let us choose generators $v_1, \dots, v_{m+1}$ of G, $m = \sum_{i=1}^{n} m_i$, so that

$$G \simeq (\mathbb{Z}/2\mathbb{Z})v_1 \oplus (\bigoplus_{i=2}^{m+1}(\mathbb{Z}/2^{\overline{r}_i}\mathbb{Z}))v_i,$$

where all $\overline{r}_i \geq 2$. Consider the $\mathbb{Z}$-submodule $\overline{G}$ of G consisting of all elements $v \in G$ of order ≤ 4. Obviously $\overline{G}$ is a Λ-submodule of G and it is generated over $\mathbb{Z}$ (and therefore over Λ) by $\overline{v}_1 = v_1$ and $\overline{v}_i = 2^{\overline{r}_i - 2}v_i$, $i = 2\dots, m+1$. It is easy to see that as an abelian group $\overline{G}$ is isomorphic to

$$\overline{G} \simeq (\mathbb{Z}/2\mathbb{Z})\overline{v}_1 \oplus (\bigoplus_{i=2}^{m+1}(\mathbb{Z}/4\mathbb{Z}))\overline{v}_i.$$

By Proposition 1.5, $\overline{G}$ is $(t-1)$-invertible Λ-module. The multiplication by t is an automorphism of $\overline{G}$. Let

$$\begin{aligned} t\overline{v}_1 = & \quad a_1\overline{v}_1 + 2\sum_{i=2}^{m+1} b_i\overline{v}_i, \\ t\overline{v}_j = & \quad a_j\overline{v}_1 + \sum_{i=2}^{m+1} c_{j,i}\overline{v}_i, \qquad j=2,\dots,m+1, \end{aligned} \tag{4}$$

where each $a_j = 0$ or 1.

Let us show that $a_1 = 1$. Indeed, assume that $a_1 = 0$. Since the multiplication by t is an automorphism and $\overline{v}_1,\dots,\overline{v}_{m+1}$ generate $\overline{G}$, we should have an equality $\overline{v}_1 = \sum d_i t\overline{v}_i$, where one of d_i is odd for some $i \geq 2$ if $a_1 = 0$. Next, the element $\overline{v}_1$ is of second order, therefore $2\sum_{i=2}^{m+1} d_i t\overline{v}_i = 0$. On the other hand, $t\overline{v}_2,\dots,t\overline{v}_{m+1}$ are linear independent over $\mathbb{Z}/4\mathbb{Z}$, since $\overline{v}_2,\dots,\overline{v}_{m+1}$ are linear independent over $\mathbb{Z}/4\mathbb{Z}$ and the multiplication by t is an isomorphism. Therefore the equality $2\sum_{i=2}^{m+1} d_i t\overline{v}_i = 0$ is impossible if some of d_i is odd, and hence a_1 in (4) should be equal to 1.

Let us show that $\overline{G}$ can not be $(t-1)$-invertible. Indeed, we have

$$t\overline{v}_1 = \overline{v}_1 + 2\sum_{i=2}^{m+1} b_i\overline{v}_i.$$

Therefore

$$(t-1)\overline{v}_1 = 2\sum_{i=2}^{m+1} b_i\overline{v}_i$$

and the above arguments show that the multiplication by $t-1$ is not an automorphism of $\overline{G}$, since $(t-1)\overline{v}_1$ is a linear combination of the elements $\overline{v}_2,\dots,\overline{v}_{m+1}$. □

Theorem 1.19. *An abelian group*

$$G = G_1 \oplus (\bigoplus_{i=1}^{n}(\mathbb{Z}/2^{r_i}\mathbb{Z})^{m_i}),$$

where $r_i \neq r_j$ for $i \neq j$ and G_1 is a group of odd order, admits a structure of $(t-1)$-invertible Λ-module if and only if all $m_i \geq 2$.

Proof. By Theorem 1.12, if $M = M_{fin}$ is a Noetherian $(t-1)$-invertible Λ-module and $d = p_1^{r_1}\dots p_n^{r_n}$ its exponent, then M is the direct sum

$$M = \bigoplus_{i=1}^{n} M(p_i)$$

of its p-submodules which are $(t-1)$-invertible by Proposition 1.5. Now, each its submodule $M(p_i)$ with odd p_i is of odd order and, by Lemma 1.18, its

2-submodule $M(2)$ is isomorphic (as an abelian group) to $\bigoplus_{i=1}^{k}(\mathbb{Z}/2^{r_i}\mathbb{Z})^{m_i}$, where all $m_i \geq 2$.

To prove the inverse statement, note, first, that the finite direct sum of $(t-1)$-invertible Λ-modules is also a $(t-1)$-invertible Λ-module. Next, for any prime $p > 2$, a $(t-1)$-invertible Λ-module $M = \Lambda/I$, where I is generated by the number p^r and polynomial $2t-1$, is isomorphic to $\mathbb{Z}/p^r\mathbb{Z}$ as an abelian group. Finally, for $n \geq 2$ the $(t-1)$-invertible Λ-module $M = \Lambda/I$, where I is generated by 2^r and $t^n - t + 1$, is isomorphic to $(\mathbb{Z}/2^r\mathbb{Z})^n$ as an abelian group. □

2. t-Unipotent $\mathbb{Z}[t, t^{-1}]$-modules

2.1. **Properties of t-unipotent Λ-modules.** The following proposition is a simple consequence of Propositions 1.5 and 1.6.

Proposition 2.1. *Any Λ-submodule M_1 and any factor module M/M_1 of a Noetherian $(t-1)$-invertible t-unipotent Λ-module M is a $(t-1)$-invertible t-unipotent Λ-module.*

Lemma 2.2. *Let $M_1, \dots, M_n$ be Noetherian $(t-1)$-invertible t-unipotent Λ-modules. Then the direct sum $M = \bigoplus_{i=1}^{n} M_i$ is a Noetherian $(t-1)$-invertible t-unipotent Λ-module.*

Proof. By Lemma 1.7, M is a Noetherian $(t-1)$-invertible Λ-module.

Since M_i is a $(t-1)$-invertible t-unipotent Λ-module, there is $k_i \in \mathbb{N}$ such that $t^{k_i} - 1 \in \mathrm{Ann}(M_i)$. It is easy to see that $t^k - 1 \in \mathrm{Ann}(M)$, where $k = k_1 \dots k_n$, since each polynomial $t^{k_i} - 1$, $i = 1, \dots, n$, divides the polynomial $t^k - 1$. □

Proposition 2.1 and Lemma 2.2 imply the following proposition.

Proposition 2.3. *A Noetherian Λ-module M_1 is $(t-1)$-invertible t-unipotent if and only if each its principle submodule M_v is $(t-1)$-invertible t-unipotent.*

Theorem 2.4. *Any Noetherian $\mathbb{Z}$-torsion $(t-1)$-invertible Λ-module is t-unipotent.*

Proof. let M be a Noetherian $\mathbb{Z}$-torsion $(t-1)$-invertible Λ-module. By Corollary 1.17, M consists of finite number of elements. Therefore the automorphism of M, defined by the multiplication by t, has a finite order, say k, that is, $t^k v = v$ for all $v \in M$, in other words, $t^k - 1 \in \mathrm{Ann}(M)$. □

The following Propositions 2.5, 2.6 describe bi-principle $(t-1)$-invertible t-unipotent modules and principle $(t-1)$-invertible t-unipotent modules of mixed type.

Proposition 2.5. *Let $M = \Lambda/I$ be a bi-principle $(t-1)$-invertible t-unipotent Λ-module, and let the ideal $I =< g(t) >$ is generated by a polynomial $g(t)$. Then*

- (i) *all roots of $g(t)$ are roots of unity,*
- (ii) *$g(t)$ has not multiple roots,*
- (iii) *if ξ is a k-th root of unity (that is, $\xi^k = 1$), were $k = p^r$ for some prime p, then ξ is not a root of $g(t)$,*
- (iv) *$g(1) = \pm 1$,*
- (v) *$\deg g(t)$ is even.*

Proof. To prove (i) and (ii), notice that there is k such that $t^k - 1 \in I$, since M is t-unipotent. Therefore $t^k - 1$ is divisible by $g(t)$.

To prove (iii) – (v), we use Theorem 1.10. By Theorem 1.10, there is a polynomial $f(t) \in I$ such that $f(1) = 1$. We have $f(t) = h(t)g(t)$ for some polynomial $h(t) \in \mathbb{Z}[t]$, since I is a principle ideal generated by $g(t)$. Therefore $g(1) = \pm 1$ (and we can assume that $g(1) = 1$), since we have

$$1 = f(1) = h(1)g(1),$$

where $h(1), g(1) \in \mathbb{Z}$.

On the other hand, if for some prime p, a primitive p^r-th root of unity ξ is a root of $g(t)$, then $g(t)$ should be divided by the p^r-th cyclotomic polynomial $\Phi_{p^r}(t)$, that is, there is a polynomial $h(t) \in \mathbb{Z}[t]$ such that $g(t) = \Phi_{p^r}(t)h(t)$. Therefore, $1 = g(1) = \Phi_{p^r}(1)h(1)$ and we obtain a contradiction, since $\Phi_{p^r}(1) = p$.

To complete the proof, notice that, by (iii) and (iv), $\xi = \pm 1$ are not roots of $g(t)$ and hence all roots of $g(t)$ are not real. Thus if ξ is a root of $g(t)$, then the number $\overline{\xi}$ complex conjugated to ξ is also a root of $g(t)$, since $g(t) \in \mathbb{Z}[t]$. Therefore $\deg g(t)$ is even, since $\overline{\xi} \neq \xi$ for all roots of unity $\neq \pm 1$. $\square$

Proposition 2.6. *Let $M = \Lambda/I$ be a principle $(t-1)$-invertible t-unipotent Λ-module of mixed type, and let $f(t) = dg(t)$ be the leading generator of the ideal I, where $d \in \mathbb{N}$ and the polynomial $g(t)$ has content 1. Then $g(t)$ satisfies properties (i) – (v) from Proposition 2.5.*

Proof. Let v be a generator of M. Denote by M_1 a Λ-submodule of M generated by $v_1 = g(t)v$. We have the exact sequence of Λ-modules

$$0 \to M_1 \to M \to M/M_1 \to 0,$$

where M_1 is a principle module of finite type and $M_2 = M/M_1$ is a bi-principle Λ-module isomorphic to $\Lambda/ < g(t) >$. By Proposition 2.1, M_2 is $(t-1)-$invertible t-unipotent. Now, we apply Proposition 2.5 to complete the proof. $\square$

Let M be a Noetherian $(t-1)$-invertible t-unipotent Λ-module. The smallest $k \in \mathbb{N}$ such that $t^k - 1 \in \operatorname{Ann}(M)$ is called the *unipotence index* of M.

Lemma 2.7. *If M is a Noetherian $(t-1)$-invertible t-unipotent Λ-module of unipotence index k, then the polynomial $\sum_{i=0}^{k-1} t^i \in Ann(M)$.*

Proof. We have $t^k - 1 = (t-1)(\sum_{i=0}^{k-1} t^i) \in \operatorname{Ann}(M)$. Therefore $(\sum_{i=0}^{k-1} t^i)v = 0$ for all $v \in M$, since M is a $(t-1)$-invertible Λ-module. □

Lemma 2.8. *A Noetherian $(t-1)$-invertible Λ-module M of unipotence index 2 is a finite $\mathbb{Z}$-module of odd order.*

Proof. It follows from Lemma 1.13 and Corollary 1.17 that M is finite. By Lemma 2.7, the polynomial $(t+1) \in \operatorname{Ann}(M)$. Therefore $tv = -v$ for all $v \in M$. In particular, if v is of order 2, then $tv = v$. This is impossible, since M is $(t-1)$-invertible. Therefore M has not elements of even order. □

Proposition 2.9. *A cyclic group G of order $n = p_1^{r_1} \dots p_m^{r_m}$, where $p_1, \dots, p_m$ are primes, possesses a structure of $(t-1)$-invertible Λ-module of unipotence index k if and only if for each $i = 1, \dots, m$ the polynomial $\sum_{i=0}^{k-1} t^i$ has a root $a_i \neq 1$ in the field $\mathbb{Z}/p_i\mathbb{Z}$.*

Proof. By Theorem 1.12, it suffices to consider only the case when $i = 1$, that is $n = p^m$ for some prime p.

Let a cyclic group G of order $n = p^m$ has a structure of $(t-1)$-invertible Λ-module of unipotence index k, then its subgroup $G_p = p^{m-1}G$ consisting of the elements of order p is also a $(t-1)$-invertible Λ-module of unipotence index k. Therefore the polynomial $\sum_{i=0}^{k-1} t^i \in \operatorname{Ann}(G_p)$. Let $v \in G_p$ be a generator of G_p, then $tv = av$ for some $a \not\equiv 1 \mod p$ since G_p is a $(t-1)$-invertible module. We have $\sum_{i=0}^{k-1} a^i v = 0$. Therefore $\sum_{i=0}^{k-1} a^i \equiv 0 \mod p$, that is, the polynomial $\sum_{i=0}^{k-1} t^i$ has a root in the field $\mathbb{Z}/p\mathbb{Z}$ not equal to 0 or 1.

Conversely, let $a \not\equiv 1 \mod p$ be a root of the polynomial $\sum_{i=0}^{k-1} t^i$ in the field $\mathbb{Z}/p_i\mathbb{Z}$, and let v be a generator of a cyclic group G of order p^r. If we define the action of t on the $\mathbb{Z}$-module G putting $t(v) = av$, we obtain a structure of $(t-1)$-invertible Λ-module on G, since $a \not\equiv 1 \mod p$. It is easy to see that $t^k - 1 \in \operatorname{Ann}(G)$. □

Theorem 2.10. *Any Noetherian $(t-1)$-invertible t-unipotent Λ-module M is finitely generated over $\mathbb{Z}$.*

Proof. Theorem follows from Proposition 1.15, since for some $k \in \mathbb{Z}$ the polynomial $t^k - 1 \in \operatorname{Ann}(M)$. □

It follows from Theorem 2.4 and Structure Theorem for finitely generated $\mathbb{Z}$-modules that a Noetherian $(t-1)$-invertible t-unipotent Λ-module M as a $\mathbb{Z}$-module is isomorphic to

$$M \simeq M_{fin} \oplus \mathbb{Z}^k, \tag{5}$$

where M_{fin} is the submodule of M consisting of the elements of finite order. The rank k of the free part of M in decomposition (5) is called *Betti number* of Noetherian $(t-1)$-invertible t-unipotent Λ-module M.

Theorem 2.11. *The Betti number of a Noetherian $(t-1)$-invertible t-unipotent Λ-module M is an even number.*

Proof. By definition, the Betti number of M coincides with Betti number of the Noetherian $(t-1)$-invertible t-unipotent Λ-module $M_{free} = M/M_{fin}$.

The module M_{free} has not non-zero elements of finite order. Therefore the annihilator Ann_v of each its element v is a principle ideal, it is generated by polynomial $g_v(t)$ satisfying properties (i) – (v) from Proposition 2.5.

Let M_{free} is generated by elements $v_1, \dots, v_m$ over Λ. Then there is a surjective Λ-homomorphism

$$f : \Lambda/ < g_{v_1}(t) > \oplus \dots \oplus \Lambda/ < g_{v_m}(t) > \longrightarrow M_{free}.$$

Consider the modules $\widetilde{M} = \bigoplus \Lambda/ < g_{v_i}(t) >$ and M_{free} as free $\mathbb{Z}$-modules and denote by $h_{\widetilde{M}}$ and $h_{M_{free}}$ the automorphisms respectively of $\widetilde{M}$ and M_{free} defined by the multiplication by t. Then it is easy to see that the characteristic polynomial $\widetilde{\Delta}(t) = \det(h_{\widetilde{M}} - t\mathrm{Id})$ coincides up to the sign with the product $g_{v_1}(t) \dots g_{v_m}(t)$. Next, the characteristic polynomial $\Delta(t) = \det(h_{M_{free}} - t\mathrm{Id})$ is a divisor of the polynomial $\widetilde{\Delta}(t)$, since the homomorphism f is surjective and t-equivariant. Therefore all roots of $\Delta(t)$ are roots of unity $\neq \pm 1$ and hence $\deg \Delta(t)$ is an even number. To complete the proof, notice that the Betti number of M_{free} coincides with $\deg \Delta(t)$. □

2.2. **Derived Alexander modules.** To a Noetherian $(t-1)$-invertible Λ-module M we associate an infinite sequence of Noetherian $(t-1)$-invertible t-unipotent Λ-modules

$$A_n(M) = M/(t^n - 1)M, \qquad n \in \mathbb{N}. \tag{6}$$

The module $A_n(M)$ is called the *n-th derived Alexander module* of Λ-module M.

Note that $A_1(M) = 0$, since M is $(t-1)$-invertible. It is also evident that $A_n(A_n(M)) = A_n(M)$.

It is obvious, that if $f : M_1 \to M_2$ is a Λ-homomorphism of $(t-1)$-invertible modules, then the sequence of Λ-homomorphisms

$$f_{n*} : A_n(M_1) \to A_n(M_2),$$

$n \in \mathbb{N}$, is well defined, that is, the map $M \mapsto \{A_n(M)\}$ is a functor from the category of Noetherian $(t-1)$-invertible Λ-modules to the category of infinite sequences of Noetherian $(t-1)$-invertible t-unipotent Λ-modules.

Proposition 2.12. *If*

$$0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0$$

is an exact sequence of Noetherian $(t-1)$-invertible Λ-modules, then

$$A_n(M_2) \simeq A_n(M)/im\, f_{n*}(A_n(M_1)).$$

If $M = \bigoplus_{i=1}^k M_i$ is the direct sum of Noetherian $(t-1)$-invertible Λ-modules M_i, then

$$A_n(M) \simeq \bigoplus_{i=1}^{k} A_n(M_i).$$

Proof. Obvious. □

Proposition 2.13. *Let p be a prime number and $r \in \mathbb{N}$, then for a Noetherian $(t-1)$-invertible Λ-module M its derived Alexander module $A_{p^r}(M)$ is finite.*

Proof. It follows from Lemma 1.13 and Corollary 1.17. □

Example 2.14. *For $M_m = \Lambda/\langle (m+1)t - m\rangle$, where $m \in \mathbb{N}$, its n-th derived Alexander module*

$$A_n(M_m) \simeq \mathbb{Z}/((m+1)^n - m^n)\mathbb{Z}$$

is a cyclic group of order $(m+1)^n - m^n$ and the multiplication by t is given by

$$tv = (-1)^{n+1} m (\sum_{i=1}^{n-1} (-1)^i \binom{n}{i} (m+1)^{n-i-1}) v$$

for all $v \in A_n(M_m)$.

Proof. The module $M_m = \Lambda/\langle (m+1)t - m\rangle$ is isomorphic to a Λ-submodule $\mathbb{Z}[\frac{m}{m+1}, \frac{m+1}{m}] \subset \mathbb{Q}$ if we put $t = \frac{m}{m+1}$ and $tv = \frac{m}{m+1} v$ for $v \in \mathbb{Q}$. Therefore we have

$$A_n(M_m) \simeq M_m/(t^n - 1)M_m \simeq \mathbb{Z}[\tfrac{m+1}{m}, \tfrac{m}{m+1}]/\langle (\tfrac{m}{m+1})^n - 1\rangle$$

and consequently,

$$A_n(M_m) \simeq \mathbb{Z}[\tfrac{m}{m+1}, \tfrac{m+1}{m}]/\langle (m+1)^n - m^n\rangle.$$

It is easy to see that the module $\mathbb{Z}[\frac{m}{m+1}, \frac{m+1}{m}]$ coincides with the sum of submodules $\mathbb{Z}[\frac{1}{m+1}]$ and $\mathbb{Z}[\frac{1}{m}] \subset \mathbb{Q}$,

$$\mathbb{Z}[\tfrac{m}{m+1}, \tfrac{m+1}{m}] = \mathbb{Z}[\tfrac{1}{m+1}] + \mathbb{Z}[\tfrac{1}{m}].$$

Indeed, it is obvious, that

$$\mathbb{Z}[\tfrac{m}{m+1},\tfrac{m+1}{m}]\subset\mathbb{Z}[\tfrac{1}{m+1}]+\mathbb{Z}[\tfrac{1}{m}].$$

Next, we have

$$(\tfrac{m+1}{m})^n=\frac{\sum_{i=0}^n\binom{n}{i}m^{n-i}}{m^n}$$

and therefore

$$\tfrac{1}{m^n}=(\tfrac{m+1}{m})^n-\textstyle\sum_{i=0}^{n-1}\binom{n}{i}\tfrac{1}{m^i}.$$

Similarly, we have

$$\tfrac{1}{(m+1)^n}=\textstyle\sum_{i=0}^{n-1}(-1)^{n+1+i}\binom{n}{i}\tfrac{1}{(m+1)^i}+(-1)^n(\tfrac{m}{m+1})^n$$

In particular, $\frac{1}{m}=\frac{m+1}{m}-1$ and $\frac{1}{m+1}=1-\frac{m}{m+1}$. Therefore, by induction, we obtain that $\frac{1}{m^n},\ \frac{1}{(m+1)^n}\in\mathbb{Z}[\frac{m}{m+1},\frac{m+1}{m}]$ for all n and hence

$$\mathbb{Z}[\tfrac{1}{m+1}]+\mathbb{Z}[\tfrac{1}{m}]\subset\mathbb{Z}[\tfrac{m}{m+1},\tfrac{m+1}{m}].$$

Let us show now that each element $v\in\mathbb{Z}[\frac{m}{m+1},\frac{m+1}{m}]$ is equivalent to some $v_{in}\in\mathbb{Z}\subset\mathbb{Z}[\frac{m}{m+1},\frac{m+1}{m}]$ modulo the ideal $I=\langle(m+1)^n-m^n\rangle$. For this, it suffices to show that for each k there are $v_{in,k},u_{in,k}\in\mathbb{Z}$ such that

$$\tfrac{1}{m^k}\equiv v_{in,k}\mod I\qquad\text{and}\qquad\tfrac{1}{(m+1)^k}\equiv u_{in,k}\mod I.$$

We prove the existence of such elements only for $\frac{1}{m^k}$ and the case $\frac{1}{(m+1)^k}$ will be left to the reader, since it is similar. We have

$$\tfrac{(m+1)^n-m^n}{m^k}=\textstyle\sum_{i=1}^n\binom{n}{i}m^{n-i-k}\equiv 0\mod I$$

and therefore

$$\tfrac{1}{m^k}\equiv-\textstyle\sum_{j=k+1-n}^{k-1}\binom{n}{n+j-k}\tfrac{1}{m^j}\mod I.$$

In particular,

$$\tfrac{1}{m}\equiv-\textstyle\sum_{j=0}^{n-2}\binom{n}{n-j-1}m^j\mod I.$$

Now the existence of desired $v_{in,k}$ is proved by induction on k.

It follows from the above consideration that

$$A_n(M_m)\simeq\mathbb{Z}[\tfrac{m}{m+1},\tfrac{m+1}{m}]/\langle(m+1)^n-m^n\rangle$$

is a cyclic group generated by the image $\overline{1}$ of $1\in\mathbb{Z}[\frac{m}{m+1},\frac{m+1}{m}]$. We have

$$((m+1)^n-m^n)\overline{1}=0$$

and hence the order of $A_n(M_m)$ is a divisor of $(m+1)^n-m^n$.

Let us show that the order of $A_n(M_m)$ is equal to $(m+1)^n-m^n$. Let $k\in\mathbb{Z}$ be such that $k\overline{1}=0$. Then

$$k=(\textstyle\sum_{i_1\le i\le i_2}a_i\tfrac{1}{(m+1)^i}+\sum_{j_1\le j\le j_2}b_j\tfrac{1}{m^j})((m+1)^n-m^n),$$

where $a_i, b_j \in \mathbb{Z}$. Multiplying by $(m+1)^{i_2}$ and m^{j_2} if $i_2 > 0$ or $j_2 > 0$, we obtain an equality

$$(m+1)^{i_2} m^{j_2} k = C((m+1)^n - m^n)$$

with some $C \in \mathbb{Z}$ which shows that $(m+1)^n - m^n$ is a divisor of k, since m, $m+1$, and $(m+1)^n - m^n$ are coprime.

To calculate the action of t on the cyclic group

$$A_n(M_m) \simeq \mathbb{Z}/((m+1)^n - m^n)\mathbb{Z},$$

notice that

$$t\overline{1} = \overline{\tfrac{m}{m+1}} = (-1)^{n+1} m (\textstyle\sum_{i=1}^{n-1} (-1)^i \binom{n}{i} (m+1)^{n-i-1})\overline{1},$$

since similar (as above) calculation gives

$$\tfrac{1}{m+1} \equiv (-1)^{n+1} \textstyle\sum_{i=1}^{n-1} (-1)^i \binom{n}{i} (m+1)^{n-i-1} \mod I. \qquad \square$$

Proposition 2.15. *An abelian group G is isomorphic (as a $\mathbb{Z}$-module) to the derived Alexander module $A_2(M)$ of some Noetherian $(t-1)$-invertible Λ-module M if and only if G is a finite group of odd order.*

Proof. By Lemma 2.8, we need only to prove that for any finite group G of odd order there is a Noetherian $(t-1)$-invertible Λ-module M for which $A_2(M) \simeq G$.

Represent G as a direct sum of cyclic groups:

$$G = \bigoplus_{i=1}^{k} G_i,$$

and let $n_i = 2m_i + 1$ be the order of G_i.

For each i, consider the Λ-module M_{m_i} from Example 2.14. We have $A_2(M_{m_i})$ is a cyclic group of order $(m_i+1)^2 - m_i^2 = 2m_i + 1$. Now, proposition follows from Proposition 2.12 if we put $M = \bigoplus_{i=1}^{k} M_{m_i}$. $\square$

Theorem 2.16. *Let M be a Noetherian $(t-1)$-invertible t-unipotent Λ-module of unipotence index k. Then the sequence of its derived Alexander modules*

$$A_1(M), \ldots, A_n(M), \ldots$$

has period k, that is, $A_n(M) \simeq A_{n+k}(M)$ for all n.

If n and k are coprime, then $A_n(M) = 0$.

Proof. Note that if k is the unipotence index of M, then, by Lemma 2.7, the polynomial $f_k(t) = \displaystyle\sum_{i=0}^{k-1} t^i \in \mathrm{Ann}(M)$. Besides, to get $A_n(M)$ from M, it suffices to factorize M by the relations $f_n(t)v = 0$ for all $v \in M$, where

$f_n(t) = \sum_{i=0}^{n-1} t^i$. Now, to prove the periodicity of sequence (6), it suffices to notice that

$$f_{n+k}(t) = t^n f_k(t) + f_n(t).$$

Let n and k be coprime and let polynomials $f_k(t)$ and $f_n(t)$ belong to $\mathrm{Ann}(M)$. Applying Euclidian algorithm to $f_k(t)$ and $f_n(t)$, it is easy to see that there are polynomials $g_k(t)$ and $g_n(t)$ such that

$$f_k(t)g_k(t) + f_n(t)g_n(t) = 1,$$

since n and k are coprime. Therefore $\mathrm{Ann}(M) = \Lambda$ and hence $A_n(M) = 0$. $\square$

Example 2.17. *The Λ-module $M = \Lambda / < t^2 - t + 1 >$ has the following derived Alexander modules:*

$$A_{6k\pm1}(M) = 0, \qquad A_{6k\pm2}(M) \simeq \mathbb{Z}/3\mathbb{Z}, \qquad A_{6k+3}(M) \simeq (\mathbb{Z}/2\mathbb{Z})^2,$$

where the multiplication by t on $\mathbb{Z}/3\mathbb{Z}$ coincides with the multiplication by 2 and the multiplication by t on $(\mathbb{Z}/2\mathbb{Z})^2$ coincides with cyclic permutation of the non-zero elements of $A_{6k+3}(M)$.

Proof. The module M has the unipotency index 6, since $t^2 - t + 1$ is a divisor of the polynomial $t^6 - 1$. Therefore $A_{6k\pm1}(M) = 0$.

To compute $A_{6k+2}(M)$, it suffices to compute $A_2(M)$. We have $A_2(M) = \Lambda / < t^2 - t + 1, t + 1 >$ and since

$$t^2 - t + 1 = (t - 2)(t + 1) + 3,$$

then $\Lambda / < t^2 - t + 1, t + 1 >= \Lambda / < t + 1, 3 >\simeq \mathbb{Z}/3\mathbb{Z}$.

To compute $A_{6k+3}(M)$, it suffices to compute $A_3(M)$. We have $A_3(M) = \Lambda / < t^2 - t + 1, t^2 + t + 1 >$ and since

$$t^2 + t + 1 = t^2 - t + 1 + 2t,$$

then $\Lambda / < t^2 - t + 1, t^2 + t + 1 >= \Lambda / < t^2 + t + 1, 2 >\simeq (Z/2\mathbb{Z})^2$.

To compute $A_{6k+4}(M)$, it suffices to compute $A_4(M)$. We have $A_4(M) = \Lambda / < t^2 - t + 1, t^3 + t^2 + t + 1 >$ and since

$$t^3 + t^2 + t + 1 = (t + 2)(t^2 - t + 1) + 2t - 1,$$

then $\Lambda / < t^2 - t + 1, t^3 + t^2 + t + 1 >= \Lambda / < t^2 - t + 1, 2t - 1 >$ is isomorphic to the quotient module $M/(2t-1)M$. Let v be a generator of bi-principle module M. It is easy to check that in the basis $v_1 = v, v_2 = tv$ of M over $\mathbb{Z}$, the module $(2t - 1)M$ is generated by the elements $2v_2 - v_1$ and $t(2v_2 - v_1) = v_2 - 2v_1$, since $tv_2 = v_2 - v_1$. In the new basis $e_1 = v_1$, $e_2 = v_2 - 2v_1$, the element $2v_2 - v_1 = 2e_2 + 3e_1$, that is, $(2t - 1)M$ is generated over $\mathbb{Z}$ by $3e_1$ and e_2. Therefore $A_4(M) \simeq \mathbb{Z}/3\mathbb{Z}$. $\square$

3. Alexander modules of irreducible C-groups

3.1. **Proof of Theorems 0.1 and 0.3.** Recall that the class of irreducible C-groups coincides with the class of fundamental groups of knotted n-manifolds V if $n \geq 2$ and the knot groups are also C-groups if they are gvien by Wirtinger presentation. Similarly, the class of irreducible Hurwitz C-groups coincides with the class of the fundamental groups of the complements of irreducible "affine" Hurwitz (resp., pseudo-holomorphic) curves and it contains the subclass of the fundamental groups of the complements of algebraic irreducible affine plane curves. Therefore to speak about the Alexander modules of knotted n-manifolds and, respectively, about the Alexander modules of irreducible Hurwitz (resp., pseudo-holomorphic) curves is the same as to speak about the Alexander modules of irreducible C-groups and, respectively, of irreducible Hurwitz C-groups. Hence Theorems 0.1 and 0.3 are equivalent to the following two theorems.

Theorem 3.1. *A Λ-module M is the Alexander module of an irreducible C-group if and only if it is Noetherian $(t-1)$-invertible.*

Theorem 3.2. *A Λ-module M is the Alexander module of an irreducible Hurwitz C-group if and only if it is Noetherian $(t-1)$-invertible t-unipotent Λ-module.*

The unipotence index of the Alexander module $A_0(G)$ of an irreducible C-group G of degree m is a divisor of m.

Proof. Let
$$G = < x_1, \dots, x_m \mid r_1, \dots, r_n > \tag{7}$$
be a C-presentation of a C-group G and $\mathbb{F}_m$ be the free group freely generated by the C-generators $x_1, \dots, x_m$. Denote by $\frac{\partial}{\partial x_i}$ the Fox derivative ([3]), that is, an endomorphism of the group ring $\mathbb{Z}[\mathbb{F}_m]$ over $\mathbb{Z}$ of the free group $\mathbb{F}_m$ into itself, such that $\frac{\partial}{\partial x_i} : \mathbb{Z}[\mathbb{F}_m] \to \mathbb{Z}[\mathbb{F}_m]$ is a $\mathbb{Z}$-linear map defined by the following properties
$$\begin{array}{rcl} \dfrac{\partial x_j}{\partial x_i} & = & \delta_{i,j} \\ \dfrac{\partial uv}{\partial x_i} & = & \dfrac{\partial u}{\partial x_i} + u\dfrac{\partial v}{\partial x_i} \end{array} \tag{8}$$
for any $u, v \in \mathbb{Z}[\mathbb{F}_m]$. The matrix
$$\mathcal{A}(G) = \nu_*\big(\tfrac{\partial r_i}{\partial x_j}\big) \in \mathrm{Mat}_{n\times m}(\mathbb{Z}[t, t^{-1}])$$
is called the *Alexander matrix* of the C-group G given by presentation (7), where r_i, $i = 1, \dots, n$, are the defining relations of G and $\nu_* : \mathbb{Z}[\mathbb{F}_m] \to \mathbb{Z}[\mathbb{F}_1] \simeq \mathbb{Z}[t, t^{-1}]$ is induced by the canonical C-epimorphism $\nu : \mathbb{F}_m \to \mathbb{F}_1$.

Lemma 3.3. *The sum of the columns of the Alexander matrix $\mathcal{A}(G)$ of a C-group G, given by presentation* (7)*, is equal to zero.*

Proof. Each relation r_i has the form

$$r = wx_jw^{-1}x_l^{-1},$$

where w is a word in letters $x_1^{\pm 1},\dots,x_m^{\pm 1}$, and x_j, x_l are some two letters.

By induction on the length $l(w)$ of the word w, let us show that

$$\sum_{k=1}^{m}\nu_*(\frac{\partial r}{\partial x_k})=0.$$

If $l(w)=0$, that is, $r := x_jx_l^{-1}$, we have

$$\nu_*(\frac{\partial r}{\partial x_k})=\begin{cases} 1 & \text{if } k=j,\\ -1 & \text{if } k=l,\\ 0 & \text{if } k\neq j \text{ and } k\neq l\end{cases}$$

and in this case we obtain $\sum_{k=1}^{m}\nu_*(\frac{\partial r}{\partial x_k})=0$.

Assume that for all words $r = wx_jw^{-1}x_l^{-1}$ we have $\sum_{k=1}^{m}\nu_*(\frac{\partial r}{\partial x_k})=0$ if $l(w)\leq n$. Consider a word $r = wx_jw^{-1}x_l^{-1}$, such that $l(w)=n+1$. Put $r_1 = w_1x_jw_1^{-1}x_l^{-1}$, where $w = x_i^{\varepsilon}w_1$, $\varepsilon=\pm 1$, and $l(w_1)=n$. We consider only the case when $i\neq j$, $i\neq l$, $j\neq l$, and $\varepsilon = 1$. All other cases are similar and the proof that $\sum_{k=1}^{m}\nu_*(\frac{\partial r}{\partial x_k})=0$ in these cases will be left to the reader.

It follows from (8) that

$$\nu_*(\frac{\partial r}{\partial x_k})=\begin{cases} t\nu_*(\frac{\partial r_1}{\partial x_k}) & \text{if } k\neq i, k\neq j, k\neq l,\\ 1+t\nu_*(\frac{\partial r_1}{\partial x_k})-t & \text{if } k=i,\\ 1+t\nu_*(\frac{\partial r_1}{\partial x_k}) & \text{if } k=j,\\ t(\nu_*(\frac{\partial r_1}{\partial x_k})+1)-1 & \text{if } k=l\end{cases}$$

and it is easy to see that $\sum_{k=1}^{m}\nu_*(\frac{\partial r}{\partial x_k})=0$. □

To each monomial $a_it^i\in\mathbb{Z}[t]$ let us associate a word

$$w_{a_it^i}(x_1,x_2)=(x_2^ix_1x_2^{-(i+1)})^{a_i}$$

if $a_i>0$ and

$$w_{a_it^i}(x_1,x_2)=(x_2^{i+1}x_1^{-1}x_2^{-i})^{-a_i}$$

if $a_i < 0$, and for $g(t) = \sum_{i=0}^{k} a_i t^i \in \mathbb{Z}$ we put

$$w_{g(t)}(x_1, x_2) = \prod_{i=0}^{k} w_{a_i t^i}(x_1, x_2).$$

Next, to a polynomial $f(t) = (1-t)g(t) + 1$ we associate a word

$$r_{f(t)}(x_1, x_2) = w_{g(t)}(x_1, x_2) x_1 w^{-1}_{g(t)}(x_1, x_2) x_2^{-1}, \tag{9}$$

and to a vector $u = (1-t)\overline{u} = ((1-t)g_1(t), \dots, (1-t)g_m(t))$, we associate a word

$$r_u(x_1, \dots, x_{m+1}) = w_u(x_1, \dots, x_{m+1}) x_{m+1} w_u^{-1}(x_1, \dots, x_{m+1}) x_{m+1}^{-1}, \tag{10}$$

where

$$w_u(x_1, \dots, x_{m+1}) = \prod_{i=1}^{m} w_{g_i(t)}(x_i, x_{m+1}).$$

Lemma 3.4. *For a polynomial $f(t) = (1-t)g(t) + 1$ and a vector*

$$u = ((1-t)g_1(t), \dots, (1-t)g_m(t))$$

we have

$$\begin{aligned} \nu_*(\frac{\partial r_{f(t)}}{\partial x_1}) &= f(t), \\ \nu_*(\frac{\partial r_u}{\partial x_i}) &= (1-t)g_i(t), \qquad i = 1, \dots, m. \end{aligned}$$

Proof. Let $f(t) = (1-t)g(t) + 1$. It follows from (8) that

$$\nu_*(\frac{\partial w_{g(t)}(x_1, x_2)}{\partial x_1}) = -\nu_*(\frac{\partial w^{-1}_{g(t)}(x_1, x_2)}{\partial x_1}) = g(t)$$

since $w_{g(t)}(x_1, x_2) w^{-1}_{g(t)}(x_1, x_2) = 1$,

$$\nu_*(w_{g(t)}(x_1, x_2)) = \nu_*(w_{a_i t^i}(x_1, x_2)) = 1,$$

and

$$\nu_*(\frac{\partial w_{a_i t^i}(x_1, x_2)}{\partial x_1}) = a_i t^i.$$

Therefore we have

$$\begin{aligned} \nu_*(\frac{\partial r_{f(t)}}{\partial x_1}) = \;& \nu_*(\frac{\partial (w_{g(t)}(x_1, x_2) x_1 w^{-1}_{g(t)}(x_1, x_2) x_2^{-1})}{\partial x_1}) = \\ & g(t) + 1 - t g(t) = f(t). \end{aligned}$$

The proof in the second case is similar and it will be left to the reader. $\square$

Proposition 3.5. *The Alexander module $A_0(G)$ of a C-group G, given by presentation (7), is isomorphic to a factor module $\Lambda^{m-1}/M(G)$, where the submodule $M(G)$ of Λ^{m-1} is generated by the rows of the matrix $\overline{\mathcal{A}}$ formed by the first $m-1$ columns of the Alexander matrix $\mathcal{A}(G)$.*

Proof. To describe the Alexander module of a C-group G, we follow [18] (see also [7]). To a C-group G given by C-presentation (7) we associate a two-dimensional complex K with a single vertex x_0 whose one dimensional skeleton is a bouquet of oriented circles s_i, $1 \leq i \leq m$, corresponding to the C-generators of G in presentation (7). Furthermore, $K \setminus (\cup s_i) = \bigsqcup_{i=1}^{n} \overset{\circ}{D}_i$ is a disjoint union of open discs. Each disc D_i corresponds to the relation $r_i = x_{j_{i,1}}^{\varepsilon_{i,1}} \dots x_{j_{i,k_i}}^{\varepsilon_{i,k_i}}$ from presentation (7), where $\varepsilon_{i,j} = \pm 1$, and it is glued to the bouquet $\bigvee s_i$ along the path $s_{j_{i,1}}^{\varepsilon_{i,1}} \dots s_{j_{i,k_i}}^{\varepsilon_{i,k_i}}$. It is clear that $\pi_1(K, x_0) \simeq G$.

The C-homomorphism $\nu : G \to \mathbb{F}_1$ defines an infinite cyclic covering $f : \widetilde{K} \to K$ such that $\pi_1(\widetilde{K}) = N$ and $H_1(\widetilde{K}, \mathbb{Z}) = N/N'$, where $N = \ker \nu$. The group $\mathbb{F}_1$ acts on $\widetilde{K}$.

Let $\widetilde{K}_0 = f^{-1}(x_0)$, and let $\widetilde{K}_1$ be the one-dimensional skeleton of the complex $\widetilde{K}$. Consider the following exact sequences of homomorphisms of homology groups with coefficients in $\mathbb{Z}$:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & H_1(\widetilde{K}) \arrow[d] & \\
 \arrow[r] & H_2(\widetilde{K}, \widetilde{K}_1) \arrow[r, "\alpha"] & H_1(\widetilde{K}_1, \widetilde{K}_0) \arrow[r, "\beta"] & H_1(\widetilde{K}, \widetilde{K}_0) \arrow[r] \arrow[d, "\partial"] & 0 \\
 & & & H_0(\widetilde{K}_0) \arrow[d] & \\
 & & & 0 &
\end{tikzcd} \tag{11}$$

The action of $\mathbb{F}_1$ on $\widetilde{K}$ turns the groups in these sequences into Λ-modules. We fix a vertex $p_0 \in \widetilde{K}_0$. Let $p_i = t^i p_0$ be the result of action of the element $t^i \in \mathbb{F}_1$ on the point p_0. Then $H_1(\widetilde{K}_1, \widetilde{K}_0)$ is a free Λ-module whose generators $\bar{s}_i$ are edges joining p_0 with p_1 which are mapped onto the loops s_i. The result of action of t^i on the generator $\bar{s}_j$ is an edge beginning at the vertex p_i which is mapped onto the loop s_j.

The free Λ-module $H_2(\widetilde{K},\widetilde{K}_1)$ is generated by the discs $\overline{D}_i$, $i=1,\dots,n$, corresponding to the relations $r_i=x_{j_{i,1}}^{\varepsilon_{i,1}}\dots x_{j_{i,k_i}}^{\varepsilon_{i,k_i}}$, where each disc $\overline{D}_i$ is glued to the one-dimensional skeleton along the product of paths

$$t^{\delta(\varepsilon_{i,1})}\bar{s}_{j_{i,1}}^{\varepsilon_{i,1}},t^{\delta(\varepsilon_{i,2})+\varepsilon_{i,1}}\bar{s}_{j_{i,2}}^{\varepsilon_{i,2}},\dots,t^{\delta(\varepsilon_{i,k_i})+\sum_{l=1}^{k_i-1}\varepsilon_{i,l}}\bar{s}_{j_{i,k_i}}^{\varepsilon_{i,k_i}},$$

where $\delta(1)=0$ and $\delta(-1)=-1$. It is easy to verify that the coordinates of elements $\alpha(\overline{D}_i)\in H_1(\widetilde{K}_1,\widetilde{K}_0)$ in the basis $\bar{s}_1,\dots,\bar{s}_m$ coincide with the rows $\mathcal{A}_i$ of the Alexander matrix $\mathcal{A}(G)$ of C-group G given by presentation (7).

It follows from the vertical exact sequence in (11) that $\partial(\beta(\bar{s}_i))=(t-1)p_0$ for each generator $\bar{s}_i$ of the module $H_1(\widetilde{K}_1,\widetilde{K}_0)$. Let us choose a new basis $e_i=\bar{s}_i-\bar{s}_m$, $i=1,\dots,m-1$, $e_m=\bar{s}_m$ in $H_1(\widetilde{K}_1,\widetilde{K}_0)$. Then $\beta(e_i)\in\ker\partial$ for $i=1,\dots,m-1$, and $\ker\partial$ is generated by $\beta(e_1),\dots,\beta(e_{m-1})$. Hence we may identify $H_1(\widetilde{K})$ with $\beta(H_1'(\widetilde{K}_1,\widetilde{K}_0))$, where $H_1'(\widetilde{K}_1,\widetilde{K}_0)$ is a free submodule of the free Λ-module $H_1(\widetilde{K}_1,\widetilde{K}_0)$ generated by the elements $e_1,\dots,e_{m-1}$.

In the basis $e_1,\dots,e_m$ the matrix formed by the coordinates of $\alpha(\overline{D}_i)$ coincides with the matrix $\widetilde{\mathcal{A}}(G)$ obtained from $\mathcal{A}(G)$ by replacing the last column by the column of zeros. Hence $H_1(\widetilde{K})$ is isomorphic to the quotient of the free Λ-module $H_1'(\widetilde{K}_1,\widetilde{K}_0)\simeq\bigoplus_{i=1}^{m-1}\Lambda e_i$ by the submodule $M(G)$ generated by the rows of the matrix $\overline{\mathcal{A}}(G)$, where $\overline{\mathcal{A}}(G)$ is the matrix formed by the first $m-1$ columns of the matrix $\mathcal{A}(G)$. □

To prove that a Noetherian $(t-1)$-invertible (resp., t-unipotent) Λ-module M is the Alexander module of an irreducible (resp., Hurwitz) C-group, we use Proposition 1.11. By Proposition 1.11, a Noetherian $(t-1)$-invertible Λ-module M is isomorphic to a factor module Λ^m/M_1 of a free Λ-module Λ^m, where the submodule M_1 is generated by elements $u_1,\dots,u_m,\dots,u_{m+k}$ of Λ^m such that

- (i) for $i=1,\dots,m$ the vector $u_i=(0,\dots,0,f_i(t),0,\dots,0)$, where a polynomial $f_i(t)$ is such that $f_i(1)=1$ and it stands on the i-th place,
- (ii) $u_{m+j}=(1-t)\overline{u}_{m+j}=((1-t)g_{j,1}(t),\dots,(1-t)g_{j,m}(t))$ for $j=1,\dots,k$, where $g_{j,l}(t)$ are polynomials,

and if M is a t-unipotent Λ-module of unipotence index n, then we can assume that

- (iii) the vector $u_{m+k+i}=(0,\dots,0,t^n-1,0,\dots,0)\in M_1$ for $i=1,\dots,m$, where the polynomial t^n-1 stands on the i-th place.

Express each polynomial $f_i(t)$ in the form $f_i(t)=(1-t)g_i(t)+1$ and consider a C-group

$$G=\langle x_1,\dots,x_{m+1}\mid r_1,\dots,r_{m+k}\rangle,$$

where $r_i := r_{f_i(t)}(x_i, x_{m+1})$ for $i = 1, \dots, m$ and $r_{m+j} := r_u(x_1, \dots, x_{m+1})$ for $j = 1, \dots, k$, where the words $r_{f(t)}$ and r_u were defined by formulas (9) and (10). Denote by $r_{m+k+i} := x^n_{m+1} x_i x^{-n}_{m+1} x^{-1}_i$ if

$$u_{m+k+i} = (0, \dots, 0, t^n - 1, 0, \dots, 0) \in M_1$$

for $i = 1, \dots, m$ and denote by

$$\overline{G} = \langle x_1, \dots, x_{m+1} \mid r_1, \dots, r_{2m+k} \rangle.$$

It follows from Lemma 3.4 that the matrix $\widetilde{\mathcal{A}}(G)$ (resp., $\widetilde{\mathcal{A}}(G)$) formed by the first m columns of the Alexander matrix $\mathcal{A}(G)$ (resp., $\mathcal{A}(G)$) coincides with the matrix $\mathcal{U}$ (resp., $\overline{\mathcal{U}}$) formed by the rows $u_1, \dots, u_{m+k}$ (resp., by $u_1, \dots, u_{2m+k}$). Therefore, by Proposition 3.5, the Alexander module $A_0(G)$ (resp., $A_0(\overline{G})$) coincides with $M = \Lambda^m/M_1$, where M_1 is generated by the rows $u_1, \dots, u_{m+k}$ (resp., by $u_1, \dots, u_{2m+k}$).

Notice that G (resp., $\overline{G}$) is an irreducible C-group, since all C-generators $x_1, \dots, x_m$ are conjugated to x_{m+1}. Moreover, $\overline{G}$ is a Hurwitz C-group. Indeed, it follows from relations r_{m+k+j}, $j = 1, \dots, m$, that x^n_{m+1} belongs to the center of $\overline{G}$. Since all x_i are conjugated to x_{m+1}, we have $x^n_i = x^n_{m+1}$ for all $i = 1, \dots, m$. Therefore the product $x^n_1 \dots x^n_{m+1}$ also belongs to the center of $\overline{G}$ and $\overline{G}$ possesses a Hurwitz presentation

$$\begin{array}{ll} \overline{G} = \langle x_1, \dots, x_{n(m+1)} \mid & r_1, \dots, r_{2m+k}, \\ & x_i x^{-1}_{i+m+1},\ i = 1, \dots, (n-1)(m+1), \\ & [x_i, (x_1 \dots x_{n(m+1)})],\ i = 1, \dots, n(m+1) \rangle. \end{array}$$

The following two lemmas complete the proof of Theorems 0.1 and 0.3.

Lemma 3.6. ([13]) *The Alexander module $A_0(G) = G'/G''$ of an irreducible C-group G is a Noetherian $(t-1)$-invertible Λ-module.*

Proof. For an irreducible C group G its commutator subgroup G' coincides with the kernel of the C-epimorphism $\nu : G \to \mathbb{F}_1$. By the Reidemeister – Schreier method, if C-generators $x_1, \dots, x_m$ generate G, then the elements $a_{i,n} = x^n_m x_i x^{-(n+1)}_m$, $i = 1, \dots, m-1$, $n \in \mathbb{Z}$, generate G'. Therefore $A_0(G) = G'/G''$ is generated by the images $\overline{a}_{i,n}$ of the elements $a_{i,n}$ under the natural epimorphism $G' \to G'/G''$. The action of t on $A_0(G)$ is defined by conjugation $a \mapsto x_m a x^{-1}_m$ for $a \in G'$. Therefore $t\overline{a}_{i,n} = \overline{a}_{i,n+1}$. Thus $A_0(G)$ is generated over Λ by $\overline{a}_{1,0} \dots, \overline{a}_{m-1,0}$ and hence it is a Noetherian Λ-module.

To show that $A_0(G)$ is a $(t-1)$-invertible Λ-module, notice, first, that any element $g \in G$ can be written in the form $g = x^k_m a$, where $a \in G'$ and $k = \nu(g)$. Therefore G' is generated by the elements of the form $[x^n_m a, x^k_m b]$, where $a, b \in G'$, and hence $A_0(G)$ is generated by their images $\overline{[x_m^n a, x_m^k b]}$. It is easily to check that

$$\begin{array}{rl} [x_m^n a, x_m^k b] = & [x_m^n, a](a x_m^{n+k}[b, a^{-1}] x_m^{-(n+k)} a^{-1})[a, x_m^{n+k}]\cdot \\ & \cdot(x_m^{n+k}[b, x_m^{-n}] x_m^{-(n+k)}). \end{array} \tag{12}$$

It follows from (12) that

$$\begin{array}{rl} \overline{[x_m^n a, x_m^k b]} = & (t^n - 1)\overline{a} + (1 - t^{n+k})\overline{a} + t^{n+k}(1 - t^{-n})\overline{b} = \\ & t^n(1 - t^k)\overline{a} + t^k(t^n - 1)\overline{b} = \\ & (t-1)(\sum_{i=0}^{n-1} t^{i+k}\overline{b} - \sum_{i=0}^{k-1} t^{i+n}\overline{a}), \end{array} \tag{13}$$

since $a x_m^{n+k}[b, a^{-1}] x_m^{-(n+k)} a^{-1} \in G''$. Now, it is easy to see that the multiplication by $t - 1$ is an epimorphism of $A_0(G)$, since the elements of the form $\overline{[x_m^n a, x_m^k b]}$ generate $A_0(G)$ over $\mathbb{Z}$. To complete the proof, we apply Lemma 1.1. $\square$

Lemma 3.7. ([10]) *The Alexander module of a Hurwitz C-group of degree m is a Noetherian $(t-1)$-invertible t-unipotent Λ-module of unipotence index d, where d is a divisor of m.*

Proof. If G is a Hurwitz group of degree m, then it is generated by C-generators $x_1, \dots, x_m$ such that the product $x_1 \dots x_m$ belongs to the center of G. By Lemma 3.6, the Alexander module $A_0(G) = G'/G''$ in a Noetherian $(t-1)$-invertible Λ-module. The multiplication by t on $A_0(G)$ is induced by conjugation $a \mapsto x_m a x_m^{-1}$ for $a \in G'$. Since $\nu(x_m^m) = \nu(x_1 \dots x_m)$, there is an element $a_0 \in G'$ such that $x_m^m = a_0 \cdot x_1 \dots x_m$ and hence the conjugation by x_m^m is an inner automorphism of G'. Therefore the induced automorphism t^m of G'/G'' is the identity. $\square$

3.2. **Alexander modules of C-products of C-groups.** Let G_1 and G_2 be two irreducible C-groups and let $x \in G_1$ (resp., $y \in G_2$) be one of C-generators of G_1 (resp., of G_2). Consider the amalgamated product $G_1 *_{\{x=y\}} G_2$. If

$$\begin{array}{l} G_1 = \langle x_1, \dots, x_n \mid \mathcal{R}_1 \rangle, \\ G_2 = \langle y_1, \dots, y_m \mid \mathcal{R}_2 \rangle \end{array} \tag{14}$$

are C-presentations of G_1 and G_2, where $x = x_n$ and $y = y_m$, then $G_1 *_{\{x=y\}} G_2$ is given by C-presentation

$$\langle x_1, \dots, x_{n-1}, y_1, \dots, y_{m-1}, z \mid \widetilde{\mathcal{R}}_1 \cup \overline{\mathcal{R}}_2 \rangle \tag{15}$$

in which each relation $\widetilde{r}_i \in \widetilde{\mathcal{R}}_1$ (resp., $\overline{r}_i \in \overline{\mathcal{R}}_2$) is obtained from the relation $r_i \in \mathcal{R}_1$ (resp., from $r_i \in \mathcal{R}_2$) by substitution of z instead of x_n (resp., instead of y_m).

If $x' \in G_1$ and $y' \in G_2$ are two another C-generators of these groups, then there are inner C-isomorphisms $f_i : G_i \to G_i$ such that $f_1(x') = x$ and $f_2(y') = y$, since all C-generators of an irreducible C-group are conjugated to each other. Therefore there is an isomorphism

$$f_1 * f_2 : G_1 *_{\{x'=y'\}} G_2 \to G_1 *_{\{x=y\}} G_2,$$

that is, the group $G_1 *_{\{x=y\}} G_2$, up to a C-isomorphism, does not depend on the choice of C-generators x and y, so we denote it by $G_1 *_C G_2$ and call the *C-product* of irreducible C-groups G_1 and G_2.

Proposition 3.8. *If a C-group $G = G_1 *_C G_2$ is the C-product of irreducible C-groups G_1 and G_2, then its Alexander module $A_0(G)$ is isomorphic to the direct sum of the Alexander modules of G_1 and G_2,*

$$A_0(G) = A_0(G_1) \oplus A_0(G_2).$$

Proof. This proposition is a simple consequence of Proposition 3.5. Indeed, if G_1 and G_2 are given by presentation (14), then, by Proposition 3.5, the Alexander module $A_0(G)$ of the C-group $G = G_1 *_C G_2$, given by presentation (15), is isomorphic to a factor module $\Lambda^{n+m-1}/M(G)$, where the submodule $M(G)$ of Λ^{n+m-1} is generated by the rows of the matrix

$$\overline{\mathcal{A}} = \begin{pmatrix} \overline{\mathcal{A}}_1 & 0 \\ 0 & \overline{\mathcal{A}}_2 \end{pmatrix},$$

where $\overline{\mathcal{A}}_1$ (resp., $\overline{\mathcal{A}}_2$) is the matrix formed by the first $n-1$ (resp., $m-1$) columns of the matrix $\mathcal{A}(G_1)$ (resp., $\mathcal{A}(G_2)$). Now, it is easy to see that $A_0(G) = A_0(G_1) \oplus A_0(G_2)$. □

Let

$$G =< x_1, \dots, x_m \mid r_1, \dots, r_n > \tag{16}$$

be a C-presentation of a C-group G. The number $d_P = m - n$ is called the *C-deficiency* of presentation (16) and $d_G = \min d_P$, where the minimum is taken over all C-presentation of a C-group G, is called the *C-deficiency* of the group G. Obviously, for a C-group consisting of k connected component, its C-deficiency $d_G \le k$ and, in particular, if G is an irreducible C-group, then $d_G \le 1$.

Lemma 3.9. *Let $G = G_1 *_C G_2$ be the C-product of irreducible C-groups G_1 and G_2. Then*

$$d_G \ge d_{G_1} + d_{G_2} - 1.$$

In particular, if $d_{G_1} = d_{G_2} = 1$, then $d_G = 1$.

Proof. It follows from formula (15).

3.3. **Presentation graphs of C-groups.** Let us associate a *presentation graph* Γ_P to each C-presentation (16) as follows. The vertices of the graph Γ_P are labeled by the generators from presentation (16) (and, in particular, they are in one to one correspondence with the generators from presentation (16)), and its edges are in one to to one correspondence with the relations r_j of the presentation (16) and if $r_j := w_j^{-1}(x_1, \dots, x_m)x_{i_1}w_j(x_1, \dots, x_m)x_{i_2}^{-1}$, then the corresponding edge connects the vertices x_{i_1} and x_{i_2}.

Obviously, the C-deficiency

$$d_P = \dim H_0(\Gamma_P, \mathbb{R}) - \dim H_1(\Gamma_P, \mathbb{R}).$$

Therefore *for an irreducible C-group G its C-deficiency $d_G = 1$ if and only if G possesses a C-presentation whose presentation graph Γ_P is a tree.*

A C-presentation

$$G =< x_1, \dots, x_m \mid r_1, \dots, r_n > \tag{17}$$

is said to be *simple* if each relation r_j in (17) is of the form:

$$r_j := x_{i_3}^{-1}x_{i_1}x_{i_3}x_{i_2}^{-1},$$

for some $i_1, i_2, i_3 \in \{1, \dots, m\}$ (that is, $x_{i_2} = x_{i_3}^{-1}x_{i_1}x_{i_3}$).

Remark 3.10. *If presentations* (14) *of irreducible C-groups G_1 and G_2 are simple, then presentation* (15) *of $G = G_1 *_C G_2$ is also simple and the presentation graph Γ_P of presentation* (15) *is the bouquet $\Gamma_P = \Gamma_{P_1} \bigvee_{z=x_n=y_m} \Gamma_{P_2}$ of the presentation graphs Γ_{P_1} and Γ_{P_2} of presentations* (14)*. In particular, if Γ_{P_1} and Γ_{P_2} are trees, then the presentation graph Γ_P is also a tree.*

Lemma 3.11. *Any C-group possesses a simple C-presentation with C-deficiency $d_P = d_G$.*

Proof. Let G be given by C-presentation of C-deficiency $d_P = d_G$ and $r := w^{-1}x_iwx_j^{-1}$ is one of its relations (that is, $w^{-1}x_iw = x_j$), where $w = x_{i_1}^{\varepsilon_1} \dots x_{i_k}^{\varepsilon_k}$ is a word in $\mathbb{F}_m$ and $\varepsilon_l = \pm 1$, then we can add $k-1$ new generators $x_{m+1}, \dots,$ x_{m+k-1} and replace the relation r by k relations:

$$\begin{array}{lcl} x_{m+1} & = & x_{i_1}^{-\varepsilon_1}x_ix_{i_1}^{\varepsilon_1}, \\ x_{m+2} & = & x_{i_2}^{-\varepsilon_2}x_{m+1}x_{i_2}^{\varepsilon_2}, \\ \dots\dots & \dots & \dots\dots\dots\dots \\ x_{m+k-1} & = & x_{i_{k-1}}^{-\varepsilon_{k-1}}x_{m+k-2}x_{i_{k-1}}^{\varepsilon_{k-1}}, \\ x_j & = & x_{i_k}^{-\varepsilon_k}x_{m+k-1}x_{i_k}^{\varepsilon_k}. \end{array}$$

Obviously, we obtain a new C-presentation of the same C-deficiency which defines the same C-group G. $\square$

3.4. **The Alexander modules of C-groups possessing C-presentations whose presentation graphs are trees.** By Lemma 3.11, an irreducible C-group G possesses a simple C-presentation whose presentation graph is a tree if and only if its C-deficiency $d_G = 1$.

Proposition 3.12. *If $M = \bigoplus_{i=1}^{m} M_i$ is the direct sum of bi-principle $(t-1)$-invertible Λ-modules $M_i = \Lambda/\langle f_i(t)\rangle$, then there is an irreducible C-group G such that $A_0(G) \simeq M$ and such that its C-deficiency $d_G = 1$.*

Proof. Note that the C-deficiency of a C-group given by presentation

$$G = \langle x_1, x_2 \mid w x_1 w^{-1} x_2^{-1}\rangle, \tag{18}$$

where $w = w(x_1, x_2)$ is a word in letters x_1, x_2 and their inverses, is equal to 1. Applying Proposition 3.5, we see that the Alexander module $A_0(G)$ of a C-group G, given by presentation (18), is a bi-principle $(t-1)$-invertible Λ-module.

Conversely, it was shown in the proof of Theorem 3.1 that any bi-principle $(t-1)$-invertible Λ-module $M = \Lambda/\langle f(t)\rangle$ is the Alexander module of some irreducible C-group given by presentation (18). To complete the proof we apply Proposition 3.8 and Remark 3.10. □

Corollary 3.13. *Let $M = \bigoplus_{i=1}^{m} M_i$ is a direct sum of bi-principle $(t-1)$-invertible Λ-modules $M_i = \Lambda/\langle f_i(t)\rangle$. Then for each $n \geq 2$ there is a knotted sphere $S^n \subset S^{n+2}$ such that the Alexander module*

$$A_0(\pi_1(S^{n+2} \setminus S^n)) \simeq M.$$

In particular, a polynomial $f(t) \in \mathbb{Z}[t]$ is the Alexander polynomial $\Delta(t)$ of some knotted sphere $S^n \subset S^{n+2}$ if and only if $f(1) = \pm 1$ and, moreover, the Jordan blocks of the Jordan canonical form of the matrix of the automorphism $h_{\mathbb{C}}$ acting on $A_0(S^n) \otimes \mathbb{C}$ can be of arbitrary size.

Proof. In [8], it was shown that if an irreducible C-group is given by a simple C-presentation which presentation graph is a tree, then for each $n \geq 2$ there is a knotted sphere $S^n \subset S^{n+2}$ such that $\pi_1(S^{n+2} \setminus S^n) \simeq G$. □

Proposition 3.14. *Let G be an irreducible C-group of C-deficiency $d_G = 1$. Then its Alexander module $A_0(G)$ has not non-zero $\mathbb{Z}$-torsion elements.*

Proof. Let

$$G = < x_1, \dots, x_m \mid r_1, \dots, r_{m-1} > \tag{19}$$

be a C-presentation of G. By Proposition 3.5, its Alexander module $A_0(G)$ is isomorphic to a factor module $\Lambda^{m-1}/M(G)$, where the submodule $M(G)$ of Λ^{m-1} is generated by the rows of the matrix $\overline{\mathcal{A}}$ formed by the first $m-1$ columns of the Alexander matrix $\mathcal{A}(G)$ of the group G given by presentation (19). The size of the matrix $\overline{\mathcal{A}}$ is $(m-1) \times (m-1)$.

Lemma 3.15. *The determinant* $\Delta(t) = \det \overline{\mathcal{A}}$ *satisfies the following property:* $\Delta(1) = \pm 1$.

Proof. It coincides with the similar statement for knot groups (see the proof, for example, in [3]). □

Denote by $\mathcal{A}_j$ the rows of the matrix $\overline{\mathcal{A}}$, $j = 1, \dots, m-1$. The module $A_0(G)$ has a non-zero $\mathbb{Z}$-torsion element if and only if there is a vector $u = (f_1(t), \dots, f_{m-1}(t))$ such that $u \notin M(G)$ and $ku \in M(G)$ for some $k \in \mathbb{N}$. Assume that there is a such vector u. Then there are $g_j(t) \in \Lambda$ such that $ku = \sum g_j(t)\mathcal{A}_j$, where for some $g_j(t)$ one of its coefficients is not divisible by k.

Without loss of generality, we can assume that all $f_i(t)$ and $g_j(t)$ belong to $\mathbb{Z}[t]$. By Cramer's theorem,

$$g_j(t) = \frac{\Delta_j(t)}{\Delta(t)},$$

where $\Delta_j(t)$ is the determinant of the matrix obtained from $\overline{\mathcal{A}}$ by substitution ku instead of the row $\mathcal{A}_j$. Therefore the coefficients of all polynomials $\frac{\Delta_j(t)}{\Delta(t)}$ are divisible by k. A contradiction. □

Remark 3.16. *If* G *is an irreducible* C*-group given by presentation of* C*-deficiency* $d_P = d_G = 1$*, then the determinant* $\Delta(t) = \det \overline{\mathcal{A}}$ *of the matrix* $\overline{\mathcal{A}}$*, obtained from the Alexander matrix* $\mathcal{A}$ *after deleting its last column, coincides with the Alexander polynomial* $\Delta_G(t)$ *of the group* G.

3.5. Finitely $\mathbb{Z}$-generated Alexander modules of irreducible C-groups.

Theorem 3.17. *Let* G *be an irreducible* C*-group. The Alexander module* $A_0(G)$ *is finitely generated over* $\mathbb{Z}$ *if and only if the leading coefficient* a_n *and the constant coefficient* a_0 *of the Alexander polynomial* $\Delta_G(t) = \sum_{i=0}^{n} a_i t^i$ *of* G *are equal to* ± 1.

Proof. By Theorem 3.1, $A_0(G)$ is a Noetherian $(t-1)$-invertibele Λ-module. Let $A_0(G)_{fin}$ be the Z-torsion submodule of the Alexander module $A_0(G)$. By Theorem 1.16, $A_0(G)_{fin}$ is finitely generated over $\mathbb{Z}$.

Consider the quotient module $M = A_0(G)/A_0(G)_{fin}$. It is free from $\mathbb{Z}$-torsion. Therefore there is a natural embedding $M \hookrightarrow M_{\mathbb{Q}} = M \otimes \mathbb{Q}$. We have $\dim_{\mathbb{Q}} M_{\mathbb{Q}} < \infty$, since M is a Noetherian Λ-torsion module.

Denote by $h_{\mathbb{Q}}$ an automorphism of $M_{\mathbb{Q}}$ induced by the multiplication by t. Then, by definition, $\Delta_G(t) = a\det(h_{\mathbb{Q}} - t\mathrm{Id})$, where $a \in \mathbb{N}$ is the smallest number such that $a\det(h_{\mathbb{Q}} - t\mathrm{Id}) \in \mathbb{Z}[t]$.

If the Alexander module $A_0(G)$ is finitely generated over $\mathbb{Z}$, then M is a free finitely generated $\mathbb{Z}$-module. Denote by h an automorphism of M induced by multiplication by t. We have $\det h = \pm 1$ and

$$\det(h - t\mathrm{Id}) = \det(h_{\mathbb{Q}} - t\mathrm{Id}) \in \mathbb{Z}[t].$$

Therefore $\Delta_G(t) = \det(h - t\mathrm{Id})$ and its leading coefficient $a_n = (-1)^n$, where $n = \mathrm{rk}\, M$, and $a_0 = \det h = \pm 1$.

Let the leading coefficient a_n and the constant coefficient a_0 of the Alexander polynomial $\Delta_G(t)$ of G be equal to ± 1. By Cayley-Hamilton's Theorem, $\Delta_G(t) \in \mathrm{Ann}(M_{\mathbb{Q}})$. Therefore $\Delta_G(t) \in \mathrm{Ann}(M)$ and M is finitely generated over $\mathbb{Z}$ by Proposition 1.15. □

Remark 3.18. *Let an irreducible C-group G is given by C-presentation $G = \langle x_1, \dots, x_m \mid r_1, \dots, r_n\rangle$ and $\mathcal{A}(G)$ its Alexander matrix. Then the Alexander polynomial $\Delta_G(t)$ coincides (up to multiplication by $\pm t^k$) with the greatest common divisor of the determinants of all $(m-1)\times(m-1)$ submatrices $\mathcal{A}_{m-1}$ of the matrix $\mathcal{A}(G)$.*

3.6. **Alexander modules of some irreducible C-groups.** In the end of this section, we compute the Alexander modules for some irreducible C-groups.

Example 3.19. *The Alexander module $A_0(Br_{m+1})$ of the braid group Br_{m+1} is trivial if $m \geq 4$ (or $m = 1$) and isomorphic to $\Lambda/\langle t^2 - t + 1\rangle$ for $m = 2$ and 3.*

This statement is well known, but for completeness, we give a proof.

Proof. The braid group Br_{m+1} is given by presentation

$$\mathrm{Br}_{m+1} = \langle x_1, \dots, x_m \mid \begin{array}{cl} [x_i, x_j] & \text{for } |i-j| \geq 2, \\ x_i x_{i+1} x_i x_{i+1}^{-1} x_i^{-1} x_{i+1}^{-1} & \text{for } i = 1, \dots, m-1 \rangle. \end{array}$$

Notice that it is a C-presentation of an irreducible C-group.

By Proposition 3.5, to calculate $A_0(\mathrm{Br}_{m+1})$ we should calculate the matrix $\overline{\mathcal{A}}(\mathrm{Br}_{m+1})$.

The relations $[x_m, x_i]$, $i = 1, \dots, m-2$, give the rows

$$(0, \dots, 0, (t-1), 0 \dots, 0), \tag{20}$$

where $t-1$ stands on the i-th place for $i = 1, \dots, m-2$, and if $m \geq 4$, then the relation $[x_{m-1}, x_1]$ gives the row

$$(t-1, 0, \dots, 0, 1-t). \tag{21}$$

If $m \geq 4$, then the rows from (20) and row (21) generate submodule $(t-1)\Lambda^{m-1}$ of the module Λ^{m-1}. On the other hand, these rows belong to the module $M(\mathrm{Br}_{m+1})$. Therefore $A_0(\mathrm{Br}_{m+1}) = 0$, since $A_0(\mathrm{Br}_{m+1}) \simeq \Lambda^{m-1}/M(\mathrm{Br}_{m+1})$ is a $(t-1)$-invertible Λ-module and $(t-1)\Lambda^{m-1} \subset M(\mathrm{Br}_{m+1})$.

If $m=2$, then we have the only one relation in the presentation of Br_3, namely,

$$r := x_1x_2x_1x_2^{-1}x_1^{-1}x_2^{-1}$$

We have $\nu_*(\frac{\partial r}{\partial x_1}) = 1+t^2-t$ and therefore $A_0(\mathrm{Br}_3)\simeq \Lambda/\langle t^2-t+1\rangle$.

If $m=3$, then we have the only three relations in the presentation of Br_4, namely,

$$\begin{array}{rl} r_1 := & x_1x_2x_1x_2^{-1}x_1^{-1}x_2^{-1}, \\ r_2 := & x_2x_3x_2x_3^{-1}x_2^{-1}x_3^{-1}, \\ r_3 := & x_1x_3x_1^{-1}x_3^{-1}. \end{array}$$

We have

$$\begin{array}{l} \nu_*(\frac{\partial r_1}{\partial x_1}) = -\nu_*(\frac{\partial r_1}{\partial x_2}) = \nu_*(\frac{\partial r_2}{\partial x_2}) = t^2-t+1 \\ \nu_*(\frac{\partial r_3}{\partial x_1}) = 1-t, \end{array}$$

Therefore $M(\mathrm{Br}_3)\subset\Lambda^2$ is generated by vectors

$$v_1=(t^2-t+1,-(t^2-t+1)),\quad v_2=(0,t^2-t+1),\quad v_3=(1-t,0),$$

and hence $A_0(\mathrm{Br}_3)\simeq\Lambda/\langle t^2-t+1\rangle$. □

Example 3.20. *The Alexander module of a C-group*

$$G_m = \langle x_1,x_2 \mid (x_1^{-1}x_2)^m x_1 (x_1^{-1}x_2)^{-m}x_2^{-1}\rangle,$$

$m\in\mathbb{N}$, *is isomorphic to* $A_0(G)\simeq\Lambda/\langle (m+1)t-m\rangle$.

These irreducible C-groups are interesting, since they are non-Hopfian if $m\geq 2$ and therefore they are not residually finite. (The group G_m is isomorphic to Baumslag – Solitar group (see [1]) $\langle a,x_1 \mid x_1^{-1}a^m x_1 a^{-(m+1)}\rangle$ if we put $x_2=x_1a$.) Note also that each of these groups can be realized as $\pi_1(S^4\setminus S^2)$ for some knotted sphere $S^2\subset S^4$.

Proof. Straightforward calculation gives

$$\nu_*(\frac{\partial r}{\partial x_1}) = -mt^{-1}+m+1,$$

where $r := (x_1^{-1}x_2)^m x_1 (x_1^{-1}x_2)^{-m}x_2^{-1}$. Therefore the Alexander module

$$A_0(G)\simeq\Lambda/\langle (m+1)t-m\rangle. \qquad \square$$

4. First homology groups of cyclic coverings

4.1. **Proof of Theorems 0.2 and 0.5.** Theorems 0.2 and 0.5 will be proved simultaneously.

In the notations from Introduction, we denote by X either the sphere S^{n+2} (*Case I*) or $\mathbb{CP}^2$ (*Case II*), and by X' respectively either the complement of a knotted n-manifold V in S^{n+2} or the complement of the union of an irreducible Hurwitz curve H and a line "at infinity" L in $\mathbb{CP}^2$. Recall that the fundamental group $G=\pi_1(X')$ is an irreducible C-group.

Consider the infinite cyclic covering $f=f_\infty : X_\infty \to X'$ corresponding to the C-epimorphism $\nu : G\to \mathbb{F}_1$ with $\ker \nu = G'$. Let $h\in \mathrm{Deck}(X_\infty/X')\simeq \mathbb{F}_1$ be a covering transformation corresponding to the C-generator $x\in\mathbb{F}_1$. We say that h is the *monodromy* respectively of the knotted manifold V and of the Hurwitz curve H. The space X' will be considered as the quotient space $X'=X_\infty/\mathbb{F}_1$. In such a situation Milnor [19] considered an exact sequence of chain complexes

$$0\to C.(X_\infty)\xrightarrow{h-id} C.(X_\infty)\xrightarrow{f_*} C.(X')\to 0$$

which gives an exact sequence of homology groups with integer coefficients:

$$\dots\to H_1(X_\infty)\xrightarrow{t-id} H_1(X_\infty)\xrightarrow{f_*} H_1(X')\xrightarrow{\partial} H_0(X_\infty)\to 0, \qquad (22)$$

where $t=h_*$.

The action h_* (resp., h_{k*}) defines on $H_1(X_\infty)\simeq G'/G''$ a structure of Λ-module such that sequence (22) is an exact sequence of Λ-modules (so that $H_1(X_\infty)$ is the Alexander module of the C-group G). The action of $t\in\Lambda$ on $H_0(X_\infty)\simeq\mathbb{Z}$ is trivial, that is, t is the identity automorphism of $H_0(X_\infty)$.

If $\langle h^k\rangle\subset\mathbb{F}_1$ is an infinite cyclic group generated by h^k, then $X'_k=X_\infty/\langle h^k\rangle$ and $X'=X'_k/\mu_k$, where $\mu_k=\mathbb{F}_1/\langle h^k\rangle$ is the cyclic group of order k. Denote by h_k an automorphism of X'_k induced by the monodromy h. Then h_k is a generator of the covering transformation group $\mathrm{Deck}(X'_k/X')=\mu_k$ acting on X'_k.

It is easy to see that in Case I the manifold X'_k can be embedded to the compact smooth manifold X_k satisfying the following properties:

(i) the action of h_k on X'_k and the map $f'_k : X'_k\to X'$ are continued to an action (denote it again by h_k) on X_k and to a smooth map

$$f_k : X_k\to X\simeq X_k/\{h_k\},$$

(ii) the set of fixed points of h_k coincides with $f_k^{-1}(V)=\overline{V}$ and the restriction $f_{k|\overline{V}}:\overline{V}\to V$ of f_k to $\overline{V}$ is a smooth isomorphism.

In Case II (in the notations of the proof of Theorem 4.1 in [2]), the covering f'_k can be extended to a map $\widetilde{f}_{k\,\mathrm{norm}} : \widetilde{X}_{k\,\mathrm{norm}} \to X$ branched along H and, maybe, along L. Let $\sigma : \overline{X}_k \to \widetilde{X}_{k,\,\mathrm{norm}}$ be a resolution of the singularities, $E = \sigma^{-1}(\mathrm{Sing}\,\widetilde{X}_{k,\,\mathrm{norm}})$, and $\overline{f}_k = \widetilde{f}_{k,\,\mathrm{norm}} \circ \sigma$. Denote by $R = \widetilde{f}^{-1}_{k,\,\mathrm{norm}}(H)$ and $R_\infty = \widetilde{f}^{-1}_{k,\,\mathrm{norm}}(L)$. Note that the restriction of $\widetilde{f}_{k,\,\mathrm{norm}}$ to R is one-to-one and the restriction of $\widetilde{f}_{k,\,\mathrm{norm}}$ to R_∞ is a k_0-sheeted cyclic covering, where $k_0 = \mathrm{GCD}(k, d)$ and the ramification index of $\widetilde{f}_{k,\,\mathrm{norm}}$ along R_∞ is equal to $k_\infty = \frac{k}{k_0}$. As in the algebraic case, it is easy to show that R_∞ is irreducible. Denote by $\overline{R} = \sigma^{-1}(R)$ the proper transform of R. Note that k_0 is a divisor of m. Put $m_0 = \frac{m}{k_0}$, we have $m_0 \in \mathbb{N}$.

Denote by $X_k = \overline{X}_k \setminus E$. We have two embeddings $i_k : X'_k \hookrightarrow X_k$ and $j_k : X_k \hookrightarrow \overline{X}_k$.

In both cases , the action of h_k on X_k induces on $H_1(X_k, \mathbb{Z})$ (resp., on $H_1(X'_k, \mathbb{Z})$) a structure of Λ-module such that the homomorphism

$$i_{k*} : H_1(X'_k, \mathbb{Z}) \to H_1(X_k, \mathbb{Z}),$$

induced by the embedding $i : X'_k \hookrightarrow X_k$, is a Λ-homomorphism. Obviously, the homomorphism i_{k*} is an epimorphism.

In Case I, let $S \subset X_k$ be a germ of a smooth surface meeting transversally $\overline{V}$ at $p \in \overline{V}$ and let $\bar{\gamma} \subset S$ be a circle of small radius with center at p. Then $\ker i_{k*}$ is generated by the homology class $[\bar{\gamma}] \in H_1(X'_k, \mathbb{Z})$ containing the cycle $\bar{\gamma}$, since $\overline{V}$ is a smooth connected codimension two submanifold of X_k.

It is obvious, that $t([\bar{\gamma}]) = [\bar{\gamma}]$, where $t = h_{k*}$, and

$$f_{k*}([\bar{\gamma}]) = \pm k[\gamma] \in H_1(X', \mathbb{Z}) \simeq \mathbb{Z},$$

where $[\gamma]$ is a generator of $H_1(X', \mathbb{Z})$ represented by a simple loop γ around V.

In Case II, let $S \subset X_k$ be a germ of a smooth surface meeting transversally R at $p \in R$ and let $\bar{\gamma} \subset S$ be a circle of small radius with center at p. Evidently, the homology class $[\bar{\gamma}] \in H_1(X'_k, \mathbb{Z})$ is invariant under the multiplication by t and $f_{k*}([\bar{\gamma}]) = k[\gamma]$, where $[\gamma]$ is a generator of $H_1(\mathbb{CP}^2 \setminus (H \cup L), \mathbb{Z}) \simeq \mathbb{Z}$.

Similarly, let a complex line $L_1 \subset \mathbb{CP}^2$ meet L transversely at $q \in L \setminus H$ and γ_∞ be a simple small loop around L lying in L_1. Then $f_k^{-1}(\gamma_\infty)$ splits into the disjoint union of k_0 simple loops $\bar{\gamma}_{\infty,i}$, $i = 1, \dots, k_0$. Since R_∞ is irreducible, each two loops $\bar{\gamma}_{\infty,i}$ and $\bar{\gamma}_{\infty,j}$ belong to the same homology class of $H_1(X'_k, \mathbb{Z})$ (denote it by $[\bar{\gamma}_\infty]$). It is easy to see that $t(\bar{\gamma}_{\infty,i}) = \bar{\gamma}_{\infty,i+1}$. Therefore the homology class $[\bar{\gamma}_\infty] \in H_1(X'_n\mathbb{Z})$ is invariant under the multiplication by t. Note also that $f_{k*}([\bar{\gamma}_\infty]) = k_\infty m[\gamma] = km_0[\gamma]$, since $[\gamma_\infty] = m[\gamma]$.

Lemma 4.1. *The Λ-module $H_1(X'_k,\mathbb{Z})$ is isomorphic to*

$$A_k(G)\oplus H_1(X'_k)_1\simeq A_k(G)\oplus\mathbb{Z},$$

where $A_k(G)$ is the k-th derived Alexander module of C-group G and

$$H_1(X'_k)_1=\{h\in H_1(X'_k,\mathbb{Z})\mid (t-1)h=0\}.$$

Proof. We apply the sequence

$$\dots\to H_1(X_\infty,\mathbb{Z})\xrightarrow{t^k-id}H_1(X_\infty,\mathbb{Z})\xrightarrow{g_{k,*}}H_1(X'_k,\mathbb{Z})\xrightarrow{\partial}H_0(X_\infty,\mathbb{Z})\to 0 \quad (23)$$

constructed in the same way as (22) to the infinite cyclic covering $g_k=g_{\infty,k}:X_\infty\to X'_k$, to analyze the group $H_1(X'_k,\mathbb{Z})$.

By (23), we have the short exact sequence

$$0\to H_1(X_\infty)/(t^k-1)H_1(X_\infty)\xrightarrow{g_{k,*}}H_1(X'_k)\xrightarrow{\partial}H_0(X_\infty)\to 0 \qquad (24)$$

which is a sequence of Λ-homomorphisms.

Denote by $M_1=\ker\partial=\mathrm{im}g_{k,*}\simeq H_1(X_\infty)/(t^k-1)H_1(X_\infty)$ and by $M_2=H_1(X'_k)_1$.

We have $H_0(X_\infty,\mathbb{Z})\simeq\mathbb{Z}$. Let us choose a generator $u\in H_0(X_\infty,\mathbb{Z})$ and let $v_1\in H_1(X'_k,\mathbb{Z})$ be an element such that $\partial(v_1)=u$. Then $(t-1)v_1\in\ker\partial$, since $H_0(X_\infty,\mathbb{Z})$ is a trivial Λ-module and ∂ is a Λ-homomorphism. We fix a such v_1.

By Theorems 0.1 and 0.3, $H_1(X_\infty,\mathbb{Z})=A_0(G)$ is a Noetherian $(t-1)$-invertible Λ-module. Therefore, by Proposition 1.6,

$$M_1\simeq H_1(X_\infty)/(t^k-1)H_1(X_\infty)=A_k(G)$$

is also a Noetherian $(t-1)$-invertible Λ-module and, by Theorem 1.10, there is a polynomial $g_1(t)\in\mathrm{Ann}(M_1)$ such that $g_1(1)=1$. We fix a such polynomial $g_1(t)$.

Consider the element $\overline{v}_1=g_1(t)v_1$. We have $\partial(\overline{v}_1)=g_1(1)u=u$ and hence

$$(t-1)\overline{v}_1=(t-1)g_1(t)v_1=g_1(t)(t-1)v_1=0,$$

since $(t-1)v_1\in M_1$. Therefore $\overline{v}_1\in M_2$.

Note that $M_1\cap M_2=0$, since M_1 is $(t-1)$-invertible. Therefore ∂ maps M_2 isomorphically onto $H_0(X_\infty,\mathbb{Z})$, that is, exact sequence (24) splits and hence $H_1(X'_k,\mathbb{Z})\simeq M_1\oplus M_2$. □

Lemma 4.2. *For $f_{k*}:H_1(X'_k,\mathbb{Z})\longrightarrow H_0(X',\mathbb{Z})$ we have*

(*i*) $\ker f_{k,*}=A_k(G)\subset H_1(X'_k,\mathbb{Z})$,

(*ii*) *$\mathrm{im}\, f_{k,*}=k\mathbb{Z}\subset\mathbb{Z}\simeq H_1(X',\mathbb{Z})$ and the restriction of f_{k*} to $H_1(X'_k)_1$ is an isomorphism of $H_1(X'_k)_1$ with its image.*

Proof. The group $H_1(X',\mathbb{Z})$ is isomorphic to $G/G'\simeq\mathbb{Z}$. Similarly, the group $H_1(X'_k,\mathbb{Z})$ is isomorphic to G_k/G'_k, where $G_k=\ker\nu_k$,

$$\nu_k = \mod{}_k\circ\nu : G\to\mu_k=\mathbb{Z}/\langle h^k\rangle,$$

and $f_{k*}:H_1(X'_k,\mathbb{Z})\to H_1(X',\mathbb{Z})$ coincides with the homomorphism

$$i_{k*}:G_k/G'_k\to G/G'$$

induced by the embedding $i_k:G_k\hookrightarrow G$.

Let the C-group G be given by C-presentation (7). To describe $\ker i_{k*}$ and $\operatorname{im} i_{k*}$, let us consider again the two-dimensional complex K described in section 3.1. The complex K has a single vertex x_0, its one dimensional skeleton is a bouquet of oriented circles s_j, $1\leq j\leq m$, corresponding to the C-generators of G from presentation (7), and $K\setminus(\cup s_i)=\bigsqcup_{j=1}^{l}\overset{\circ}{D}_j$ is a disjoint union of open discs, where each disc D_j corresponds to the relation r_i from presentation (7) (we denote here by l the number of relations r_i in presentation (7)).

The embedding $i_k:G_k\hookrightarrow G$ defines an un-ramified covering $f_k:K_k\to K$, where K_k is a two-dimensional complex consisting of k vertices $p_1,\dots,p_k$, $f_k(p_j)=x_0$; the preimage $f^{-1}(s_j)=\bigsqcup_{s=1}^{k}\overline{s}_{j,s}$ is the disjoint union of k edges $\overline{s}_{j,s}$, $1\leq s\leq k$; and the preimage $f^{-1}(\overset{\circ}{D}_j)=\bigsqcup_{s=1}^{k}\overset{\circ}{\overline{D}}_{j,s}$ is also the disjoint union of k open discs $\overset{\circ}{\overline{D}}_{j,s}$, $1\leq s\leq k$.

Let h_k be a generator of the covering transformation group $\mathrm{Deck}(K_k/K)=\mu_k$ acting on K_k. The homeomorphism h_k induces an action h_{k*} on the chain complex $C.(K_k)$ and an action t on $H_i(K_k,\mathbb{Z})$ so that this action defines on $H_i(K_k,\mathbb{Z})$ a structure of Λ-module. It is easy to see that this structure on $H_1(K_k,\mathbb{Z})$ coincides with one on $H_1(X'_k,\mathbb{Z})$ defined above if we identify $H_1(K_k,\mathbb{Z})$ and $H_1(X'_k,\mathbb{Z})$ by means of isomorphisms $H_1(K_k,\mathbb{Z})\simeq G_k/G'_k$ and $H_1(X'_k,\mathbb{Z})\simeq G_k/G'_k$.

Consider the sequence of chain complexes

$$C.(K_k)\xrightarrow{h_{k*}-id}C.(K_k)\xrightarrow{f_{k*}}C.(K)\to 0.$$

It is easy to see that $\operatorname{im}(h_{k*}-id)=\ker f_{k*}$ and

$$\ker(h_{k*}-id)=(\sum_{j=0}^{k-1}h_{k*}^{j})C.(K_k).$$

Now the proof of Lemma 4.2 follows from the exact sequence

$$\begin{array}{l}\ldots\to H_1(C.(K_k/\ker(h_{k*}-id))\xrightarrow{t^k-1}H_1(K_k)\xrightarrow{f_{k*}}H_1(K)\xrightarrow{\partial}\\ \xrightarrow{\partial}H_0(C.(K_k/\ker(h_{k*}-id))\xrightarrow{t^k-1}H_0(K_k)\xrightarrow{f_{k*}}H_0(K)\to 0,\end{array}\tag{25}$$

since

$$\begin{aligned}
&im[H_1(C.(K_k/\ker(h_{k*}-id)) \stackrel{t^k-1}{\longrightarrow} H_1(K_k)] = A_k(G),\\
&H_1(K) \simeq \mathbb{Z},\\
&H_0(C.(K_k/\ker(h_{k*}-id)) \simeq \mathbb{Z}/k\mathbb{Z},\\
&H_0(K_k) \stackrel{f_{k*}}{\simeq} H_0(K) \simeq \mathbb{Z},
\end{aligned}$$

are Λ-modules with trivial action of t and exact sequence (25) is a sequence of Λ-homomorphisms of Λ-modules. □

Now Theorem 0.2 follows from Lemmas 4.1 and 4.2, since $\ker i_{k*}$ is generated by $[\bar\gamma] \in H_1(X')_1$ and $f_{k*}([\bar\gamma]) = k[\gamma]$.

Similarly, in Case II, we have $\ker i_{k*} = H_1(X'_k)_1$. Indeed, $\ker i_{k*}$ is generated by $\bar\gamma$ and $\bar\gamma_\infty \in H_1(X'_k)_1 \simeq \mathbb{Z}$ and $f_{k*}([\bar\gamma]) = k[\gamma]$. Therefore $H_1(X'_k)_1$ is generated by $[\bar\gamma]$.

As a consequence, we obtain that the restriction of i_{k*} to the submodule $A_k(G)$ of $H_1(X'_k,\mathbb{Z})$ is an isomorphism of $A_k(G)$ with $H_1(X_k,\mathbb{Z})$. Therefore the following lemma implies Theorem 0.5.

Lemma 4.3. ([2]) *The homomorphism $j_{k*} : H_1(X_k,\mathbb{Q}) \to H_1(\overline{X}_k,\mathbb{Q})$ is an isomorphism.*

4.2. **Corollaries of Theorems 0.2 and 0.5.**

Corollary 4.4. *Let V be a knotted n-manifold, $n \geq 1$, and $f_k : X_k \to S^{n+2}$ the cyclic covering branched along V, $\deg f_k = k$. Then*

- (i) *the first Betti number $b_1(X_k)$ of X_k is an even number;*
- (ii) *if $k = p^r$, where p is prime, then $H_1(X_k,\mathbb{Z})$ is finite;*
- (iii) *a finitely generated abelian group G can be realized as $H_1(X_k,\mathbb{Z})$ for some knotted n-manifold V, $n \geq 2$, if and only if there is an automorphism $h \in Aut(G)$ such that $h^k = Id$ and $h - Id$ is also an automorphism of G; in particular, $H_1(X_2,\mathbb{Z})$ is a finite abelian group of odd order and any finite abelian group G of odd order can be realized as $H_1(X_2,\mathbb{Z})$ for some knotted n-sphere, $n \geq 2$.*

Proof. It follows from Theorems 0.1, 0.2, 2.11, Propositions 2.13, Corollary 3.13, and Examples 2.14, 3.20. □

Corollary 0.4 follows from Theorems 0.3 and 2.10.

Corollary 0.6 is a simple consequence of Lemma 4.3 and the following corollary, since the homomorphism $j_{k*} : H_1(X_k,\mathbb{Z}) \to H_1(\overline{X}_k,\mathbb{Z})$ is an epimorphism and $H_1(\overline{X}_k,\mathbb{Q}) \simeq A_k(H) \otimes \mathbb{Q}$.

Corollary 4.5. *Let H be an algebraic (resp, Hurwitz or pseudo-holomorphic) irreducible curve in $\mathbb{CP}^2$, $\deg H = m$, and $\overline{f}_k : \overline{X}_k \to \mathbb{CP}^2$ be a resolution of singularities of the cyclic covering of degree k branched along H and, maybe, alone a line "at infinity" L, and let $X_k = \overline{X}_k \setminus E$. Then*

(*i*) *the sequence of groups*
$$H_1(X_1, \mathbb{Z}), \dots, H_1(X_k, \mathbb{Z}), \dots$$
has period m, that is, $H_1(X_k, \mathbb{Z}) \simeq H_1(X_{k+m}, \mathbb{Z})$;

(*ii*) *the first Betti number $b_1(\overline{X}_k)$ of $\overline{X}_k$ is an even number;*

(*iii*) *if $k = p^r$, where p is prime, then $H_1(X_k, \mathbb{Z})$ and $H_1(\overline{X}_k, \mathbb{Z})$ are finite groups;*

(*iv*) *if k and m are coprime, then $H_1(\overline{X}_k, \mathbb{Z}) = 0$;*

(*v*) *a finitely generated abelian group G can be realized as $H_1(X_k, \mathbb{Z})$ for some Hurwitz (resp., pseudo-holomorphic) curve H if and only if there is an automorphism $h \in Aut(G)$ such that $h^d = Id$ and $h - Id$ is also an automorphism of G, where d is a divisor of k, and, moreover, if G is realized as $H_1(X_k, \mathbb{Z})$ for a curve H, then d is a divisor of $\deg H$; in particular, $H_1(\overline{X}_2, \mathbb{Z})$ is a finite abelian group of odd order and any finite abelian group G of odd order can be realized as $H_1(X_2, \mathbb{Z})$ for some Hurwitz (resp., pseudo-holomorphic) curve H of even degree.*

Proof. It follows from Theorems 0.3, 0.5, 2.11, 2.16 and Propositions 2.13, 2.15. □

Note that there are plane algebraic curves H for which the homomorphisms $j_{k*} : H_1(X_k, \mathbb{Z}) \to H_1(\overline{X}_k, \mathbb{Z})$ are not isomorphisms.

Example 4.6. *Let $H \subset \mathbb{CP}^2$ be a curve of degree 6 given by equation*
$$Q^3(z_0, z_1, z_2)C^2(z_0, z_1, z_2) = 0,$$
where Q and C are homogeneous forms of $\deg Q = 2$, $\deg C = 3$ and the conic and cubic, given by equations $Q = 0$ and $C = 0$, meet transversally at 6 points. Then $A_2(H) \simeq \mathbb{Z}/3\mathbb{Z}$, but $H_1(\overline{X}_2, \mathbb{Z}) = 0$.

Proof. It is known (see [20]) that $\pi_1(\mathbb{CP}^2 \setminus (H \cup L)) \simeq \mathrm{Br}_3$ as a C-group. Therefore $A_2(H) \simeq \mathbb{Z}/3\mathbb{Z}$ (see Examples 2.17 and 3.19).

It is also well known that the minimal resolution of singularities of two-sheeted covering of $\mathbb{CP}^2$ branched along H is a $K3$-surface which is simply connected. □

Note also that in the case of knotted n-manifold $V \subset S^{n+2}$ the sequence of homology groups $H_1(X_k, \mathbb{Z})$, $k \in \mathbb{N}$, is not necessary to be periodic. For example, if $S^2 \subset S^4$ is a knotted sphere for which $\pi_1(S^4 \setminus S^2) \simeq G_m$, where

G_m is a group considered in Example 3.20 (by Corollary 3.13, this group can be realized as a group of knotted sphere), then $H_1(X_k, \mathbb{Z})$ is the cyclic group of order $(m+1)^k - m^k$ (see Example 2.14).

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