KOWALEVSKI TOP AND GENUS TWO CURVES

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We studied in [LM] two curves of genus 2, such that the equations of motion of the Kowalevski top are linearized on their Jacobians: Kowalewski's [Kow],

$$C_1: \qquad u^2 = x((x-H)^2 - \frac{1}{4}I_2)((x-H)^2 - \frac{1}{4}I_2 + 1),$$

and the Bobenko–Reyman–Semenov-Tian-Shansky (BRS) curve [BRS]

$$C_2: \qquad u^2 = x(x^2 + 2Hx + \frac{1}{4}I_2)(x^2 + 2Hx + \frac{1}{4}I_2 - 1),$$

where H is the Hamiltonian, I_2 Kowalevski's integral of motion. Both equations are written for the case when the angular momentum l is orthogonal to the gravity vector g, because only in this case the procedure of Bobenko–Reyman–Semenov-Tian-Shansky leads to a genus 2 curve. We established an isogeny between the Jacobians $J(C_i)$ by purely algebraic means, using Richelot's transformation of a genus 2 curve.

The curve C_2 arises in a quite natural way when one applies the method of spectral curves to the Lax representation of the Kowalevski top discovered by Reyman–Semenov-Tian-Shansky [R-STS]. The curve C_1 was found by Kowalevski as a result of an ingenious change of variables, and it is quite surprising that no simple explanation of her trick, putting it in the framework of a more general procedure, has been suggested during the period of more than 100 years. The results of [LM] show how Kowalevski's curve appears from the spectral curve of the Lax representation of Reyman–Semenov-Tian-Shansky. This gives a partial answer to the above question. Now we will explain the relation of C_1, C_2 to the spectral curve B of another Lax representation: the one of Kuznetsov–Tsiganov [KT]. The latter exists under the same condition as the one for the existence of C_2: the scalar product (l, g) should be zero.

This relation is based on the transformation of Kowalevski's curve into a genus 3 curve introduced by Kuznetsov [K1], [K2], [KK] in his

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solution to the problem of separation of variables for the Kowalevski top. Remark, that it exists for arbitrary (l, g), but we use it only in the case $(l, g) = 0$. It yields the following diagram of morphisms of (smooth compact complex) curves:

$$\begin{array}{ccc} C_1 & \xleftarrow{f} & B_1 \\ & & \pi\big\downarrow \\ & & E_1 \end{array}, \qquad (1)$$

where $g(C_1) = 2, g(B_1) = 3, g(E_1) = 1$, and f, π are double coverings; f is unramified, and π is ramified at 4 points.

We prove that the diagram induces an isogeny $P(B_1/E_1) \longrightarrow J(C_1)$ with kernel of order 8 and another one $J(C_1) \longrightarrow P(B_1/E_1)$ in the opposite direction with kernel of order 2. Next, we fulfil a parallel construction for another curve of genus 3, namely, the spectral curve $B_2 = B$ of the Kuznetsov–Tsiganov L-matrix:

$$\begin{array}{ccc} C_2' & \xleftarrow{\tilde{f}} & B_2 \\ & & \tilde{\pi}\big\downarrow \\ & & E_2 \end{array}, \qquad (2)$$

where the genera of curves and the properties of the morphisms are the same as in the preceding diagram. It turns out that C_2' is nothing else but the BRS curve C_2. So, we get another pair of isogenies between $P(B_2/E_2)$, $J(C_2)$ with kernels of orders 2 and 8. Combining them with the Richelot isogeny $J(C_2) \longrightarrow J(C_1)$ (with kernel of order 4), we obtain an isogeny between the Prym varieties $P(B_i/E_i)$ with kernel of order 64. It factors through the multiplication by 2, hence there also exists an isogeny between $P(B_i/E_i)$ of degree 4.

To summarize, we have the following relations between the above curves. The curves B_2, C_2 are "natural" in the sense that they come from a general technique of integrable systems – method of spectral curves, – but this general technique is applied to different Lax matrices. B_2, C_2 are related to each other by the analog of Kuznetsov's transformation, which he originally applied to the other pair of curves B_1, C_1. Now, B_1, C_1 cannot be obtained by the method of spectral curves from the known Lax representations. Nonetheless, C_1, C_2 are connected by Richelot's transformation, so B_1, C_1 become indirectly related to B_2, C_2. Remark, that for generic constants of motion H, I_2, there are no non-trivial maps between any two curves from different triples B_i, C_i, E_i, $i = 1, 2$, but there exists a $(2, 2)$-correspondence $C_1 \longleftrightarrow C_2$. One can see that the curves E_1, E_2, as well as the Jacobian threefolds

$J(B_1), J(B_2)$, are not isogeneous, though the latter ones contain isogeneous abelian surfaces $P(B_i/E_i)$.

One can think of Kuznetsov's transformation as a way to pass from the Jacobian of a curve of genus 2 to an isogeneous Prym variety of a double covering $B \longrightarrow E$ of a genus 3 curve over an elliptic curve. As such, it represents a particular case of Audin's construction, described in Section 5.3 of her book [Au].

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1. Generalities

We will use the notations and the normalization of dimensional parameters from [BRS]. The top represents a solid with a fixed point O in a constant gravity field. There are 6 dynamical variables: 3 components of the angular momentum $l = (l_1, l_2, l_3)$ and 3 components of the gravity vector $g = (g_1, g_2, g_3)$, everything with respect to a moving orthonormal frame, attached to the solid. The motion of the top can be described by the following system:

$$\begin{cases} \frac{dl}{dt} = [l, \omega] + [c, g] \\ \frac{dg}{dt} = [g, \omega] \\ l = I\omega \end{cases} , \tag{3}$$

where c is the constant vector of the center of mass, I the inertia tensor. The system (3) is Hamiltonian with respect to the Poisson brackets

$$\{l_i, l_j\} = \epsilon_{ijk} l_k, \ \{l_i, g_j\} = \epsilon_{ijk} g_k, \ \{g_i, g_j\} = 0.$$

In Kowalevski's integrable case, the principal inertia moments have the ratio $2m : 2m : m$ and c lies in the plane, perpendicular to the symmetry axis. One can normalize the parameters so that $I = \mathrm{diag}(1, 1, 1/2)$, $c = (1, 0, 0)$ and $|g| = 1$. The problem possesses a trivial integral of motion, the scalar product (l, g), which we will consider as a parameter. Fixing its value $(l, g) = \delta$, we obtain a 4-dimensional phase space, which is a symplectic manifold. The Hamiltonian

$$H = \frac{1}{2}(l_1^2 + l_2^2 + 2l_3^2) - g_1$$

Poisson commutes with another, Kowalevski's first integral

$$I_2 = \left({l_1}^2 - {l_2}^2 + 2\, g_1\right)^2 + 4\ (l_1 l_2 + g_2)^2\,,$$

and the common level sets of H, I_2 represent the Liouville tori in our symplectic 4-fold.

The first step of Kowalevski's solution to the problem is the complexification: she considers $x = l_1 + il_2, y = l_1 - il_2$ as independent complex variables. Next she makes her famous change of variables $\xi_1 = \xi_1(x, y), \xi_2 = \xi_2(x, y)$, which we will not specify here. It brings the system (3) to the form

$$\frac{d\xi_1/dt}{\eta_1} + \frac{d\xi_2/dt}{\eta_2} = 0 \ , \quad \frac{\xi_1 d\xi_1/dt}{\eta_1} + \frac{\xi_2 d\xi_2/dt}{\eta_2} = \sqrt{-2} \tag{4}$$

where the two points (ξ_i, η_i), $i = 1, 2$, belong to *Kowalevski's curve* C_1 of genus 2

$$\eta^2 = [\xi((\xi - H)^2 + 1 - \frac{1}{4}I_2) - \delta^2]((\xi - H)^2 - \frac{1}{4}I_2).$$

The equations (4) represent thus a system of ODE on the symmetric square $\mathrm{Sym}^2(C_1)$ of C_1. The Abel–Jacobi map

$$\begin{array}{rcl} AJ : \mathrm{Sym}^2 C_1 & \longrightarrow & J(C_1) = \Omega^*_{C_1}/H_1(C_1, \mathbb{Z}) \\ (P_1, P_2) & \mapsto & \int_{P_0}^{P_1} + \int_{P_0}^{P_2} \mod H_1(C_1, \mathbb{Z}), \end{array}$$

maps birationally the symmetric square onto the Jacobian $J(C_1)$; here the integrals are considered as linear functionals on the space Ω_{C_1} of holomorphic 1-forms, defined modulo the period lattice $H_1(C_1, \mathbb{Z})$. This allows to consider (4) as the linearized system $\dot{X} = V_0 = \text{const}$ on the Jacobian, where $V_0 = (0, \sqrt{-2})$ in the basis of $T_0 J(C_1) = \Omega^*_{C_1}$, dual to the basis $(d\xi/\eta, \xi d\xi/\eta)$ of Ω_{C_1}.

The method of spectral curves, applied to the Lax representation of Reyman–Semenov-Tian-Shansky, brings us to a similar linearized equation $\dot{Y} = W_0 = \text{const}$ on the Prym variety $P(C_2/E_2)$ of a double unramified covering of curves $C_2 \xrightarrow{2:1} E_2$. Generically, the genera of C_2, E_2 are equal resp. to $3, 1$. However, if we restrict ourselves to the symplectic leaf $\delta = 0$, we have $g(C_2) = 2, g(E_2) = 0$, so the Prymian $P(C_2/E_2)$ becomes simply the Jacobian $J(C_2)$. In this case we find two different curves of genus 2 with the same property that the Hamiltonian flow of the Kowalevski top is linearized on their Jacobians: C_1 and C_2. The authors of [BRS] wrote out the equation of C_2 and raised the question about the relation between the two curves. We will call C_2 the Bobenko–Reyman–Semenov-Tian-Shansky (BRS) curve. Remark that none of the Jacobians is isomorphic to the complex Liouville torus, but only isogeneous to it.

The following theorem is the main result of [LM]:

Theorem 1. *Assume $\delta = 0$, that is $l \perp g$. Then C_1, C_2 are related by the Richelot transformation which induces an isogeny ψ :*

$J(C_2)\longrightarrow J(C_1)$ with kernel $\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$. This isogeny transforms the flow of W_0 into that of V_0.

We will describe briefly the way in which appears the curve C_2 and postpone the definition of the Richelot isogeny until the next section.

The Lax pair of Reyman–Semenov-Tyan-Shansky yields a more general system, called Kowalevsky gyrostat. If we put the gyrostatic terms to 0, we will obtain the equation

$$\frac{dL}{dt} = [L, M] \tag{5}$$

with Lax matrix

$$L(\lambda) = \frac{1}{\lambda}\begin{bmatrix} g_1 & g_2 & g_3 & 0 \\ g_2 & -g_1 & 0 & -g_3 \\ g_3 & 0 & -g_1 & g_2 \\ 0 & -g_3 & g_2 & g_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -l_2 & -l_1 \\ 0 & 0 & l_1 & -l_2 \\ l_2 & -l_1 & 0 & -2l_3 \\ l_1 & l_2 & 2l_3 & 0 \end{bmatrix} + \lambda\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

where $[L, M] = LM - ML$ for some matrix M, which we will not explicitize here. The functions H, δ^2 and I_2 belong to the algebra generated by the coefficients of λ^{-2} and λ^0 in the Laurent expansions of $\mathrm{Tr}(L(\lambda)^2)$ and $\mathrm{Tr}(L(\lambda)^4)$. Since these coefficients are invariant under the flow of (5), the spectral curve $P(\lambda,\mu) = 0$ is also invariant, where

$$P(\lambda,\mu) = \det\,(L(\lambda) - \mu).$$

Let Γ be the non-singular compactification of the spectral curve, and $L(t)$ a solution of (5). Then we have the line bundle E_t of eigenvectors of $L(t)$ on Γ. It is defined a priori on a Zariski open subset of Γ, but it is uniquely extended to all of Γ as a line subbundle of the trivial vector bundle $\mathbb{C}^4\times\Gamma$. According to [Au], Section III.2.3, the degree of E_t is 8, so that the isomorphism class $[E_t]$ is an element of $\mathrm{Pic}^8(\Gamma)$. Choosing a reference point $P_0 \in C_2$, we can write $\mathrm{Pic}^8(\Gamma) = 8[P_0] + J(\Gamma)$, which identifies $\mathrm{Pic}^8(\Gamma)$ with $J(\Gamma)$, hence the evolution equation for $[E_t]$ can be written on $J(\Gamma)$. The following theorem is proved in [BRS].

Theorem 2 (Bobenko–Reyman–Semenov-Tyan-Shansky)**.** *The evolution equation for $[E_t]$ has the form*

$$\frac{d[E_t]}{dt} = W_0 = \text{const}, \tag{6}$$

where the vector $W_0 \in \Omega^*_\Gamma$ is given via its values on the elements ω of the dual space:

$$< \omega | W_0 >= \sum_{P:\lambda(P)=\infty} \mathrm{Res}_P \left(\frac{1}{2}\mu\omega \right).$$

Further, Γ possesses 2 commuting involutions, changing the signs of coordinates:

$$\tau_1 : (\lambda, \mu) \mapsto (-\lambda, \mu), \;\; \tau_2 : (\lambda, \mu) \mapsto (\lambda, -\mu).$$

Let $C_2 = \Gamma / < \tau_1 >$, $E = \Gamma / < \tau_1, \tau_2 >$ be the quotients. Then the flow of (6) is costraint to the image $\pi^* \operatorname{Pic}^4(C_2) \subset \operatorname{Pic}^8(\Gamma)$ and tangent to $P(C_2/E)$, where $\pi : \Gamma \xrightarrow{2:1} C_2$ is the quotient map.

From now on, we assume $\delta = 0$. Then, as we mentioned above, the genus of C_2 is 2 and $P(C_2/E) = J(C_2)$. By an appropriate change of variables $(\lambda^2, \mu) \to (x, y)$, the authors of [BRS] bring the equation of C_2 to the standard form

$$y^2 = x(x^2 + 2Hx + \frac{1}{4}I_2)(x^2 + 2Hx + \frac{1}{4}I_2 - 1) \tag{7}$$

The application of the formula for W_0 from the above Theorem yields $W_0 = (0, -\sqrt{2})$ in the basis of $\Omega^*_{C_2}$, dual to $(dx/y, xdx/y)$.

2. Richelot's transformation of the BRS curve

Richelot's transformation was first introduced in 1836 by Richelot [R1], [R2] in the problem of the approximate calculation of hyperelliptic integrals of genus 2. By an appropriate iteration of this transformation, he obtained a sequence of genus 2 curves converging very rapidly to a rational curve, which allowed to approximate the hyperelliptic integrals by integrals of rational functions. Later the Richelot transform reappeared in 1901, in the work of Humbert [H], where he studied its action on 2-dimensional theta-functions. See [BM] and [CF] for more details.

Let C be a genus 2 curve defined by an equation

$$y^2 = P_1(x)P_2(x)P_3(x),$$

where $P_j(x)$ are quadratic polynomials without multiple or common roots in $\mathbb{P}^1(\mathbb{C})$. The fact that we are looking at the roots in $\mathbb{P}^1(\mathbb{C})$ rather than those in $\mathbb{C}$ means that we admit the case when one of the quadratic polynomials is in fact linear, and then ∞ is one of its roots. Let C' be the curve defined by the equation

$$\Delta y_1^2 = U_1(z)U_2(z)U_3(z),$$

where

$$U_{i+2}(z) = [P_i, P_{i+1}] = P_i'(z)P_{i+1}(z) - P_i(z)P_{i+1}'(z)$$

(addition of subscripts modulo 3) and $\Delta = \det(P_1P_2P_3)$ is the determinant of the 3×3 matrix of coefficients of the P_i. C' is called the *Richelot transform* of C. It depends on the partition of the 6 roots of the degree 6 polynomial defining C into three pairs. If one reapplies the Richelot transform to C' with the natural partition into pairs, one will get again C up to the scaling of the variable y, so that the Richelot transform is an involution in this case. But if one changes the partition on each step, one can get an infinite sequence of nonisomorphic curves.

The ubiquity of the Richelot transform is in the possibility to transport the abelian differentials from one curve to the other. The standard way to transport the 1-forms from one variety to another is the pullback with respect to a map between them. In our case, this does not work: there are no non-constant maps between C and C'. However, there is a nice substitute for a map – a $(2,2)$-correspondence. It is a curve $Z \subset C \times C'$ with double projections to both factors (one can think of it as the graph of a bivalued map from C to C', or in the opposite direction, from C' to C). It is defined by the explicit equations

$$Z \;=\; \begin{cases} P_1(x)U_1(z) + P_2(x)U_2(z) = 0, \\ yy_1 = P_1(x)U_1(z)(x-z). \end{cases}$$

Some other equations satisfied on the same set Z

$$\begin{array}{lcl} y_1P_2P_3 = U_1y(x-z) & , & y_1P_1P_3 = -U_2y(x-z) \\ yU_2U_3 = \Delta P_1y_1(x-z) & , & yU_1U_3 = -\Delta P_2y_1(x-z) \end{array}$$

and the identity

$$P_1(x)U_1(z) + P_2(x)U_2(z) + P_3(x)U_3(z) + (x-z)^2\Delta \equiv 0 \;\forall\; (x,z) \in \mathbb{C}^2$$

are useful in the proofs.

The induced map $\delta_Z : \Omega_{C'} \longrightarrow \Omega_C$ between the 2-dimensional vector spaces of 1-forms is defined by $\delta_Z = p_{1*}p_2^*$, where p_1, p_2 are the natural projections from Z to C, resp. C', p_2^* is the usual pullback of exterior forms and p_{1*} the trace map for the unramified double covering p_1, a particular case of the integration over the fibers of a smooth map. This map δ_Z is calculated in [BM]: $\delta_Z\left(S(z)\frac{dz}{y_1}\right) = S(x)\frac{dx}{y}$ for S a polynomial of degree ≤ 1.

Besides the transformation of Abelian differentials, the correspondence induces an isogeny $\psi_Z : J(C) \longrightarrow J(C')$ between the Jacobians of the two curves. For the definition, represent $J(C)$ as the abelian group of divisor classes $[\sum n_iP_i]$ modulo divisors of meromorphic functions, where $P_i \in C$ are points of the curve, and n_i are integers such that $\sum n_i = 0$. Then define

$$\psi_Z([\sum n_i P_i]) = [\sum n_i p_2 p_1^{-1} P_i].$$

The Abel-Jacobi isomorphism

$$\begin{array}{rcl} AJ : J(C) & \longrightarrow & \Omega_C^*/H_1(C,\mathbb{Z}) \\ [\sum n_i P_i] & \mapsto & \sum n_i \int_{P_0}^{P_i} \mod H_1(C,\mathbb{Z}), \end{array}$$

together with the Newton–Leibnitz rule of derivation of the integral with respect to the upper limit implies that $\delta_Z = (d_0\psi_Z)^*$, the adjoint of the differential of ψ_Z. This proves the assertion of Theorem 1 about the transformation of the flows of V_0, W_0.

The following lemma gives the kernel of ψ_Z which will be used in Section 4:

Lemma 1. *Let a_i, a_i' be the roots of the polynomial $P_i(x)$, $i = 1, 2, 3$. Then the kernel of ψ_Z is the subgroup $\{0, [(a_1, 0) - (a_1', 0)], [(a_2, 0) - (a_2', 0)], [(a_3, 0) - (a_3', 0)]\} \simeq (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ of the group $J(C)_2 \simeq (\mathbb{Z}/2\mathbb{Z})^{\oplus 4}$ of points of order two in $J(C)$.*

Proof. The inclusion "$\supset$" is obvious. Indeed, $p_2p_1^{-1}((a_i, 0)) = p_2p_1^{-1}((a_i', 0)) = \{(u_i, 0), (u_i', 0)\}$, where u_i, u_i' are the roots of $U_i(z)$, hence $p_2p_1^{-1}((a_i, 0) - (a_i', 0)) = 0$. The opposite inclusion follows from the observation that the iteration of the Richelot isogeny $J(C) \xrightarrow{\psi} J(C') \xrightarrow{\psi'} J(C'')$ produces the map of multiplication by 2 on the Jacobian of $C'' \simeq C$, and its kernel $J(C)_2$ contains 16 elements. Hence $|\ker \psi| = |\ker \psi'| = 4$. $\square$

Apply now the Richelot transform to the BRS curve $C = C_2$ with $P_1(x) = x$, $P_2(x) = x^2 + 2Hx + \frac{1}{4}I_2$, $P_3(x) = x^2 + 2Hx + \frac{1}{4}I_2 - 1$. We will obtain the curve

$$y_1^2 = -2(z + H)(z^2 + 1 - \frac{1}{4}I_2)(z^2 - \frac{1}{4}I_2)$$

which differs from Kowalevski's curve for $\delta = 0$ only by transformation $\eta = \sqrt{-2}y_1, \xi = z + H$.

3. Kuznetsov–Tsiganov Lax representation

The authors of [KT] start from the (2×2) Lax matrix for the Neumann system with one spectral parameter u

$$L_N = \left[\begin{array}{cc} u^2 - 2\, l_3\, u - {l_1}^2 - {l_2}^2 - \frac{2\alpha}{g_3^2} & ib\left((g_1 + ig_2)\, u - g_3\ (l_1 + il_2)\right) \\ ib\left((g_1 - ig_2)\, u - g_3\ (l_1 - il_2)\right) & b^2 {g_3}^2 \end{array} \right]$$

having for Hamiltonian

$$H_N = \frac{1}{2}({l_1}^2 + {l_2}^2 - b^2{g_3}^2) + \frac{\alpha}{g_3^2}.$$

They construct from it a Lax matrix for the Kowalevsky– Goryachev–Chaplygin top (as before, $\delta = (l, g) = 0$) by the following formula:

$$L_{KGCT} = K_+(u + 2i\kappa)L_N(u)K_-(u - 2i\kappa)\sigma_2 L_N^t(-u)\sigma_2,$$

where

$$\kappa = H^2 - \frac{1}{4}I_2 + \frac{1}{2},$$

$$K_-(u) = \begin{bmatrix} \alpha_1 & u \\ -\beta_1 u & \alpha_1 \end{bmatrix}, \quad K_+(u) = \begin{bmatrix} \alpha_2 & \beta_2 u \\ -u & \alpha_2 \end{bmatrix}, \quad \alpha_i, \beta_i \in \mathbb{C}.$$

The Hamiltonian has the form

$$H_{KGCT} = \frac{1}{2}({l_1}^2 + {l_2}^2 + 2l_3^2) + c_1 g_1 + c_2 g_2 + c_3(g_1^2 - g_2^2) + c_4 g_1 g_2 + \frac{c_5}{g_3^2},$$

where

$$c_1 = \frac{ib}{2}(\alpha_2 - \alpha_1), \; c_2 = \frac{b}{2}(\alpha_1 + \alpha_2), \; c_3 = -\frac{b^2}{4}(\beta_1 + \beta_2),$$
$$c_4 = \frac{ib^2}{2}(\beta_2 - \beta_1), \; c_5 = \alpha.$$

One obtains the Kowalevski top from this by putting

$$\alpha_1 = i, \; \alpha_2 = -i, \; b = \frac{1}{2}, \; \beta_1 = \beta_2 = \alpha = 0.$$

It yields the following spectral curve:

$$B = \{\lambda + \frac{1}{\lambda} = -4(u^4 - 2Hu^2 + \tilde{\kappa})\}, \; \tilde{\kappa} = \frac{1}{4}I_2 - \frac{1}{2}. \tag{8}$$

In the next section, we will relate it to the curves of genus 2 C_1, C_2 introduced above.

4. Kuznetsov's transformation

Let

$$C_1 = \{\eta^2 = P_5(\xi)\},$$

$$P_5(\xi) = \xi((\xi - H)^2 + 1 - \frac{1}{4}I_2)((\xi - H)^2 - \frac{1}{4}I_2),$$

be the the Kowalevski curve for the case $(l, g) = 0$. Kowalevski's equations (4) represent a system of two ODE in two independent variables ξ_i, and η_i can be eliminated by $\eta_i = \sqrt{P_5(\xi_i)}$. Kuznetsov introduces the new unknown functions – the canonically conjugated momenta $p_i = 2\int \frac{\partial H}{\partial \xi_i} \frac{d\dot{\xi}_i}{\dot{\xi}_i}$, which allow to separate variables in (4). The relation between p_i, ξ_i is transcendental, but if we introduce $\lambda_i = \exp(\pm 2\sqrt{-2\xi_i} p_i)$, then the pair $\lambda_i, \sqrt{\pm\xi_i}$ will satisfy an algebraic relation which is an equation of a genus 3 curve. Let us fix the choices of signs, which determine the change of variables and our new genus 3 curve B_1:

$$\left\{ \begin{array}{l} u = \sqrt{\xi} \\ \lambda = \frac{2\eta + \sqrt{4\eta^2 + \xi}}{\xi} \end{array} \right. ,$$

$$B_1 = \{\lambda + \frac{1}{\lambda} = 4(u^4 - 2Hu^2 + \kappa)\},$$

where $\kappa = H^2 + \frac{1}{2} - \frac{1}{4} I_2$. We solve the quadratic equation in λ to write down the equation of B_1 in the standard hyperelliptic form $\mu^2 = P_8(u)$, where $\mu = \lambda - 2(u^4 - 2Hu^2 + \kappa)$ and $4P_8(u)$ is the discriminant. We have:

$$\left\{ \begin{array}{l} u = \sqrt{\xi} \\ \mu = \frac{2\eta}{\sqrt{\xi}} \end{array} \right. , \qquad \left\{ \begin{array}{l} \xi = u^2 \\ \eta = \frac{\mu u}{2} \end{array} \right. , \tag{9}$$

$$B_1 = \{\mu^2 = 4(u^4 - 2Hu^2 + \kappa - \frac{1}{2})(u^4 - 2Hu^2 + \kappa + \frac{1}{2})\} \tag{10}$$

The elliptic curve E_1 and the map π from the diagram (1) are obtained by substituting $\xi = u^2$ in (10), whereas formulas (9) define the map f. One can immediately verify that f is unramified and π is ramified at 4 points $(0, \pm\mu_0), (\infty, \pm\infty)$, where $\mu_0 = \sqrt{4\kappa^2 - 1}$. The Prym variety $P(B_1/E_1)$, by definition, is the connected component of 0 in the kernel of the map $\pi_* : J(B_1) \longrightarrow J(E_1)$ and can be understood as the subvariety of $J(B_1)$ of divisor classes $[\sum_k ((u_k, \mu_k) - (-u_k, \mu_k))]$. As $\dim P(B_1/E_1) = 2$ in our case, the sums with k taking only two values 1, 2 will suffice. One can easily see that $P(B_1/E_1)$ coincides with the image of $f^* : J(C_1) \longrightarrow J(B_1)$. Indeed, the inclusion $\operatorname{im} f^* \subset \ker \pi_*$ follows from the observation that the class of the divisor $\pi_* \circ f^*((\xi, \eta) + (\xi', \eta'))$ is constant in $\operatorname{Pic}^4(E_1)$, namely, it is the lift of the class of degree 2 from $\mathbb{P}^1$ via the double covering map $E_1 \longrightarrow \mathbb{P}^1$, $(\xi, \mu) \mapsto \xi$. The opposite inclusion follows from the fact that the kernel of f^* is finite, which we will now prove.

Let $f_* : J(B_1) \longrightarrow J(C_1)$ be the natural map and $\varphi = f_*|_{P(B_1/E_1)}$. Then $\varphi \circ f^* = f_* \circ f^* = 2\,\mathrm{id}_{J(C_1)}$, hence the kernels of φ, f^* are sums of a number of copies of $\mathbb{Z}/2\mathbb{Z}$, so that $|\ker\varphi| \cdot |\ker f^*| = |\ker 2\,\mathrm{id}_{J(C_1)}| = 16$.

It is easy to determine directly the kernel of f^*. We have $f^*(K_{C_1}) = K_{B_1}$, where K stands for the canonical divisor class of a curve. In order to deal with positive divisors, we will look at $(f^*)^{-1}K_{B_1}$, rather than $(f^*)^{-1}(0)$. We have $(f^*)^{-1}K_{B_1} \subset \frac{1}{2}(2K_{C_1})$, where the last expression is a notation for the set of all the 16 halves of the divisor class $2K_{C_1}$. They can be represented by K_{C_1} itself and 15 other divisors $(\xi_i, 0) + (\xi_j, 0)$, $1 \leq i, j \leq 6$, $i \neq j$, where ξ_i runs over 5 roots of $P_5(\xi)$ and ∞; let us number them so that $\xi_5 = 0, \xi_6 = \infty$. If we apply f^* to any of these 15 divisors, we will get a divisor of degree 4, and we have to determine whether it lies in the canonical linear system $|K_{B_1}|$ or not. Since any 1-form on B_1 can be written as $(a + bu + cu^2)\frac{du}{\mu}$, the positive canonical divisors are exactly the sums of 4 points of the form $(u_1, \mu_1) + (u_2, \mu_2) + (u_1, -\mu_1) + (u_2, -\mu_2)$. Only $(0,0) + (\infty, 0)$, among the 15 nontrivial halves of $2K_{C_1}$, provides a divisor of such a form. Hence $\ker f^* \simeq \mathbb{Z}/2\mathbb{Z}$ and $|\ker\varphi| = 8$.

One can identify the kernel of f^* in a different way. The following assertion reproduces Lemma 5.3.2 from the Appendix to [Au]:

Lemma 2. *Let Z be a nonsingular curve, $\mathcal{D}$ an element of order 2 in $\mathrm{Pic}^0(Z)$. Let $f : Y \longrightarrow Z$ be the unramified double covering defined by $\mathcal{D}$. Then the kernel of the map*

$$f^* : \mathrm{Pic}(Z) \longrightarrow \mathrm{Pic}(Y)$$

is the subgroup of order 2, generated by $\mathcal{D}$.

This lemma implies that our double covering $f : B_1 \longrightarrow C_1$ is defined by the 2-torsion element $\mathcal{D} = [(0,0) - (\infty, 0)] \in J(C_1)$. Remark, that Audin mentions Kuznetsov's curve B_1 in the last lines of her book, but does not identify it as such (her notation for B_1 is $\tilde{X}$). Audin provides also a coordinate free construction of B_1: it is the normalization of the cartesian product $E_1 \times_{\mathbb{P}^1} C_1$ (see loc.cit., Proposition 5.3.1).

Corollary 1. *The Prym variety $P(B_1/E_1)$ is isomorphic to the quotient of $J(C_1)$ by the subgroup generated by the element $\mathcal{D}$ of order 2 which defines the unramified covering map f.*

The kernel of φ can be identified as the set of divisor classes $[f^*((\xi_i, 0) - (\xi_j, 0))]$, $1 \leq i, j \leq 6$, $i \neq j$, among which there are only 8 distinct ones. They form a subgroup $(\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$ of the group $P(B_1/E_1)_2 \simeq (\mathbb{Z}/2\mathbb{Z})^{\oplus 4}$ of points of order 2 in $P(B_1/E_1)$. Remark, that $\varphi(P(B_1/E_1)_2) = \{0, \mathcal{D}\}$, and that the Richelot isogeny $R: J(C_1)\longrightarrow J(C_2)$ transforms $\mathcal{D}$ into 0. Hence $P(B_1/E_1)_2 \subset \ker(R\circ\varphi)$. The composition $R\circ\varphi$ thus factors through the multiplication by 2 on $P(B_1/E_1)$.

Now, similarly, consider the spectral curve $B=B_2$ from the previous section and complete it to the diagram (2) in using formulas, analogous to (9), (10). We are using the same characters for the notation of variables, but mark them with tildes. We obtain:

$$C_2' = \{\tilde\eta^2 = \tilde\xi(\tilde\xi^2 - 2H\tilde\xi + \frac{1}{4}I_2)(\tilde\xi^2 - 2H\tilde\xi + \frac{1}{4}I_2 - 1)\}\ , \qquad (11)$$

$$\begin{cases} \tilde u = \sqrt{\tilde\xi} \\ \tilde\mu = \frac{2\tilde\eta}{\sqrt{\tilde\xi}} \end{cases} , \qquad \begin{cases} \tilde\xi = \tilde u^2 \\ \tilde\eta = \frac{\tilde\mu\tilde u}{2} \end{cases} , \qquad (12)$$

$$B_2 = \{\tilde\mu^2 = 4(\tilde u^4 - 2H\tilde u^2 + \tilde\kappa - \frac{1}{2})(\tilde u^4 - 2H\tilde u^2 + \tilde\kappa + \frac{1}{2})\}\ , \qquad (13)$$

The change of the equation of B_2 from the form (8) to (13) is given by $\tilde\mu = \lambda + 2(u^4 - 2Hu^2 + \tilde\kappa), \tilde u = u$. The equation (11) differs from the BRS equation (7) only by the signs of terms with H. We can transform one to the other by the change of variables $y = i\tilde\eta, x = -\tilde\xi$; thus $C_2' \simeq C_2$. The equation of E_2 is obtained by the substitution of $\tilde u^2 = \tilde\xi$ in (13). We can repeat all the above arguments in replacing (1) (9), (10) by (2), (11)-(13).

We summarize the relations between the above curves in the following statement.

Theorem 3. *Let the curves B_i, C_i, E_i and the maps $f, \tilde f, \varphi, \tilde\varphi$ be defined as above. Then we have the following sequence of isogenies*

$$P(B_1/E_1) \xrightarrow{f_*} J(C_1) \xrightarrow{Richelot} J(C_2) \xrightarrow{\tilde f^*} P(B_2/E_2)$$

of degrees respectively $8, 4, 2$, their kernels being direct sums of copies of $\mathbb{Z}/2\mathbb{Z}$. The following assertions are verified:

(i) The composition $P(B_1/E_1) \xrightarrow{f_*} J(C_1) \xrightarrow{Richelot} J(C_2)$ is an isogeny of degree 32 whose kernel contains all the points of order 2 of $P(B_1/E_1)$. Hence it factors through the multiplication by 2 on $P(B_1/E_1)$, and there exists an isogeny $P(B_1/E_1)\longrightarrow J(C_2)$ of degree 2.

(ii) The composition of all the three isogenies is an isogeny of degree 64 which factors through the multiplication by 2, hence there exists also an isogeny of order 4 $P(B_1/E_1) \longrightarrow P(B_2/E_2)$ with kernel $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

The same assertions hold for the maps in the opposite direction.

(iii) The elliptic curves E_1, E_2 are non-isogeneous for generic constants of motion I_2 and H.

The fact that the elliptic curves are non-isogeneous for generic constants of motion follows from the observation that if we fix the value of I_2 and vary H, then the j-invariant of E_1 will be constant, but that of E_2 will vary. We can obtain a generic curve E_2 by such a variation in keeping E_1 fixed, hence they are non-isogeneous.

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