ON THREE-DIMENSIONAL COMPLEX ASSOCIATIVE ALGEBRAS

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ABSTRACT. In this paper, we study the moduli space of 3-dimensional complex associative algebras. We use extensions to compute the moduli space, and then give a decomposition of this moduli space into strata consisting of complex projective orbifolds, glued together through jump deformations. The moduli space was previously computed. The main purpose of this paper is to give a better organized description of the moduli space, and to give an explicit description of how the moduli space is constructed by extensions.

1. INTRODUCTION

The classification of associative algebras was instituted by Benjamin Peirce in the 1870's [11], who gave a partial classification of the complex associative algebras of dimension up to 6, although in some sense, one can deduce the complete classification from his results, with some additional work. The classification method relied on the following remarkable fact:

Theorem 1.1. *Every finite dimensional algebra which is not nilpotent contains a nontrivial idempotent element.*

A nilpotent algebra A is one which satisfies $A^n = 0$ for some n, while an idempotent element a satisfies $a^2 = a$. This observation of Peirce eventually leads to two important theorems in the classification of finite dimensional associative algebras. Recall that an algebra is said to be simple if it has no nontrivial proper ideals, and it is not the 1-dimensional nilpotent algebra over $\mathbb{K}$, given by the trivial product.

Theorem 1.2 (Fundamental Theorem of Finite Dimensional Associative Algebras). *Suppose that A is a finite dimensional algebra over a field $\mathbb{K}$. Then A has a maximal nilpotent ideal N, called its radical. If A is not nilpotent, then A/N is a semisimple algebra, that is, a direct sum of simple algebras.*

In fact, in the literature, the definition of a semisimple algebra is often given as one whose radical is trivial, and then it is a theorem that semisimple algebras are direct sums of simple algebras. Moreover, when A/N satisfies a property called separability over $\mathbb{K}$, then A is a semidirect product of its radical and a semisimple algebra.

A simple algebra is separable iff its center is a separable field over $\mathbb{K}$, and a semisimple algebra is separable iff its simple components are separable. Therefore, over the complex numbers every semisimple algebra is separable. To apply

the theorem above to construct algebras by extension, we will use the following characterization of simple algebras [16].

Theorem 1.3 (Wedderburn). *If A is a finite dimensional algebra over $\mathbb{K}$, then A is simple iff A is isomorphic to a tensor product $M \otimes D$, where $M = \mathfrak{gl}(n, \mathbb{K})$ and D is a division algebra over $\mathbb{K}$.*

One can also say that A is a matrix algebra with coefficients in a division algebra over $\mathbb{K}$. An associative division algebra is a unital associative algebra where every nonzero element has a multiplicative inverse. (One has to modify this definition in the case of graded algebras, but we will not address this issue in this paper.)

Over the complex numbers, the only division algebra is $\mathbb{C}$ itself, so Wedderburn's theorem says that the only simple algebras are the matrix algebras. In particular, there can be no 3-dimensional simple algebras, so the only semisimple 3-dimensional algebra is $\mathbb{C}^3$, the direct sum of three copies of the 1-dimensional simple algebra.

The main goal of this paper is to give a complete description of the moduli space of 3-dimensional associative algebras, including a computation of the miniversal deformation of every element. Moreover, we will give a canonical stratification of the moduli space into projective orbifolds of a very simple type, so that the strata are connected only by deformations factoring through jump deformations, and the elements of a particular stratum are given by neighborhoods determined by smooth deformations.

2. Preliminaries

Suppose that V is a vector space, defined over a field $\mathbb{K}$ whose characteristic is not 2 or 3, equipped with an associative multiplication structure $m : V \otimes V \to V$. The associativity relation can be given in the form

$$m \circ (m \otimes 1) = m \circ (1 \otimes m).$$

The notion of isomorphism or *equivalence* of associative algebra structures is given as follows. If g is a linear automorphism of V, then define

$$g^*(m) = g^{-1} \circ m \circ (g \otimes g).$$

Two algebra structures m and m' are equivalent if there is an automorphism g such that $m' = g^*(m)$. The set of equivalence classes of algebra structures on V is called the *moduli space* of associative algebras on V.

We wish to transform this classical viewpoint into the more modern viewpoint of associative algebras as being given by codifferentials on a certain coalgebra. To do this, we first introduce the *parity reversion* ΠV of a $\mathbb{Z}_2$-graded vector space V. If $V = V_e \oplus V_o$ is the decomposition of V into its even and odd parts, then $W = \Pi V$ is the $\mathbb{Z}_2$-graded vector space given by $W_e = V_o$ and $W_o = V_e$. In other words, W is just the space V with the parity of elements reversed.

Given an ordinary associative algebra, we can view the underlying space V as being $\mathbb{Z}_2$-graded, with $V = V_e$. Then its parity reversion W is again the same space, but now all elements are considered to be odd. One can avoid this gyration for ordinary spaces, by introducing a grading by exterior degree on the tensor coalgebra of V, but the idea of parity reversion works equally well when the algebra is $\mathbb{Z}_2$-graded, whereas the method of grading by exterior degree does not.

Denote the tensor (co)-algebra of W by $\mathcal{T}(W) = \bigoplus_{k=0}^{\infty} W^k$, where W^k is the k-th tensor power of W and $W^0 = \mathbb{K}$. For brevity, the element in W^k given by the

tensor product of the elements w_i in W will be denoted by $w_1 \cdots w_k$. The coalgebra structure on $\mathcal{T}(W)$ is given by

$$\Delta(w_1 \cdots w_n) = \sum_{i=0}^{n} w_1 \cdots w_i \otimes w_{i+1} \cdots w_n.$$

Define $d : W^2 \to W$ by $d = \pi \circ m \circ (\pi^{-1} \otimes \pi^{-1})$, where $\pi : V \to W$ is the identity map, which is odd, because it reverses the parity of elements. Note that d is an odd map. The space $C(W) = \mathrm{Hom}(\mathcal{T}(W), W)$ is naturally identifiable with the space of coderivations of $\mathcal{T}(W)$. In fact, if $\varphi \in C^k(W) = \mathrm{Hom}(W^k, W)$, then φ is extended to a coderivation of $\mathcal{T}(W)$ by

$$\varphi(w_1 \cdots w_n) = \sum_{i=0}^{n-k} (-1)^{(w_1 + \cdots + w_i)\varphi} w_1 \cdots w_i \varphi(w_{i+1} \cdots w_{i+k}) w_{i+k+1} \cdots w_n.$$

The space of coderivations of $\mathcal{T}(W)$ is equipped with a $\mathbb{Z}_2$-graded Lie algebra structure given by

$$[\varphi, \psi] = \varphi \circ \psi - (-1)^{\varphi\psi} \psi \circ \varphi.$$

The reason that it is more convenient to work with the structure d on W rather than m on V is that the condition of associativity for m translates into the codifferential property $[d, d] = 0$.

3. Construction of algebras by extensions

In [2], the theory of extensions of an algebra W by an algebra M is described in the language of codifferentials, and so is well adapted to our constructions here, but there is a long history of the study of extensions, which we do not mention. Consider the diagram

$$0 \to M \to V \to W \to 0$$

of associative $\mathbb{K}$-algebras, so that $V = M \oplus W$ as a $\mathbb{K}$-vector space, M is an ideal in the algebra V, and $W = V/M$ is the quotient algebra. Suppose that $\delta \in C^2(W)$ and $\mu \in C^2(M)$ represent the algebra structures on W and M respectively. We can view μ and δ as elements of $C^2(V)$. Let $T^{k,l}$ be the subspace of $T^{k+l}(V)$ given recursively by

$$T^{0,0} = \mathbb{K}$$
$$T^{k,l} = M \otimes T^{k-1,l} \oplus V \otimes T^{k,l-1}$$

Let $C^{k,l} = \mathrm{Hom}(T^{k,l}, M) \subseteq C^{k+l}(V)$. If we denote the algebra structure on V by d, we have

$$d = \delta + \mu + \lambda + \psi,$$

where $\lambda \in C^{1,1}$ and $\psi \in C^{0,2}$. Note that in this notation, $\mu \in C^{2,0}$. Then the condition that d is associative: $[d, d] = 0$ gives the following relations:

$$[\delta, \lambda] + \tfrac{1}{2}[\lambda, \lambda] + [\mu, \psi] = 0, \quad \text{The Maurer-Cartan equation}$$
$$[\mu, \lambda] = 0, \quad \text{The compatibility condition}$$
$$[\delta + \lambda, \psi] = 0, \quad \text{The cocycle condition}$$

Since μ is an algebra structure, $[\mu, \mu] = 0$, so if we define D_μ by $D_\mu(\varphi) = [\mu, \varphi]$, then $D_\mu^2 = 0$. Thus D_μ is a differential on $C(V)$. Moreover $D_\mu : C^{k,l} \to C^{k+1,l}$. Let

$$Z_\mu^{k,l} = \ker(D_\mu : C^{k,l} \to C^{k+1,l}), \quad \text{the } (k,l)\text{-cocycles}$$

$$B_\mu^{k,l} = \operatorname{Im}(D_\mu : C^{k-1,l} \to C^{k,l}), \quad \text{the } (k,l)\text{-coboundaries}$$

$$H_\mu^{k,l} = Z_\mu^{k,l}/B_\mu^{k,l}, \quad \text{the } D_u \ (k,l)\text{-cohomology}$$

Then the compatibility condition means that $\lambda \in Z^{1,1}$. If we define $D_{\delta+\lambda}(\varphi) = [\delta + \lambda, \varphi]$, then it is not true that $D_{\delta+\lambda}^2 = 0$, but $D_{\delta+\lambda}D_\mu = -D_\mu D_{\delta+\lambda}$, so that $D_{\delta+\lambda}$ descends to a map $D_{\delta+\lambda} : H_\mu^{k,l} \to H_\mu^{k,l+1}$, whose square is zero, giving rise to the $D_{\delta+\lambda}$-cohomology $H_{\mu,\delta+\lambda}^{k,l}$. If the pair (λ, ψ) give rise to a codifferential d, and (λ, ψ') give rise to another codifferential d', then if we express $\psi' = \psi + \tau$, it is easy to see that $[\mu, \tau] = 0$, and $[\delta + \lambda, \tau] = 0$, so that the image $\bar{\tau}$ of τ in $H_\mu^{0,2}$ is a $D_{\delta+\lambda}$-cocycle, and thus τ determines an element $\{\bar{\tau}\} \in H_{\mu,\delta+\lambda}^{0,2}$.

If $\beta \in C^{0,1}$, then $g = \exp(\beta) : \mathcal{T}(V) \to \mathcal{T}(V)$ is given by $g(m,w) = (m + \beta(w), w)$. Furthermore $g^* = \exp(-\operatorname{ad}_\beta) : C(V) \to C(V)$ satisfies $g^*(d) = d'$, where $d' = \delta + \mu + \lambda' + \psi'$ with $\lambda' = \lambda + [\mu, \beta]$ and $\psi' = \psi + [\delta + \lambda + \frac{1}{2}[\mu, \beta], \beta]$. In this case, we say that d and d' are equivalent extensions in the restricted sense. Such equivalent extensions are also equivalent as codifferentials on $\mathcal{T}(V)$. Note that λ and λ' differ by a D_μ-coboundary, so $\bar{\lambda} = \bar{\lambda}'$ in $H_\mu^{1,1}$. If λ satisfies the MC-equation for some ψ, then any any element λ' in $\bar{\lambda}$ also gives a solution of the MC equation, for the ψ' given above. The cohomology classes of those λ for which a solution of the MC equation exists determine distinct restricted equivalence classes of extensions.

Let $G_{M,W} = \mathbf{GL}(M) \times \mathbf{GL}(W) \subseteq \mathbf{GL}(V)$. If $g \in G_{M,W}$ then $g^* : C^{k,l} \to C^{k,l}$, and $g^* : C^k(W) \to C^k(W)$, so $\delta' = g^*(\delta)$ and $\mu' = g^*(\mu)$ are codifferentials on $\mathcal{T}(M)$ and $\mathcal{T}(W)$ respectively. The group $G_{\delta,\mu}$ is the subgroup of $G_{M,W}$ consisting of those elements g such that $g^*(\delta) = \delta$ and $g^*(\mu) = \mu$. Then $G_{\delta,\mu}$ acts on the restricted equivalence classes of extensions, giving the equivalence classes of general extensions. Also $G_{\delta,\mu}$ acts on $H_\mu^{k,l}$, and induces an action on the classes $\bar{\lambda}$ of λ giving a solution to the MC equation.

Next, consider the group $G_{\delta,\mu,\lambda}$ consisting of the automorphisms h of V of the form $h = g\exp(\beta)$, where $g \in G_{\delta,\mu}$, $\beta \in C^{0,1}$ and $\lambda = g^*(\lambda) + [\mu, \beta]$. If $d = \delta + \mu + \lambda + \psi + \tau$, then $h^*(d) = \delta + \mu + \lambda + \psi + \tau'$ where

$$\tau' = g^*(\psi) - \psi + [\delta + \lambda - \tfrac{1}{2}[\mu, \beta], \beta] + g^*(\tau).$$

Thus the group $G_{\delta,\mu,\lambda}$ induces an action on $H_{\mu,\delta+\lambda}^{0,2}$ given by $\{\bar{\tau}\} \to \{\overline{\tau'}\}$.

The general group of equivalences of extensions of the algebra structure δ on W by the algebra structure μ on M is given by the group of automorphisms of V of the form $h = \exp(\beta)g$, where $\beta \in C^{0,1}$ and $g \in G_{\delta,\mu}$. Then we have the following classification of such extensions up to equivalence.

Theorem 3.1. *The equivalence classes of extensions of δ on W by μ on M is classified by the following:*

(1) *Equivalence classes of $\bar{\lambda} \in H_\mu^{1,1}$ which satisfy the MC equation*

$$[\delta, \lambda] + \tfrac{1}{2}[\lambda, \lambda] + [\mu, \psi] = 0$$

for some $\psi \in C^{0,2}$, under the action of the group $G_{\delta,\mu}$.

(2) *Equivalence classes of* $\{\bar{\tau}\} \in H^{0,2}_{\mu,\delta+\lambda}$ *under the action of the group* $G_{\delta,\mu,\lambda}$.

Equivalent extensions will give rise to equivalent codifferentials on V, but it may happen that two codifferentials arising from nonequivalent extensions are equivalent. This is because the group of equivalences of extensions is the group of invertible block upper triangular matrices on the space $V = M \oplus W$, whereas the the equivalence classes of codifferentials on V are given by the group of all invertible matrices, which is larger.

The fundamental theorem of finite dimensional algebras allows us to restrict our consideration of extensions to two cases. First, we can consider those extensions where δ is a semisimple algebra structure on W, and μ is a nilpotent algebra structure on M. In this case, because we are working over $\mathbb{C}$, we can also assume that $\psi = \tau = 0$. Thus the classification of the extension reduces to considering equivalence classes of λ.

Secondly, we can consider extensions of the trivial algebra structure $\delta = 0$ on a 1-dimensional space W by a nilpotent algebra μ. This is because a nilpotent algebra has a codimension 1 ideal M, and the restriction of the algebra structure to M is nilpotent. However, in this case, we cannot assume that ψ or τ vanish, so we need to use the classification theorem above to determine the equivalence classes of extensions. In many cases, in solving the MC equation for a particular λ, if there is any ϕ yielding a solution, then $\psi = 0$ also gives a solution, so the action of $G_{\delta,\mu,\lambda}$ on $H^{0,2}_{\mu}$ takes on a simpler form than the general action we described above.

Consider the general setup, where an n-dimensional space $W = \langle v_{m+1}, \dots v_{m+n} \rangle$ is extended by an m-dimensional space $M = \langle v_1, \cdots, v_m \rangle$. Then the module structure is of the form

$$\lambda = \psi^{kj}_i (L_k)^i_j + \psi^{jk}_i (R_k)^i_j, \quad i,j = 1, \dots m, k = m+1 \dots m+n,$$

and we can consider L_k and R_k to be $m \times m$ matrices. Then we can express the bracket $\frac{1}{2}[\lambda, \lambda]$, which appears in the MC equation in terms of matrix multiplication.

$$(1) \qquad \tfrac{1}{2}[\lambda,\lambda] = \psi^{jkl}_i (R_l R_k)^i_j + \psi^{kjl}_i (L_k R_l - R_k L_l)^i_j - \psi^{klj}_i (L_k L_l)^i_j,$$

where $i, j = 1, \dots m$, and $k, l = m+1, \dots m+n$.

Next, suppose that $\delta = \psi^{m,m}_m + \cdots + \psi^{m+n,m+n}_{m+n}$ is the semisimple algebra structure $\mathbb{C}^n$ on W. Then we can also express $[\delta, \lambda]$ in terms of matrix multiplication.

$$(2) \qquad\qquad [\delta, \lambda] = \psi^{kkj}_i (L_k)^i_j - \psi^{jkk}_i (R_k)^i_j.$$

Since δ is semisimple, one can ignore the cocycle ψ in constructing an extension, so the MC equation is completely determined by the equations (2) and 1, so we obtain the conditions Therefore, the MC-equation holds precisely when

$$L^2_k = L_k, \quad R^2_k = R_k, \quad L_k L_l = R_k R_l = 0 \text{ if } k \neq l, \quad L_k R_l = R_l L_k.$$

As a consequence, both L_k and R_k must be commuting nondefective matrices whose eigenvalues are either 0 or 1, which limits the possibilities. Moreover, it can be shown that G_δ, the group of automorphisms of W preserving δ is just the group of permutation matrices. Thus if $G = \mathrm{diag}(G_1, G_2)$ is a block diagonal element of $G_{\delta,\mu}$, the matrix G_2 is a permutation matrix. The action of G on λ is given by simultaneous conjugation of the matrices L_k and R_k by G_1, and a simultaneous permutation of the k-indices determined by the permutation associated to G_2.

It is well known that a collection of commuting diagonalizable matrices can be simultaneously diagonalized. When $\mu = 0$, this means we can assume that all of the matrices L_k and R_k are diagonal. When $\mu \neq 0$, the matrices G_1 are required to preserve μ, and the compatibility condition $[\mu, \lambda]$ also complicates the picture.

It is important to note that given an m and a nilpotent element μ on an m-dimensional space M, there is an n beyond which the extensions of the semisimple codifferential on an N dimensional space with N greater than n are simply direct sums of the extensions of the n-dimensional semisimple algebra $\mathbb{C}^n$ and the semisimple algebra $\mathbb{C}^{N-n}$. We say that the extension theory becomes stable at n. Moreover, the deformation picture also stabilizes as well.

When $\mu = 0$, we can easily see that at most m matrices L_k could be nonzero, since we can assume they are diagonal matrices. Similarly, at most m matrices R_k can be nonzero. By rearranging using the fact that G_2 is an arbitrary permutation matrix, one can assume that all L_k matrices when $k > 2m$ vanish, and then we can assume that all R_k matrices vanish if $k > 3m$. Thus the extension theory becomes stable when $n = 2m$.

When $\mu \neq 0$, the matrices L_k and R_k are still diagonalizable, so it also follows that the extension theory is stable for $n \geq 2m$, but in fact it may become stable for a much smaller value of n.

In higher dimensions, there are semisimple algebras which are not of the form $\mathbb{C}^n$. Also, as m increases, the complexity of the nontrivial nilpotent elements μ increases as well. However, all 3-dimensional nonnilpotent algebras are extensions of a semisimple algebra of the type $\mathbb{C}^n$, for $n = 1, 2$, and there is only one nontrivial nilpotent algebra in dimension 2, so the discussion above will suffice to characterize the nonnilpotent 3-dimensional algebras in a straightforward manner.

4. ASSOCIATIVE ALGEBRA STRUCTURES ON A 3-DIMENSIONAL VECTOR SPACE

In our classification of the algebra structure on a 3-dimensional space, we will refer to the classification of algebras on a 2-dimensional space [2, 1], which is given in Table 4 below.

Codifferential	H^0	H^2	H^1	H^3	H^4
$d_1 = \psi_1^{11} + \psi_2^{22}$	2	0	0	0	0
$d_2 = \psi_2^{22} + \psi_1^{12}$	0	0	0	0	0
$d_3 = \psi_2^{22} + \psi_1^{21}$	0	0	0	0	0
$d_4 = \psi_2^{22} + \psi_1^{12} + \psi_1^{21}$	2	1	1	1	1
$d_5 = \psi_2^{22}$	2	1	1	1	1
$d_6 = \psi_1^{22}$	2	2	2	2	2

TABLE 1. Cohomology of the six codifferentials on a 2-dimensional space

In constructing the elements of the moduli space by extensions, we need to consider two possibilities, extensions of the semisimple algebra structure on a 2-dimensional space W by the trivial algebra structure on a 1-dimensional space M, or extensions of an algebra structure on a 1-dimensional space W by a nilpotent algebra structure on a 2-dimensional space M. In this second case, we need to study

both extensions of the simple algebra structure on W and the trivial structure. Moreover, there are two nilpotent algebra structures on M.

5. Extensions of the 2-dimensional semisimple algebra $\mathbb{C} \oplus \mathbb{C}$ by the 1-dimensional trivial algebra $\mathbb{C}_0$

Let $W = \langle v_2, v_3 \rangle$ and $M = \langle v_1 \rangle$. The group $G_{M,W}$ is given by matrices of the form

$$G = \begin{bmatrix} g_{1,1} & 0 & 0 \\ 0 & g_{2,2} & g_{2,3} \\ 0 & g_{3,2} & g_{3,3} \end{bmatrix} = \begin{bmatrix} g_{1,1} & 0 \\ 0 & \tilde{G} \end{bmatrix}, \text{ where } \tilde{G} = \begin{bmatrix} g_{2,2} & g_{2,3} \\ g_{3,2} & g_{3,3} \end{bmatrix} \in \mathbf{GL}(2, \mathbb{C}).$$

The semisimple 2-dimensional algebra structure δ on W is given by the codifferential $\delta = \psi_2^{22} + \psi_3^{33}$. The trivial 1-dimensional algebra structure on M is just $\mu = 0$.

The module structure λ on M is given by

$$\lambda = \psi_1^{21}(L_1)_1^1 + \psi_1^{12}(R_1)_1^1 + \psi_1^{31}(L_2)_1^1 + \psi_1^{13}(R_2)_1^1.$$

In other words, the matrices L_k and R_k are just numbers. Since $m = 1$ and $n = 2$, this is the stable case. There are exactly 5 solutions, given below.

$$
\begin{array}{ccccc}
L_1 = 1 & L_1 = 1 & L_1 = 1 & L_1 = 0 & L_1 = 0 \\
R_1 = 1 & R_1 = 0 & R_1 = 0 & R_1 = 1 & R_1 = 0 \\
L_2 = 0 \text{,} & L_2 = 0 \text{,} & L_2 = 0 \text{,} & L_2 = 0 \text{,} & L_2 = 0 \\
R_2 = 0 & R_2 = 1 & R_2 = 0 & R_2 = 0 & R_2 = 0
\end{array}
$$

These extensions give the nonequivalent codifferentials d_2, d_3, d_4, d_5, and d_6.

6. Extensions of a 1-dimensional algebra by a 2-dimensional nilpotent algebra

Let $W = \langle v_3 \rangle$ and $M = \langle v_1, v_2 \rangle$. The group $G_{M,W}$ is given by matrices of the form

$$G = \begin{bmatrix} g_{1,1} & g_{1,2} & 0 \\ g_{2,1} & g_{2,2} & 0 \\ 0 & 0 & g_{3,3} \end{bmatrix} = \begin{bmatrix} \tilde{G} & 0 \\ 0 & g_{3,3} \end{bmatrix}, \text{ where } \tilde{G} = \begin{bmatrix} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{2,2} \end{bmatrix} \in \mathbf{GL}(2, \mathbb{C}).$$

The general form of the module structure λ is given by the matrices

$$L_1 = \begin{bmatrix} L_1^1 & L_2^1 \\ L_1^2 & L_2^2 \end{bmatrix}, \quad R_1 = \begin{bmatrix} R_1^1 & R_2^1 \\ R_1^2 & R_2^2 \end{bmatrix}.$$

6.1. Extensions of the 1-dimensional simple algebra $\mathbb{C}$.
In this case, we have $\delta = \psi_3^{33}$ There are two choices of μ in this case, depending on whether the algebra structure on M is the trivial or nontrivial nilpotent structure.

6.1.1. *Extensions by the nontrivial nilpotent algebra.* In this case $\mu = \psi_1^{22}$. The condition $[\mu, \lambda] = 0$ forces λ to have the following form:

$$L_1 = \begin{bmatrix} L_1^1 & L_2^1 \\ 0 & L_1^1 \end{bmatrix}, \quad R_1 = \begin{bmatrix} L_1^1 & R_2^1 \\ 0 & L_1^1 \end{bmatrix}.$$

Since the matrices cannot be defective, it follows that either both matrices are the identity or both the zero matrix. This gives exactly two possible choices for λ, corresponding to the codifferentials d_7 and d_8. It is interesting to note that this is the stable case even though $n < 2m$.

6.1.2. *Extensions by the trivial nilpotent algebra.* In this case, L_1 is given by the identity matrix, $\mathrm{diag}(1, 0)$, or the zero matrix. If L_1 is the identity matrix or the zero matrix, then R_1 has the same possibilities, but when L_1 is given by the matrix $\mathrm{diag}(1, 0)$, we obtain one additional case, where R_1 is given by the matrix $\mathrm{diag}(0, 1)$.

This gives a total of 10 possibilities for λ, which gives rise to the codifferentials d_9 through d_{18}. This is not the stable case.

6.2. Extensions of the 1-dimensional trivial algebra $\mathbb{C}_0$.

Because any extension of the trivial algebra structure $\delta = 0$ by a nilpotent algebra is nilpotent, we cannot assume that the "cocycle" $\psi = c_1 \psi_1^{33} + c_2 \psi_2^{33}$ vanishes. Since δ vanishes, the Maurer-Cartan equation takes the following form:

$$\tfrac{1}{2}[\lambda, \lambda] + [\mu, \psi] = 0.$$

6.2.1. *Extensions by the trivial nilpotent algebra.* In this case, the group $G_{\delta, \mu}$ coincides with $G_{M,W}$. The MC condition becomes simply $[\lambda, \lambda] = 0$, which forces $L_1^2 = 0$, $R_1^2 = 0$ and $L_1 R_1 = R_1 L_1$. As a consequence, up to equivalence, we can assume that lambda has the following form:

$$L_1 = \begin{bmatrix} 0 & p \\ 0 & 0 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 0 & q \\ 0 & 0 \end{bmatrix}.$$

Thus we have $\lambda = \psi_2^{13} p + \psi_2^{31} q$. Also, if $p' = tp$ and $q' = tq$, for some $t \neq 0$, then $\lambda' = \psi_2^{13} p' + \psi_2^{31} q'$ is equivalent to λ under the action of $G_{\delta, \mu}$, so the λ's form a projective family labeled by projective coordinates $(p : q)$.

Any element of $C^{0,1}$ is of the form $\beta = \psi_1^3 b_1 + \psi_2^3 b_2$, and $[\lambda, \beta] = \psi_2^{33}(p + q) b_1$. Therefore, unless $p = -q$, it does not matter, up to equivalence, what the value of c_2 is. It may seem natural to choose $c_2 = 0$ as the generic value, but our experience has shown that this is not the correct choice in order to correctly align the family.

The reason for this is as follows. Let us consider a neighborhood of the codifferentials given by λ labeled by $(1 : -1)$. For all nearby values, choosing $c_2 = 1$ yields an equivalent codifferential to the one resulting from choosing $c_2 = 0$. However, at exactly this point, the two codifferentials are not equivalent. Since every value of c_2 except $c_2 = 0$ yields an equivalent codifferential to the one with $c_2 = 1$, the codifferential given by choosing $c_2 = 0$ has a jump deformation to the one given by $c_2 = 1$. Thus the element with $c_2 = 1$ should belong to the family instead of the one with $c_2 = 0$.

Moreover, we have $[\lambda, \psi] = \psi_2^{333}(p - q) c_1$, and since $[\lambda, \psi] = 0$, this forces $c_1 = 0$ unless $p = q$. In this case, when $p = q = 1$, we obtain a special case for $c_1 = 1$. This gives the element d_{19}, while the generic case $c_1 = 0$ and p and q being arbitrary, with $c_2 = 1$ gives a projective family $d_{20}(p : q)$. When $p = 1$, $q = -1$ and $c_2 = 0$, we obtain the special element d_{21}.

In the case when $p = q = 0$ we obtain no condition on either c_2 or c_1. The group $G_{\mu,\delta,\lambda}$ coincides with the group $G_{M,W}$, and the action of this group on $C^{0,2}$ is just the multiplication of $\mathbf{GL}(2, \mathbb{C})$ on $\mathbb{C}^2$, which has exactly 2 orbits. Thus we can reduce to the case when $c_1 = 0$ and $c_2 = 0$ or $c_2 = 1$. The case $c_2 = 0$ gives the zero codifferential, while the case $c_2 = 1$ corresponds to the element $d_{20}(0 : 0)$, which fits the generic pattern.

6.2.2. *Extensions by the nontrivial nilpotent algebra.* Suppose that $\mu = \psi_1^{22}$. Then $H_\mu^{1,1} = \langle \psi_1^{32}, \psi_1^{31} + \psi_2^{32} + \psi_1^{13} + \psi_2^{23} \rangle$ as before.

From the MC equation, we obtain that $b_1^1 = 0$ and $c_2 = 0$, Thus we can assume that $\lambda = \psi_1^{32} q$ and that $\psi = \psi_1^{33} c_1$. Moreover, if $q \neq 0$, we can assume that $q = 1$. However, we cannot make such a simplifying assumption for c_1. Nevertheless, the case when $c_1 = 0$ will be treated separately.

If $c_1 = 0$ and $q = 1$, then $\lambda = \psi_1^{32}$ and $\psi = 0$, so $d = \psi_1^{22} + \psi_1^{32}$. It can be shown that $d \sim d_{20}(1 : 0)$. In fact, the permutation $(1, 2, 3)$ can be seen to give this transformation. Note that the permutation matrix for this transformation does not lie in $G_{M,W}$. Thus the two codifferentials do not give equivalent extensions, but give isomorphic algebra structures on V.

If $\lambda = \psi_1^{32}$ and $\psi = \psi_1^{33} c_1$, with $c_1 \neq 0$, then $d = \psi_1^{22} + \psi_1^{32} + \psi_1^{33} c_1$. It is convenient to express $c_1 = (1 - u)u$, where $u \neq 0$ and $u \neq 1$. Let $q \neq 0$ and g be given by the matrix $\begin{bmatrix} 0 & -\frac{u}{q(-1+u)} & 0 \\ -\frac{u}{-1+u} & 0 & 0 \\ 0 & 1 & u \end{bmatrix}$. Then $g^*(d) = d_{20}(p : q)$, where $p = q(u - 1)/u$.

If $\lambda = 0$ and $\psi = \psi_1^{33}$ then $d = \psi_1^{22} + \psi_1^{33}$. Then $g^*(d) = d_{20}(1 : 1)$, if g is the automorphism given by the matrix $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & i \end{bmatrix}$.

Finally, if $\lambda = 0$ and $\psi = 0$, then $d = \psi_1^{22}$, and the permutation $(1, 2, 3)$ transforms d into $d_{20}(0 : 0)$. Thus the extensions by the nontrivial nilpotent algebra just give some special cases of the family $d_{20}(p : q)$, which already occur as extensions of the trivial algebra.

This completes the classification of the moduli space of 3-dimensional complex associative algebras by means of extensions. We summarize the information in Table 6.2.2 below, including the special cases of the family $d_{20}(p : q)$.

7. COHOMOLOGY AND DEFORMATION THEORY

Hochschild cohomology was introduced in [9], and used to classify infinitesimal deformations of associative algebras. Suppose that

$$m_t = m + t\varphi,$$

is an infinitesimal deformation of m. By this we mean that the structure m_t is associative up to first order. From an algebraic point of view, this means that we assume that $t^2 = 0$, and then check whether associativity holds. It is not difficult to show that is equivalent to the following.

$$a\varphi(b, c) - \varphi(ab, c) + \varphi(a, bc) - \varphi(a, b)c = 0,$$

where, for simplicity, we denote $m(a, b) = ab$. Moreover, if we let

$$g_t = I + t\lambda$$

Codifferential	H^0	H^2	H^1	H^3	H^4
$d_1 = \psi_3^{33} + \psi_2^{22} + \psi_1^{11}$	3	0	0	0	0
$d_2 = \psi_2^{22} + \psi_3^{33} + \psi_1^{21} + \psi_1^{13}$	1	0	0	0	0
$d_3 = \psi_2^{22} + \psi_3^{33} + \psi_1^{12}$	1	0	0	0	0
$d_4 = \psi_2^{22} + \psi_3^{33} + \psi_1^{21}$	1	0	0	0	0
$d_5 = \psi_2^{22} + \psi_3^{33} + \psi_1^{21} + \psi_1^{12}$	3	1	1	1	1
$d_6 = \psi_2^{22} + \psi_3^{33}$	3	1	1	1	1
$d_7 = \psi_3^{33} + \psi_1^{22} + \psi_1^{31} + \psi_2^{32} + \psi_1^{13} + \psi_2^{23}$	3	2	2	2	2
$d_8 = \psi_3^{33} + \psi_1^{22}$	3	2	2	2	2
$d_9 = \psi_3^{33} + \psi_1^{31} + \psi_2^{32}$	0	3	0	0	0
$d_{10} = \psi_3^{33} + \psi_1^{13} + \psi_2^{23}$	0	3	0	0	0
$d_{11} = \psi_3^{33} + \psi_1^{31} + \psi_2^{23}$	0	1	0	1	0
$d_{12} = \psi_3^{33} + \psi_1^{13} + \psi_2^{32} + \psi_2^{23}$	1	1	1	1	1
$d_{13} = \psi_3^{33} + \psi_1^{31} + \psi_2^{32} + \psi_2^{23}$	1	1	1	1	1
$d_{14} = \psi_3^{33} + \psi_2^{32}$	1	1	2	2	2
$d_{15} = \psi_3^{33} + \psi_2^{23}$	1	1	2	2	2
$d_{16} = \psi_3^{33} + \psi_2^{32} + \psi_2^{23}$	3	2	2	2	2
$d_{17} = \psi_3^{33} + \psi_1^{31} + \psi_1^{13} + \psi_2^{32} + \psi_2^{23}$	3	4	6	12	24
$d_{18} = \psi_3^{33}$	3	4	8	16	32
$d_{19} = \psi_2^{13} + \psi_2^{31} + \psi_1^{33}$	3	3	3	3	3
$d_{20}(0:0) = \psi_2^{33}$	3	5	9	17	33
$d_{20}(1:0) = \psi_2^{13} + \psi_2^{33}$	1	2	5	8	11
$d_{20}(1:1) = \psi_2^{13} + \psi_2^{31} + \psi_2^{33}$	3	4	5	7	8
$d_{20}(1:-1) = \psi_2^{13} - \psi_2^{31} + \psi_2^{33}$	1	2	3	4	5
$d_{20}(p:q) = \psi_2^{13}p + \psi_2^{31}q + \psi_2^{33}$	1	2	3	3	4
$d_{21} = \psi_2^{13} - \psi_2^{31}$	1	4	5	8	9

TABLE 2. Cohomology of the 21 families of codifferentials on a
3-dimensional space

be an infinitesimal automorphism of V, where $\lambda \in \mathrm{Hom}(V, V)$, then it is easily
checked that

$$g_t^*(m)(a, b) = ab + t(a\lambda(b) - \lambda(ab) + \lambda(a)b).$$

This naturally leads to a definition of the Hochschild coboundary operator D on
$\mathrm{Hom}(\mathcal{T}(V), V)$ by

$$D(\varphi)(a_0, \cdots, a_n) = a_0\varphi(a_1, \cdots, a_n) + (-1)^{n+1}\varphi(a_0, \cdots, a_{n-1})a_n$$

$$+ \sum_{i=0}^{n-1}(-1)^{i+1}\varphi(a_0, \cdots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \cdots, a_n).$$

If we set $C^n(V) = \mathrm{Hom}(V^n, V)$, then $D : C^n(V) \to C^{n+1}(V)$. One obtains the
following classification theorem for infinitesimal deformations.

Theorem 7.1. *The equivalence classes of infinitesimal deformations m_t of an as-
sociative algebra structure m under the action of the group of infinitesimal auto-
morphisms on the set of infinitesimal deformations are classified by the Hochschild*

cohomology group

$$H^2(m) = \ker(D : C^2(V) \to C^3(V)) / \operatorname{Im}(D : C^1(V) \to C^2(V)).$$

Moreover, the Hochschild coboundary operation translates into the coboundary operator D on $C(W)$, given by

$$D(\varphi) = [d, \varphi].$$

This point of view on Hochschild cohomology first appeared in [15]. The fact that the space of Hochschild cochains is equipped with a graded Lie algebra structure was noticed much earlier [4, 5, 6, 7, 8].

For notational purposes, we introduce a basis of $C^n(W)$ as follows. Suppose that $W = \langle w_1, \cdots, w_m \rangle$. Then if $I = (i_1, \cdots, i_n)$ is a *multi-index*, where $1 \leq i_k \leq m$, denote $w_I = w_{i_1} \cdots w_{i_n}$. Define $\varphi_i^I \in C^n(W)$ by

$$\varphi_i^I(w_J) = \delta_J^I w_i,$$

where δ_J^I is the Kronecker delta symbol. In order to emphasize the parity of the element, we will denote φ_i^I by ψ_i^I when it is an odd coderivation.

For a multi-index $I = (i_1, \cdots, i_k)$, denote its *length* by $\ell(I) = k$. Then since W is a completely odd space, the parity of φ_i^I is given by $|\varphi_i^I| = k + 1 \pmod{2}$. If K and L are multi-indices, then denote $KL = (k_1, \cdots, k_{\ell(K)}, l_l, \cdots, l_{\ell(L)})$. Then

$$(\varphi_i^I \circ \varphi_j^J)(w_K) = \sum_{K_1 K_2 K_3 = K} (-1)^{w_{K_1} \varphi_j^J} \varphi_i^I(w_{K_1}, \varphi_j^J(w_{K_2}), w_{K_3})$$

$$= \sum_{K_1 K_2 K_3 = K} (-1)^{w_{K_1} \varphi_j^J} \delta_{K_1 j K_3}^I \delta_{K_2}^J w_i,$$

from which it follows that

$$(3) \qquad \varphi_i^I \circ \varphi_j^J = \sum_{k=1}^{\ell(I)} (-1)^{(w_{i_1} + \cdots + w_{i_{k-1}}) \varphi_j^J} \delta_j^k \varphi_i^{(I,J,k)},$$

where (I, J, k) is given by inserting J into I in place of the k-th element of I; *i.e.*, $(I, J, k) = (i_1, \cdots, i_{k-1}, j_1, \cdots, j_{\ell(J)}, i_{k+1}, \cdots, i_{\ell(I)})$.

Let us recast the notion of an infinitesimal deformation in terms of the language of coderivations. We say that

$$d_t = d + t\psi$$

is an infinitesimal deformation of the codifferential d precisely when $[d_t, d_t] = 0$ mod t^2. This condition immediately reduces to the cocycle condition $D(\psi) = 0$. Note that we require d_t to be odd, so that ψ must be an odd coderivation. One can introduce a more general idea of parameters, allowing both even and odd parameters, in which case even coderivations play an equal role, but we will not adopt that point of view in this paper.

For associative algebras, we require that d and ψ lie in $C^2(W)$. Since in this paper, our algebras are be ordinary algebras, so that the parity of an element in $C^n(W)$ is $n+1$, and thus elements of $C^2(W)$ are automatically odd. More generally, one could require that d be an arbitrary odd codifferential, in which case we would obtain an A_∞ algebra, a natural generalization of an associative algebra.

We need the notion of a versal deformation, in order to understand how the moduli space is glued together. To explain versal deformations we introduce the notion of a deformation with a local base. A local base A is a $\mathbb{Z}_2$-graded commutative, unital $\mathbb{K}$-algebra with an augmentation $\epsilon : A \to \mathbb{K}$, whose kernel $\mathfrak{m}$ is

the unique maximal ideal in A, so that A is a local ring. It follows that A has a unique decomposition $A = \mathbb{K} \oplus \mathfrak{m}$ and ϵ is just the projection onto the first factor. Let $W_A = W \otimes A$ equipped with the usual structure of a right A-module. Let $T_A(W_A)$ be the tensor algebra of W_A over A, that is $T_A(W_A) = \bigoplus_{k=0}^{\infty} T_A^k(W_A)$ where $T_A^0(W_A) = A$ and $T_A^{k+1}(W_A) = T^k(W_A)_A \otimes_A W_A$. It is a standard fact that $T_A^k(W_A) = T^k(W) \otimes A$ in a natural manner, and thus $T_A(W_A) = T(W) \otimes A$.

Any A-linear map $f : T_A(W) \to T_A(W)$ is induced by its restriction to $T(W) \otimes \mathbb{K} = T(W)$ so we can view an A-linear coderivation δ_A on $T_A(W_A)$ as a map $\delta_A : T(W) \to T(W) \otimes A$. A morphism $f : A \to B$ induces a map

$$f_* : \mathrm{Coder}_A(T_A(W_A)) \to \mathrm{Coder}_B(T_B(W_B))$$

given by $f_*(\delta_A) = (1 \otimes f)\delta_A$, moreover if δ_A is a codifferential then so is $f_*(A)$. A codifferential d_A on $T_A(W_A)$ is said to be a deformation of the codifferential d on $T(W)$ if $\epsilon_*(d_A) = d$.

If d_A is a deformation of d with base A then we can express

$$d_A = d + \varphi$$

where $\varphi : T(W) \to T(W) \otimes \mathfrak{m}$. The condition for d_A to be a codifferential is the Maurer-Cartan equation,

$$D(\varphi) + \frac{1}{2}[\varphi, \varphi] = 0$$

If $\mathfrak{m}^2 = 0$ we say that A is an infinitesimal algebra and a deformation with base A is called infinitesimal.

A typical example of an infinitesimal base is $\mathbb{K}[t]/(t^2)$; moreover, the classical notion of an infinitesimal deformation: $d_t = d + t\varphi$ is precisely an infinitesimal deformation with base $\mathbb{K}[t]/(t^2)$.

A local algebra A is complete if

$$A = \varprojlim_k A/\mathfrak{m}^k$$

A complete, local augmented $\mathbb{K}$-algebra will be called formal and a deformation with a formal base is called a formal deformation. An infinitesimal base is automatically formal, so every infinitesimal deformation is a formal deformation.

An example of a formal base is $A = \mathbb{K}[[t]]$ and a deformation of d with base A can be expressed in the form

$$d_t = d + t\psi_1 + t^2\psi_2 + \dots$$

This is the classical notion of a formal deformation. It is easy to see that the condition for d_t to be a formal deformation reduces to

$$D(\psi_{n+1}) = -\frac{1}{2}\sum_{k=1}^{n}[\psi_k, \psi_{n+1-k}]$$

An automorphism of W_A over A is an A-linear isomorphism $g_A : W_A \to W_A$ making the diagram below commute:

The map g_A is induced by its restriction to $T(W) \otimes \mathbb{K}$ so we can view g_A as a map

$$g_A : T(W) \to T(W) \otimes A$$

so we ca express g_A in the form

$$g_A = I + \lambda$$

$$\begin{array}{ccc} W_A & \xrightarrow{g_A} & W_A \\ \downarrow{\epsilon_*} & & \downarrow{\epsilon_*} \\ W & \xrightarrow{I} & W \end{array}$$

where $\lambda : T(W) \to T(W) \otimes \mathfrak{m}$. If A is infinitesimal then $g_A^{-1} = I - \lambda$.

Two deformations d_A and d'_A are said to be equivalent over A if there is an automorphism g_A of W_A over A such that $g_A^*(d_A) = d'_A$. In this case we write $d'_A \sim d_A$.

An infinitesimal deformation d_A with base A is called universal if whenever d_B is an infinitesimal deformation with base B, there is a unique morphism $f : A \to B$ such that $f_*(d_A) \sim d_B$.

Theorem 7.2. *If* $\dim H^2_{odd}(d) < \infty$ *then there is a universal infinitesimal deformation* d^{inf} *of* d. *Given by*

$$d^{inf} = d + \delta^i t_i$$

where $H^2_{odd}(d) = \langle \bar{\delta}^i \rangle$ *and* $A = \mathbb{K}[t_i]/(t_i t_j)$ *is the base of deformation.*

A formal deformation d_A with base A is called versal if given any formal deformation of d_B with base B there is a morphism $f : A \to B$ such that $f_*(d_A) \sim d_B$. Notice that the difference between the versal and the universal property of infinitesimal deformations is that f need not be unique. A versal deformation is called *miniversal* if f is unique whenever B is infinitesimal. The basic result about versal deformation is:

Theorem 7.3. *If* $\dim H^2_{odd}(d) < \infty$ *then a miniversal deformation of* d *exists.*

In this paper we will only need the following result to compute the versal deformations.

Theorem 7.4. *Suppose* $H^2_{odd}(d) = \langle \bar{\delta}^i \rangle$ *and* $[\delta^i, \delta^j] = 0$ *for all* i, j *then the infinitesimal deformation*

$$d^{inf} = d + \delta^i t_i$$

is miniversal, with base $A = \mathbb{K}[[t_i]]$.

The construction of the moduli space as a geometric object is based on the idea that codifferentials which can be obtained by deformations with small parameters are "close" to each other. From the small deformations, we can construct 1-parameter families or even multi-parameter families, which are defined for small values of the parameters, except possibly when the parameters vanish.

If d_t is a one parameter family of deformations, then two things can occur. First, it may happen that d_t is equivalent to a certain codifferential d' for every small value of t except zero. Then we say that d_t is a jump deformation from d to d'. It will never occur that d' is equivalent to d, so there are no jump deformations from a codifferential to itself. Otherwise, the codifferentials d_t will all be nonequivalent if t is small enough. In this case, we say that d_t is a smooth deformation.

In [3], it was proved for Lie algebras that given three codifferentials d, d' and d'', if there are jump deformations from d to d' and from d' to d'', then there is a jump deformation from d to d''. The proof of the corresponding statement for associative algebras is essentially the same.

Similarly, if there is a jump deformation from d to d', and a family of smooth deformations d'_t, then there is a family d_t of smooth deformations of d, such that every deformation in the image of d'_t lies in the image of d_t, for sufficiently small values of t. In this case, we say that the smooth deformation of d factors through the jump deformation to d'.

In the examples of complex moduli spaces of Lie and associative algebras which we have studied, it turns out that there is a natural stratification of the moduli space of n-dimensional algebras by orbifolds, where the codifferentials on a given strata are connected by smooth deformations, which don't factor through jump deformations. These smooth deformations determine the local neighborhood structure.

The strata are connected by jump deformations, in the sense that any smooth deformation from a codifferential on one strata to another strata factors through a jump deformation. Moreover, all of the strata are given by projective orbifolds. In fact, in all the complex examples we have studied, the orbifolds either are single points, or $\mathbb{CP}^n$ quotiented out by either Σ_{n+1} or a subgroup, acting on $\mathbb{CP}^n$ by permuting the coordinates.

We don't have a proof at this time, but we conjecture that this pattern holds in general. In other words, we believe the following conjecture.

Conjecture 7.5 (Fialowski-Penkava). *The moduli space of Lie or associative algebras of a fixed finite dimension n are stratified by projective orbifolds, with jump deformations and smooth deformations factoring through jump deformations providing the only deformations between the strata.*

One might say that in this paper, we prove this conjecture is true for the moduli space of 3-dimensional complex associative algebras.

8. Deformations of the elements in the moduli space

8.1. $d_1 := \psi_3^{33} + \psi_2^{22} + \psi_1^{11}$. The matrix of this codifferential is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

This is the only 3-dimensional complex semisimple algebra, and is the direct sum of three copies of $\mathbb{C}$. The algebra is both unital and commutative. We have $h^0 = 3$ and $h^n = 0$ otherwise, so this algebra is rigid, as is always the case with semisimple algebras. Recall that H^0 is the center of an associative algebra, so the commutativity of the algebra given by d_1 is reflected in the dimension of H^0. It is a nice fact that one can detect the commutativity of an algebra in terms of its cohomology.

8.2. $d_2 = \psi_3^{33} + \psi_2^{22} + \psi_1^{21} + \psi_1^{13}$. The matrix of this codifferential is

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

This algebra is an extension of the complex semisimple algebra $\mathbb{C}^2$ by the trivial algebra $\mathbb{C}_0$. It cannot be decomposed as a direct sum. We have $h^0 = 1$, and its center is spanned by $v_1 + v_2$, which is also the identity of this algebra, so this algebra

is unital. Since $h^n = 0$ for $n > 0$, the algebra is rigid. This algebra is isomorphic to its opposite algebra.

8.3. $d_3 = \psi_3^{33} + \psi_2^{22} + \psi_1^{21}$. The matrix of this codifferential is

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The algebra is an extension of the complex semisimple algebra $\mathbb{C}_2$ by the trivial algebra $\mathbb{C}_0$. It also is the direct sum of the simple 1-dimensional algebra C with the rigid 2-dimensional algebra d_3. We have $h^0 = 1$ and its center is spanned by v_3. Since this algebra is a direct sum of two rigid algebras, it is not surprising that $h^n = 0$ for $n > 0$, so that this algebra is rigid. Its opposite algebra is d_4.

8.4. $d_4 = \psi_3^{33} + \psi_2^{22} + \psi_1^{12}$. The matrix of this codifferential is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The algebra is an extension of the complex semisimple algebra $\mathbb{C}_2$ by the trivial algebra $\mathbb{C}_0$. It also is the direct sum of the simple 1-dimensional algebra C with the rigid 2-dimensional algebra d_2. We have $h^0 = 1$ and its center is spanned by v_3. Since this is the opposite algebra to d_3, the dimensions of its cohomology spaces H^n are the same as for d_3. In other words, $h^n = 0$ for $n > 0$, so this algebra is rigid.

8.5. $d_5 = \psi_3^{33} + \psi_2^{22} + \psi_1^{21} + \psi_1^{12}$. The matrix of this codifferential is

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The algebra is an extension of the complex semisimple algebra $\mathbb{C}_2$ by the trivial algebra $\mathbb{C}_0$. It is also the direct sum of the simple 1-dimensional algebra C with the unital commutative 2-dimensional algebra d_4, so the algebra d_5 is both commutative and unital. Since the 2-dimensional algebra d_4 deforms to the 2-dimensional semisimple algebra, it is obvious that the 3-dimensional algebra d_5 deforms to the semisimple algebra d_1. In fact, the versal deformation of d_5 is given by

$$d_5^\infty = d_5 + \psi_1^{22} t.$$

If g is given by the matrix $G = \begin{bmatrix} 0 & 0 & 1 \\ -\sqrt{t} & 1 & 0 \\ \sqrt{t} & 1 & 0 \end{bmatrix}$, then $g^*(d_5^\infty) = d_1$, as long as $t \neq 0$. In general, $h^n = 1$ for $n > 0$. In fact, a basis of H^n is just

$$H^0 = \langle \psi_1, \psi_2, \psi_3 \rangle$$
$$H^n = \langle \varphi_1^{1^n} \rangle \quad \text{if } n \text{ is odd}$$
$$H^n = \langle \psi_2^{1^n} \rangle \quad \text{if } n \text{ is even}$$

In particular, d_5 will have nontrivial extensions as an A_∞ algebra.

8.6. $d_6 = \psi_3^{33} + \psi_2^{22}$. The matrix of this codifferential is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The algebra is the direct sum of the complex semisimple algebra $\mathbb{C}_2$ and the trivial algebra $\mathbb{C}_0$. Since the algebra is commutative $h^0 = 3$. This algebra is not unital. The algebra can also be seen as the direct sum of the simple 1-dimensional algebra and the 2-dimensional algebra d_5, and its cohomology coincides with that of the the 2-dimensional algebra. Its versal deformation is given by

$$d_6^\infty = d_6 + \psi_1^{11} t.$$

If $t \neq 0$, then $g^*(d_6^\infty) = d_1$ where g is given by the matrix $G = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ t & 0 & 0 \end{bmatrix}$. In general, $h^n = 1$ for $n > 0$. In fact, a basis of H^n is just

$$H^0 = \langle \psi_1, \psi_2, \psi_3 \rangle$$
$$H^n = \langle \varphi_1^{1^n} \rangle \quad \text{if } n > 0$$

8.7. $d_7 = \psi_3^{33} + \psi_1^{22} + \psi_1^{31} + \psi_1^{13} + \psi_2^{32} + \psi_2^{23}$. The matrix of this codifferential is

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The algebra is an extension of the simple 1-dimensional algebra by the nontrivial nilpotent 2-dimensional algebra. It is the unique such extension of this nilpotent algebra as a unital algebra, and it is also commutative.

Its versal deformation is given by

$$d_7^\infty = d_7 + \psi_3^{22} t_1 - \psi_1^{11} t_1 + \psi_2^{11} t_2 + \psi_3^{12} t_2 + \psi_3^{21} t_2.$$

Note that the versal deformation is given by the infinitesimal deformation, and that it remains both unital and commutative. Therefore, it is not surprising that this codifferential deforms to d_1 and d_5. In fact, since d_5 deforms to d_1, the fact that d_7 deforms to d_5 implies that it will deform to d_1.

In fact, $d_7^\infty \sim d_1$, on the surface given by $t_1 = u^2 + uv + v^2$, $t_2 = -uv(u + v)$, except when $v = u$, $v = -u/2$ and $v = -2u$, where it is equivalent to d_5, unless $u = v = 0$.

In general,it appears that $h^n = 2$ for $n > 0$, while $h^0 = 3$, since the algebra is commutative. It might be difficult to compute a basis for the cohomology, but the pattern looks similar to the codifferential d_6 of the 2-dimensional algebras (in which case it might be difficult to compute this cohomology).

8.8. $d_8 = \psi_3^{33} + \psi_1^{22}$. The matrix of this codifferential is

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The algebra is the direct sum of the simple 1-dimensional algebra with the nontrivial nilpotent 2-dimensional algebra. It is nonunital, but is commutative.

Its versal deformation is given by

$$d_8^\infty = d_8 + \psi_1^{22} t_1 + \psi_1^{11} t_2 + \psi_2^{12} t_2 + \psi_2^{21} t_2.$$

Note that the versal deformation is given by the infinitesimal deformation, and that it remains commutative. Moreover, the versal deformation is just the direct sum of the simple 1-dimensional algebra and the versal deformation of the nontrivial nilpotent 2-dimensional algebra.

Except on the curves $t_1 = 0$ and $t_1 = -t_2^2/4$, $d_8^\infty \sim d_1$. Along the curve $t_1 = 0$ (except when $t_2 = 0$), it is equivalent to d_5, and on the punctured curve $t_1^2 = -t_2^2/4$, the deformation is equivalent to d_6. Note that all three of the codifferentials to which d_8 deforms are in fact direct sums of the simple 1-dimensional algebra (given by the codifferential ψ_3^{33}) and a 2-dimensional algebra to which the nilpotent 2-dimensional algebra deforms.

We have $h^n = 2$ when $n > 1$ and the cohomology is precisely the same cohomology as the cohomology of the nilpotent 2-dimensional algebra. It has a complicated basis, which was explicitly computed in [1]. Since d_8 is commutative, $h^0 = 3$.

8.9. $d_9 = \psi_3^{33} + \psi_1^{31} + \psi_2^{32}$**.** The matrix of this codifferential is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The algebra is an extension of the simple 1-dimensional algebra by the trivial nilpotent 2-dimensional algebra. It is neither unital nor commutative. Its opposite algebra is d_{10}. It is one of a family of 3 rigid 2-dimensional algebras. In fact, in dimension n, there is always a family of n rigid algebras which are extensions of the simple 1-dimensional algebra.

This algebra has a trivial center, but $h^1 = 3$, reflecting that the space of outer derivations is 3-dimensional. We have $H^1 = \langle \varphi_2^1, \varphi_1^2, \varphi_2^2 \rangle$ and $h^n = 0$ unless $n = 1$.

8.10. $d_{10} = \psi_3^{33} + \psi_1^{13} + \psi_2^{23}$**.** The matrix of this codifferential is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The algebra is an extension of the simple 1-dimensional algebra by the trivial nilpotent 2-dimensional algebra. It is neither unital nor commutative. Its opposite algebra is d_9.

This algebra has a trivial center, but $h^1 = 3$, reflecting that the space of outer derivations is 3-dimensional. We have $H^1 = \langle \varphi_2^1, \varphi_1^2, \varphi_2^2 \rangle$ and $h^n = 0$ unless $n = 1$. This is the second element in the family of 3 rigid extensions of the simple 1-dimensional algebra.

8.11. $d_{11} = \psi_3^{33} + \psi_1^{31} + \psi_2^{23}$**.** The matrix of this codifferential is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The algebra is an extension of the simple 1-dimensional algebra by the trivial nilpotent 2-dimensional algebra. It is neither unital nor commutative. It is isomorphic to its opposite algebra.

This algebra has a trivial center, so $h^0 = 0$. In fact, the pattern for cohomology seems to be

$$H^1 = \langle \varphi_2^2 \rangle.$$
$$H^{2n} = 0.$$
$$H^{2n+1} = \langle \varphi_1^{(12)^n 1} \rangle, \quad \text{if } n > 0.$$

This algebra is the third element in the family of 3 rigid extensions of the simple 1-dimensional algebra.

8.12. $d_{12} = \psi_3^{33} + \psi_1^{13} + \psi_2^{32} + \psi_2^{23}.$ The matrix of this codifferential is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The algebra is an extension of the simple 1-dimensional algebra by the trivial nilpotent 2-dimensional algebra. It is neither unital nor commutative, and its center is spanned by v_3. Its opposite algebra is d_{13}.

The versal deformation is given by

$$d_{12}^\infty = d_{12} + \psi_2^{22} t,$$

and the versal deformation is equivalent to d_3 when $t \neq 0$.

Its cohomology is given by

$$H^{2n} = \langle \psi_2^{2^n} \rangle$$
$$H^{2n+1} = \langle \varphi_2^{2^{n+1}} \rangle.$$

8.13. $d_{13} = \psi_3^{33} + \psi_1^{31} + \psi_2^{32} + \psi_2^{23}.$ The matrix of this codifferential is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The algebra is an extension of the simple 1-dimensional algebra by the trivial nilpotent 2-dimensional algebra. It is neither unital nor commutative, and its center is spanned by v_3. Its opposite algebra is d_{12}.

The versal deformation is given by

$$d_{12}^\infty = d_{12} + \psi_2^{22} t,$$

and the versal deformation is equivalent to d_4 when $t \neq 0$.

Its cohomology is given by

$$H^{2n} = \langle \psi_2^{2^n} \rangle$$
$$H^{2n+1} = \langle \varphi_2^{2^{n+1}} \rangle.$$

8.14. $d_{14} = \psi_3^{33} + \psi_2^{32}$. The matrix of this codifferential is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The algebra is an extension of the simple 1-dimensional algebra by the trivial nilpotent 2-dimensional algebra. It is also a direct sum of the 2-dimensional rigid algebra d_2 and the trivial 1-dimensional algebra. It is neither unital nor commutative, and its center is spanned by v_1. Its opposite algebra is d_{15}.

The versal deformation is given by

$$d_{12}^\infty = d_{12} + \psi_1^{11}t_1 + \psi_2^{21}t_2.$$

However, in this case, there are relations on the base of the versal deformation. The relation is $t_2(t_2 - t_1) = 0$, which has solutions $t_2 = 0$ or $t_2 = t_1$. For the first solution, $t_2 = 0$, the deformation is equivalent to d_4. For the second solution, $t_2 = t_1$, the deformation is equivalent to d_2. Notice how different the deformation situation is when there are relations. Neither d_2 nor d_4 deform into the other, so the deformations given by the two different solutions to the relations are not "near to" each other.

The cohomology is given by

$$H^0 = \langle \psi_1 \rangle.$$
$$H^1 = \langle \varphi_1^1 \rangle.$$
$$H^{2n} = \langle \psi_1^{1^{2n}}, \psi_2^{21^{2n-1}} \rangle, \quad n > 0.$$
$$H^{2n+1} = \langle \varphi_1^{1^{2n+1}}, \varphi_2^{21^{2n}} \rangle, \quad n > 0.$$

8.15. $d_{15} = \psi_3^{33} + \psi_2^{23}$. The matrix of this codifferential is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The algebra is an extension of the simple 1-dimensional algebra by the trivial nilpotent 2-dimensional algebra. It is also a direct sum of the 2-dimensional rigid algebra d_2 and the trivial 1-dimensional algebra. It is neither unital nor commutative, and its center is spanned by v_1. Its opposite algebra is d_{14}.

The versal deformation is given by

$$d_{12}^\infty = d_{12} + \psi_1^{11}t_1 + \psi_2^{12}t_2.$$

As in the previous case, there are relations on the base of the versal deformation. The relation is $t_2(t_2 - t_1) = 0$, which has solutions $t_2 = 0$ or $t_2 = t_1$. For the first solution, $t_2 = 0$, the deformation is equivalent to d_3. For the second solution, $t_2 = t_1$, the deformation is equivalent to d_2.

The cohomology is given by

$$H^0 = \langle \psi_1 \rangle.$$
$$H^1 = \langle \varphi_1^1 \rangle.$$
$$H^{2n} = \langle \psi_1^{1^{2n}}, \psi_2^{1^{2n-1}2} \rangle, \quad n > 0.$$
$$H^{2n+1} = \langle \varphi_1^{1^{2n+1}}, \varphi_2^{1^{2n}2} \rangle, \quad n > 0.$$

8.16. $d_{16} = \psi_3^{33} + \psi_2^{32} + \psi_2^{23}$. The matrix of this codifferential is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The algebra is an extension of the simple 1-dimensional algebra by the trivial nilpotent 2-dimensional algebra. It is also a direct sum of the 2-dimensional rigid algebra d_4 and the trivial 1-dimensional algebra. It is not unital, but it is commutative.

The versal deformation is given by

$$d_{12}^\infty = d_{12} + \psi_1^{11} t_1 + \psi_3^{22} t_2.$$

On the punctured curve $t_1 = 0$, the deformation is equivalent to d_6, while on the punctured curve $t_2 = 0$, the deformation is equivalent to d_5. Otherwise, the deformation is equivalent to d_1.

The cohomology is given by

$$H^0 = \langle \psi_1, \psi_2, \psi_3 \rangle.$$
$$H^{2n} = \langle \psi_1^{1^{2n}}, \psi_2^{2^{2n}} \rangle, \quad n > 0.$$
$$H^{2n+1} = \langle \varphi_1^{1^{2n+1}}, \varphi_2^{2^{2n+1}} \rangle.$$

8.17. $d_{17} = \psi_3^{33} + \psi_1^{13} + \psi_1^{31} + \psi_2^{32} + \psi_2^{23}$. The matrix of this codifferential is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The algebra is the unique unital extension of the simple 1-dimensional algebra by the trivial nilpotent 2-dimensional algebra, and it is commutative.

The matrix of the versal deformation is

$$\begin{bmatrix} 0 & 0 & t_4 & t_2 & 1 & 0 & 1 & 0 & 0 \\ t_5 & t_6 & t_1 & t_3 & 0 & 1 & 0 & 1 & 0 \\ -t_3 t_5 + t_6{}^2 & t_2 t_5 & t_2 t_5 - t_1 t_4 & t_2 t_6 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Given that $h^3 = 12$, it is not surprising that there are a lot of relations on the base of the versal deformation. Removing duplicate relations, we obtain the following 8 relations:

$$-t_1 t_2 + t_3 t_4 - t_4{}^2 + t_2 t_6 = 0, \quad (t_6 - t_1)(-t_3 + t_4) = 0, \quad t_1{}^2 + t_4 t_5 - t_6{}^2 = 0$$

$$t_2(t_1 - t_6) = 0, \quad t_5(t_1 - t_6) = 0, \quad t_4(t_6 + t_1) = 0, \quad t_2 t_4 = 0, \quad t_4 t_5 = 0$$

The solutions to these relations are given by

$$t_1 = t_6, t_4 = 0 \quad \text{or} \quad t_2 = 0, t_5 = 0, t_3 = t_4, t_1 = -t_6$$

For the first solution to the relations, the deformation is equivalent to d_1, except on some hypersurfaces, where it is either equivalent to d_5 or d_7. For the second solution, there are two parameters to the deformation, but the versal deformation isequivalent to d_2 unless both parameters vanish). In fact, the transformation g given by the matrix $G = \begin{bmatrix} \overline{t_1} & -\overline{t_4} & 0 \\ t_4 & t_1 & 1 \\ -t_4 & 0 & 1 \end{bmatrix}$ satisfies $g^*(d_{17}^\infty) = d_2$, whenever not both t_1 and t_4 vanish. This is an example of a two parameter family of deformations that are all equivalent to the same deformation.

It is difficult to compute the cohomology in general. We have $h^0 = 3$, $h^1 = 3$, $h^2 = 6$, $h^3 = 12$, $h^4 = 24$, and it seems likely that $h^n = 3 * 2^{n-1}$ when $n > 1$. Determining a basis for H^n in general appears to be a difficult problem.

8.18. $d_{18} = \psi_3^{33}$. The matrix of this codifferential is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The algebra is the direct of the simple 1-dimensional algebra and the trivial nilpotent 2-dimensional algebra, and it is commutative but not unital.

The matrix of the versal deformation is

$$\begin{bmatrix} t_7 & t_4 & t_5 & t_2 & 0 & 0 & 0 & 0 & 0 \\ t_8 & t_6 & t_1 & t_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

This means that the deformations of d_{18} are precisely the elements in the moduli space which are direct sums of the simple 1-dimensional algebra, and an arbitrary nontrivial 2-dimensional algebra, so the deformation theory of this codifferential captures the complete deformation theory of the moduli space of 2-dimensional algebras.

There are 16 nontrivial relations on the base of the versal deformation, corresponding to the fact that $h^3 = 16$, and the solution to these relations decomposes into 7 distinct solutions. The deformations which occur are d_1, d_3, d_4, d_5, d_6 and d_8, the six elements of the moduli space which correspond to the six nontrivial 2-dimensional algebras.

It is easy to compute the cohomology of this codifferential. We have $h^0 = 3$, and $h^n = 2^{n+1}$ for $n > 0$. In fact, a basis of H^n is given by those elements in C^n whose indices are either 1 or 2, which accounts for the dimension. This is a pattern which should be repeated in every dimension. That is, the extension of the simple 1-dimensional algebra by a trivial m-dimensional algebra should have have H^n coinciding with C^n of the m-dimensional subspace, and therefore we will have $h^n = m^{n+1}$. Moreover, the codifferential will deform to those elements which are direct sums of the simple algebra and a nontrivial element of the m-dimensional moduli space.

Actually, we can say a similar thing about the direct sum of the 1-dimensional simple algebra and any m-dimensional algebra. Its cohomology coincides with the cohomology of the m-dimensional algebra, and its deformations correspond to the deformations of the m-dimensional algebra. However, many of the extensions

of the 1-dimensional simple algebra by a trivial m-dimensional algebra are not decomposable as direct sums of the simple algebra and an m-dimensional algebra.

8.19. $d_{19} = \psi_1^{33} + \psi_2^{13} + \psi_2^{31}$. The matrix of this codifferential is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This is an extension of the of the trivial 1 dimensional algebra by the trivial 2 dimensional nilpotent algebra. It is nilpotent, and therefore not unital, but it is commutative.

The matrix of the versal deformation is

$$\begin{bmatrix} t_3 & t_2 & t_2 & 0 & 0 & 0 & 0 & 0 & 1 \\ -t_1 & 0 & 0 & t_2 & 1 & 0 & 1 & 0 & 0 \\ t_2 & 0 & 0 & 0 & t_3 & t_2 & t_3 & t_2 & t_1 \end{bmatrix}.$$

There are no relations on the base of the versal deformation, which means that the infinitesimal deformation is versal. However, given that there are 3 deformation parameters, it is not surprising that the codifferential has a lot of deformations. It deforms to d_1, d_5, d_6, d_7, d_8 and d_{16}.

8.20. $d_{20}(p:q) = \psi_2^{33} + \psi_2^{13}p + \psi_2^{31}q$. The matrix of this codifferential is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p & 0 & q & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This is a projective family of extensions of the of the trivial 1 dimensional algebra by the trivial 2 dimensional nilpotent algebra. By projective, we mean that the codifferential labeled by $(p : q)$ is isomorphic to the element labeled by $(tp : tq)$ for any nonzero t. Moreover, $d_{20}(p : q) \sim d_{20}(q : p)$, so that actually, the family is parameterized by the projective orbifold $\mathbb{P}^1/\Sigma_2$, where the action of Σ_2 on $\mathbb{P}^1$ is given by permuting the projective coordinates. Note that we include the generic point $(0 : 0)$ of $\mathbb{P}^1$ in our consideration. We shall describe this special point in more detail below.

The algebra $d_{20}(p : q)$ is nilpotent, and it is not commutative except when $p = q$, in other words, the special points $d_{20}(0 : 0)$ and $d_{20}(1 : 1)$.

There are some special points for which the cohomology of the codifferential is not generic, the points $d_{20}(0 : 0)$, $d_{20}(1 : 0)$, and the two *orbifold points* $d_{20}(1 : 1)$ and $d_{20}(1 : -1)$. (An element of an orbifold is called an orbifold point if its isotropy group is nontrivial.) We will describe the deformation theory of these special points separately.

8.20.1. *Generic Case.* Let us suppose that $(p : q)$ is not one of the four spacial points listed above. Then $h^2 = 3$, and $h^3 = 3$, so there are three parameters to the versal deformation and 3 relations on the base. The versal deformation is a bit complicated, so we do not give the general form. However, the relations on the

base are given by

$$(3\,t_2 + t_1 q + t_3 t_1)\,t_1 = 0.$$
$$(2\,t_1 q + 3\,t_2 + 2\,t_3 t_1 - p t_1)\,t_1 = 0.$$
$$(3\,t_2 + 2\,t_1 q + 2\,t_3 t_1)\,(3\,t_2 + t_1 q + t_3 t_1) = 0.$$

When we solve these relations, we obtain two solutions. The first is given by $t_1 = t_2 = 0$, and the second by the two conditions $p = q + t_3$ and $3t_3 = -t_1(q + t_3)$. For a solution to the relations to be "local", it is necessary that the solution is well defined if all the t parameters can be set equal to zero. However, in the second solution, if we substitute $t_3 = 0$ in the first equation we obtain $p = q$, which is not local unless $p = q$, which only happens at a special point. Thus generically it is not true. As a consequence, generically only the first solution applies and in this case, the versal deformation has a very simple format:

$$d_{20}(p : q)^\infty = d_{20}(p : q + t_3).$$

This means that generically, $d_{20}(p : q)$ only has deformations along the family!

Generically, $h^0 = 1$ and the center is spanned by v_2. We have $h^1 = 2$, $h^2 = h^3 = 3$ and $h^4 = 4$. It seems likely that it would be very difficult to compute the cohomology for this codifferential.

8.20.2. *The special point $d_{20}(1 : -1)$.* In this case, $h^2 = 3$ as before, but $h^3 = 4$, and the versal deformation has four relations on the base, with exactly one solution, for which two of the parameters vanish. The solution gives a codifferential $d_{20}(1 : -1)^\infty = d_{20}(1 : -1) + \psi_2^{11} t$, which is equivalent to $d_{20}(1 + t_3 : -1 + t_3 + \sqrt{-t_3})$. This is also just a deformation along the family. Thus the special point $(1 : -1)$ is not too special. The cohomology is not generic though, because $h^3 = 4$ and $h^4 = 5$ which are not the generic numbers. The center is still spanned by v_2.

8.20.3. *The special point $d_{20}(1 : 0)$.* In this case, $h^2 = 5$ and $h^3 = 8$, so there are 5 parameters to the versal deformation, and 8 relations on the base. We will not give them here, but mention that one of the relations has a denominator $1 - t_5$. It is interesting to note that while there is no guarantee in our method of computation of the versal deformation that the form of the versal deformation or the relations on the base should be given by rational functions of the parameters (theoretically, they are given by formal power series in the parameters), computationally, in the cases we have studied it has always been possible to find a basis of the cohomology for which both of these conditions hold.

In the case here, there are four solutions of the relations on the base:

(1) $t_1 = 0$, $t_2 = 0$, $t_3 = 0$, $t_4 = 0$
(2) $t_2 = 0$, $t_3 = 0$, $t_5 = 0$
(3) $t_4 = 0$, $t_5 = 0$, $t_1 = t_3 - t_2$
(4) $t_3 = 0$, $t_5 = 0$, $t_1 = t_4 - t_2$

For the first solution, the versal deformation is $d_{20}(1 : 0)^\infty = d_{20}(1 : t_5)$, which gives a deformation along the family.

For the second solution, the versal deformation has matrix

$$\begin{bmatrix} -t_1 + 2\,t_4 & 0 & t_4(t_4 - t_1) & 0 & 0 & 0 & t_4 & 0 & 0 \\ 0 & t_4 - t_1 & 0 & t_4(t_4 - t_1) & 1 & 0 & 0 & 0 & 1 \\ 0 & t_4(t_4 - t_1) & 0 & 0 & t_4 & t_4(t_4 - t_1) & 0 & t_4(t_4 - t_1) & t_1 \end{bmatrix}.$$

Except on the curves $t_1 = t_4$ and $t_1 = 2t_4$, the deformation is equivalent to d_4. On the punctured curve $t_1 = t_4$, the deformation is equivalent to d_{14} and on the punctured curve $t_1 = 2t_4$ it is equivalent to d_{13}.

For the third solution, the versal deformation has matrix

$$\begin{bmatrix} t_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_2 \\ 0 & 0 & t_3 & 0 & 1 & -t_2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & t_3 & 0 & t_3 & 0 & -t_2 + t_3 \end{bmatrix}$$

Except on the curves $t_3 = 0$ and $t_3 = -t_2$, the deformation is equivalent to d_3. On the punctured curve $t_3 = 0$ the deformation is equivalent to d_{15} and on the punctured curve $t_3 = -t_2$ it is equivalent to d_{12}.

For the fourth solution, the versal deformation has matrix

$$\begin{bmatrix} t_4 & 0 & 0 & 0 & 0 & 0 & t_4 & -t_2 t_4 & t_2 \\ 0 & 0 & 0 & t_2 t_4 & 1 & -t_2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & t_4 & 0 & 0 & t_2 t_4 & -t_2 + t_4 \end{bmatrix}.$$

Except on the curves $t_2 = 0$ and $t_4 = 0$, the deformation is equivalent to d_2. On the punctured curve $t_2 = 0$ the deformation is equivalent to d_{14} and on the punctured curve $t_4 = 0$ it is equivalent to d_{15}.

The center of $d_{20}(1 : 0)$ is spanned by v_2, $h^1 = 2$, and we have $h^4 = 8$, which is not generic.

8.20.4. *The special point $d_{20}(1 : 1)$.* We have $h^2 = 5$ and $h^3 = 7$, so there are 5 parameters and 7 relations on the base of the versal deformation. There are two solutions to the relations

(1) $t_1 = 0$, $t_3 = 0$, $t_4 = 0$, $t_5 = 0$
(2) $t_2 = 0$

The first solution gives the deformation $d_{20}(1 : 1 + t_2)$, which is just a deformation along the family. The second solution is a bit complicated, so we will not give the precise form. Since there are 4 nonzero parameters in the expression for the versal deformation, it is not surprising that the deformation can be equivalent to several different types, d_1, d_5, d_6, d_7, d_8, d_{16} and d_{19}. Notice that d_{19} is another nilpotent algebra.

8.20.5. *The generic point $d_{20}(0 : 0)$.* Unfortunately, the point $(0 : 0)$ is traditionally called the generic point in the literature, because its behavior is anything but generic. In projective geometry, the closure of the generic point is the entire space, which is reflected here in that $d_{20}(0 : 0)$ has jump deformations to any point $d_{20}(x : y)$ (except itself).

We have $h^2 = 9$ and $h^3 = 17$, so there are 9 parameters and 17 relations on the base of the versal deformation, giving 9 solutions to the relations on the base, which we will omit here for brevity. The versal deformation can be equivalent to d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8, d_{12}, d_{13}, d_{14}, d_{15}, d_{16}, d_{17}, d_{18}, d_{19}, and it has jump deformations to $d_{20}(x : y)$ for any $(x : y)$ except $(0 : 0)$. Most of these jumps follow from the fact that it jumps to $d_{20}(x : y)$, because an algebra always deforms to any algebra to which an algebra that it jumps to deforms.

This algebra is commutative. We have $h^1 = 5$ and $h^4 = 33$, which are not generic.

8.21. $d_{21} = \psi_2^{13} - \psi_2^{31}$. The matrix of this codifferential is

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

This is an extension of the of the trivial 1 dimensional algebra by the trivial 2 dimensional nilpotent algebra. It is nilpotent and its center is spanned by v_2.

We have $h^2 = 5$ and $h^3 = 8$, so there are 5 parameters and 8 relations on the base of the versal deformation. We do not give the formula for the versal deformation or the relations on the base, but give the solutions to the relations below:

(1) $t_3 = 0$, $t_4 = 0$, $t_5 = 0$
(2) $t_1 = 0$, $t_2 = 0$
(3) $t_1 = -t_2$, $t_4 = -1$

The third solution is not local, since $t_4 = -1$.

The first solution has matrix

$$
\begin{bmatrix}
t_2 & 0 & t_1 t_2 & 0 & 0 & 0 & t_1 & 0 & 0 \\
0 & t_2 & 0 & t_1 t_2 & 1 & t_1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & t_2 & t_1 t_2 & t_1
\end{bmatrix}
$$

and gives a jump to d_{11}.

The second solution has matrix

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
t_3 & 0 & 0 & 0 & 1 + t_4 & 0 & -1 & 0 & t_5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

This gives a jump deformation to $d_{20}(1 : -1)$ and smooth deformations nearby. Whenever a codifferential has a jump deformation to another codifferential, it always has smooth deformations to codifferentials which are near the codifferential to which it jumps.

If we restrict our study of the deformations of the nilpotent elements to just the nilpotent elements, we see a completely parallel pattern to the moduli space of 3-dimensional Lie algebras.

9. UNITAL ALGEBRAS

It is well known that finite dimensional unital algebras only deform to unital algebras. Let us identify the unital algebras and their deformations which occur in this moduli space. It is not hard to see that if an algebra is unital, then in terms of the decomposition of the algebra as a semidirect sum of its radical and its semisimple part, that the identity of the algebra must be equal to the identity in the semisimple part. Moreover, no nilpotent algebra can be unital. Therefore, it is easy to identify the unital algebras in the moduli space. They are d_1, d_2, d_5, d_7 and d_{17}. We have jump deformations

$$
d_{17} \rightsquigarrow d_7 \rightsquigarrow d_5 \rightsquigarrow d_1,
$$

while d_2, the only noncommutative unital algebra, neither deforms to any other algebra, nor does any unital algebra deform to it. (It is possible for a nonunital

algebra to deform to a unital one, and in fact, there are algebras which deform to d_2.).

It is known that any n-dimensional algebra can be realized as an ideal in an $n + 1$-dimensional unital algebra in a natural manner, by simply adding a new basis element which becomes the identity in the extended algebra. Thus, every 2-dimensional algebra can be realized as an ideal of a 3-dimensional algebra in this form. One might imagine that one could therefore identify the 2-dimensional algebras as forming a subset of the unital algebras in a natural manner, but this is not the case, as the process of constructing a unital extension does not produce an injective map of algebras.

For example, of the six 2-dimensional algebras $d_1, \cdots, d_6$, the 2-dimensional algebra d_1 extends to the 3-dimensional algebra d_1, the 2-dimensional algebras d_2 and d_3 extend to the 3-dimensional algebra d_2, the 2-dimensional algebras d_4 and d_5 extend to the 3-dimensional algebra d_5, and the nilpotent 2-dimensional algebra extends to the 3-dimensional d_7. Thus it is not possible to reconstruct the 2-dimensional algebras from the unital 3-dimensional algebras in a simple manner.

10. Commutative algebras

If an algebra deforms to a commutative algebra, then it must be commutative. On the other hand, commutative algebras can deform to noncommutative ones, as is well known. The commutative algebras which occur in this moduli space are d_1, d_5, d_6, d_7, d_8, d_{16}, d_{17}, d_{18}, $d_{20}(1:1)$ and $d_{20}(0:0)$ (and of course, the zero algebra). All commutative algebras deform to d_1, which is just $\mathbb{C}^3$, while every algebra except d_1 and d_6 also deform to d_5. There is a long sequence of jumps

$$d_{20}(0:0) \rightsquigarrow d_{17} \rightsquigarrow d_7 \rightsquigarrow d_5 \rightsquigarrow d_1.$$

The algebras d_{18}, d_{16} and d_8 have jumps to both d_5 and d_6, while $d_{20}(1:1)$ jumps to d_{16} and d_{18}. The algebra $d_{20}(0:0)$ has jump deformations to every other commutative algebra (except the trivial one).

11. Conclusions

We have tried in this paper to give a completely logical classification of the elements in the moduli space of 3-dimensional associative algebras. The only elements which will play a role in constructing higher dimensional algebras are the nilpotent algebras, since every algebra is either semisimple, an extension of a semisimple algebra by a nilpotent algebra, or an extension of a trivial 1-dimensional algebra by a nilpotent algebra, so in some sense, it is not so important what order the non-nilpotent algebras are presented in. We have been careful to construct the algebras in such a manner that the deformations always are in a "downward direction"; that is, the label of an algebra is never lower than the label of an algebra to which it deforms.

One sees from our description of the algebras that there are certain families of algebras, which have counterparts in higher dimensions. Up to dimension 3, the only complex simple algebra is $\mathbb{C}$, but in dimension 4, the matrix algebra $\mathfrak{gl}(2, \mathbb{C})$, which is simple, adds some new features to the description of the moduli space. We are hoping that a systematic construction of the moduli spaces of higher dimension is possible. However, the computational complexity will increase dramatically in

higher dimensions, making it difficult to compute the versal deformations, so that there are practical limitations to our approach.

We have shown that for 3-dimensional complex associative algebras, the conjecture of Fialowski-Penkava, that the moduli space decomposes into strata given by projective orbifolds, connected by jump deformations holds. It remains to be seen whether the conjecture will hold in higher dimensions.

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