ON FROBENIUS TRACES

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ABSTRACT. In this paper we discuss one diophantine property of the Frobenius traces associated to an abelian variety over a number field k and give its application to the proof of the Mumford - Tate conjecture for $4p$-dimensional abelian variety J over k, where p is a prime number, $p \geq 17$, $\mathrm{Cent}(\mathrm{End}(J \otimes \overline{k})) = \mathbb{Z}$ or (under some weak assumptions) $\mathrm{End}^0(J \otimes \overline{k})$ is an imaginary quadratic extension of $\mathbb{Q}$.

§0. INTRODUCTION

0.1. Let J be an abelian variety over a number field $k \subset \mathbb{C}$, $[k : \mathbb{Q}] < \infty$. Suppose that l is a prime number,

$$\rho_l : \mathrm{Gal}(\overline{k}/k) \to \mathrm{GL}(H^1_{et}(J \otimes \overline{k}, \mathbb{Q}_l))$$

is the natural l-adic representation.

It is well known that ρ_l is unramified outside a finite set T of non-Archimedean places of k. We denote by $F_{\overline{v}} \in \mathrm{Gal}(\overline{k}/k)$ the Frobenius element associated with a place $\overline{v}$ of $\overline{\mathbb{Q}}$ lying over an unramified place v of k. It is well known that the conjugacy class of $\rho_l(F_{\overline{v}}^{-1})$ depends only on v, the characteristic polynomial of $\rho_l(F_{\overline{v}}^{-1})$ lies in $\mathbb{Z}[t] \subset \mathbb{Q}_l[t]$, and all its roots are of absolute value $(\mathrm{Norm}_{k/\mathbf{Q}}(v))^{1/2}$.

Let S be a set of non-Archimedean places of k. We recall that the Dirichlet density of S in the set of all non-Archimedean places of k is defined as

$$\lim_{x \to \infty} \frac{\log x}{x} \mathrm{Card}\{v \in S \mid \mathrm{Norm}_{k/\mathbf{Q}}(v) \leq x\}$$

(if such limit exists). It is well known that the density of $\{v \mid \mathrm{Norm}_{k/\mathbf{Q}}(v) = p_v\}$ equals 1 [4, ch.8, sect.2.4].

The following result is well known.

0.2. **N. Katz theorem** [6,sect.2.1], [13,sect.2.7.1]. *Assume that for some natural number n $l^n > 2 \cdot \dim_k J$ and each l^n-torsion point of $J(\overline{k})$ is rational over k. Then the set $\{ v \mid v$ is unramified place of k, $\mathrm{Norm}_{k/\mathbf{Q}}(v) = p_v$ is a prime number and p_v does not divide $\mathrm{Tr}(\rho_l(F_{\overline{v}}^{-1}))\}$ is of density 1.*

This theorem plays an important role in the proof of the following theorem.

0.3. **J.-P. Serre theorem**[6,sect.6]. *Let J be a simple abelian variety over a number field k . If $\dim_k J$ is an odd integer and $\mathrm{End}(J \otimes \overline{k}) = \mathbb{Z}$, then the Hodge* [9],[10] *Tate* [21] *and Mumford - Tate conjectures* [11] *hold for J.*

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The survey of Serre's technique is contained in [6].

We want to extend Serre theorem into the area of even dimensions.

Let Δ be the set of all eigenvalues of $\rho_l(F_{\overline{v}}^{-1})$ (without counting multiplicities). The Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts in a natural way on Δ and on $\Delta \cdot \Delta$. For each element $\eta \in \Delta \cdot \Delta$ we define a map $T_\eta : \Delta \to \overline{\mathbb{Q}}^\times$ by the formula $T_\eta(\delta) = \eta\delta^{-1}$. This map is a modification of the corresponding map $T_\gamma^0 : \Delta \to \overline{\mathbb{Q}}^\times$ in [6,sect.5.2], which is defined by the formula $T_\gamma^0(\delta) = \gamma^2\delta^{-1}$ for $\gamma \in \Delta$. It is evident that for each $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

$$\mathrm{Card}(T_{\sigma(\eta)}(\Delta) \cap \Delta) = \mathrm{Card}(T_{\sigma(\eta)}(\sigma(\Delta)) \cap \sigma(\Delta)) = \mathrm{Card}(T_\eta(\Delta) \cap \Delta),$$

and hence for any constant c the set

$$\{\eta \in \Delta \cdot \Delta | \mathrm{Card}(T_\eta(\Delta) \cap \Delta) = c\} \textit{ is } \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) - \textit{invariant.} \qquad (0.3.1)$$

So we have a good instrument of computing the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant subsets of $\Delta \cdot \Delta$, which is used, for example, in [18],[19],[20].

On the other hand, some (not all) elements of $\Delta \cdot \Delta$ are the eigenvalues of $\rho_l^{\wedge 2}(F_{\overline{v}}^{-1})$, where

$$\rho_l^{\wedge 2} : \mathrm{Gal}(\overline{k}/k) \to \mathrm{GL}(H^2_{et}(J \otimes \overline{k}, \mathbb{Q}_l)) = \mathrm{GL}(\wedge^2 H^1_{et}(J \otimes \overline{k}, \mathbb{Q}_l))$$

is the natural l-adic representation. Hence there is a reason to look at the trace of $\rho_l^{\wedge 2}(F_{\overline{v}}^{-1})$.

0.4. **Theorem**. *Let J be a d-dimensional abelian variety over a number field k. Assume that for some natural number n $l^n > d(2d-1)$, each l^n-torsion point of $J(\overline{k})$ is rational over k and k contains all the $(l^n)^{th}$ roots of unity. If $d \geq 2$ then the set $\{ v \mid v$ is unramified place of k, $\mathrm{Norm}_{k/\mathbb{Q}}(v) = p_v$ is a prime number and p_v does not divide $\mathrm{Tr}(\rho_l^{\wedge 2}(F_{\overline{v}}^{-1}))\}$ is of density 1.*

The main idea of the proof is similar to the idea which is used in the proof of A.Ogus result concerning the existence of many ordinary reductions of an abelian surface over a number field [13, sect.2.7-2.10]. Of course, we use G.Faltings theorems [8].

Assume that J has a good reduction J_v at the non-Archimedean place v of k. It is evident that $\mathrm{Tr}(\rho_l^{\wedge 2}(F_{\overline{v}}^{-1}))$ coincides with the trace of linear operator in the 2nd homology of J_v induced by the Frobenius endomorphism of abelian variety J_v over a prime field $\mathbb{F}_{p_v}$.

0.5. We recall that J has an ordinary reduction at a non-Archimedean place v of k with a residue field $k(v) = \mathbb{F}_{q_v}$ of characteristic $p_v \Leftrightarrow$ the special fibre J_v of the Neron minimal model of J is an abelian variety and the following equivalent conditions hold:

(0.5.1) p_v-rank of J_v equals $\dim_{k(v)} J_v$;

(0.5.2) for any eigenvalue δ of the Frobenius endomorphism of l-adic Tate module $T_l(J_v \otimes_{k(v)} \overline{k(v)})(l \neq p_v)$ and for any place w of $\overline{\mathbb{Q}}$ over p_v the following relation holds:

$$\frac{w(\delta)}{w(q_v)} \in \{0, 1\}$$

[7,sect.2].

0.6. **Definition**. An abelian variety J over a number field k has many ordinary reductions $\Leftrightarrow$ there exists a set S of non-Archimedean places of k such that J has an ordinary reduction at each place $v \in S$ and the density of S is *positive*.

Now we consider some consequences of the theorem 0.4. We denote by (Lie $\mathrm{Im}(\rho_l))^{ss}$ the semisimple part of the reductive Lie algebra Lie $\mathrm{Im}(\rho_l)$. The following result is initially proved in [20, th.1.14, 1.16, 1.17] under the additional assumption that J has many ordinary reductions.

0.7. **Theorem**. *Assume that p is a prime number. Let J be a simple 4p-dimensional abelian variety over a number field k.*

Assume that $\mathrm{End}(J \otimes \overline{k}) = \mathbb{Z}$ *and* $p \geq 17$. *Then* (Lie $\mathrm{Im}(\rho_l))^{ss} \otimes \overline{\mathbb{Q}_l}$ *is a semisimple Lie algebra of type* C_{4p}, *the general Hodge, Tate and Mumford - Tate conjectures hold for J.*

Assume that $\mathrm{End}^0(J \otimes \overline{k})$ *is an indefinite quaternion division algebra over* $\mathbb{Q}$ *and* $p \neq 2$, $p \neq 5$. *Then* (Lie $\mathrm{Im}(\rho_l))^{ss} \otimes \overline{\mathbb{Q}_l}$ *is a semisimple Lie algebra of type* C_{2p}, *the general Hodge, Tate and Mumford - Tate conjectures hold for J.*

Assume that $\mathrm{End}^0(J \otimes \overline{k})$ *is a definite quaternion division algebra over* $\mathbb{Q}$ *and* $p \neq 2$. *Then* (Lie $\mathrm{Im}(\rho_l))^{ss} \otimes \overline{\mathbb{Q}_l}$ *is a semisimple Lie algebra of type* D_{2p}, *the Mumford - Tate conjecture holds for J.*

0.8. Assume that $\mathrm{End}^0(J \otimes \overline{k}) = K$ is an imaginary quadratic extension of $\mathbb{Q}$. Then $K \otimes_{\mathbb{Q}} \mathbb{C}$ is the direct sum of two copies of $\mathbb{C}$ indexed by two different embeddings σ, τ of the field K to $\mathbb{C}$, $\mathrm{Lie}(J_{\mathbb{C}})=M_\sigma \oplus M_\tau$, where $M_\sigma = \{v \in \mathrm{Lie}(J_{\mathbb{C}}) \mid e \cdot v = \sigma(e)v \text{ for all } e \in K\}$, $M_\tau = \{v \in \mathrm{Lie}(J_{\mathbb{C}}) \mid e \cdot v = \tau(e)v \text{ for all } e \in K\}$, $\dim_k J = n_\sigma + n_\tau$, $n_\sigma = \dim_{\mathbb{C}} M_\sigma \geq 1$, $n_\tau = \dim_{\mathbb{C}} M_\tau \geq 1$ [16, th.5]. The following result is a strong variant of [20, th.1.18].

0.9. **Theorem**. *Assume that p is a prime number,* $p \geq 17$. *Let J be an absolutely simple 4p-dimensional abelian variety over a number field k. Assume that* $\mathrm{End}^0(J \otimes \overline{k})$ *is an imaginary quadratic extension of* $\mathbb{Q}$. *Then* (Lie $\mathrm{Im}(\rho_l))^{ss} \otimes \overline{\mathbb{Q}_l}$ *is a semisimple Lie algebra of type* A_{2p-1} *or* $A_3 \times A_{p-1}$. *If J has many ordinary reductions, then* (Lie $\mathrm{Im}(\rho_l))^{ss} \otimes \overline{\mathbb{Q}_l}$ *is a semisimple Lie algebra of type* A_{2p-1}, *there exists a canonical isomorphism of semisimple parts*

$$[\mathrm{Lie}\ \mathrm{Im}(\rho_l)]^{ss} \simeq [\mathrm{Lie}[\mathrm{MT}(J_{\mathbb{C}})(\mathbb{Q}_l)]]^{ss};$$

moreover, for $n_\sigma \neq 2p$ *it extends to an isomorphism*

$$\mathrm{Lie}\ \mathrm{Im}(\rho_l) \simeq \mathrm{Lie}[\mathrm{MT}(J_{\mathbb{C}})(\mathbb{Q}_l)]$$

and in this case the $\mathbb{Q}_l$*-space* $H^{2r}_{et}(J \otimes \overline{k}, \mathbb{Q}_l(r))^{\mathrm{Gal}(\overline{k}/k)}$ *is spanned by the cohomology classes of intersections of divisors.*

§1. On the Frobenius trace in the 2nd homology of an abelian variety

1.1. We give here a proof of theorem 0.4.

Assume that $\mathrm{Tr}(\rho_l^{\wedge 2}(F_{\overline{v}}^{-1})) = p_v \cdot b_v$, where b_v is an integer. By A.Weil theorem [12, ch.4, §21, th.4] the absolute value of b_v is less than or equal to

$$d(2d-1) = \dim_{\mathbb{Q}_l} H^2_{et}(J \otimes \overline{k}, \mathbb{Q}_l).$$

Moreover, we may assume that p_v is unramified in k. From the relation $\mathrm{Norm}_{k/\mathbf{Q}}(v) = p_v$ it follows that p_v splits completely in k. Hence we have

$$k \otimes_{\mathbf{Q}} \mathbb{Q}_{p_v} \simeq \mathbb{Q}_{p_v} \times \ldots \times \mathbb{Q}_{p_v}. \tag{1.1.1}$$

By the condition of the theorem k containes all the $(l^n)^{th}$ roots of unity. On the other hand it is well known that $(\mathbb{Q}_{p_v}^{\times})_{tors} \simeq \mathbb{Z}/(p_v - 1)\mathbb{Z}$. It follows from (1.1.1) that $l^n | (p_v - 1)$ and hence

$$p_v = 1(\mathrm{mod}\ l^n). \tag{1.1.2}$$

By the condition of the theorem all the l^n-torsion points of $J(\overline{k})$ are rational over k. It follows that $\rho_l|_{\mathrm{Gal}(\overline{k}/k)}$ is trivial mod l^n and hence $\rho_l^{\wedge 2}|_{\mathrm{Gal}(\overline{k}/k)}$ is also trivial mod l^n. So

$$p_v \cdot b_v = d(2d-1)(\mathrm{mod}\ l^n). \tag{1.1.3}$$

It is clear that the relations (1.1.1)-(1.1.3) and the inequality $l^n > d(2d-1)$ imply the relations

$$d(2d-1)(\mathrm{mod}\ l^n) = b_v(\mathrm{mod}\ l^n),$$

$$b_v = d(2d-1),$$

$$\mathrm{Tr}(\rho_l^{\wedge 2}(F_{\overline{v}}^{-1})) = p_v \cdot b_v = p_v \cdot d(2d-1). \tag{1.1.4}$$

This trace is the sum of $d(2d-1)$ complex numbers of absolute value p_v. Hence the relation (1.1.4) implies that each number actually equals p_v and all eigenvalues of $\rho_l^{\wedge 2}(F_{\overline{v}}^{-1})$ are equal to p_v. In this situation

$$\mathrm{Tr}(\rho_l^{\wedge 2}(F_{\overline{v}}^{-1})) = d(2d-1)\mathrm{Tr}(\chi_l^{-1}(F_{\overline{v}}^{-1})), \tag{1.1.5}$$

where

$$\chi_l : \mathrm{Gal}(\overline{k}/k) \to \mathbb{Z}_l^{\times}$$

is the cyclotomic character, defined by the natural action of $\mathrm{Gal}(\overline{k}/k)$ on the 1-dimensional Tate module $\mathbb{Z}_l(1) = T_l(\mu)$ attached to the group of l-power of roots of unity in $\overline{k}$.

1.2. Fix a k-polarization on J once for all. We denote by

$$\Psi : H^1_{et}(J \otimes \overline{k}, \mathbb{Q}_l) \times H^1_{et}(J \otimes \overline{k}, \mathbb{Q}_l) \to \mathbb{Q}_l(-1)$$

the induced nondegenerate alternating form on $H^1_{et}(J \otimes \overline{k}, \mathbb{Q}_l)$. Since the Weil pairing is known to be $\mathrm{Gal}(\overline{k}/k)$-equivariant, one has

$$\Psi(\sigma x, \sigma y) = \chi_l^{-1}(\sigma) \cdot \Psi(x, y)$$

for all $\sigma \in \mathrm{Gal}(\overline{k}/k); x, y \in H^1_{et}(J \otimes \overline{k}, \mathbb{Q}_l)$. This relation implies $\mathrm{Ker}(\rho_l) \subset \mathrm{Ker}(\chi_l)$. Hence we may consider χ_l as a character of $\mathrm{Im}(\rho_l)$. So

$$f(s) = \mathrm{Tr}(s^{\wedge 2}) - d(2d-1)\mathrm{Tr}(\chi_l^{-1}(s))$$

is an analytic function on the compact l-adic Lie group $\mathrm{Im}(\rho_l)$. In virtue of (1.1.5) this function vanishes on the set $\{\rho_l(F_{\overline{v}}^{-1})|v$ is unramified place of k, $\mathrm{Norm}_{k/\mathbf{Q}}(v) = p_v$ is a prime number and p_v divides $\mathrm{Tr}(\rho_l^{\wedge 2}(F_{\overline{v}}^{-1}))\}$.

Assume that the zero locus Z_f of f is the group $\mathrm{Im}(\rho_l)$. From G.Faltings theorems [8] it follows that ρ_l and $\rho_l^{\wedge 2}$ are semisimple representations. Since two semisimple representations in characteristic 0 are isomorphic if and only if they have the same trace, we have the relations $\rho_l^{\wedge 2} = (\chi_l^{-1})^{\oplus d(2d-1)}$ and

$$\dim_{\mathbf{Q}_l} H^2_{et}(J \otimes \overline{k}, \mathbf{Q}_l(1))^{\mathrm{Im}(\rho_l)} = d(2d-1). \tag{1.2.1}$$

On the other hand, by G.Faltings results [8]

$$\mathrm{End}_{\mathrm{Im}(\rho_l)} H^1_{et}(J \otimes \overline{k}, \mathbf{Q}_l) \simeq \mathrm{End}(J) \otimes \mathbf{Q}_l.$$

By the well known Tate theorems [22, th.3, th.4] this relation implies

$$H^2_{et}(J \otimes \overline{k}, \mathbf{Q}_l(1))^{\mathrm{Im}(\rho_l)} \simeq \mathrm{NS}(J) \otimes \mathbf{Q}_l. \tag{1.2.2}$$

It follows from (1.2.1)-(1.2.2) that the $\mathbf{Q}_l$-space $H^2_{et}(J \otimes \overline{k}, \mathbf{Q}_l(1))$ is generated by algebraic cohomology classes. Since $\mathrm{rank}(\mathrm{NS}(J \otimes_k \mathbf{C})) \leq h^{1,1}(J \otimes_k \mathbf{C})$, we have the equality $h^{2,0}(J \otimes_k \mathbf{C}) = 0$. This is possible only if J is an elliptic curve.

In our situation $\dim_k J \geq 2$. Hence the zero locus Z_f of f is an analytic hypersurface on the compact l-adic Lie group $\mathrm{Im}(\rho_l)$ and it is stable by conjugation. Let μ be the Haar measure on $\mathrm{Im}(\rho_l)$ such that the total mass of $\mathrm{Im}(\rho_l)$ is 1. It is well known that $\mu(Z_f) = 0$ [15, ch.1, sect.2.2, exercise]. Hence by Chebotarev density theorem the set $\{v|v$ is unramified place of k, $\mathrm{Norm}_{k/\mathbf{Q}}(v) = p_v$ is a prime number and $\rho_l(F_{\overline{v}}^{-1}) \in Z_f\}$ is of density 0. Hence $\{v|v$ is unramified place of k, $\mathrm{Norm}_{k/\mathbf{Q}}(v) = p_v$ is a prime number and p_v divides $\mathrm{Tr}(\rho_l^{\wedge 2}(F_{\overline{v}}^{-1}))\}$ is of density 0. This proves the claim.

1.3. **Corollary**. *Let J be a d-dimensional abelian variety over a number field k. Assume that for some natural number n $l^n > d(2d-1)$, each l^n-torsion point of $J(\overline{k})$ is rational over k and k contains all the $(l^n)^{th}$ roots of unity. If $d \geq 2$ then the set $\{ v \mid v$ is unramified place of k, $\mathrm{Norm}_{k/\mathbf{Q}}(v) = p_v$ is a prime number and p_v does not divide the trace of linear operator in the 2nd homology of J_v induced by the Frobenius endomorphism of abelian variety J_v over a prime field $\mathbb{F}_{p_v}\}$ is of density 1.*

§2. Cycles on simple abelian variety of dimension 4p over a number field

2.1. We give here the proof of theorem 0.7.

Suppose that $\mathrm{End}(J \otimes \overline{k}) = \mathbb{Z}$ and p$\geq$ 17. Let $\mathbb{N}^+ = \{1, 2, ...\}$ be the set of all positive natural numbers. First of all we introduce the set of *exceptional* numbers

$$Ex(1) = \{4^l, \frac{1}{2}\binom{4l+2}{2l+1}^{2m-1}, 2^{8lm+4l-4m-3}, 4^l(m+1)^{2l+1} \mid l, m \in \mathbb{N}^+\} =$$

={4,10,16,32,64,108,126,256,500,512,864,1024,1372,1716,2048,...}. It is easy to verify that for each prime number p the number $4p$ is not exceptional [20, lemma

4.3]. Hence Lie $\mathrm{Hg}(J_{\mathbb{C}}) \otimes_{\mathbb{Q}} \mathbb{C} = sp_{8p}$ and the general Hodge conjecture holds for $J_{\mathbb{C}} \times \ldots \times J_{\mathbb{C}}$, where $\mathrm{Hg}(J_{\mathbb{C}})$ is the Hodge group of the complex abelian variety $J_{\mathbb{C}} = J \otimes_k \mathbb{C}$ [19, th.1.1].

We also recall the folowing classification result.

2.2. **Theorem**[17, th.2], [18, th.2.2], [19, th.4.1], [20, th.2.2]. *Assume that g is a simple Lie algebra of rank m over an algebraically closed field of characteristic zero, $\omega_1, \omega_2, \ldots, \omega_m$ are fundamental weights, $E = E(n_1\omega_1 + \ldots + n_m\omega_m)$ is an irreducible g-module with the highest weight $n_1\omega_1 + \ldots + n_m\omega_m$, where $n_i \in \mathbb{N}$.*

Let p be a prime number.

Suppose that $\deg E = p$.

If E is an orthogonal representation, then the triple (type of g, E, p) assumes the following values:

$$(A_1, E((p-1)\omega_1), p)(p \geq 3); \tag{2.2.1}$$

$$(B_{(p-1)/2}, E(\omega_1), p)(p \geq 5); \tag{2.2.2}$$

$$(G_2, E(\omega_1), 7). \tag{2.2.3}$$

Moreover, the highest weight of g-module E is a radical weight.

If E is a symplectic representation, then the triple (type of g, E, p) assumes the following value:

$$(A_1, E(\omega_1), 2). \tag{2.2.4}$$

If $E \neq E^*$, then the triple (type of g, E,p) assumes the following values:

$$(A_{p-1}, E(\omega_1), p), (A_{p-1}, E(\omega_{p-1}), p)(p \geq 3). \tag{2.2.5}$$

Suppose that $\deg E = 2p$.

If E is an orthogonal representation, then the triple (type of g, E, p) assumes the following values:

$$(B_2, E(2\omega_2), 5); \tag{2.2.6}$$

$$(B_2, E(2\omega_1), 7); \tag{2.2.7}$$

$$(C_3, E(\omega_2), 7); \tag{2.2.8}$$

$$(G_2, E(\omega_2), 7); \tag{2.2.9}$$

$$(F_4, E(\omega_4), 13); \tag{2.2.10}$$

$$(D_p, E(\omega_1), p)(p \geq 3). \tag{2.2.11}$$

Moreover, in $(2.2.6)-(2.2.10)$ *the highest weight of g-module E is a radical weight.*

If E is a symplectic representation, then the triple (type of g, E,p) assumes the following values:

$$(A_1, E((2p-1)\omega_1), p); \tag{2.2.12}$$

$$(C_3, E(\omega_3), 7); \tag{2.2.13}$$

$$(C_p, E(\omega_1), p). \tag{2.2.14}$$

If $E \neq E^*$, then the triple (type of g, E, p) assumes the following values:

$$(A_2, E(2\omega_1), 3), (A_2, E(2\omega_2), 3); \tag{2.2.15}$$

$$(A_2, E(3\omega_1), 5), (A_2, E(3\omega_2), 5); \tag{2.2.16}$$

$$(A_3, E(2\omega_1), 5), (A_3, E(2\omega_3), 5); \tag{2.2.17}$$

$$(A_4, E(\omega_2), 5), (A_4, E(\omega_3), 5); \tag{2.2.18}$$

$$(A_{2p-1}, E(\omega_1), p), (A_{2p-1}, E(\omega_{2p-1}), p). \tag{2.2.19}$$

Suppose that $\deg E = 4p$.

If E is an orthogonal representation, then the triple (type of g, E, p) assumes the following values:

$$(A_2, E(\omega_1 + \omega_2), 2); \tag{2.2.20}$$

$$(A_3, E(2\omega_2), 5); \tag{2.2.21}$$

$$(B_3, E(\omega_3), 2); \tag{2.2.22}$$

$$(B_4, E(2\omega_1), 11); \tag{2.2.23}$$

$$(C_5, E(\omega_2), 11); \tag{2.2.24}$$

$$(D_4, E(\omega_2), 7); \tag{2.2.25}$$

$$(D_4, E(\omega_3), 2); \tag{2.2.26}$$

$$(D_4, E(\omega_4), 2); \tag{2.2.27}$$

$$(F_4, E(\omega_1), 13); \tag{2.2.28}$$

$$(D_{2p}, E(\omega_1), p). \tag{2.2.29}$$

Moreover, in (2.2.20), (2.2.21), (2.2.23), (2.2.24), (2.2.25), (2.2.28) *the highest weight of g-module E is a radical weight.*

If E is a symplectic representation, then the triple (type of g, E,p) assumes the following values:

$$(A_1, E((4p-1)\omega_1), p); \tag{2.2.30}$$

$$(A_5, E(\omega_3), 5); \tag{2.2.31}$$

$$(B_2, E(3\omega_2), 5); \tag{2.2.32}$$

$$(C_{2p}, E(\omega_1), p). \tag{2.2.33}$$

If $E \neq E^*$, then the triple (type of g, E,p) assumes the following values:

$$(A_2, E(6\omega_1), 7), (A_2, E(6\omega_2), 7); \tag{2.2.34}$$

$$(A_3, E(\omega_1 + \omega_2), 5), (A_3, E(\omega_2 + \omega_3), 5); \tag{2.2.35}$$

$$(A_3, E(3\omega_1), 5), (A_3, E(3\omega_3), 5); \tag{2.2.36}$$

$$(A_6, E(2\omega_1), 7), (A_6, E(2\omega_6), 7); \tag{2.2.37}$$

$$(A_7, E(\omega_2), 7), (A_7, E(\omega_6), 7); \tag{2.2.38}$$

$$(A_{4p-1}, E(\omega_1), p), (A_{4p-1}, E(\omega_{4p-1}), p). \tag{2.2.39}$$

This theorem follows from H.Weyl formula [3, ch.8].

2.3. Let G_{V_l} be the algebraic envelope of $\mathrm{Im}(\rho_l) \subset \mathrm{GL}(V_l)$, where

$$V_l = H^1_{et}(J \otimes \overline{k}, \mathbb{Q}_l).$$

By F.A.Bogomolov theorem [1] Lie $\mathrm{Im}(\rho_l)$=Lie(G_{V_l}) and G_{V_l} containes the group G_m of homotheties. By G.Faltings theorems [8] G_{V_l} is reductive and

$$\mathrm{End}_{G_{V_l}}(V_l) = \mathrm{End}(J) \otimes \mathbb{Q}_l.$$

Let g_l=Lie $\mathrm{Im}(\rho_l)$. We shall denote by g_l^{ss} the semisimple part of g_l. By J.-P.Serre theorem [6,th.3.10] the rank of G_{V_l} (resp.g_l) is independent of l. In the case under consideration we may assume that $G_{V_l}=S_{V_l}\cdot\mathrm{G}_m$, where $S_{V_l}=[G_{V_l}, G_{V_l}]$ is the commutator subgroup of G_{V_l} [6, sect.1.2.2b].

2.4. Assume that v is a non-Archimedean place of k at which J has a good reduction. Let $\overline{v}$ be any extension of v to $\overline{k}$ and let $F_{\overline{v}} \in \mathrm{Gal}(\overline{k}/k)$ be the corresponding Frobenius element. It is well known that the characteristic polynomial of $\rho_l(F_{\overline{v}}^{-1})$ coincides with the characteristic polynomial of the Frobenius endomorphism π_v of the reduction J_v of J at v. We denote by Δ the set of all eigenvalues of $\rho_l(F_{\overline{v}}^{-1})$ (without counting multiplicities). Let Γ_v be a multiplicative subgroup of $\overline{\mathbb{Q}}^{\times}$ generated by Δ.

It is well known that $\mathbb{Q}[\pi_v]=\prod K_i$, K_i are number fields. The multiplicative group $\mathbb{Q}[\pi_v]^{\times}$ defines a $\mathbb{Q}$-torus $T_{\pi_v} = \prod R_{K_i/\mathbb{Q}}(\mathrm{G}_{mK_i})$, where $R_{K_i/\mathbb{Q}}$ are the Weil restrictions of scalar functors. Let H_v be the smallest algebraic subgroup of T_{π_v} defined over $\mathbb{Q}$, such that $\pi_v \in H_v(\mathbb{Q})$. As is well-known, H_v is a group of multiplicative type. The connected component of the identity in H_v is called the *Frobenius torus* T_v. It can be regarded as the $\mathbb{Q}$-model of the connected component of 1 in the Zariski closure of the set $\{\rho_l(F_{\overline{v}}^{-1})^n | n \in \mathbb{Z}\}$ in G_{V_l} [6, sect.3b].

2.5. As an easy consequence of the theorem 0.4 and [6, prop.3.6, 5.2.1, lemma 2.1, cor.3.8] we have the following result.

After replacing k by some finite extension we may assume that for some set S of density 1 in the set of all non-Archimedean places of k and for each $v \in S$ the following conditions hold:

1) *for a fixed integer $n \geq 2$ such that $l^n > (2\dim_k J)^2$, the l^n-torsion points of $J(\overline{k})$ are rational points over k and k containes all the $(l^n)^{th}$ roots of unity;*

2) $p_v = \mathrm{char}(k(v)) > (2\dim_k J)^2$;

3) $\mathrm{Norm}_{k/\mathbb{Q}}(v) = p_v$;

4) *the Frobenius traces* $\mathrm{Tr}(\rho_l(F_{\overline{v}}^{-1}))$ *and* $\mathrm{Tr}(\rho_l^{\wedge 2}(F_{\overline{v}}^{-1}))$ *are not divisible by* p_v;

5) Γ_v *is torsion-free,* G_{V_l} *is connected and* $\rho_l(F_{\overline{v}}^{-1}) \in T_v(\overline{\mathbb{Q}_l})$;

6) *the Frobenius torus* T_v *is a maximal torus of* G_{V_l} *and*

$$\mathrm{rank}(\Gamma_v) = \dim(T_v) = \mathrm{rank}(G_{V_l}).$$

2.6. It is well known that $V_l \otimes \overline{\mathbb{Q}_l}$ is an absolutely irreducible symplectic $g_l^{ss} \otimes \overline{\mathbb{Q}_l}$-module.

Assume that the Lie algebra $g_l^{ss} \otimes \overline{\mathbb{Q}_l}$ is simple. From the relation $\dim_k J = 4p \notin Ex(1)$ it follows that $g_l^{ss} \otimes \overline{\mathbb{Q}_l}$ is the Lie algebra of type C_{4p} [18, sect.1.3-1.8].

On the other hand, Lie $\mathrm{Hg}(J_{\mathbf{C}})\otimes\overline{\mathbb{Q}_l} \subset sp(V_l \otimes \overline{\mathbb{Q}_l})$. By Piatetski-Shapiro -Deligne - Borovoi theorem [14], [2] there exists a canonical embedding

$$\text{Lie Im}(\rho_l) \subset \text{Lie}[\mathrm{MT}(J_{\mathbf{C}})(\mathbb{Q}_l)] = \mathbb{Q}_l \times \text{Lie}[\mathrm{Hg}(J_{\mathbf{C}})(\mathbb{Q}_l)].$$

So there exists a canonical isomorphism of Lie algebras

$$\text{Lie Im}(\rho_l) \simeq \text{Lie}[\mathrm{MT}(J_{\mathbf{C}})(\mathbb{Q}_l)].$$

This relation implies the equivalence of the usual Hodge conjecture for $J_{\mathbf{C}}$ and the Tate conjecture for J.

2.7. Now we may assume that the Lie algebra $g_l^{ss} \otimes \overline{\mathbb{Q}_l}$ is not simple.

Let $f : S \to S_{V_l} \otimes \overline{\mathbb{Q}_l}$ be the universal covering, where $S = S_1 \times S_2 \times ... \times S_q$ is a product of simple simply connected algebraic $\overline{\mathbb{Q}_l}$-groups. An isogeny f extends to an isogeny

$$f : \mathrm{G}_m \times S_1 \times ... \times S_q \to \mathrm{G}_m \cdot (S_{V_l} \otimes \overline{\mathbb{Q}_l}) = G_{V_l} \otimes \overline{\mathbb{Q}_l},$$

defined by the formula $f((a,s)) = a \cdot f(s)$ for $a \in \mathrm{G}_m, s \in S_1 \times ... \times S_q$.

By (2.5.6) the Frobenius torus T_v is a maximal torus of G_{V_l}. Hence

$$T = (f^{-1}(T_v \otimes \overline{\mathbb{Q}_l}))^0 \subset \mathrm{G}_m \times S_1 \times ... \times S_q$$

is a maximal subtorus. Consider the canonical projections

$$\mathrm{pr}_0 : \mathrm{G}_m \times S_1 \times ... \times S_q \to \mathrm{G}_m$$

$$\mathrm{pr}_i : \mathrm{G}_m \times S_1 \times ... \times S_q \to S_i.$$

It is evident that $T = \mathrm{pr}_0(T) \times \mathrm{pr}_1(T) \times ... \times \mathrm{pr}_q(T)$.

On the other hand,

$$V_l \otimes \overline{\mathbb{Q}_l} = W_1 \otimes ... \otimes W_q,$$

where W_1 is an irreducible $\mathrm{G}_m \times S_1$-module, W_2 is an irreducible S_2-module,...,W_q is an irreducible S_q-module. Let

$$\rho_1 : \mathrm{G}_m \times S_1 \to \mathrm{GL}(W_1),$$

$$\rho_i : S_i \to \mathrm{GL}(W_i)(i \geq 2)$$

are the corresponding representations. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{G}_m \times S_1 \times ... \times S_q & \stackrel{\rho_1 \otimes ... \otimes \rho_q}{\longrightarrow} & \mathrm{GL}(W_1 \otimes ... \otimes W_q) \\ \downarrow f & & \| \\ G_{V_l} \otimes \overline{\mathbb{Q}_l} & \subset & \mathrm{GL}(W_1 \otimes ... \otimes W_q) \end{array}$$

By (2.5.5) $\rho_l(F_{\overline{v}}^{-1}) \in T_v(\overline{\mathbb{Q}_l})$, hence there exists an element

$$\tau_{\overline{v}} = (\tau_0, \tau_1, ..., \tau_q) \in \mathrm{pr}_0(T) \times \mathrm{pr}_1(T) \times ... \times \mathrm{pr}_q(T)$$

such that

$$(\rho_1 \otimes ... \otimes \rho_q)(\tau_{\overline{v}}) = f(\tau_{\overline{v}}) = \rho_l(F_{\overline{v}}^{-1}).$$

We see that each eigenvalue of $\rho_l(F_{\overline{v}}^{-1})$ is of the form $\chi_0^{(0)}(\tau_0) \cdot \chi_i^{(1)}(\tau_1)...\chi_j^{(q)}(\tau_q)$, where $\chi_k^{(m)} \in \mathrm{X}(\mathrm{pr}_m(T))$ are some characters.

2.8. By (2.5.1) $\mathrm{Im}(\rho_l) \subset \{x \in \mathrm{End}\ T_l(J \otimes \overline{k}) \mid x \in 1 + l^n \mathrm{End}\, T_l(J \otimes \overline{k})\}$. Hence for any $x \in \mathrm{Im}(\rho_l)$ the l-adic logarithm $\log x$ is defined.

Let μ be the Haar measure on $\mathrm{Im}(\rho_l)$ normalized by the equality $\mu(\mathrm{Im}(\rho_l)) = 1$. It is well known that $X = \{x \in \mathrm{Im}(\rho_l) \mid \log x$ is a regular element in Lie $\mathrm{Im}(\rho_l)\}$ is open and everywhere dense in $\mathrm{Im}(\rho_l)$. Its boundary ∂X is a closed analytic subset. So $\mu(\partial X) = 0$ [15, sect.2.2]. Moreover, the set X is invariant under conjugation in $\mathrm{Im}(\rho_l)$. By Chebotarev theorem the density of $\{v | \rho_l(F_{\overline{v}}^{-1}) \in X\}$ is equal to $\mu(X) = 1 - \mu(\partial X) = 1$ [15, sect.2.2, corollary 2]. Hence we may assume that for v the conditions (2.5.1)-(2.5.6) hold and $\log \rho_l(F_{\overline{v}}^{-1})$ is a regular element in Lie $\mathrm{Im}(\rho_l)$.

On the other hand, each W_i is a symplectic or orthogonal S_i-module. Theorem 2.2 and the inequality $p \geq 17$ imply that the pair (type of $g_l^{ss} \otimes \overline{\mathbb{Q}_l}, V_l \otimes \overline{\mathbb{Q}_l}$) assumes one of the following values:

$$(C_2 \times D_p, E(\omega_1^{(1)} + \omega_1^{(2)})), \tag{2.8.1}$$

$$(A_1 \times A_1 \times C_p, E(\omega_1^{(1)} + \omega_1^{(2)} + \omega_1^{(3)})), \tag{2.8.2}$$

$$(A_1 \times D_{2p}, E(\omega_1^{(1)} + \omega_1^{(2)})), \tag{2.8.3}$$

$$(A_1 \times D_p, E(3\omega_1^{(1)} + \omega_1^{(2)})), \tag{2.8.4}$$

$$(A_1 \times A_1, E(7\omega_1^{(1)} + (p-1)\omega_1^{(2)})), \tag{2.8.5}$$

$$(A_1 \times B_{(p-1)/2}, E(7\omega_1^{(1)} + \omega_1^{(2)})), \tag{2.8.6}$$

$$(C_4 \times A_1, E(\omega_1^{(1)} + (p-1)\omega_1^{(2)})), \tag{2.8.7}$$

$$(C_4 \times B_{(p-1)/2}, E(\omega_1^{(1)} + \omega_1^{(2)})), \tag{2.8.8}$$

$$(A_1 \times A_1 \times A_1 \times A_1, E(\omega_1^{(1)} + \omega_1^{(2)} + \omega_1^{(3)} + (p-1)\omega_1^{(4)})), \tag{2.8.9}$$

$$(A_1 \times A_1 \times A_1 \times B_{(p-1)/2}, E(\omega_1^{(1)} + \omega_1^{(2)} + \omega_1^{(3)} + \omega_1^{(4)})), \tag{2.8.10}$$

$$(A_1 \times A_1 \times A_1, E(\omega_1^{(1)} + \omega_1^{(2)} + (2p-1)\omega_1^{(3)})), \tag{2.8.11}$$

where an index (i) shows that the corresponding fundamental weight relates to the i-th factor.

2.9. Consider the case $(C_2 \times D_p, E(\omega_1^{(1)} + \omega_1^{(2)}))$. In virtue of (2.5.6) we may assume that $\Delta = \{\lambda \alpha_{1,2}^{\pm 1} \beta_j^{\pm 1} | j = 1, ..., p\}$, where $\lambda, \alpha_1, \alpha_2, \beta_1, ..., \beta_p$ are multiplicatively independent (in other words, these numbers generate the multiplicative subgroup of $\overline{\mathbb{Q}}^{\times}$ of rank $p+3$).

2.10. **Lemma** . *The following subsets of* $\Delta \cdot \Delta$ *are* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-invariant:*

$$\{\lambda^2\}; \{\lambda^2 \alpha_1^{\pm 1} \alpha_2^{\pm 1}\}; \{\lambda^2 \alpha_{1,2}^{\pm 2}\}; \{\lambda^2 \beta_i^{\pm 1} \beta_j^{\pm 1} | i \neq j\};$$

$$\{\lambda^2\beta_i^{\pm 2}\}; \{\lambda^2\alpha_1^{\pm 1}\alpha_2^{\pm 1}\beta_i^{\pm 1}\beta_j^{\pm 1}|i \neq j\};$$

$$\{\lambda^2\alpha_1^{\pm 1}\alpha_2^{\pm 1}\beta_i^{\pm 2}\}; \{\lambda^2\alpha_{1,2}^{\pm 2}\beta_i^{\pm 1}\beta_j^{\pm 1}|i \neq j\};$$

$$\{\lambda^2\alpha_{1,2}^{\pm 2}\beta_i^{\pm 2}\}.$$

Moreover, $\lambda^2 = \pm p_v$.

Proof. Let $\eta \in \Delta \cdot \Delta$. We may assume that

$$\eta \in \{\lambda^2, \lambda^2\alpha_1^2, \lambda^2\alpha_1\alpha_2, \lambda^2\beta_1^2, \lambda^2\alpha_1^2\beta_1^2, \lambda^2\alpha_1\alpha_2\beta_1^2, \lambda^2\beta_1\beta_2, \lambda^2\alpha_1^2\beta_1\beta_2, \lambda^2\alpha_1\alpha_2\beta_1\beta_2\}.$$

If $\delta \in \Delta$, then $T_\eta(\delta) = \eta\delta^{-1}$, hence

$$T_{\lambda^2}(\delta) \in \Delta \Leftrightarrow \delta \in \Delta; \quad T_{\lambda^2\alpha_1^2}(\delta) \in \Delta \Leftrightarrow \delta \in \{\lambda\alpha_1\beta_j^{\pm 1} \mid j = 1, ..., p\};$$

$$T_{\lambda^2\alpha_1\alpha_2}(\delta) \in \Delta \Leftrightarrow \delta \in \{\lambda\alpha_{1,2}\beta_j^{\pm 1} \mid j = 1, ..., p\}; \quad T_{\lambda^2\beta_1^2}(\delta) \in \Delta \Leftrightarrow \delta \in \{\lambda\alpha_{1,2}^{\pm 1}\beta_1\};$$

$$T_{\lambda^2\alpha_1^2\beta_1^2}(\delta) \in \Delta \Leftrightarrow \delta \in \{\lambda\alpha_1\beta_1\}; \quad T_{\lambda^2\alpha_1\alpha_2\beta_1^2}(\delta) \in \Delta \Leftrightarrow \delta \in \{\lambda\alpha_{1,2}\beta_1\};$$

$$T_{\lambda^2\beta_1\beta_2}(\delta) \in \Delta \Leftrightarrow \delta \in \{\lambda\alpha_{1,2}^{\pm 1}\beta_{1,2}\}; \quad T_{\lambda^2\alpha_1^2\beta_1\beta_2}(\delta) \in \Delta \Leftrightarrow \delta \in \{\lambda\alpha_1\beta_{1,2}\};$$

$$T_{\lambda^2\alpha_1\alpha_2\beta_1\beta_2}(\delta) \in \Delta \Leftrightarrow \delta \in \{\lambda\alpha_{1,2}\beta_{1,2}\}.$$

It is clear that

$$\mathrm{Card}(T_{\lambda^2}(\Delta) \cap \Delta) = \mathrm{Card}(\Delta) = 8p; \quad \mathrm{Card}(T_{\lambda^2\alpha_1^2}(\Delta) \cap \Delta) = 2p;$$

$$\mathrm{Card}(T_{\lambda^2\alpha_1\alpha_2}(\Delta) \cap \Delta) = 4p; \quad \mathrm{Card}(T_{\lambda^2\beta_1^2}(\Delta) \cap \Delta) = 4;$$

$$\mathrm{Card}(T_{\lambda^2\alpha_1^2\beta_1^2}(\Delta) \cap \Delta) = 1; \quad \mathrm{Card}(T_{\lambda^2\alpha_1\alpha_2\beta_1^2}(\Delta) \cap \Delta) = 2;$$

$$\mathrm{Card}(T_{\lambda^2\beta_1\beta_2}(\Delta) \cap \Delta) = 8; \quad \mathrm{Card}(T_{\lambda^2\alpha_1^2\beta_1\beta_2}(\Delta) \cap \Delta) = 2;$$

$$\mathrm{Card}(T_{\lambda^2\alpha_1\alpha_2\beta_1\beta_2}(\Delta) \cap \Delta) = 4.$$

The $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariance of $\{\lambda^2\}, \{\lambda^2\alpha_1^{\pm 1}\alpha_2^{\pm 1}\}, \{\lambda^2\alpha_{1,2}^{\pm 2}\}, \{\lambda^2\beta_i^{\pm 1}\beta_j^{\pm 1}|i \neq j\}$, $\{\lambda^2\alpha_{1,2}^{\pm 2}\beta_i^{\pm 2}\}$, $\{\lambda^2\beta_i^{\pm 2}\} \cup \{\lambda^2\alpha_1^{\pm 1}\alpha_2^{\pm 1}\beta_i^{\pm 1}\beta_j^{\pm 1}|i \neq j\}, \{\lambda^2\alpha_1^{\pm 1}\alpha_2^{\pm 1}\beta_i^{\pm 2}\} \cup$ $\{\lambda^2\alpha_{1,2}^{\pm 2}\beta_i^{\pm 1}\beta_j^{\pm 1}|i \neq j\}$ follows from (0.3.1).

Since $\{\lambda^2\alpha_1^{\pm 1}\alpha_2^{\pm 1}\beta_i^{\pm 1}\beta_j^{\pm 1}|i \neq j\} = \{\lambda^2\alpha_1^{\pm 1}\alpha_2^{\pm 1}\} \cdot \{\lambda^2\beta_i^{\pm 1}\beta_j^{\pm 1}|i \neq j\} \cdot \{\lambda^{-2}\}$ and each factor of this decomposition is $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant, we get the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariance of the sets $\{\lambda^2\alpha_1^{\pm 1}\alpha_2^{\pm 1}\beta_i^{\pm 1}\beta_j^{\pm 1}|i \neq j\}$ and $\{\lambda^2\beta_i^{\pm 2}\}$. The $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariance of another sets follows from the fact that each factor of the decompositions $\{\lambda^2\alpha_1^{\pm 1}\alpha_2^{\pm 1}\beta_i^{\pm 2}\} = \{\lambda^2\alpha_1^{\pm 1}\alpha_2^{\pm 1}\} \cdot \{\lambda^2\beta_i^{\pm 2}\} \cdot \{\lambda^{-2}\}$, $\{\lambda^2\alpha_{1,2}^{\pm 2}\beta_i^{\pm 1}\beta_j^{\pm 1}|i \neq j\} =$ $\{\lambda^2\alpha_{1,2}^{\pm 2}\} \cdot \{\lambda^2\beta_i^{\pm 1}\beta_j^{\pm 1}|i \neq j\} \cdot \{\lambda^{-2}\}$ is $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant.

The relation $\lambda^2 = \pm p_v$ follows from the fact that $\lambda^2 \in \mathbb{Q}$ is of absolute value p_v. Lemma 2.10 is proved.

2.11. It is evident that the set of all eigenvalues of $\rho_l^{\wedge 2}(F_{\overline{v}}^{-1})$ is equal to $\Delta \cdot \Delta -$ $\{\lambda^2\alpha_{1,2}^{\pm 2}\beta_i^{\pm 2}\}$.

By theorem 0.4 we may assume that p_v does not divide $\mathrm{Tr}(\rho_l^{\wedge 2}(F_{\overline{v}}^{-1}))$. The symmetry implies that the multiplicity of $\eta \in \{\lambda^2\alpha_1^{\pm 1}\alpha_2^{\pm 1}\}$ as an eigenvalue of

$\rho_l^{\wedge 2}(F_{\overline{v}}^{-1})$ is independent of the choice of η. This is valid for another $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant subsets which are defined in the statement of lemma 2.10. We deduce from this lemma that $\mathrm{Tr}(\rho_l^{\wedge 2}(F_{\overline{v}}^{-1}))$ is a sum of integers of the following types:

$$\lambda^2, \sum \lambda^2\alpha_1^{\pm 1}\alpha_2^{\pm 1}, \sum \lambda^2\alpha_{1,2}^{\pm 2}, \sum_{i\neq j} \lambda^2\beta_i^{\pm 1}\beta_j^{\pm 1}, \sum \lambda^2\beta_i^{\pm 2}, \sum_{i\neq j} \lambda^2\alpha_1^{\pm 1}\alpha_2^{\pm 1}\beta_i^{\pm 1}\beta_j^{\pm 1},$$

$$\sum \lambda^2\alpha_1^{\pm 1}\alpha_2^{\pm 1}\beta_i^{\pm 2}, \sum_{i\neq j} \lambda^2\alpha_{1,2}^{\pm 2}\beta_i^{\pm 1}\beta_j^{\pm 1}.$$

Hence p_v does not divide at least one of the sums above.

2.12. Assume that p_v does not divide $\sum \lambda^2\alpha_1^{\pm 1}\alpha_2^{\pm 1}$. Then for each place w of $\overline{\mathbb{Q}}$ lying over p_v

$$w(\sum \lambda^2\alpha_1^{\pm 1}\alpha_2^{\pm 1}) = 0.$$

It follows that there exists $x_w \in \{\lambda^2\alpha_1^{\pm 1}\alpha_2^{\pm 1}\}$ such that $w(x_w) = 0$. Hence

$$0 = w(x_w) = \frac{1}{2}\{w(x_w\beta_1^2) + w(x_w\beta_1^{-2})\}.$$

Since both summands in the last brackets are nonnegative in virtue of the relations $x_w\beta_1^{\pm 2} \in \Delta \cdot \Delta$, we have the equalities

$$w(x_w\beta_1^2) = w(x_w\beta_1^{-2}) = 0.$$

So $w(\beta_1) = 0$ for *all* $w|p_v$. It follows that β_1 is a root of 1 [23, sublemma 3.4.0] contrary to the assumption that $\lambda, \alpha_1, \alpha_2, \beta_1, ..., \beta_p$ are multiplicatively independent.

Hence p_v divides $\sum \lambda^2\alpha_1^{\pm 1}\alpha_2^{\pm 1}$. From the relation $\lambda^2 = \pm p_v$ we deduce that $\sum \alpha_1^{\pm 1}\alpha_2^{\pm 1}$ is an integer.

By the similar arguments we prove that $\sum \alpha_{1,2}^{\pm 2}$ is an integer.

2.13. Assume that p_v does not divide $\sum_{i\neq j} \lambda^2\beta_i^{\pm 1}\beta_j^{\pm 1}$. Then for each place w of $\overline{\mathbb{Q}}$ lying over p_v

$$w(\sum_{i\neq j} \lambda^2\beta_i^{\pm 1}\beta_j^{\pm 1}) = 0.$$

It follows that there exists $x_w \in \{\lambda^2\beta_i^{\pm 1}\beta_j^{\pm 1}\}$ such that $w(x_w) = 0$. Hence

$$0 = w(x_w) = \frac{1}{2}\{w(x_w\alpha_1^2) + w(x_w\alpha_1^{-2})\}.$$

Since both summands in the last brackets are nonnegative in virtue of the relations $x_w\alpha_1^{\pm 2} \in \Delta \cdot \Delta$, we have the equalities

$$w(x_w\alpha_1^2) = w(x_w\alpha_1^{-2}) = 0.$$

So $w(\alpha_1) = 0$ for *all* $w|p_v$. It follows that α_1 is a root of 1 [23, sublemma 3.4.0] contrary to the assumption that $\lambda, \alpha_1, \alpha_2, \beta_1, ..., \beta_p$ are multiplicatively independent.

Hence p_v divides $\sum_{i\neq j} \lambda^2\beta_i^{\pm 1}\beta_j^{\pm 1}$, and $\sum_{i\neq j} \beta_i^{\pm 1}\beta_j^{\pm 1}$ is an integer.

By the similar arguments we prove that $\sum \beta_i^{\pm 2}$ is an integer.

2.14. Consider the decomposition

$$\sum_{i\neq j} \lambda^2\alpha_1^{\pm 1}\alpha_2^{\pm 1}\beta_i^{\pm 1}\beta_j^{\pm 1} = (\sum \lambda^2\alpha_1^{\pm 1}\alpha_2^{\pm 1})\cdot(\sum_{i\neq j}\beta_i^{\pm 1}\beta_j^{\pm 1}).$$

We have proved that p_v divides the first factor of this decomposition, and the second factor is an integer. Hence p_v divides the product.

The decompositions

$$\sum \lambda^2\alpha_1^{\pm 1}\alpha_2^{\pm 1}\beta_i^{\pm 2} = (\sum \lambda^2\alpha_1^{\pm 1}\alpha_2^{\pm 1})\cdot(\sum \beta_i^{\pm 2}),$$

$$\sum_{i\neq j} \lambda^2\alpha_{1,2}^{\pm 2}\beta_i^{\pm 1}\beta_j^{\pm 1} = (\sum \lambda^2\alpha_{1,2}^{\pm 2})\cdot(\sum_{i\neq j}\beta_i^{\pm 1}\beta_j^{\pm 1})$$

show that p_v divides each left side.

Hence p_v divides $\mathrm{Tr}(\rho_l^{\wedge 2}(F_{\overline{v}}^{-1}))$ contrary to our assumption. This excludes the case $(C_2\times D_p, E(\omega_1^{(1)}+\omega_1^{(2)}))$.

2.15. Since the structure of Δ does not distinguish the cases $(C_2\times D_p, E(\omega_1^{(1)}+\omega_1^{(2)}))$ and $(A_1\times A_1\times C_p, E(\omega_1^{(1)}+\omega_1^{(2)}+\omega_1^{(3)})) = (D_2\times C_p, E(\omega_1^{(1)}+\omega_1^{(2)}))$, we exclude the case $(A_1\times A_1\times C_p, E(\omega_1^{(1)}+\omega_1^{(2)}+\omega_1^{(3)}))$ by the same procedure.

2.16. Consider the case $(A_1\times D_{2p}, E(\omega_1^{(1)}+\omega_1^{(2)}))$. In virtue of (2.5.6) we may assume that $\Delta = \{\lambda\alpha_1^{\pm 1}\beta_i^{\pm 1} | i = 1,...,2p\}$, where $\lambda, \alpha_1, \beta_1, ..., \beta_{2p}$ are multiplicatively independent. The proof of the following result is similar to the proof of lemma 2.10.

2.17. **Lemma.** *The following subsets of* $\Delta\cdot\Delta$ *are* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-invariant:*

$$\{\lambda^2\}; \{\lambda^2\alpha_1^{\pm 2}\}; \{\lambda^2\beta_i^{\pm 1}\beta_j^{\pm 1} | i\neq j\};\ \{\lambda^2\beta_i^{\pm 2}\};$$

$$\{\lambda^2\alpha_1^{\pm 2}\beta_i^{\pm 1}\beta_j^{\pm 1} | i\neq j\};\ \{\lambda^2\alpha_1^{\pm 2}\beta_i^{\pm 2}\}.$$

Moreover, $\lambda^2 = \pm p_v$.

2.18. It is evident that the set of all eigenvalues of $\rho_l^{\wedge 2}(F_{\overline{v}}^{-1})$ is equal to $\Delta\cdot\Delta - \{\lambda^2\alpha_1^{\pm 2}\beta_i^{\pm 2}\}$.

By theorem 0.4 we may assume that p_v does not divide $\mathrm{Tr}(\rho_l^{\wedge 2}(F_{\overline{v}}^{-1}))$. It is evident that this trace is a sum of integers of the following types:

$$\lambda^2, \sum \lambda^2\alpha_1^{\pm 2}, \sum_{i\neq j}\lambda^2\beta_i^{\pm 1}\beta_j^{\pm 1}, \sum \lambda^2\beta_i^{\pm 2}, \sum_{i\neq j}\lambda^2\alpha_1^{\pm 2}\beta_i^{\pm 1}\beta_j^{\pm 1}.$$

Hence p_v does not divide at least one of the sums above. By the arguments of sections 2.12-2.14 we prove that

$$\sum \alpha_1^{\pm 2}, \sum_{i\neq j}\beta_i^{\pm 1}\beta_j^{\pm 1}, \sum \beta_i^{\pm 2}, \sum_{i\neq j}\alpha_1^{\pm 2}\beta_i^{\pm 1}\beta_j^{\pm 1}$$

are integers. Hence p_v divides $\mathrm{Tr}(\rho_l^{\wedge 2}(F_{\overline{v}}^{-1}))$ contrary to our assumption. This excludes the case $(A_1\times D_{2p}, E(\omega_1^{(1)}+\omega_1^{(2)}))$.

2.19. Consider the case $(A_1 \times D_p, E(3\omega_1^{(1)}+\omega_1^{(2)}))$. In virtue of (2.5.6) we may assume that $\Delta = \{\lambda\alpha_1^{\pm1,\pm3}\beta_i^{\pm1} | i = 1, ..., p\}$, where $\lambda, \alpha_1, \beta_1, ..., \beta_p$ are multiplicatively independent.

2.20. **Lemma**. *The following subsets of $\Delta \cdot \Delta$ are $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant:*

$$\{\lambda^2\}; \{\lambda^2\alpha_1^{\pm2}\}; \{\lambda^2\alpha_1^{\pm4}\}; \{\lambda^2\alpha_1^{\pm6}\}; \{\lambda^2\beta_i^{\pm1}\beta_j^{\pm1} | i \neq j\}; \{\lambda^2\alpha_1^{\pm2}\beta_i^{\pm1}\beta_j^{\pm1} | i \neq j\};$$

$$\{\lambda^2\alpha_1^{\pm4}\beta_i^{\pm1}\beta_j^{\pm1} | i \neq j\}; \{\lambda^2\alpha_1^{\pm6}\beta_i^{\pm1}\beta_j^{\pm1} | i \neq j\}; \{\lambda^2\beta_i^{\pm2}\};$$

$$\{\lambda^2\alpha_1^{\pm2}\beta_i^{\pm2}\}; \{\lambda^2\alpha_1^{\pm4}\beta_i^{\pm2}\}; \{\lambda^2\alpha_1^{\pm6}\beta_i^{\pm2}\}.$$

Moreover, $\lambda^2 = \pm p_v$.

Proof. Let $\eta \in \Delta \cdot \Delta$. We may assume that

$$\eta \in \{\lambda^2, \lambda^2\alpha_1^2, \lambda^2\alpha_1^4, \lambda^2\alpha_1^6, \lambda^2\beta_1^2, \lambda^2\alpha_1^2\beta_1^2, \lambda^2\alpha_1^4\beta_1^2, \lambda^2\alpha_1^6\beta_1^2,$$

$$\lambda^2\beta_1\beta_2, \lambda^2\alpha_1^2\beta_1\beta_2, \lambda^2\alpha_1^4\beta_1\beta_2, \lambda^2\alpha_1^6\beta_1\beta_2\}.$$

If $\delta \in \Delta$, then $T_\eta(\delta) = \eta\delta^{-1}$, hence

$$T_{\lambda^2}(\delta) \in \Delta \Leftrightarrow \delta \in \Delta; \quad T_{\lambda^2\alpha_1^2}(\delta) \in \Delta \Leftrightarrow \delta \in \{\lambda\alpha_1^{\pm1,3}\beta_i^{\pm1} | i = 1, ..., p\};$$

$$T_{\lambda^2\alpha_1^4}(\delta) \in \Delta \Leftrightarrow \delta \in \{\lambda\alpha_1^{1,3}\beta_i^{\pm1} | i = 1, ..., p\}$$

$$T_{\lambda^2\alpha_1^6}(\delta) \in \Delta \Leftrightarrow \delta \in \{\lambda\alpha_1^{3}\beta_i^{\pm1} | i = 1, ..., p\};$$

$$T_{\lambda^2\beta_1^2}(\delta) \in \Delta \Leftrightarrow \delta \in \{\lambda\alpha_1^{\pm1,\pm3}\beta_1\}; \quad T_{\lambda^2\alpha_1^2\beta_1^2}(\delta) \in \Delta \Leftrightarrow \delta \in \{\lambda\alpha_1^{\pm1,3}\beta_1\};$$

$$T_{\lambda^2\alpha_1^4\beta_1^2}(\delta) \in \Delta \Leftrightarrow \delta \in \{\lambda\alpha_1^{1,3}\beta_1\}; \quad T_{\lambda^2\alpha_1^6\beta_1^2}(\delta) \in \Delta \Leftrightarrow \delta \in \{\lambda\alpha_1^{3}\beta_1\};$$

$$T_{\lambda^2\beta_1\beta_2}(\delta) \in \Delta \Leftrightarrow \delta \in \{\lambda\alpha_1^{\pm1,\pm3}\beta_{1,2}\}; \quad T_{\lambda^2\alpha_1^2\beta_1\beta_2}(\delta) \in \Delta \Leftrightarrow \delta \in \{\lambda\alpha_1^{\pm1,3}\beta_{1,2}\};$$

$$T_{\lambda^2\alpha_1^4\beta_1\beta_2}(\delta) \in \Delta \Leftrightarrow \delta \in \{\lambda\alpha_1^{1,3}\beta_{1,2}\}; T_{\lambda^2\alpha_1^6\beta_1\beta_2}(\delta) \in \Delta \Leftrightarrow \delta \in \{\lambda\alpha_1^{3}\beta_{1,2}\}.$$

It is clear that

$$\mathrm{Card}(T_{\lambda^2}(\Delta) \cap \Delta) = \mathrm{Card}(\Delta) = 8p; \quad \mathrm{Card}(T_{\lambda^2\alpha_1^2}(\Delta) \cap \Delta) = 6p;$$

$$\mathrm{Card}(T_{\lambda^2\alpha_1^4}(\Delta) \cap \Delta) = 4p; \quad \mathrm{Card}(T_{\lambda^2\alpha_1^6}(\Delta) \cap \Delta) = 2p;$$

$$\mathrm{Card}(T_{\lambda^2\beta_1^2}(\Delta) \cap \Delta) = 4; \quad \mathrm{Card}(T_{\lambda^2\alpha_1^2\beta_1^2}(\Delta) \cap \Delta) = 3;$$

$$\mathrm{Card}(T_{\lambda^2\alpha_1^4\beta_1^2}(\Delta) \cap \Delta) = 2; \quad \mathrm{Card}(T_{\lambda^2\alpha_1^6\beta_1^2}(\Delta) \cap \Delta) = 1;$$

$$\mathrm{Card}(T_{\lambda^2\beta_1\beta_2}(\Delta) \cap \Delta) = 8; \quad \mathrm{Card}(T_{\lambda^2\alpha_1^2\beta_1\beta_2}(\Delta) \cap \Delta) = 6;$$

$$\mathrm{Card}(T_{\lambda^2\alpha_1^4\beta_1\beta_2}(\Delta) \cap \Delta) = 4; \quad \mathrm{Card}(T_{\lambda^2\alpha_1^6\beta_1\beta_2}(\Delta) \cap \Delta) = 2.$$

The $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariance of $\{\lambda^2\}, \{\lambda^2\alpha_1^{\pm2}\}, \{\lambda^2\alpha_1^{\pm4}\}, \{\lambda^2\alpha_1^{\pm6}\}$,

$$\{\lambda^2\beta_i^{\pm2}\} \cup \{\lambda^2\alpha_1^{\pm4}\beta_i^{\pm1}\beta_j^{\pm1} | i \neq j\}, \{\lambda^2\alpha_1^{\pm2}\beta_i^{\pm2}\},$$

$$\{\lambda^2\alpha_1^{\pm4}\beta_i^{\pm2}\}\cup\{\lambda^2\alpha_1^{\pm6}\beta_i^{\pm1}\beta_j^{\pm1}|i\neq j\},\{\lambda^2\alpha_1^{\pm6}\beta_i^{\pm2}\},$$
$$\{\lambda^2\beta_i^{\pm1}\beta_j^{\pm1}|i\neq j\},\{\lambda^2\alpha_1^{\pm2}\beta_i^{\pm1}\beta_j^{\pm1}|i\neq j\}$$

follows from (0.3.1). Since

$$\{\lambda^2\alpha_1^{\pm4}\beta_i^{\pm1}\beta_j^{\pm1}|i\neq j\}=\{\lambda^2\alpha_1^{\pm4}\}\cdot\{\lambda^2\beta_i^{\pm1}\beta_j^{\pm1}|i\neq j\}\cdot\{\lambda^{-2}\}$$

and each factor of this decomposition is $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant, we get the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariance of the sets $\{\lambda^2\alpha_1^{\pm4}\beta_i^{\pm1}\beta_j^{\pm1}|i\neq j\}$ and $\{\lambda^2\beta_i^{\pm2}\}$. The decomposition $\{\lambda^2\alpha_1^{\pm6}\beta_i^{\pm1}\beta_j^{\pm1}|i\neq j\}=\{\lambda^2\alpha_1^{\pm6}\}\cdot\{\lambda^2\beta_i^{\pm1}\beta_j^{\pm1}|i\neq j\}\cdot\{\lambda^{-2}\}$ gives the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariance of the sets $\{\lambda^2\alpha_1^{\pm6}\beta_i^{\pm1}\beta_j^{\pm1}|i\neq j\}$ and $\{\lambda^2\alpha_1^{\pm4}\beta_i^{\pm2}\}$. This proves the claim.

2.21. It is evident that the set of all eigenvalues of $\rho_l^{\wedge2}(F_{\overline{v}}^{-1})$ is equal to $\Delta\cdot\Delta-\{\lambda^2\alpha_1^{\pm6}\beta_i^{\pm2}\}$.

By theorem 0.4 we may assume that p_v does not divide $\mathrm{Tr}(\rho_l^{\wedge2}(F_{\overline{v}}^{-1}))$. It is evident that this trace is a sum of integers of the following types:

$$\lambda^2,\sum\lambda^2\alpha_1^{\pm2},\sum\lambda^2\alpha_1^{\pm4},\sum\lambda^2\alpha_1^{\pm6},\sum_{i\neq j}\lambda^2\beta_i^{\pm1}\beta_j^{\pm1},\sum_{i\neq j}\lambda^2\alpha_1^{\pm2}\beta_i^{\pm1}\beta_j^{\pm1},$$

$$\sum_{i\neq j}\lambda^2\alpha_1^{\pm4}\beta_i^{\pm1}\beta_j^{\pm1},\sum_{i\neq j}\lambda^2\alpha_1^{\pm6}\beta_i^{\pm1}\beta_j^{\pm1},\sum\lambda^2\beta_i^{\pm2},\sum\lambda^2\alpha_1^{\pm2}\beta_i^{\pm2},\sum\lambda^2\alpha_1^{\pm4}\beta_i^{\pm2}.$$

Hence p_v does not divide at least one of the sums above. By the arguments of section 2.12 we prove that

$$\sum\alpha_1^{\pm2},\sum\alpha_1^{\pm4},\sum\alpha_1^{\pm6}$$

are integers. By the arguments of section 2.13 we prove that

$$\sum_{i\neq j}\beta_i^{\pm1}\beta_j^{\pm1},\sum_{i\neq j}\alpha_1^{\pm2}\beta_i^{\pm1}\beta_j^{\pm1},\sum_{i\neq j}\alpha_1^{\pm4}\beta_i^{\pm1}\beta_j^{\pm1},\sum\beta_i^{\pm2},\sum\alpha_1^{\pm2}\beta_i^{\pm2},\sum\alpha_1^{\pm4}\beta_i^{\pm2}$$

are integers. The decomposition

$$\sum_{i\neq j}\lambda^2\alpha_1^{\pm6}\beta_i^{\pm1}\beta_j^{\pm1}=\lambda^2\cdot(\sum\alpha_1^{\pm6})\cdot(\sum_{i\neq j}\beta_i^{\pm1}\beta_j^{\pm1})$$

shows that p_v divides the left side. Hence p_v divides $\mathrm{Tr}(\rho_l^{\wedge2}(F_{\overline{v}}^{-1}))$ contrary to our assumption. This excludes the case $(A_1\times D_p,E(3\omega_1^{(1)}+\omega_1^{(2)}))$.

2.22. Consider the cases (2.8.5)-(2.8.10). It is clear that $\dim_{\overline{\mathbb{Q}}}W_q=p$ and $\dim_{\overline{\mathbb{Q}}}W_1\otimes\ldots\otimes W_{q-1}=8$.

2.23. **Lemma.** *In the notations of section 2.7 assume that one of the following conditions hold:*

1) *S_1 is a simple simply connected Lie group of type A_1, $W_1=E(7\omega_1^{(1)})$;*

2) *S_1 is a simple simply connected Lie group of type C_4, $W_1=E(\omega_1^{(1)})$;*

3) *$S_1\times S_2\times S_3$ is a semisimple simply connected Lie group of type* $A_1\times A_1\times A_1$, $W_1\otimes W_2\otimes W_3 = E(\omega_1^{(1)}+\omega_1^{(2)}+\omega_1^{(3)})$.

Then the highest weight of S_q-module W_q is not a radical weight.

2.24. *Proof.* It is evident that

$$\Delta = \{\lambda\alpha_i^{\pm 1} | i = 1, ..., 4\}\cdot\{\beta_j \mid j = 1, ..., p\},$$

where $\lambda, \alpha_1, ..., \alpha_4$ correspond to $\mathrm{G}_m\times S_1\times ...\times S_{q-1}$, and $\beta_1, ..., \beta_p$ correspond to S_q. We do not suppose that these numbers are multiplicatively independent. Let $\Phi = \{\lambda\alpha_i^{\pm 1} | i = 1, ..., 4\}$, $\Psi = \{\beta_j \mid j = 1, ..., p\}$.

Assume that the highest weight of S_q-module W_q is a radical weight. In this case 0 is a weight of Lie(S_q)-module W_q [3, ch.8,§ 7, exercise 3]. Hence $1\in\Psi$ and $\Phi\subset\Gamma_v$. On the other hand, $\beta_j = (\lambda\alpha_1\beta_j)/(\lambda\alpha_1)\in\Gamma_v$. Hence $\Psi\subset\Gamma_v$. We denote by Γ_Φ (resp. Γ_Ψ) the multiplicative subgroup of Γ_v generated by Φ (resp. Ψ). In virtue of (2.5.5)-(2.5.6) Γ_Φ and Γ_Ψ are torsion-free abelian groups of positive rank.

It is clear that $\mathrm{rank}(\Gamma_\Phi)\le 1+\mathrm{rank}\ \mathrm{X}(\mathrm{pr}_1(T)\times ...\times\mathrm{pr}_{q-1}(T))$, $\mathrm{rank}(\Gamma_\Psi)\le\mathrm{rank}\ \mathrm{X}(\mathrm{pr}_q(T))$, $\Gamma_v\subset\Gamma_\Phi\cdot\Gamma_\Psi$. Hence the relations

$$\mathrm{rank}(\Gamma_v)\le\mathrm{rank}(\Gamma_\Phi\cdot\Gamma_\Psi)\le\mathrm{rank}(\Gamma_\Phi)+\mathrm{rank}(\Gamma_\Psi)\le$$

$$1+\mathrm{rank}\ \mathrm{X}(\mathrm{pr}_1(T)\times ...\times\mathrm{pr}_{q-1}(T))+\mathrm{rank}\ \mathrm{X}(\mathrm{pr}_q(T)) = \mathrm{rank}\ \mathrm{X}(T) =$$

$$\mathrm{rank}\ \mathrm{X}(T_v\otimes\overline{\mathbb{Q}_l}) = \mathrm{rank}(\Gamma_v)$$

imply the relations

$$\mathrm{rank}(\Gamma_\Phi) = 1+\mathrm{rank}\ \mathrm{X}(\mathrm{pr}_1(T)\times ...\times\mathrm{pr}_{q-1}(T)) =$$

$$\mathrm{rank}\ \mathrm{G}_m\times S_1\times ...\times S_{q-1}\ge 2, \qquad (2.24.1)$$

$$\mathrm{rank}(\Gamma_\Psi) = \mathrm{rank}\ \mathrm{X}(\mathrm{pr}_q(T)) = \mathrm{rank} S_q, \qquad (2.24.2)$$

$$\mathrm{rank}(\Gamma_v) = \mathrm{rank}(\Gamma_\Phi\cdot\Gamma_\Psi) = \mathrm{rank}(\Gamma_\Phi)+\mathrm{rank}(\Gamma_\Psi).$$

Hence

$$\Gamma_\Phi\cap\Gamma_\Psi = (1). \qquad (2.24.3)$$

Due to [6, sect. 5.2] for each $\gamma\in\Delta$ we define $T_\gamma^0 : \Delta\to\overline{\mathbb{Q}}^\times$ by the formula $T_\gamma^0(\delta) = \gamma^2\delta^{-1}$. It is evident that for each $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

$$\mathrm{Card}(T^0_{\sigma(\gamma)}(\Delta)\cap\Delta) = \mathrm{Card}(T^0_{\sigma(\gamma)}(\sigma(\Delta))\cap\sigma(\Delta)) = \mathrm{Card}(T^0_\gamma(\Delta)\cap\Delta),$$

and hence for any constant c the set

$$\{\gamma\in\Delta \mid \mathrm{Card}(T^0_\gamma(\Delta)\cap\Delta) = c\}\ \textit{is}\ \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) - \textit{invariant}. \qquad (2.24.4)$$

Consider the case (2.23.1). In this situation we may assume that

$$\Phi = \{\lambda\alpha_1^{\pm 1,\pm 3,\pm 5,\pm 7}\}.$$

We claim that

$$\gamma \in \{\lambda\alpha_1^{\pm 1}\} \Leftrightarrow \mathrm{Card}(T^0_\gamma(\Delta) \cap \Delta) \geq 7p. \tag{2.24.5}$$

Indeed, we may assume that $\gamma \in \{\lambda\alpha_1\beta_i, \lambda\alpha_1^3\beta_i, \lambda\alpha_1^5\beta_i, \lambda\alpha_1^7\beta_i \mid i = 1, ..., p\}$. For $\delta \in \Delta$ we have $T^0_\gamma(\delta) = \gamma^2\delta^{-1}$. Hence the multiplicative independence of λ, α_1 (which follows from (2.24.1)) and (2.24.3) imply the relation

$$T^0_{\lambda\alpha_1\beta_i}(\delta) = (\lambda\alpha_1\beta_i)^2 \cdot \delta^{-1} \in \Delta \Leftrightarrow \delta \in \{\lambda\alpha_1^{\pm 1, \pm 3, \pm 5, 7} \cdot \beta_i^2 \cdot \Psi\} \cap \Delta.$$

It is evident that

$$\mathrm{Card}(T^0_{\lambda\alpha_1\beta_i}(\Delta) \cap \Delta) \geq 7p \Leftrightarrow \beta_i^2 \cdot \Psi = \Psi.$$

Since $1 \in \Psi$, we have the relation $(\beta_i^2 \cdot \Psi = \Psi) \Rightarrow (\beta_i^2 \in \Psi, \beta_i^3 \in \Psi, ..., \beta_i^r \in \Psi$ for all natural $r)$. Hence β_i is a root of unity, $\beta_i = 1$ because Γ_Ψ is torsion-free, $\gamma = \lambda\alpha_1$. On the other hand,

$$T^0_{\lambda\alpha_1^3\beta_i}(\delta) = (\lambda\alpha_1^3\beta_i)^2 \cdot \delta^{-1} \in \Delta \Leftrightarrow \delta \in \{\lambda\alpha_1^{\pm 1, 3, 5, 7} \cdot \beta_i^2 \cdot \Psi\} \cap \Delta,$$

and $\mathrm{Card}(T^0_{\lambda\alpha_1^3\beta_i}(\Delta) \cap \Delta) \leq 5p$. It is clear that for $r \geq 3$ $\mathrm{Card}(T^0_{\lambda\alpha_1^r\beta_i}(\Delta) \cap \Delta) < 7p$. So the claim (2.24.5) is proved. In virtue of (2.24.4) the set $\{\lambda\alpha_1^{\pm 1}\}$ is $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant. Hence $\lambda\alpha_1 + \lambda\alpha_1^{-1} \in \mathbb{Z}$. Moreover, $\lambda\alpha_1 + \lambda\alpha_1^{-1} \neq 0$: otherwise we would have $\lambda\alpha_1 = (-1)\lambda\alpha_1^{-1} \in \Gamma_v$ and $(-1) \in \Gamma_v$ contrary to the condition (2.5.5). On the other hand, the absolute value of $\lambda\alpha_1 + \lambda\alpha_1^{-1}$ is less than or equal to $2\sqrt{p_v}$. Hence for $p_v >> 0$ we get the relation $\lambda\alpha_1 + \lambda\alpha_1^{-1} \not\equiv 0 (\mathrm{mod}\ p_v)$. Then for each place w of $\overline{\mathbb{Q}}$ lying over p_v

$$w(\sum \lambda\alpha_1^{\pm 1}) = 0.$$

It follows that there exists $x_w \in \{\lambda\alpha_1^{\pm 1}\}$ such that $w(x_w) = 0$. Hence

$$0 = w(x_w) = \frac{1}{2}\{w(x_w\beta_j) + w(x_w\beta_j^{-1})\}. \tag{2.24.6}$$

Since both summands in the last brackets are nonnegative in virtue of the relations $x_w\beta_j^{\pm 1} \in \Delta$, we have the equalities

$$w(x_w\beta_j) = w(x_w\beta_j^{-1}) = 0.$$

So $w(\beta_j) = 0$ for *all* $w|p_v$. It follows that $\forall j$ β_j is a root of 1 [23, sublemma 3.4.0] contrary to the relation (2.24.2).

Consider the case (2.23.2). In this situation we may assume that

$$\Phi = \{\lambda\alpha_i^{\pm 1} \mid i = 1, ..., 4\},$$

where $\lambda, \alpha_1, ..., \alpha_4$ are multiplicatively independent. We claim that

$$\gamma \in \Phi \Leftrightarrow \mathrm{Card}(T^0_\gamma(\Delta) \cap \Delta) \geq p. \tag{2.24.7}$$

Indeed, we may assume that $\gamma = \lambda\alpha_1\beta_i$, $\delta = \lambda\alpha_j^a\beta_k^b$, where $a, b \in \{\pm 1\}$. We have $T^0_\gamma(\delta) = \gamma^2\delta^{-1}$. Hence the multiplicative independence of $\lambda, \alpha_1, ..., \alpha_4$ (which follows from (2.24.1)) and (2.24.3) imply the relation

$$T^0_{\lambda\alpha_1\beta_i}(\delta) = (\lambda\alpha_1\beta_i)^2 \cdot (\lambda\alpha_j^a\beta_k^b)^{-1} = \lambda\alpha_1^2\alpha_j^{-a}\beta_i^2\beta_k^{-b} \in \Delta \Leftrightarrow$$

$$(\lambda\alpha_1^2\alpha_j^{-a} \in \Phi \text{ and } \beta_i^2\beta_k^{-b} \in \Psi) \Leftrightarrow (\alpha_j^a = \alpha_1 \text{ and } \beta_i^2\beta_k^{-b} \in \Psi) \Leftrightarrow$$

$$(\delta = \lambda\alpha_1\beta_k^b \text{ and } \beta_i^2\beta_k^{-b} \in \Psi).$$

It is clear that

$$\mathrm{Card}(T^0_\gamma(\Delta) \cap \Delta) \geq p = \mathrm{Card}(\Psi) \Leftrightarrow \beta_i^2 \cdot \Psi = \Psi.$$

By the arguments above we know that $\beta_i^2 \cdot \Psi = \Psi \Leftrightarrow \beta_i = 1$. This proves the claim (2.24.7). Moreover, $\Phi = \{\gamma \in \Delta \mid \mathrm{Card}(T^0_\gamma(\Delta) \cap \Delta) \geq p\}$ is the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant set. By the condition (2.5.4) $\mathrm{Tr}(\rho_l(\mathbb{F}_{\overline{v}}^{-1})) = \sum_{\beta\in\Psi}(\sum_{x\in\Phi} x)\beta \neq 0$. Hence $\sum_{x\in\Phi} x \neq 0$, and the absolute value of this integer is less than or equal to $8\sqrt{p_v}$. For $p_v >> 0$ we get the relation $\sum_{x\in\Phi} x \neq 0 (\mathrm{mod}\ p_v)$. Then for each place w of $\overline{\mathbb{Q}}$ lying over p_v

$$w(\sum_{x\in\Phi} x) = 0.$$

It follows that there exists $x_w \in \Phi$ such that $w(x_w) = 0$. Hence we get the relation (2.24.6) in the new situation. By the arguments above we know that this relation implies the relation $\mathrm{rank}(\Psi) = 0$ contrary to the relation (2.24.2).

Consider the case (2.23.3). In this situation we may assume that

$$\Phi = \{\lambda\alpha_1^{\pm 1}\alpha_2^{\pm 1}\alpha_3^{\pm 1} \mid i = 1, 2, 3\},$$

where $\lambda, \alpha_1, \alpha_2, \alpha_3$ are multiplicatively independent. We claim that

$$\gamma \in \Phi \Leftrightarrow \mathrm{Card}(T^0_\gamma(\Delta) \cap \Delta) \geq p. \tag{2.24.8}$$

Indeed, we may assume that

$$\gamma = \lambda\alpha_1\alpha_2\alpha_3\beta_i, \ \delta = \lambda\alpha_1^{a_1}\alpha_2^{a_2}\alpha_3^{a_3}\beta_k^b,$$

where $a_1, a_2, a_3, b \in \{\pm 1\}$. We have $T^0_\gamma(\delta) = \gamma^2\delta^{-1}$. Hence the multiplicative independence of $\lambda, \alpha_1, \alpha_2, \alpha_3$ (which follows from (2.24.1)) and (2.24.3) imply the relation

$$T^0_{\lambda\alpha_1\alpha_2\alpha_3\beta_i}(\delta) = (\lambda\alpha_1\alpha_2\alpha_3\beta_i)^2 \cdot (\lambda\alpha_1^{a_1}\alpha_2^{a_2}\alpha_3^{a_3}\beta_k^b)^{-1} = \lambda\alpha_1^{2-a_1}\alpha_2^{2-a_2}\alpha_3^{2-a_3}\beta_i^2\beta_k^{-b} \in \Delta$$

$$\Leftrightarrow (\lambda\alpha_1^{2-a_1}\alpha_2^{2-a_2}\alpha_3^{2-a_3} \in \Phi \text{ and } \beta_i^2\beta_k^{-b} \in \Psi)$$

$$\Leftrightarrow (\alpha_j^{a_j} = \alpha_j \text{ for all } j \text{ and } \beta_i^2\beta_k^{-b} \in \Psi)$$

$$\Leftrightarrow (\delta = \lambda\alpha_1\alpha_2\alpha_3\beta_k^b \text{ and } \beta_i^2\beta_k^{-b} \in \Psi).$$

It is clear that

$$\mathrm{Card}(T_\gamma^0(\Delta)\cap\Delta)\geq p=\mathrm{Card}(\Psi)\Leftrightarrow\beta_i^2\cdot\Psi=\Psi.$$

By the arguments above we know that $\beta_i^2\cdot\Psi=\Psi\Leftrightarrow\beta_i=1$. This proves the claim (2.24.8). Moreover, $\Phi=\{\gamma\in\Delta\mid\mathrm{Card}(T_\gamma^0(\Delta)\cap\Delta)\geq p\}$ is the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant set. We obtain the contradiction by the same arguments as before. Lemma 2.23 is proved.

Thus the cases (2.8.5)-(2.8.10) are excluded.

2.25. Consider the case

$$(A_1\times A_1\times A_1,E(\omega_1^{(1)}+\omega_1^{(2)}+(2p-1)\omega_1^{(3)}))=(D_2\times A_1,E(\omega_1^{(1)}+(2p-1)\omega_1^{(2)})).$$

In virtue of (2.5.6) we may assume that $\Delta=\{\lambda\alpha_{1,2}^{\pm1}\beta_1^{\pm1,\pm3,\ldots,\pm(2p-1)}\}$, where $\lambda,\alpha_1,\alpha_2,\beta_1$ are multiplicatively independent.

2.26. **Lemma**. *For each natural number m $(0\leq m\leq 2p-1)$ the following subsets of $\Delta\cdot\Delta$ are $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant:*

$$\{\lambda^2\beta_1^{\pm2m}\};\{\lambda^2\alpha_1^{\pm1}\alpha_2^{\pm1}\beta_1^{\pm2m}\};\{\lambda^2\alpha_{1,2}^{\pm2}\beta_1^{\pm2m}\}.$$

Moreover, $\lambda^2=\pm p_v$.

Proof. Let $\eta\in\Delta\cdot\Delta$. We may assume that

$$\eta\in\{\lambda^2\beta_1^{2m},\lambda^2\alpha_1^2\beta_1^{2m},\lambda^2\alpha_1\alpha_2\beta_1^{2m}\}.$$

If $\delta\in\Delta$, then $T_\eta(\delta)=\eta\delta^{-1}$, hence

$$T_{\lambda^2\beta_1^{2m}}(\delta)\in\Delta\Leftrightarrow\delta\in\{\lambda\alpha_{1,2}^{\pm1}\beta_1^r\mid r\text{ is an odd integer},|r|\leq2p-1,|2m-r|\leq2p-1\},$$

$$\mathrm{Card}(T_{\lambda^2\beta_1^{2m}}(\Delta)\cap\Delta)=8p-4m;$$

$$T_{\lambda^2\alpha_1^2\beta_1^{2m}}(\delta)\in\Delta\Leftrightarrow\delta\in\{\lambda\alpha_1\beta_1^r\mid r\text{ is an odd integer},|r|\leq2p-1,|2m-r|\leq2p-1\},$$

$$\mathrm{Card}(T_{\lambda^2\alpha_1^2\beta_1^{2m}}(\Delta)\cap\Delta)=2p-m;$$

$$T_{\lambda^2\alpha_1\alpha_2\beta_1^{2m}}(\delta)\in\Delta\Leftrightarrow\delta\in\{\lambda\alpha_{1,2}\beta_1^r\mid$$

$$r\text{ is an odd integer},|r|\leq2p-1,|2m-r|\leq2p-1\},$$

$$\mathrm{Card}(T_{\lambda^2\alpha_1\alpha_2\beta_1^{2m}}(\Delta)\cap\Delta)=4p-2m.$$

It is easy to see that

$$\eta\in\{\lambda^2\}\Leftrightarrow\mathrm{Card}(T_\eta(\Delta)\cap\Delta)=8p,$$

$$\eta\in\{\lambda^2\beta_1^{\pm2}\}\Leftrightarrow\mathrm{Card}(T_\eta(\Delta)\cap\Delta)=8p-4.$$

In virtue of (0.3.1) $\{\lambda^2\}$ and $\{\lambda^2\beta_1^{\pm2}\}$ are $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant sets. The relation $\lambda^2=\pm p_v$ follows from the fact that $\lambda^2\in\mathbb{Q}$ is of absolute value p_v. Moreover, $\{\beta_1^{\pm2}\}$ is a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant set. Hence $\{\beta_1^{\pm2m}\}$ is a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant set for each natural m.

On the other hand,

$$\eta \in \{\lambda^2\alpha_1^{\pm1}\alpha_2^{\pm1}, \lambda^2\beta_1^{\pm2p}\} \Leftrightarrow \mathrm{Card}(T_\eta(\Delta)\cap\Delta) = 4p.$$

Since $\{\lambda^2\beta_1^{\pm2p}\}$ is a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant set, we see that $\{\lambda^2\alpha_1^{\pm1}\alpha_2^{\pm1}\}$ is also invariant under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

It is clear that

$$\eta \in \{\lambda^2\alpha_{1,2}^{\pm2}, \lambda^2\alpha_1^{\pm1}\alpha_2^{\pm1}\beta_1^{\pm2p}\} \Leftrightarrow \mathrm{Card}(T_\eta(\Delta)\cap\Delta) = 2p.$$

Since $\{\lambda^2\beta_1^{\pm2p}\}$ and $\{\lambda^2\alpha_1^{\pm1}\alpha_2^{\pm1}\}$ are $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant sets, we see that $\{\lambda^2\alpha_{1,2}^{\pm2}\}$ is also $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant. Lemma 2.26 is proved.

2.27. It is evident that the set of all eigenvalues of $\rho_l^{\wedge2}(F_{\overline{v}}^{-1})$ is equal to $\Delta\cdot\Delta - \{\lambda^2\alpha_{1,2}^{\pm2}\beta_1^{\pm(4p-2)}\}$.

By theorem 0.4 we may assume that p_v does not divide $\mathrm{Tr}(\rho_l^{\wedge2}(F_{\overline{v}}^{-1}))$. It is evident that this trace is a sum of integers of the following types:

$$\lambda^2,\ \sum\lambda^2\alpha_1^{\pm1}\alpha_2^{\pm1},\ \sum\lambda^2\alpha_{1,2}^{\pm2},$$

$$\lambda^2(\beta_1^{2m}+\beta_1^{-2m}),\ (\sum\lambda^2\alpha_1^{\pm1}\alpha_2^{\pm1})(\beta_1^{2m}+\beta_1^{-2m})\ (1\le m\le 2p-1),$$

$$(\sum\lambda^2\alpha_{1,2}^{\pm2})(\beta_1^{2m}+\beta_1^{-2m})\ (1\le m\le 2p-2).$$

Hence p_v does not divide at least one of the sums above. By the arguments of sections 2.12-2.13 we prove that

$$\sum\alpha_1^{\pm1}\alpha_2^{\pm1},\ \sum\alpha_{1,2}^{\pm2},\ (\beta_1^{2m}+\beta_1^{-2m})\ (1\le m\le 2p-1)$$

are integers. Hence p_v divides $\mathrm{Tr}(\rho_l^{\wedge2}(F_{\overline{v}}^{-1}))$ contrary to our assumption. This excludes the case $(A_1\times A_1\times A_1, E(\omega_1^{(1)}+\omega_1^{(2)}+(2p-1)\omega_1^{(3)}))$.

2.28. Now we may assume that $\mathrm{End}^0(J\otimes\overline{k})$ is a quaternion division algebra over $\mathbb{Q}$. This case is completely investigated in [20, th.1.16, th.1.17]. Theorem 0.7 is proved.

§3. On the l-adic representation associated to an abelian variety of the 4th type by Albert's classification

3.1. We give here a proof of theorem 0.9. After replacing k by some finite extension we may assume that for J the conditions (2.5.1)-(2.5.6) hold. In this case $G_{V_l}=S_{V_l}\cdot\mathrm{G}_m$, where S_{V_l} is the connected component of the identity of $G_{V_l}\cap\mathrm{SL}(V_l)$. By G.Faltings results [8]

$$\mathrm{End}_{S_{V_l}}(V_l) = \mathrm{End}_{G_{V_l}}(V_l) \simeq \mathrm{End}(J)\otimes\mathbb{Q}_l. \tag{3.1.1}$$

By the well known Tate theorems [22, th.3, th.4] this relation implies

$$1 = \mathrm{rank}\ \mathrm{NS}(J) = \dim_{\mathbb{Q}_l}(\wedge^2V_l)^{S_{V_l}}. \tag{3.1.2}$$

Schur's lemma and (3.1.1)-(3.1.2) give the decomposition $V_l \otimes \overline{\mathbb{Q}_l} = U \oplus U^*$, where U and U^* are $4p$-dimensional irreducible nonisomorphic dual Lie $S_{V_l} \otimes \overline{\mathbb{Q}_l}$-moduli,

$$\dim_{\mathbb{Q}} \mathrm{Cent}(\mathrm{Lie}(S_{V_l})) \leq 1. \tag{3.1.3}$$

3.2. Theorem 2.2 and the inequality $p \geq 17$ imply that the pair (type of $g_l^{ss} \otimes \overline{\mathbb{Q}_l}, U$) assumes one of the following values:

$$(A_{4p-1}, E(\omega_{1,4p-1})); \tag{3.2.1}$$

$$(D_{2p}, E(\omega_1)); \tag{3.2.2}$$

$$(C_{2p}, E(\omega_1)); \tag{3.2.3}$$

$$(A_1, E((4p-1)\omega_1)); \tag{3.2.4}$$

$$(A_1 \times C_p, E(\omega_1^{(1)} + \omega_1^{(2)})); \tag{3.2.5}$$

$$(A_1 \times D_p, E(\omega_1^{(1)} + \omega_1^{(2)})); \tag{3.2.6}$$

$$(A_1 \times A_1, E(\omega_1^{(1)} + (2p-1)\omega_1^{(2)})); \tag{3.2.7}$$

$$(A_1 \times A_1 \times A_1, E(\omega_1^{(1)} + \omega_1^{(2)} + (p-1)\omega_1^{(3)})); \tag{3.2.8}$$

$$(A_1 \times A_1 \times B_{(p-1)/2}, E(\omega_1^{(1)} + \omega_1^{(2)} + \omega_1^{(3)})); \tag{3.2.9}$$

$$(A_1 \times A_1, E(3\omega_1^{(1)} + (p-1)\omega_1^{(2)})); \tag{3.2.10}$$

$$(A_1 \times B_{(p-1)/2}, E(3\omega_1^{(1)} + \omega_1^{(2)})); \tag{3.2.11}$$

$$(C_2 \times A_1, E(\omega_1^{(1)} + (p-1)\omega_1^{(2)})); \tag{3.2.12}$$

$$(C_2 \times B_{(p-1)/2}, E(\omega_1^{(1)} + \omega_1^{(2)})); \tag{3.2.13}$$

$$(A_1 \times A_{2p-1}, E(\omega_1^{(1)} + \omega_{1,2p-1}^{(2)})); \tag{3.2.14}$$

$$(A_1 \times A_1 \times A_{p-1}, E(\omega_1^{(1)} + \omega_1^{(2)} + \omega_{1,p-1}^{(3)})); \tag{3.2.15}$$

$$(C_2 \times A_{p-1}, E(\omega_1^{(1)} + \omega_{1,p-1}^{(2)})); \tag{3.2.16}$$

$$(A_1 \times A_{p-1}, E(3\omega_1^{(1)} + \omega_{1,p-1}^{(2)})); \tag{3.2.17}$$

$$(A_3 \times A_1, E(\omega_{1,3}^{(1)} + (p-1)\omega_1^{(2)})); \tag{3.2.18}$$

$$(A_3 \times B_{(p-1)/2}, E(\omega_{1,3}^{(1)} + \omega_1^{(2)})); \tag{3.2.19}$$

$$(A_3 \times A_{p-1}, E(\omega_{1,3}^{(1)} + \omega_{1,p-1}^{(2)})). \tag{3.2.20}$$

3.3. Consider the case (3.2.1). We see that $g_l^{ss} \otimes \overline{\mathbb{Q}_l}$ is the Lie algebra of type A_{4p-1} and Lie $\mathrm{Hg}(J_{\mathbb{C}}) \otimes \overline{\mathbb{Q}_l} \subset gl(U)$. By Piatetski-Shapiro -Deligne - Borovoi theorem [14], [2] there exists a canonical embedding

$$\mathrm{Lie}\ \mathrm{Im}(\rho_l) \subset \mathrm{Lie}[\mathrm{MT}(J_{\mathbb{C}})(\mathbb{Q}_l)] = \mathbb{Q}_l \times \mathrm{Lie}[\mathrm{Hg}(J_{\mathbb{C}})(\mathbb{Q}_l)].$$

So there exists a canonical isomorphism of semisimple parts

$$[\text{Lie Im}(\rho_l)]^{ss} \simeq [\text{Lie}[\text{MT}(J_{\mathbf{C}})(\mathbb{Q}_l)]]^{ss}. \qquad (3.3.1)$$

Moreover, if $n_\sigma \neq 2p$ then $n_\sigma \neq n_\tau$ and $\dim_{\mathbb{Q}_l} \text{Cent}(G_{V_l}) \geq 2$ [5, sect.3.1, step 3]. Hence the isomorphism (3.3.1) extends to the isomorphism

$$\text{Lie Im}(\rho_l) \simeq \text{Lie}[\text{MT}(J_{\mathbf{C}})(\mathbb{Q}_l)].$$

So the Mumford - Tate conjecture holds for J, $\dim_{\mathbb{Q}_l} \text{Cent}(S_{V_l}) = 1 = \dim_{\mathbb{Q}_l} \text{Cent}[\text{Hg}(J_{\mathbf{C}})(\mathbb{Q}_l)]$. Hence

$$\dim_{\mathbb{Q}_l} H^{2r}_{et}(J \otimes \overline{k}, \mathbb{Q}_l(r))^{\text{Gal}(\overline{k}/k)} \leq 1.$$

[19, th.1.1]. It follows that the $\mathbb{Q}_l$-space $H^{2r}_{et}(J \otimes \overline{k}, \mathbb{Q}_l(r))^{\text{Gal}(\overline{k}/k)}$ is spanned by the cohomology classes of intersections of divisors.

3.4. We want to exclude all cases (3.2.2)-(3.2.19) by the following procedure.

First of all we note that in cases (3.2.2)-(3.2.13) there exists an isomorphism of $g_l^{ss} \otimes \overline{\mathbb{Q}_l}$-moduli $U \simeq U^*$. Hence the relation (3.1.3) implies the relations

$$U = E(\omega)(\chi),\ U^* = E(\omega)(-\chi), \qquad (3.4.1)$$

where ω is the highest weight of $g_l^{ss} \otimes \overline{\mathbb{Q}_l}$-module U and $\chi \neq 0$ is the highest weight of $\text{Cent}(\text{Lie}(S_{V_l} \otimes \overline{\mathbb{Q}_l}))$-module U.

3.5. Consider the cases (3.2.2)-(3.2.3). It follows from (3.4.1) that

$$\Delta = \{\lambda\alpha_1^{\pm 1}\beta_i^{\pm 1} \,|\, i = 1, ..., 2p\},$$

where $\lambda, \alpha_1, \beta_1, ..., \beta_{2p}$ are multiplicatively independent. Hence the structure of Δ in these cases coincides with the structure of Δ in the case (2.8.3). This excludes the cases (3.2.2)-(3.2.3).

3.6. Consider the case (3.2.4). In virtue of (3.4.1) we may assume that $\Delta = \{\lambda\alpha_1^{\pm 1}\beta_1^{\pm 1, \pm 3, ..., \pm(4p-1)}\}$, where $\lambda, \alpha_1, \beta_1$ are multiplicatively independent.

3.7. **Lemma**. *For each natural number m $(0 \leq m \leq 4p-1)$ the following subsets of $\Delta \cdot \Delta$ are* $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-invariant:*

$$\{\lambda^2\beta_1^{\pm 2m}\}; \{\lambda^2\alpha_1^{\pm 2}\beta_1^{\pm 2m}\}.$$

Moreover, $\lambda^2 = \pm p_v$.

Proof. Let $\eta \in \Delta \cdot \Delta$. We may assume that

$$\eta \in \{\lambda^2\beta_1^{2m}, \lambda^2\alpha_1^2\beta_1^{2m}\}.$$

If $\delta \in \Delta$, then $T_\eta(\delta) = \eta\delta^{-1}$, hence

$$T_{\lambda^2\beta_1^{2m}}(\delta) \in \Delta \Leftrightarrow \delta \in \{\lambda\alpha_1^{\pm 1}\beta_1^r \mid r \text{ is an odd integer}, |r| \leq 4p-1, |2m-r| \leq 4p-1\},$$

$$\text{Card}(T_{\lambda^2\beta_1^{2m}}(\Delta) \cap \Delta) = 8p - 2m;$$

$T_{\lambda^2\alpha_1^2\beta_1^{2m}}(\delta) \in \Delta \Leftrightarrow \delta \in \{\lambda\alpha_1\beta_1^r \mid r \text{ is an odd integer}, |r| \leq 4p-1, |2m-r| \leq 4p-1\}$,

$$\mathrm{Card}(T_{\lambda^2\alpha_1^2\beta_1^{2m}}(\Delta) \cap \Delta) = 4p - m.$$

It is easy to see that

$$\eta \in \{\lambda^2\} \Leftrightarrow \mathrm{Card}(T_\eta(\Delta) \cap \Delta) = 8p,$$

$$\eta \in \{\lambda^2\beta_1^{\pm 2}\} \Leftrightarrow \mathrm{Card}(T_\eta(\Delta) \cap \Delta) = 8p - 2.$$

In virtue of (0.3.1) $\{\lambda^2\}$ and $\{\lambda^2\beta_1^{\pm 2}\}$ are $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant sets. The relation $\lambda^2 = \pm p_v$ follows from the fact that $\lambda^2 \in \mathbb{Q}$ is of absolute value p_v. Moreover, $\{\beta_1^{\pm 2}\}$ is a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant set. Hence $\{\beta_1^{\pm 2m}\}$ is a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant set for each natural m.

On the other hand,

$$\eta \in \{\lambda^2\alpha_1^{\pm 2}, \lambda^2\beta_1^{\pm 4p}\} \Leftrightarrow \mathrm{Card}(T_\eta(\Delta) \cap \Delta) = 4p.$$

Since $\{\lambda^2\beta_1^{\pm 4p}\}$ is a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant set, we see that $\{\lambda^2\alpha_1^{\pm 2}\}$ is also $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant. Hence $\{\lambda^2\alpha_1^{\pm 2}\beta_1^{\pm 2m}\}$ is a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant set. Lemma 3.7 is proved.

3.8. It is evident that the set of all eigenvalues of $\rho_l^{\wedge 2}(F_{\overline{v}}^{-1})$ is equal to $\Delta \cdot \Delta - \{\lambda^2\alpha_1^{\pm 2}\beta_1^{\pm(8p-2)}\}$.

By theorem 0.4 we may assume that p_v does not divide $\mathrm{Tr}(\rho_l^{\wedge 2}(F_{\overline{v}}^{-1}))$. It is evident that this trace is a sum of integers of the following types:

$$\lambda^2,\ \sum \lambda^2\alpha_1^{\pm 2},\ \lambda^2(\beta_1^{2m} + \beta_1^{-2m}),\ (1 \leq m \leq 4p-1),$$

$$(\sum \lambda^2\alpha_1^{\pm 2})(\beta_1^{2m} + \beta_1^{-2m})\ (1 \leq m \leq 4p-2).$$

Hence p_v does not divide at least one of the sums above. By the arguments of sections 2.12-2.13 we prove that

$$\sum \alpha_1^{\pm 2},\ (\beta_1^{2m} + \beta_1^{-2m})\ (1 \leq m \leq 4p-1)$$

are integers. Hence p_v divides $\mathrm{Tr}(\rho_l^{\wedge 2}(F_{\overline{v}}^{-1}))$ contrary to our assumption. This excludes the case (3.2.4).

3.9. Consider the cases (3.2.5), (3.2.6). In virtue of (3.4.1) we may assume that $\Delta = \{\lambda\alpha_1^{\pm 1}\alpha_2^{\pm 1}\beta_i^{\pm 1} \mid i = 1, ..., p\}$, where $\lambda, \alpha_1, \alpha_2, \beta_1, ..., \beta_p$ are multiplicatively independent. Let $\gamma_1 = \alpha_1\alpha_2$, $\gamma_2 = \alpha_1^{-1}\alpha_2$. It is evident that $\Delta = \{\lambda\gamma_{1,2}^{\pm 1}\beta_i^{\pm 1} \mid i = 1, ..., p\}$, where $\lambda, \gamma_1, \gamma_2, \beta_1, ..., \beta_p$ are multiplicatively independent. Hence the structure of Δ is identical to the structure of Δ in the case (2.8.1). This excludes the cases (3.2.5),(3.2.6).

3.10. Consider the case (3.2.7). In virtue of (3.4.1) we may assume that $\Delta = \{\lambda\alpha_1^{\pm 1}\alpha_2^{\pm 1}\beta_1^{\pm 1, \pm 3, ..., \pm(2p-1)}\}$, where $\lambda, \alpha_1, \alpha_2, \beta_1$ are multiplicatively independent. Let $\gamma_1 = \alpha_1\alpha_2$, $\gamma_2 = \alpha_1^{-1}\alpha_2$. It is evident that $\Delta = \{\lambda\gamma_{1,2}^{\pm 1}\beta_1^{\pm 1, \pm 3, ..., \pm(2p-1)}\}$, where $\lambda, \gamma_1, \gamma_2, \beta_1$ are multiplicatively independent. Hence we may exclude this case by the arguments of sections 2.26-2.27.

3.11. Consider the cases (3.2.8),(3.2.9). In this situation we may assume that $\Phi = \{\lambda\alpha_1^{\pm 1}\alpha_2^{\pm 1}\alpha_3^{\pm 1}\}$, where $\lambda, \alpha_1, \alpha_2, \alpha_3$ are multiplicatively independent, $\Psi = \{\beta_j \mid j = 1, ..., p\}$. We may exclude the cases (3.2.8)-(3.2.9) by the arguments of section 2.24.

3.12. Consider the cases (3.2.10)-(3.2.11). In virtue of (3.4.1) we may assume that $\Delta = \{\lambda\alpha_1^{\pm 1}\alpha_2^{\pm 1,\pm 3}\} \cdot \{\beta_1, ..., \beta_p\}$, where $\lambda, \alpha_1, \alpha_2$ are multiplicatively independent, $1 \in \{\beta_1, ..., \beta_p\}$. Let $\Phi = \{\lambda\alpha_1^{\pm 1}\alpha_2^{\pm 1,\pm 3}\}$, $\Psi = \{\beta_1, ..., \beta_p\}$. We know that $1 \in \Psi$. Hence $\Phi \subset \Gamma_v$, $\Psi \subset \Gamma_v$. In the notations of section 2.24 we have: Γ_Φ and Γ_Ψ are torsion-free abelian groups of positive rank,

$$\Gamma_\Phi \cap \Gamma_\Psi = (1). \tag{3.12.1}$$

We claim that

$$\gamma \in \{\lambda\alpha_1^{\pm 1}\alpha_2^{\pm 1}\} \Leftrightarrow \mathrm{Card}(T_\gamma^0(\Delta) \cap \Delta) \geq 3p. \tag{3.12.2}$$

Indeed, we may assume that $\gamma \in \{\lambda\alpha_1\alpha_2\beta_i, \lambda\alpha_1\alpha_2^3\beta_i \mid i = 1, ..., p\}$. For $\delta \in \Delta$ we have $T_\gamma^0(\delta) = \gamma^2\delta^{-1}$. Hence the multiplicative independence of $\lambda, \alpha_1, \alpha_2$ and (3.12.1) imply the relation

$$T^0_{\lambda\alpha_1\alpha_2\beta_i}(\delta) = (\lambda\alpha_1\alpha_2\beta_i)^2 \cdot \delta^{-1} \in \Delta \Leftrightarrow \delta \in \{\lambda\alpha_1\alpha_2^{\pm 1,3} \cdot \beta_i^2 \cdot \Psi\} \cap \Delta.$$

It is evident that

$$\mathrm{Card}(T^0_{\lambda\alpha_1\alpha_2\beta_i}(\Delta) \cap \Delta) \geq 3p \Leftrightarrow \beta_i^2 \cdot \Psi = \Psi.$$

From the arguments of section 2.24 it follows that the relation $\beta_i^2 \cdot \Psi = \Psi$ implies $\beta_i = 1$. Hence $\gamma = \lambda\alpha_1\alpha_2$. On the other hand,

$$T^0_{\lambda\alpha_1\alpha_2^3\beta_i}(\delta) = (\lambda\alpha_1\alpha_2^3\beta_i)^2 \cdot \delta^{-1} \in \Delta \Leftrightarrow \delta \in \{\lambda\alpha_1\alpha_2^3 \cdot \beta_i^2 \cdot \Psi\} \cap \Delta,$$

and $\mathrm{Card}(T^0_{\lambda\alpha_1\alpha_2^3\beta_i}(\Delta)\cap\Delta) \leq p$. So the claim (3.12.2) is proved. In virtue of (2.24.4) the set $\{\lambda\alpha_1^{\pm 1}\alpha_2^{\pm 1}\}$ is $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant. Hence $\sum \lambda\alpha_1^{\pm 1}\alpha_2^{\pm 1} \in \mathbb{Z}$. Moreover, $\sum \lambda\alpha_1^{\pm 1}\alpha_2^{\pm 1} \neq 0$: otherwise we would have

$$0 = \lambda(\alpha_1 + \alpha_1^{-1})(\alpha_2 + \alpha_2^{-1})$$

and hence $\alpha_1 = -\alpha_1^{-1}$ or $\alpha_2 = -\alpha_2^{-1}$; assume, for example, that $\alpha_1 = -\alpha_1^{-1}$; then

$$\lambda\alpha_1\alpha_2 = -\lambda\alpha_1^{-1}\alpha_2 \in \Delta,\ \lambda\alpha_1^{-1}\alpha_2 \in \Delta,\ -1 = \lambda\alpha_1\alpha_2/(\lambda\alpha_1^{-1}\alpha_2) \in \Gamma_v$$

contrary to the condition (2.5.5).

On the other hand, the absolute value of $\sum \lambda\alpha_1^{\pm 1}\alpha_2^{\pm 1}$ is less than or equal to $4\sqrt{p_v}$. Hence for $p_v >> 0$ we get the relation $\sum \lambda\alpha_1^{\pm 1}\alpha_2^{\pm 1} \neq 0(\mathrm{mod}\ p_v)$. Then for each place w of $\overline{\mathbb{Q}}$ lying over p_v

$$w(\sum \lambda\alpha_1^{\pm 1}\alpha_2^{\pm 1}) = 0.$$

It follows that there exists $x_w \in \{\lambda\alpha_1^{\pm 1}\alpha_2^{\pm 1}\}$ such that $w(x_w) = 0$. Hence we obtain the relation (2.24.6) in the new situation. It follows that $\forall j\ \beta_j$ is a root of 1 contrary to the relation $\mathrm{rank}(\Gamma_\Psi) \geq 1$. Thus the cases (3.2.10)-(3.2.11) are excluded.

3.13. Consider the cases (3.2.12)-(3.2.13). In virtue of (3.4.1) we may assume that $\Delta = \{\lambda\alpha_1^{\pm 1}\alpha_{2,3}^{\pm 1}\} \cdot \{\beta_1, ..., \beta_p\}$, where $\lambda, \alpha_1, \alpha_2, \alpha_3$ are multiplicatively independent, $1 \in \{\beta_1, ..., \beta_p\}$. Let $\Phi = \{\lambda\alpha_1^{\pm 1}\alpha_{2,3}^{\pm 1}\}$, $\Psi = \{\beta_1, ..., \beta_p\}$. In the notations of section 2.24 we have: Γ_Φ and Γ_Ψ are torsion-free abelian groups of positive rank,

$$\Gamma_\Phi \cap \Gamma_\Psi = (1). \tag{3.13.1}$$

We claim that the relation (2.24.8) is true in this new situation. Indeed, we may assume that $\gamma \in \{\lambda\alpha_1\alpha_2\beta_i \mid i = 1, ..., p\}$. For $\delta \in \Delta$ we have $T^0_\gamma(\delta) = \gamma^2\delta^{-1}$. Hence the multiplicative independence of $\lambda, \alpha_1, \alpha_2, \alpha_3$ and (3.13.1) imply the relation

$$T^0_{\lambda\alpha_1\alpha_2\beta_i}(\delta) = (\lambda\alpha_1\alpha_2\beta_i)^2 \cdot \delta^{-1} \in \Delta \Leftrightarrow \delta \in \{\lambda\alpha_1\alpha_2 \cdot \beta_i^2 \cdot \Psi\} \cap \Delta.$$

It is easy to see that

$$\mathrm{Card}(T^0_{\lambda\alpha_1\alpha_2\beta_i}(\Delta) \cap \Delta) \geq p \Leftrightarrow \beta_i^2 \cdot \Psi = \Psi \Leftrightarrow \beta_i = 1.$$

Hence the claim (2.24.8) is proved. Thus we may exclude the cases (3.2.12)-(3.2.13) by the arguments of section 2.24.

3.14. Consider the case (3.2.14). We may assume that $U = E(\omega_1^{(1)} + \omega_1^{(2)})(\chi)$, $U^* = E(\omega_1^{(1)} + \omega_{2p-1}^{(2)})(-\chi)$, where χ is the highest weight of $\mathrm{Cent}(\mathrm{Lie}(S_{V_l} \otimes \overline{\mathbb{Q}_l}))$-module U. After replacing p by $2p$ we may use the arguments of [19, sect.4.20-4.26] in order to exclude this variant.

3.15. Consider the cases (3.2.15)-(3.2.16).

Assume, for example, that $\chi \neq 0$. In this situation

$$\Delta = \{\lambda\alpha_1\beta_{1,2}^{\pm 1}\} \cdot \{\delta_1, ..., \delta_{p-1}, (\delta_1\delta_2...\delta_{p-1})^{-1}\} \cup$$
$$\{\lambda\alpha_1^{-1}\beta_{1,2}^{\pm 1}\} \cdot \{\delta_1^{-1}, ..., \delta_{p-1}^{-1}, \delta_1\delta_2...\delta_{p-1})\},$$

where $\lambda, \alpha_1, \beta_1, \beta_2, \delta_1, ..., \delta_{p-1}$ are multiplicatively independent. The next lemma follows from (0.3.1).

3.16. **Lemma.** *The following subsets of* $\Delta \cdot \Delta$ *are* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))$*-invariant:*

$$\{\lambda^2\};\ \{\lambda^2\beta_1^{\pm 1}\beta_2^{\pm 1}\};\ \{\lambda^2\beta_{1,2}^{\pm 2}\};$$

$$\{\lambda^2\delta_i\delta_j^{-1} \mid i \neq j\} \cup \{\lambda^2(\delta_1...\delta_i^2...\delta_{p-1})^{\pm 1}\} \cup$$
$$\{\lambda^2(\alpha_1^2\delta_i\delta_j)^{\pm 1} \mid i \neq j\} \cup \{\lambda^2(\alpha_1^2(\delta_1...\delta_i^\wedge...\delta_{p-1})^{-1})^{\pm 1}\};$$
$$\{\lambda^2\beta_1^{\pm 1}\beta_2^{\pm 1}\delta_i\delta_j^{-1} \mid i \neq j\} \cup \{\lambda^2\beta_1^{\pm 1}\beta_2^{\pm 1}(\delta_1...\delta_i^2...\delta_{p-1})^{\pm 1}\} \cup$$
$$\{\lambda^2\beta_1^{\pm 1}\beta_2^{\pm 1}(\alpha_1^2\delta_i\delta_j)^{\pm 1} \mid i \neq j\} \cup \{\lambda^2\beta_1^{\pm 1}\beta_2^{\pm 1}(\alpha_1^2(\delta_1...\delta_i^\wedge...\delta_{p-1})^{-1})^{\pm 1}\};$$
$$\{\lambda^2(\alpha_1^2\delta_i^2)^{\pm 1}\} \cup \{\lambda^2(\alpha_1^2(\delta_1...\delta_{p-1})^{-2})^{\pm 1}\};$$

$$\{\lambda^2\beta_{1,2}^{\pm 2}\delta_i\delta_j^{-1} \mid i \neq j\} \cup \{\lambda^2\beta_{1,2}^{\pm 2}(\delta_1...\delta_i^2...\delta_{p-1})^{\pm 1}\}\cup$$
$$\{\lambda^2\beta_{1,2}^{\pm 2}(\alpha_1^2\delta_i\delta_j)^{\pm 1} \mid i \neq j\} \cup \{\lambda^2\beta_{1,2}^{\pm 2}(\alpha_1^2(\delta_1...\delta_i^\wedge...\delta_{p-1})^{-1})^{\pm 1}\};$$
$$\{\lambda^2\beta_1^{\pm 1}\beta_2^{\pm 1}(\alpha_1^2\delta_i^2)^{\pm 1}\} \cup \{\lambda^2\beta_1^{\pm 1}\beta_2^{\pm 1}(\alpha_1^2(\delta_1...\delta_{p-1})^{-2})^{\pm 1}\};$$
$$\{\lambda^2\beta_{1,2}^{\pm 2}(\alpha_1^2\delta_i^2)^{\pm 1}\} \cup \{\lambda^2\beta_{1,2}^{\pm 2}(\alpha_1^2(\delta_1...\delta_{p-1})^{-2})^{\pm 1}\},$$

where $\delta_i^\wedge$ *means that* δ_i *is omited. Moreover* $\lambda^2 = \pm p_v$.

Indeed, it is easy to see that

$$\mathrm{Card}(T_{\lambda^2}(\Delta) \cap \Delta) = 8p;$$
$$\mathrm{Card}(T_{\lambda^2\beta_1^{\pm 1}\beta_2^{\pm 1}}(\Delta) \cap \Delta) = 4p;$$
$$\mathrm{Card}(T_{\lambda^2\beta_{1,2}^{\pm 2}}(\Delta) \cap \Delta) = 2p;$$
$$\mathrm{Card}(T_{\lambda^2\delta_i\delta_j^{-1}}(\Delta) \cap \Delta) = \mathrm{Card}(T_{\lambda^2(\delta_1...\delta_i^2...\delta_{p-1})^{\pm 1}}(\Delta) \cap \Delta) =$$
$$\mathrm{Card}(T_{\lambda^2(\alpha_1^2\delta_i\delta_j)^{\pm 1}}(\Delta) \cap \Delta) = \mathrm{Card}(T_{\lambda^2(\alpha_1^2(\delta_1...\delta_i^\wedge...\delta_{p-1})^{-1})^{\pm 1}}(\Delta) \cap \Delta) = 8;$$
$$\mathrm{Card}(T_{\lambda^2\beta_1^{\pm 1}\beta_2^{\pm 1}\delta_i\delta_j^{-1}}(\Delta) \cap \Delta) = \mathrm{Card}(T_{\lambda^2\beta_1^{\pm 1}\beta_2^{\pm 1}(\delta_1...\delta_i^2...\delta_{p-1})^{\pm 1}}(\Delta) \cap \Delta) =$$
$$\mathrm{Card}(T_{\lambda^2\beta_1^{\pm 1}\beta_2^{\pm 1}(\alpha_1^2\delta_i\delta_j)^{\pm 1}}(\Delta)\cap\Delta) = \mathrm{Card}(T_{\lambda^2\beta_1^{\pm 1}\beta_2^{\pm 1}(\alpha_1^2(\delta_1...\delta_i^\wedge...\delta_{p-1})^{-1})^{\pm 1}}(\Delta)\cap\Delta) =$$
$$4 = \mathrm{Card}(T_{\lambda^2(\alpha_1^2\delta_i^2)^{\pm 1}}(\Delta) \cap \Delta) = \mathrm{Card}(T_{\lambda^2(\alpha_1^2(\delta_1...\delta_{p-1})^{-2})^{\pm 1}}(\Delta) \cap \Delta);$$
$$\mathrm{Card}(T_{\lambda^2\beta_{1,2}^{\pm 2}\delta_i\delta_j^{-1}}(\Delta) \cap \Delta) = \mathrm{Card}(T_{\lambda^2\beta_{1,2}^{\pm 2}(\delta_1...\delta_i^2...\delta_{p-1})^{\pm 1}}(\Delta) \cap \Delta) =$$
$$\mathrm{Card}(T_{\lambda^2\beta_{1,2}^{\pm 2}(\alpha_1^2\delta_i\delta_j)^{\pm 1}}(\Delta) \cap \Delta) = \mathrm{Card}(T_{\lambda^2\beta_{1,2}^{\pm 2}(\alpha_1^2(\delta_1...\delta_i^\wedge...\delta_{p-1})^{-1})^{\pm 1}}(\Delta) \cap \Delta) =$$
$$2 = \mathrm{Card}(T_{\lambda^2\beta_1^{\pm 1}\beta_2^{\pm 1}(\alpha_1^2\delta_i^2)^{\pm 1}}(\Delta) \cap \Delta) = \mathrm{Card}(T_{\lambda^2\beta_1^{\pm 1}\beta_2^{\pm 1}(\alpha_1^2(\delta_1...\delta_{p-1})^{-2})^{\pm 1}}(\Delta) \cap \Delta);$$
$$\mathrm{Card}(T_{\lambda^2\beta_{1,2}^{\pm 2}(\alpha_1^2\delta_i^2)^{\pm 1}}(\Delta) \cap \Delta) = \mathrm{Card}(T_{\lambda^2\beta_{1,2}^{\pm 2}(\alpha_1^2(\delta_1...\delta_{p-1})^{-2})^{\pm 1}}(\Delta) \cap \Delta) = 1.$$

Then we may use some decompositions which are similar to the decompositions of section 2.10.

3.17. It is evident that the set of all eigenvalues of $\rho_l^{\wedge 2}(F_{\overline{v}}^{-1})$ is equal to $\Delta \cdot \Delta - \{\lambda^2\beta_{1,2}^{\pm 2}(\alpha_1^2\delta_i^2)^{\pm 1}\} - \{\lambda^2\beta_{1,2}^{\pm 2}(\alpha_1^2(\delta_1...\delta_{p-1})^{-2})^{\pm 1}\}$.

By theorem 0.4 we may assume that p_v does not divide $\mathrm{Tr}(\rho_l^{\wedge 2}(F_{\overline{v}}^{-1}))$. The symmetry implies that the multiplicity of $\eta \in \{\lambda^2\beta_1^{\pm 1}\beta_2^{\pm 1}\}$ as an eigenvalue of $\rho_l^{\wedge 2}(F_{\overline{v}}^{-1})$ is independent of the choice of η. This is valid for another $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant subsets which are defined in the statement of lemma 3.16. We deduce from this lemma that $\mathrm{Tr}(\rho_l^{\wedge 2}(F_{\overline{v}}^{-1}))$ is a sum of integers of the following types:

$$\lambda^2, \sum \lambda^2\beta_1^{\pm 1}\beta_2^{\pm 1}, ..., \sum \lambda^2\beta_1^{\pm 1}\beta_2^{\pm 1}(\alpha_1^2\delta_i^2)^{\pm 1} + \sum \lambda^2\beta_1^{\pm 1}\beta_2^{\pm 1}(\alpha_1^2(\delta_1...\delta_{p-1})^{-2})^{\pm 1}.$$

Hence p_v does not divide at least one of the sums above.

Assume that p_v does not divide $\sum \lambda^2\beta_1^{\pm 1}\beta_2^{\pm 1}$. Then for each place w of $\overline{\mathbb{Q}}$ lying over p_v there exists $x_w \in \{\lambda^2\beta_1^{\pm 1}\beta_2^{\pm 1}\}$ such that $w(x_w) = 0$. Hence

$$0 = w(x_w) = \frac{1}{2}\{w(x_w\delta_1\delta_2^{-1}) + w(x_w\delta_1^{-1}\delta_2)\}.$$

Since both summands in the last brackets are nonnegative in virtue of the relations $x_w(\delta_1\delta_2^{-1})^{\pm1} \in \Delta\cdot\Delta$, we have the equalities

$$w(x_w\delta_1\delta_2^{-1}) = w(x_w\delta_1^{-1}\delta_2) = 0.$$

So $w(\delta_1\delta_2^{-1}) = 0$ for *all* $w|p_v$. It follows that $\delta_1\delta_2^{-1}$ is a root of 1 [23, sublemma 3.4.0] contrary to the assumption that δ_1, δ_2 are multiplicatively independent.

Hence p_v divides $\sum\lambda^2\beta_1^{\pm1}\beta_2^{\pm1}$. From the relation $\lambda^2 = \pm p_v$ we deduce that $\sum\beta_1^{\pm1}\beta_2^{\pm1}$ is an integer.

By the similar arguments we prove that $\sum\beta_{1,2}^{\pm2}$ is an integer.

Using the relation

$$w(x_w) = \frac{1}{2}\{w(x_w\beta_1^2) + w(x_w\beta_1^{-2})\}$$

for $x_w \in \{\lambda^2\delta_i\delta_j^{-1} \mid i \neq j\} \cup \{\lambda^2(\delta_1...\delta_i^2...\delta_{p-1})^{\pm1}\} \cup \{\lambda^2(\alpha_1^2\delta_i\delta_j)^{\pm1} \mid i \neq j\} \cup \{\lambda^2(\alpha_1^2(\delta_1...\delta_i^{\wedge}...\delta_{p-1})^{-1})^{\pm1}\}$ we deduce that

$$\sum_{i\neq j}\delta_i\delta_j^{-1} + \sum(\delta_1...\delta_i^2...\delta_{p-1})^{\pm1} + \sum_{i\neq j}(\alpha_1^2\delta_i\delta_j)^{\pm1} + \sum(\alpha_1^2(\delta_1...\delta_i^{\wedge}...\delta_{p-1})^{-1})^{\pm1}$$

is an integer. By the similar arguments we show that

$$\sum(\alpha_1^2\delta_i^2)^{\pm1} + \sum(\alpha_1^2(\delta_1...\delta_{p-1})^{-2})^{\pm1}$$

is an integer. Hence $\sum\beta_1^{\pm1}\beta_2^{\pm1}(\alpha_1^2\delta_i^2)^{\pm1} + \sum\beta_1^{\pm1}\beta_2^{\pm1}(\alpha_1^2(\delta_1...\delta_{p-1})^{-2})^{\pm1}$ is an integer. Thus p_v divides $\mathrm{Tr}(\rho_l^{\wedge 2}(F_{\bar v}^{-1}))$ contrary to our assumptions. We may exclude the case $\chi = 0$ by the same arguments.

3.18. Consider the case (3.2.17). Assume, for example, that $\chi \neq 0$. In this situation

$$\Delta = \{\lambda\alpha_1\beta_1^{\pm1,\pm3}\}\cdot\{\delta_1, ..., \delta_{p-1}, (\delta_1\delta_2...\delta_{p-1})^{-1}\}\cup$$
$$\{\lambda\alpha_1^{-1}\beta_1^{\pm1,\pm3}\}\cdot\{\delta_1^{-1}, ..., \delta_{p-1}^{-1}, \delta_1\delta_2...\delta_{p-1}\},$$

where $\lambda, \alpha_1, \beta_1, \delta_1, ..., \delta_{p-1}$ are multiplicatively independent. The next lemma follows from (0.3.1).

3.19. **Lemma**. *The following subsets of* $\Delta\cdot\Delta$ *are* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))$*-invariant:*

$$\{\lambda^2\};\ \{\lambda^2\beta_1^{\pm2}\};\ ;\{\lambda^2\beta_1^{\pm4}\};\ \{\lambda^2\beta_1^{\pm6}\};$$

$$\{\lambda^2\delta_i\delta_j^{-1} \mid i\neq j\}\cup\{\lambda^2(\delta_1...\delta_i^2...\delta_{p-1})^{\pm1}\}\cup$$
$$\{\lambda^2(\alpha_1^2\delta_i\delta_j)^{\pm1} \mid i\neq j\}\cup\{\lambda^2(\alpha_1^2(\delta_1...\delta_i^{\wedge}...\delta_{p-1})^{-1})^{\pm1}\};$$

$$\{\lambda^2\beta_1^{\pm2}\delta_i\delta_j^{-1} \mid i\neq j\}\cup\{\lambda^2\beta_1^{\pm2}(\delta_1...\delta_i^2...\delta_{p-1})^{\pm1}\}\cup$$
$$\{\lambda^2\beta_1^{\pm2}(\alpha_1^2\delta_i\delta_j)^{\pm1} \mid i\neq j\}\cup\{\lambda^2\beta_1^{\pm2}(\alpha_1^2(\delta_1...\delta_i^{\wedge}...\delta_{p-1})^{-1})^{\pm1}\};$$

$$\{\lambda^2\beta_1^{\pm4}\delta_i\delta_j^{-1} \mid i\neq j\}\cup\{\lambda^2\beta_1^{\pm4}(\delta_1...\delta_i^2...\delta_{p-1})^{\pm1}\}\cup$$

$$\{\lambda^2\beta_1^{\pm 4}(\alpha_1^2\delta_i\delta_j)^{\pm 1} \mid i \neq j\} \cup \{\lambda^2\beta_1^{\pm 4}(\alpha_1^2(\delta_1...\delta_i^{\wedge}...\delta_{p-1})^{-1})^{\pm 1}\};$$

$$\{\lambda^2\beta_1^{\pm 6}\delta_i\delta_j^{-1} \mid i \neq j\} \cup \{\lambda^2\beta_1^{\pm 6}(\delta_1...\delta_i^2...\delta_{p-1})^{\pm 1}\} \cup$$

$$\{\lambda^2\beta_1^{\pm 6}(\alpha_1^2\delta_i\delta_j)^{\pm 1} \mid i \neq j\} \cup \{\lambda^2\beta_1^{\pm 6}(\alpha_1^2(\delta_1...\delta_i^{\wedge}...\delta_{p-1})^{-1})^{\pm 1}\};$$

$$\{\lambda^2(\alpha_1^2\delta_i^2)^{\pm 1}\} \cup \{\lambda^2(\alpha_1^2(\delta_1...\delta_{p-1})^{-2})^{\pm 1}\};$$

$$\{\lambda^2\beta_1^{\pm 2}(\alpha_1^2\delta_i^2)^{\pm 1}\} \cup \{\lambda^2\beta_1^{\pm 2}(\alpha_1^2(\delta_1...\delta_{p-1})^{-2})^{\pm 1}\};$$

$$\{\lambda^2\beta_1^{\pm 4}(\alpha_1^2\delta_i^2)^{\pm 1}\} \cup \{\lambda^2\beta_1^{\pm 4}(\alpha_1^2(\delta_1...\delta_{p-1})^{-2})^{\pm 1}\};$$

$$\{\lambda^2\beta_1^{\pm 6}(\alpha_1^2\delta_i^2)^{\pm 1}\} \cup \{\lambda^2\beta_1^{\pm 6}(\alpha_1^2(\delta_1...\delta_{p-1})^{-2})^{\pm 1}\};$$

where $\delta_i^{\wedge}$ *means that* δ_i *is omited. Moreover* $\lambda^2 = \pm p_v$.

3.20. It is evident that the set of all eigenvalues of $\rho_l^{\wedge 2}(F_{\overline{v}}^{-1})$ is equal to $\Delta \cdot \Delta - \{\lambda^2\beta_1^{\pm 6}(\alpha_1^2\delta_i^2)^{\pm 1}\} - \{\lambda^2\beta_1^{\pm 6}(\alpha_1^2(\delta_1...\delta_{p-1})^{-2})^{\pm 1}\}$. Hence we may exclude the case (3.2.17) by the procedure of section 3.17.

3.21. Consider the cases (3.2.18)-(3.2.19).

Assume, for example, that $\chi \neq 0$. In this situation

$$\Delta = \{\lambda\alpha_1\{\alpha_2, \alpha_3, \alpha_4, (\alpha_2\alpha_3\alpha_4)^{-1}\} \cup \lambda\alpha_1^{-1}\{\alpha_2^{-1}, \alpha_3^{-1}, \alpha_4^{-1}, \alpha_2\alpha_3\alpha_4\}\} \cdot \{\beta_1, ..., \beta_p\},$$

where $\lambda, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ are multiplicatively independent, $1 \in \{\beta_1, ..., \beta_p\}$. Let $\Phi = \lambda\alpha_1\{\alpha_2, \alpha_3, \alpha_4, (\alpha_2\alpha_3\alpha_4)^{-1}\} \cup \lambda\alpha_1^{-1}\{\alpha_2^{-1}, \alpha_3^{-1}, \alpha_4^{-1}, \alpha_2\alpha_3\alpha_4\}$, $\Psi = \{\beta_1, ..., \beta_p\}$. In the notations of section 2.24 we have: Γ_Φ and Γ_Ψ are torsion-free abelian groups of positive rank, $\Gamma_\Phi \cap \Gamma_\Psi = (1)$. We claim that

$$\gamma \in \Phi \Leftrightarrow \mathrm{Card}(T_\gamma^0(\Delta) \cap \Delta) \geq p. \tag{3.21.1}$$

Indeed, we may assume that $\gamma \in \{\lambda\alpha_1\alpha_2\beta_i,\ \lambda\alpha_1(\alpha_2\alpha_3\alpha_4)^{-1}\beta_i \mid i = 1, ..., p\}$. Hence the multiplicative independence of $\lambda, \alpha_1, ..., \alpha_4$ and the relation $\Gamma_\Phi \cap \Gamma_\Psi = (1)$ imply the relation

$$T^0_{\lambda\alpha_1\alpha_2\beta_i}(\delta) = (\lambda\alpha_1\alpha_2\beta_i)^2 \cdot \delta^{-1} \in \Delta \Leftrightarrow$$

$$\delta \in \{\lambda\alpha_1\alpha_2\beta_i^2 \cdot \Psi\} \cap \Delta.$$

It is clear that

$$\mathrm{Card}(T^0_{\lambda\alpha_1\alpha_2\beta_i}(\Delta) \cap \Delta) \geq p = \mathrm{Card}(\Psi) \Leftrightarrow \beta_i^2 \cdot \Psi = \Psi \Leftrightarrow \beta_i = 1$$

and

$$\mathrm{Card}(T^0_{\lambda\alpha_1(\alpha_2\alpha_3\alpha_4)^{-1}\beta_i}(\Delta) \cap \Delta) \geq p = \mathrm{Card}(\Psi) \Leftrightarrow \beta_i^2 \cdot \Psi = \Psi \Leftrightarrow \beta_i = 1.$$

Hence the claim (3.21.1) is proved. We see that the relation (2.24.7) holds in this new situation. Hence we may exclude the cases (3.2.18)-(3.2.19) by the arguments of section 2.24.

3.22. Finally we consider the case (3.2.20). Assume, for example, that $\chi \neq 0$. In this situation

$$\Delta = \lambda\alpha_1 \cdot \{\beta_1, \beta_2, \beta_3, (\beta_1\beta_2\beta_3)^{-1}\} \cdot \{\delta_1, ..., \delta_{p-1}, (\delta_1\delta_2...\delta_{p-1})^{-1}\} \cup$$

$$\lambda\alpha_1^{-1}\cdot\{\beta_1^{-1},\beta_2^{-1},\beta_3^{-1},\beta_1\beta_2\beta_3\}\cdot\{\delta_1^{-1},...,\delta_{p-1}^{-1},\delta_1\delta_2...\delta_{p-1}\},$$

where $\lambda,\alpha_1,\beta_1,\beta_2,\beta_3,\delta_1,...,\delta_{p-1}$ are multiplicatively independent. The next result follows from (0.3.1).

3.23. **Lemma**. *The following subsets of* $\Delta\cdot\Delta$ *are* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))$*-invariant:*

$$\{\lambda^2\};\ \{\lambda^2\beta_k\beta_l^{-1}\mid k\neq l\}\cup\{\lambda^2(\beta_1\beta_k^2\beta_3)^{\pm1}\};$$

$$\{\lambda^2\delta_i\delta_j^{-1}\mid i\neq j\}\cup\{\lambda^2(\delta_1...\delta_i^2...\delta_{p-1})^{\pm1}\};$$

$$\{\lambda^2(\beta_k\beta_l)^{\pm1}(\alpha_1^2\delta_i\delta_j)^{\pm1}\mid k\neq l,i\neq j\}\cup\{\lambda^2(\beta_k\beta_l)^{\pm1}(\alpha_1^2(\delta_1...\delta_i^\wedge...\delta_{p-1})^{-1})^{\pm1}\mid k\neq l\};$$

$$\{\lambda^2\beta_k\beta_l^{-1}\delta_i\delta_j^{-1}\mid k\neq l,i\neq j\}\cup\{\lambda^2\beta_k\beta_l^{-1}(\delta_1...\delta_i^2...\delta_{p-1})^{\pm1}\mid k\neq l\};$$

$$\{\lambda^2(\beta_1\beta_k^2\beta_3)^{\pm1}\delta_i\delta_j^{-1}\mid i\neq j\}\cup\{\lambda^2(\beta_1\beta_k^2\beta_3)^{\pm1}(\delta_1...\delta_i^2...\delta_{p-1})^{\pm1}\};$$

$$\{\lambda^2(\alpha_1^2\beta_k^2\delta_i\delta_j)^{\pm1}\mid i\neq j\}\cup\{\lambda^2(\alpha_1^2\beta_k^2(\delta_1...\delta_i^\wedge...\delta_{p-1})^{-1})^{\pm1}\}\cup$$

$$\{\lambda^2(\beta_k\beta_l)^{\pm1}(\alpha_1^2\delta_i^2)^{\pm1}\mid k\neq l\}\cup\{\lambda^2(\beta_k\beta_l)^{\pm1}(\alpha_1^2(\delta_1...\delta_{p-1})^{-2})^{\pm1}\mid k\neq l\}\cup$$

$$\{\lambda^2(\alpha_1^2(\beta_1\beta_2\beta_3)^{-2}\delta_i\delta_j)^{\pm1}\mid i\neq j\}\cup\{\lambda^2(\alpha_1^2(\beta_1\beta_2\beta_3)^{-2}(\delta_1...\delta_i^\wedge...\delta_{p-1})^{-1})^{\pm1}\};$$

$$\{\lambda^2(\alpha_1^2\beta_k^2\delta_i^2)^{\pm1}\}\cup\{\lambda^2(\alpha_1^2\beta_k^2(\delta_1...\delta_{p-1})^{-2})^{\pm1}\}\cup$$

$$\{\lambda^2(\alpha_1^2(\beta_1\beta_2\beta_3)^{-2}\delta_i^2)^{\pm1}\}\cup\{\lambda^2(\alpha_1^2(\beta_1\beta_2\beta_3)^{-2}(\delta_1...\delta_{p-1})^{-2})^{\pm1}\};$$

where $\delta_i^\wedge$ *means that* δ_i *is omited. Moreover* $\lambda^2=\pm p_v$.

Indeed, it is easy to see that

$$\mathrm{Card}(T_{\lambda^2}(\Delta)\cap\Delta)=8p;$$

$$\mathrm{Card}(T_{\lambda^2\beta_k\beta_l^{-1}}(\Delta)\cap\Delta)=\mathrm{Card}(T_{\lambda^2(\beta_1\beta_k^2\beta_3)^{\pm1}}(\Delta)\cap\Delta)=2p;$$

$$\mathrm{Card}(T_{\lambda^2\delta_i\delta_j^{-1}}(\Delta)\cap\Delta)=\mathrm{Card}(T_{\lambda^2(\delta_1...\delta_i^2...\delta_{p-1})^{\pm1}}(\Delta)\cap\Delta)=8;$$

$$\mathrm{Card}(T_{\lambda^2(\beta_k\beta_l)^{\pm1}(\alpha_1^2\delta_i\delta_j)^{\pm1}}(\Delta)\cap\Delta)=$$

$$\mathrm{Card}(T_{\lambda^2(\beta_k\beta_l)^{\pm1}(\alpha_1^2(\delta_1...\delta_i^\wedge...\delta_{p-1})^{-1})^{\pm1}}(\Delta)\cap\Delta)=4;$$

$$\mathrm{Card}(T_{\lambda^2\beta_k\beta_l^{-1}\delta_i\delta_j^{-1}}(\Delta)\cap\Delta)=\mathrm{Card}(T_{\lambda^2(\beta_1\beta_k^2\beta_3)^{\pm1}\delta_i\delta_j^{-1}}(\Delta)\cap\Delta)=$$

$$\mathrm{Card}(T_{\lambda^2\beta_k\beta_l^{-1}(\delta_1...\delta_i^2...\delta_{p-1})^{\pm1}}(\Delta)\cap\Delta)=\mathrm{Card}(T_{\lambda^2(\beta_1\beta_k^2\beta_3)^{\pm1}(\delta_1...\delta_i^2...\delta_{p-1})^{\pm1}}(\Delta)\cap\Delta)$$

$$=\mathrm{Card}(T_{\lambda^2(\alpha_1^2\beta_k^2\delta_i\delta_j)^{\pm1}}(\Delta)\cap\Delta)=\mathrm{Card}(T_{\lambda^2(\alpha_1^2\beta_k^2(\delta_1...\delta_i^\wedge...\delta_{p-1})^{-1})^{\pm1}}(\Delta)\cap\Delta)=$$

$$\mathrm{Card}(T_{\lambda^2(\beta_k\beta_l)^{\pm1}(\alpha_1^2\delta_i^2)^{\pm1}}(\Delta)\cap\Delta)=\mathrm{Card}(T_{\lambda^2(\beta_k\beta_l)^{\pm1}(\alpha_1^2(\delta_1...\delta_{p-1})^{-2})^{\pm1}}(\Delta)\cap\Delta)=$$

$$\mathrm{Card}(T_{\lambda^2(\alpha_1^2(\beta_1\beta_2\beta_3)^{-2}\delta_i\delta_j)^{\pm1}}(\Delta)\cap\Delta)=$$

$$\mathrm{Card}(T_{\lambda^2(\alpha_1^2(\beta_1\beta_2\beta_3)^{-2}(\delta_1...\delta_i^\wedge...\delta_{p-1})^{-1})^{\pm1}}(\Delta)\cap\Delta)=2;$$

$$\mathrm{Card}(T_{\lambda^2(\alpha_1^2\beta_k^2\delta_i^2)^{\pm1}}(\Delta)\cap\Delta)=$$

$$\mathrm{Card}(T_{\lambda^2(\alpha_1^2\beta_k^2(\delta_1...\delta_{p-1})^{-2})^{\pm1}}(\Delta)\cap\Delta)=$$

$$\mathrm{Card}(T_{\lambda^2(\alpha_1^2(\beta_1\beta_2\beta_3)^{-2}\delta_i^2)^{\pm1}}(\Delta)\cap\Delta)=$$

$$\mathrm{Card}(T_{\lambda^2(\alpha_1^2(\beta_1\beta_2\beta_3)^{-2}(\delta_1...\delta_{p-1})^{-2})^{\pm1}}(\Delta)\cap\Delta)=1.$$

3.24. It is evident that the set of all eigenvalues of $\rho_l^{\wedge 2}(F_{\overline{v}}^{-1})$ is equal to

$$\Delta\cdot\Delta-\{\lambda^2(\alpha_1^2\beta_k^2\delta_i^2)^{\pm1}\}-\{\lambda^2(\alpha_1^2\beta_k^2(\delta_1...\delta_{p-1})^{-2})^{\pm1}\}-$$

$$\{\lambda^2(\alpha_1^2(\beta_1\beta_2\beta_3)^{-2}\delta_i^2)^{\pm1}\}-\{\lambda^2(\alpha_1^2(\beta_1\beta_2\beta_3)^{-2}(\delta_1...\delta_{p-1})^{-2})^{\pm1}\}.$$

On the other hand, the elements of the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))$-invariant set

$$\{\lambda^2(\beta_k\beta_l)^{\pm1}(\alpha_1^2\delta_i\delta_j)^{\pm1}\mid k\neq l, i\neq j\}\cup\{\lambda^2(\beta_k\beta_l)^{\pm1}(\alpha_1^2(\delta_1...\delta_i^{\wedge}...\delta_{p-1})^{-1})^{\pm1}\mid k\neq l\}$$

have the very "mixed" multiplicative structure. So we can't use here the usual technique of sections 2.12-2.13, 3.17.

3.25. Assume that J has many ordinary reductions. In this situation we may choose v such that the conditions (2.5.1)-(2.5.6) and an additional condition

$$\frac{w(\Delta\cdot\Delta)}{w(p_v^2)}\subset\{0,\frac{1}{2},1\}. \qquad (3.25.1)$$

hold, where w is an arbitrary place of $\overline{\mathbb{Q}}$ lying over p_v. It follows from (0.5.2).

Suppose that

$$\frac{w(\lambda^2\beta_1\beta_2^{-1})}{w(p_v^2)}=0$$

for some place $w|p_v$. Then for each $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

$$\frac{(\sigma w)(\sigma(\lambda^2\beta_1\beta_2^{-1}))}{(\sigma w)(p_v^2)}=0,$$

hence from the relation

$$\sigma(\lambda^2\beta_1\beta_2^{-1})\in\{\lambda^2\beta_k\beta_l^{-1}\mid k\neq l\}\cup\{\lambda^2(\beta_1\beta_k^2\beta_3)^{\pm1}\}$$

obtained above and from the transitivity of the natural action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\{\ w|w$ is a place of $\overline{\mathbb{Q}}$ over $p_v\ \}$ it follows that $\forall w|p_v\ \exists x_w\in\{\lambda^2\beta_k\beta_l^{-1}\mid k\neq l\}\cup\{\lambda^2(\beta_1\beta_k^2\beta_3)^{\pm1}\}$ such that $w(x_w)=0$.

So, $\forall w\mid p_v$

$$0=w(x_w)=\frac{1}{2}\{w(x_w\delta_1\delta_2^{-1})+w(x_w\delta_1^{-1}\delta_2)\}.$$

Since both summands in the last brackets are nonnegative in virtue of the relations $x_w(\delta_1\delta_2^{-1})^{\pm1}\in\Delta\cdot\Delta$, we have the equalities

$$w(x_w\delta_1\delta_2^{-1})=w(x_w\delta_1^{-1}\delta_2)=0.$$

So $w(\delta_1\delta_2^{-1})=0$ for *all* $w|p_v$. It follows that $\delta_1\delta_2^{-1}$ is a root of 1 [23, sublemma 3.4.0] contrary to the assumption that δ_1,δ_2 are multiplicatively independent.

Suppose that

$$\frac{w(\lambda^2\beta_1\beta_2^{-1})}{w(p_v^2)} = 1$$

for some place w. Let ρ be a complex conjugation defined by some fixed embedding $\overline{\mathbb{Q}} \subset \mathbb{C}$. It is well known that

$$\frac{w(\lambda^2\beta_1\beta_2^{-1})}{w(p_v^2)} + \frac{(\rho w)(\lambda^2\beta_1\beta_2^{-1})}{(\rho w)(p_v^2)} = 1$$

[18,(3.16.2)]. So in our situation we have the impossible relation

$$\frac{(\rho w)(\lambda^2\beta_1\beta_2^{-1})}{(\rho w)(p_v^2)} = 0.$$

Hence

$$\frac{w(\lambda^2(\beta_1\beta_2^{-1})^{\pm 1})}{w(p_v^2)} = \frac{1}{2}$$

for all places $w|p_v$. It follows that $\beta_1\beta_2^{-1}$ is a root of 1 [23, sublemma 3.4.0] contrary to the assumption that β_1, β_2 are multiplicatively independent. So $g_l^{ss} \otimes \overline{\mathbb{Q}_l}$ is not the Lie algebra of type $A_3 \times A_{p-1}$. Theorem 0.9 is proved.

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