# ON SOME SYSTEMS OF DIFFERENCE EQUATIONS. Part 8.

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*To 100th birthday of*
*Professor A.O.Gelfond.*

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## §8.0. Foreword.

Let

$$|z| \geq 1, -3\pi/2 < \arg(z) \leq \pi/2, \log(z) = \ln(|z|) + i\arg(z).$$

Then $\log(-z) = \log(z) - i\pi$, if $\Re(z) > 0$ and $\log(z) = \log(-z) - i\pi$, if $\Re(z) < 0$.

Let

$$(1) \qquad f^{\vee}_{l,1}(z,\nu) = f_{l,1}(z,\nu) = \sum_{k=0}^{\nu} (-1)^{(\nu+k)l} (z)^k \binom{\nu}{k}^{2+l} \binom{\nu+k}{\nu}^{2+l},$$

where $l = 0,\ 1,\ 2,\ \nu \in [0,+\infty) \cap \mathbb{Z}$. Let

$$(2) \qquad R(t,\nu) = \frac{\prod\limits_{j=1}^{\nu}(t-j)}{\prod\limits_{j=0}^{\nu}(t+j)},$$

where $\nu \in [0,+\infty) \cap \mathbb{Z}$,

(3) $$f^{\vee}_{l,2}(z,\nu) = f_{l,2}(z,\nu) = \sum_{t=1+\nu}^{+\infty} z^{-t}(R(t,\nu))^{2+l},$$

where $l = 0,\ 1,\ 2$ and $\nu \in [0, +\infty) \cap \mathbb{Z}$, and since $(R(t,\nu))^{2+l}$ for $\nu \in \mathbb{N}$ has in the points $t = 1,\ \ldots,\ \nu$, the zeros of the order $2 + l$, it follows that

(4) $$f_{l,2}(z,\nu) = \sum_{t=1}^{+\infty} z^{-t}(R(t,\nu))^{2+l},$$

for $l = 0,\ 1,\ 2$ and $\nu \in [0, +\infty) \cap \mathbb{Z}$. Let

(5) $$f^{\vee}_{l,3}(z,\nu) = f_{l,3}(z,\nu) = (\log(z))f_{l,2}(z,\nu) + f_{l,4}(z,\nu),$$

where

(6) $$f_{l,4}(z,\nu) = -\sum_{t=1+\nu}^{+\infty} z^{-t}\left(\frac{\partial}{\partial t}(R^{2+l})\right)(t,\nu),$$

$l = 0,\ 1,\ 2$ and $\nu \in [0, +\infty) \cap \mathbb{Z}$, and since $(R(t,\nu))^{2+l}$ for $\nu \in \mathbb{N}$ has in the points $t = 1,\ \ldots,\ \nu$, the zeros of the order $2 + l$, it follows that

(7) $$f_{l,4}(z,\nu) = -\sum_{t=1}^{+\infty} z^{-t}\left(\frac{\partial}{\partial t}(R^{2+l})\right)(t,\nu)$$

for $l = 0,\ 1,\ 2$ and $\nu \in [0, +\infty) \cap \mathbb{Z}$. Let

(8) $$f^{\vee}_{l,5}(z,\nu) = -i\pi f_{l,3}(z,\nu) + f_{l,5}(z,\nu),$$

with $l = 1,\ 2,\ \nu \in [0, +\infty) \cap \mathbb{Z}$ and

(9) $$f_{l,5}(z,\nu) =$$
$$2^{-1}(\log(z))^2 f_{l,2}(z,\nu) + (\log(z))f_{l,4}(z,\nu) + f_{l,6}(z,\nu) =$$
$$= -2^{-1}(\log(z))^2 f_{l,2}(z,\nu) + (\log(z))f_{l,3}(z,\nu) + f_{l,6}(z,\nu),$$

where

(10) $$f_{l,6}(z,\nu) = 2^{-1}\sum_{t=1+\nu}^{\infty} z^{-t}\left(\left(\frac{\partial}{\partial t}\right)^2(R^{2+l})\right)(t,\nu),$$

and since $(R(t,\nu))^{2+l}$ for $\nu \in \mathbb{N}$ has in the points $t = 1,\ \ldots,\ \nu$, the zeros of the order $2 + l$, and $l = 1,\ 2$ now, it follows that

(11) $$f_{l,6}(z,\nu) = 2^{-1}\sum_{t=1+\nu}^{\infty} z^{-t}\left(\left(\frac{\partial}{\partial t}\right)^2(R^{2+l})\right)(t,\nu)$$

for $l = 1,\ 2$ and $\nu \in [0, +\infty) \cap \mathbb{Z}$. Let

(12) $$f^{\vee}_{l,7}(z,\nu) = f_{l,7}(z,\nu) + (2\pi^2/3) f_{l,3}(z,\nu).$$

with $l = 2$, $\nu \in [0, +\infty) \cap \mathbb{Z}$ and

(13) $$f_{l,7}(z,\nu) =$$

$$-3^{-1}(\log(z))^3 f_{l,2}(z,\nu) + 2^{-1}(\log(z))^2 f_{l,3}(z,\nu) + f_{l,8}(z,\nu)+$$

$$(\log(z))(f_{l,5}(z,\nu) + 2^{-1}(\log(z))^2 f_{l,2}(z,\nu) - (\log(z)) f_{l,3}(z,\nu)) =$$

$$6^{-1}(\log(z))^3 f_{l,2}(z,\nu) - 2^{-1}(\log(z))^2 f_{l,3}(z,\nu) + (\log(z)) f_{l,5}(z,\nu) + f_{l,8}(z,\nu) =$$

$$(1/6)(\log(z))^3 f_{l,2}(z,\nu) + (1/2)(\log(z))^2 f_{l,4}(z,\nu)+$$

$$(\log(z)) f_{l,6}(z,\nu) + f_{l,8}(z,\nu),$$

where

(14) $$f_{l,8}(z,\nu) = -6^{-1} \sum_{t=\nu+1}^{\infty} z^{-t} \left( \left( \frac{\partial}{\partial t} \right)^3 (R^{2+l}) \right)(t,\nu),$$

and, since $(R(t,\nu))^{2+l}$ for $\nu \in \mathbb{N}$ have in the points $t = 1, \ldots, \nu$, the zeros of the order $2+l$, and $l = 2$ now, it follows that

(15) $$f_{l,8}(z,\nu) = -6^{-1} \sum_{t=1}^{\infty} z^{-t} \left( \left( \frac{\partial}{\partial t} \right)^3 (R^{2+l}) \right)(t,\nu).$$

Let

$$\mathfrak{K}_0 = \{1,\ 2,\ 3\},\ \mathfrak{K}_1 = \{1,\ 2,\ 3,\ 5\},\ \mathfrak{K}_2 = \{1,\ 2,\ 3,\ 5,\ 7\}.$$

Let $\lambda$ be a variable. We denote by $T_{n,\lambda}$ the diagonal $n \times n$-matrix, $i$-th diagonal element of which is equal to $\lambda^{i-1}$ for $i = 1, \ldots, n$. We denote by $\delta$ the operator $z\frac{d}{dz}$. Let further $l = 0, 1, 2$, $k \in \mathfrak{K}_l$, $|z| > 1, \nu \in \mathbb{N}$, and let $Y_{l,k}(z;\nu)$ be the columnn with $4+2l$ elements, $i$-th of which is equal to $(\nu^{-1}\delta)^{i-1} f^{\vee}_{l,k}(z,\nu)$ for $i = 1, \ldots, 4+2l$.

**Theorem 1.** *The following equalities hold*

(16) $$A^{\sim}_l(z;\nu) Y_{l,k}(z;\nu) = T_{4+2l, 1-\nu^{-1}} Y_{l,k}(z;\nu-1),$$

(17) $$Y_{l,k}(z;\nu) = T_{4+2l,-1} A^{\sim}_l(z;-\nu) T_{4+2l,-1+\nu^{-1}} Y_{l,k}(z;\nu-1),$$

*where* $l = 0, 1, 2$, $k \in \mathfrak{K}_l$, $|z| > 1, \nu \in \mathbb{N}, \nu \geq 2$,

(18) $$A^{\sim}_l(z;\nu) = S^{\sim}_l + z \sum_{i=0}^{1+l} \nu^{-i} V^{\sim *}_l(i)$$

*with*

(19) $$S^{\sim}_0 = \begin{pmatrix} 1 & -4 & 8 & -12 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$S_1^{\sim} = \begin{pmatrix} -1 & 6 & -18 & 38 & -66 & 102 \\ 0 & -1 & 6 & -18 & 38 & -66 \\ 0 & 0 & -1 & 6 & -18 & 38 \\ 0 & 0 & 0 & -1 & 6 & -18 \\ 0 & 0 & 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \tag{20}$$

$$S_2^{\sim} = \begin{pmatrix} 1 & -8 & 32 & -88 & 192 & -360 & 608 & -952 \\ 0 & 1 & -8 & 32 & -88 & 192 & -360 & 608 \\ 0 & 0 & 1 & -8 & 32 & -88 & 192 & -360 \\ 0 & 0 & 0 & 1 & -8 & 32 & -88 & 192 \\ 0 & 0 & 0 & 0 & 1 & -8 & 32 & -88 \\ 0 & 0 & 0 & 0 & 0 & 1 & -8 & 32 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{21}$$

$$V_0^{\sim *}(0) = 4 \begin{pmatrix} 4 & -5 & -2 & 3 \\ -3 & 4 & 1 & -2 \\ 2 & -3 & 0 & 1 \\ -1 & 2 & -1 & 0 \end{pmatrix},$$

$$V_0^{\sim *}(1) = 4 \begin{pmatrix} 3 & -6 & 3 & 0 \\ -2 & 4 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$V_1^{\sim *}(0) = \begin{pmatrix} 146 & -198 & -180 & 268 & 66 & -102 \\ -102 & 146 & 108 & -180 & -38 & 66 \\ 66 & -102 & -52 & 108 & 18 & -38 \\ -38 & 66 & 12 & -52 & -6 & 18 \\ 18 & -38 & 12 & 12 & 2 & -6 \\ -6 & 18 & -20 & 12 & -6 & 2 \end{pmatrix},$$

$$V_1^{\sim *}(1) = \begin{pmatrix} 240 & -516 & 108 & 372 & -204 & 0 \\ -160 & 348 & -84 & -236 & 132 & 0 \\ 96 & -212 & 60 & 132 & -76 & 0 \\ -48 & 108 & -36 & -60 & 36 & 0 \\ 16 & -36 & 12 & 20 & -12 & 0 \\ 0 & -4 & 12 & -12 & 4 & 0 \end{pmatrix},$$

$$V_1^{\sim *}(2) = \begin{pmatrix} 102 & -306 & 306 & -102 & 0 & 0 \\ -66 & 198 & -198 & 66 & 0 & 0 \\ 38 & -114 & 114 & -38 & 0 & 0 \\ -18 & 54 & -54 & 18 & 0 & 0 \\ 6 & -18 & 18 & -6 & 0 & 0 \\ -2 & 6 & -6 & 2 & 0 & 0 \end{pmatrix},$$

$$V_2^{\sim *}(0) = 8\begin{pmatrix} 176 & -249 & -364 & 545 & 280 & -431 & -76 & 119 \\ -119 & 176 & 227 & -364 & -169 & 280 & 45 & -76 \\ 76 & -119 & -128 & 227 & 92 & -169 & -24 & 45 \\ -45 & 76 & 61 & -128 & -43 & 92 & 11 & -24 \\ 24 & -45 & -20 & 61 & 16 & -43 & -4 & 11 \\ -11 & 24 & -1 & -20 & -5 & 16 & 1 & -4 \\ 4 & -11 & 8 & -1 & 4 & -5 & 0 & 1 \\ -1 & 4 & -7 & 8 & -7 & 4 & -1 & 0 \end{pmatrix},$$

$$V_2^{\sim *}(1) = 8\begin{pmatrix} 455 & -1020 & -113 & 1552 & -603 & -628 & 357 & 0 \\ -300 & 682 & 44 & -996 & 404 & 394 & -228 & 0 \\ 185 & -428 & -3 & 592 & -253 & -228 & 135 & 0 \\ -104 & 246 & -16 & -316 & 144 & 118 & -72 & 0 \\ 51 & -124 & 19 & 144 & -71 & -52 & 33 & 0 \\ -20 & 50 & -12 & -52 & 28 & 18 & -12 & 0 \\ 5 & -12 & 1 & 16 & -9 & -4 & 3 & 0 \\ 0 & -2 & 8 & -12 & 8 & -2 & 0 & 0 \end{pmatrix},$$

$$V_2^{\sim *}(2) = 8\begin{pmatrix} 400 & -1243 & 972 & 542 & -1028 & 357 & 0 & 0 \\ -259 & 808 & -642 & -332 & 653 & -228 & 0 & 0 \\ 156 & -489 & 396 & 186 & -384 & 135 & 0 & 0 \\ -85 & 268 & -222 & -92 & 203 & -72 & 0 & 0 \\ 40 & -127 & 108 & 38 & -92 & 33 & 0 & 0 \\ -15 & 48 & -42 & -12 & 33 & -12 & 0 & 0 \\ 4 & -13 & 12 & 2 & -8 & 3 & 0 & 0 \\ -1 & 4 & -6 & 4 & -1 & 0 & 0 & 0 \end{pmatrix},$$

$$V_2^{\sim *}(3) = 8\begin{pmatrix} 119 & -476 & 714 & -476 & 119 & 0 & 0 & 0 \\ -76 & 304 & -456 & 304 & -76 & 0 & 0 & 0 \\ 45 & -180 & 270 & -180 & 45 & 0 & 0 & 0 \\ -24 & 96 & -144 & 96 & -24 & 0 & 0 & 0 \\ 11 & -44 & 66 & -44 & 11 & 0 & 0 & 0 \\ -4 & 16 & -24 & 16 & -4 & 0 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

*The above matices $A_l^{\sim}(z;\nu), S_l^{\sim}$ and $V_l^{\sim *}(i)$ have the following properties:*

(22) $$A_l^{\sim}(z;-\nu)T_{4+2l,-1}A_l^{\sim}(z;\nu) = T_{4+2l,-1},$$

(23) $$S_l^{\sim}T_{4+2l,-1} = (S_l^{\sim}T_{4+2l,-1})^{-1}$$

(24) $$S_l^{\sim}T_{4+2l,-1}V_l^{\sim *}(i) = -(-1)^i V_l^{\sim *}(i)T_{4+2l,-1}S_l^{\sim},$$

(25) $$V_l^{\sim *}(i)T_{4+2l,-1}V_l^{\sim *}(k) = 0T_{4+2l,-1},$$

*where*

$$l = 0,\, 1,\, 2,\, i \in [0, 1+l] \cap \mathbb{Z},\, k \in [0, 1+l] \cap \mathbb{Z}.$$

**Proof.** Full proof can be found in [[56]] – [[60]]. In [60] I had promised to give arithmetical applications of the Theorem 1. In [62] I had given short deduction of the Apéry's equation from the Theorem 1. In Part 7 of this work I begin the proof of the Theorem 2, which joins the Apéry's Theorem and my result in [23], [43] in one Theorem. Now I complete this proof. Let me to formulate the Theorem 2 again. Let

$$z \in \mathbb{Q},\ |z| \geq 1,\ x = 1/z,\ b \in \mathbb{N}, a = bz \in \mathbb{Z},$$

(26)
$$\tilde{\eta}_i(z) = \left( \sum_{k=0}^{1} \sqrt{\sqrt{|z|} + k(-1)^i} \right)^2 =$$

$$2\sqrt{|z|} + (-1)^i + 2\sqrt{|z| + (-1)^i \sqrt{|z|}}$$

for $i = 0, 1$,

(27)
$$\tilde{\eta}_2(z) = \sqrt{|z|} + \sqrt{|z| + 1} +$$

$$\sum_{k=0}^{1} \sqrt{\sqrt{|z|^2 + |z|} + (-1)^k \sqrt{|z|}} =$$

$$\sqrt{|z|} + \sqrt{|z| + 1} + \sqrt{2(\sqrt{|z|^2 + |z|} + |z|)} =$$

$$r + \sqrt{r^2 + 1} + \sqrt{2(\sqrt{r^4 + r^2} + r^2)},$$

where $r = \sqrt{|z|}$.

(28)
$$\beta_k(z) = \frac{\ln((\tilde{\eta}_{2[k/2]}(z))^2 e^3 b)}{\ln((\tilde{\eta}_k(z))^2 / e^3 b)},$$

where $k = 0,\ 1,\ 2$

(29)
$$\alpha_k(z) = \beta_k + \frac{(1 - (-1)^k)(\ln(\tilde{\eta}_0(z)/\tilde{\eta}_1(z)}{\ln((\tilde{\eta}_1(z))^2/e^3 b)},$$

(30)
$$D_k(b) =$$

$$\{y \in \mathbb{R}\colon (-1)^{[k/2]} y > ((\sqrt{e^3 b} + 1)^4 / (e^3 b + 1)^2)^{[k/2]} / 16 e^3 b\},$$

where $k = 0,\ 1,\ 2,$

(31)
$$L_{i,s}(x) = (i/x + (-1)^i) \sum_{n=1}^{+\infty} x^n / n^s,$$

where $i = 0,\ 1,\ s \in \mathbb{N},\ |x| \leq 1,\ |x - 1| + s > 1,$

$$L_{1,1}(1) = 0,$$

$$x_1 \in \mathbb{R},\ x_2 \in \mathbb{Q},\ |x_1| + |x_2| > 0,$$

(32)
$$\varphi_i = \phi_i(x_1, x_2, x) = \tilde{\varphi}_i(z, x_1, x_2) =$$

$$x_1 L_{2-i,i}(x) + i x_2 L_{2-i,i+1}(x) =$$

$$x_1 L_{2-i,i}(1/z) + i x_2 L_{2-i,i+1}(1/z),$$

where $i = 1,\ 2$. Let further,

$$\varphi_3(x_1, x_2, x) = \tilde{\varphi}_3(z, x_1, x_2, x) = x_1,\ \hat{\alpha}_0(x) = \alpha_0(z), \tag{33}$$

$$\tilde{\alpha}_i(x) = \alpha_1(z) \text{ for } i = 1,\ 2$$

$\tilde{\alpha}_0(x) = \alpha_2(z)$, $\varepsilon > 0$, and $\|\psi\|$ denotes the distance from $\psi$ to $\mathbb{Z}$.

**Theorem 2.** *There exsist effective positive*

$$\hat{\gamma}_i(x_1, x_2, x, \varepsilon) = \gamma_i^*(z, x_1, x_2 \varepsilon),$$

*where* $i = 1, 2$,

$$\hat{\gamma}_0(x, \varepsilon) = \gamma_0(z, \varepsilon),$$

$$\tilde{\gamma}_1(x, \varepsilon) = \gamma_1(z, \varepsilon),\ \tilde{\gamma}_0(x, \varepsilon) = \gamma_2(z, \varepsilon),$$

*such that,*

*if*

$$z \in D_0(b),\ x_1 = \ln(z),\ x_2 = 1,$$

*then*

$$\max_{i=1,2,3} \|q\phi_i\| q^{\alpha_0(z)+\varepsilon} \geq \gamma_0(z, \varepsilon) \tag{34}$$

*for any* $q \in \mathbb{N}$;

*if* $k = 1,\ 2$,

$$z \in D_k(b),\ x_1 \in \mathbb{Z},\ x_2 \in \mathbb{Z},\ x_2 \neq 0,$$

*then*

$$\max_{i=1,2,3} \|q\phi_i\| q^{beta_k(z)+\varepsilon} \geq \gamma_k^*(z, x_1, x_2, \varepsilon) \tag{35}$$

*for any* $q \in \mathbb{N}$,

$$\max_{i=1,2} \|\tilde{\phi}_i(z, x_1, x_2)\| (|x_1| + |x_2|)^{\alpha_k(z)+\varepsilon} \geq \gamma_0(z, \varepsilon) \tag{36}$$

*for any* $x_1 \in \mathbb{Z}, x_2 \in \mathbb{Z}$, *for which*

$$|x_1| + |x_2| > 0.$$

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## §8.1. Properties of the roots of characteristic polynomial in the case $l = 0$.

The considered in the §7.3 of [63] difference equation

$$\sum_{\kappa=-2}^{2} -1/(16(4z-1)(\nu+1/2)^{11})a_{0,\kappa}(z;\nu)y(z;\nu+\kappa)=0, \tag{37}$$

where $|z|\geq 1$, $\nu\in\mathbb{N}_0$, is a difference eqution of Poincaré type, and characteristic polynomial of this equation is equal to

$$T_0(z;\lambda)= \tag{38}$$

$$1-4(8z+1)\lambda+(256z^2-192z+6)\lambda^2-4(8z+1)\lambda^3+\lambda^4.$$

When $\eta$ runs trough the roots of the polynomial

$$D^\wedge(z;\eta)=(\eta+1)^4-2^4z\eta^2= \tag{39}$$

$$\eta^4+4\eta^3+(6-16z)\eta^2+4\eta+1,$$

then $\lambda=\eta^2$ runs trough the roots of the polynomial $T_0(z;\lambda)$ If

$$r\geq 1,\ \varphi\in[0,\pi],\ z=r^2\exp(i2\varphi), \tag{40}$$

then we can represent polynomial $D^\wedge(z;\eta)$ in the form

$$D^\wedge(z;\eta)=\prod_{\kappa=0,1}(\eta+1)^2+4\sqrt{|z|}\exp i(\varphi-\kappa\pi)\eta. \tag{41}$$

So, we must study the roots of the polynomial

$$D_1^\vee(r,\psi;\eta)=(\eta+1)^2+4r\exp i\psi\eta= \tag{42}$$

$$\eta^2+2(1+2r\exp i\psi)\eta+1=(\eta+1+2r\exp i\psi)^2-4r\exp(i\psi)(1+r\exp(i\psi),$$

where $r=\sqrt{|z|}\geq 1$ and $\psi\in[-\pi,\pi]$. If $\varphi=0$ in (41) then we must consider for $\psi$ two values $\psi=0$ and $\psi=\pi$. If $\varphi=\pi/2$ in (41) then we must consider for $\psi$ two values $\psi=-\frac{\pi}{2}$ and $\psi=\frac{\pi}{2}$.

In the case $r=1$ the roots $\eta$ of the polynomial $D_1^\vee(r,\psi;\eta)$ are studied in §2 of [43]. Let

$$r>1,\ \phi_1(r,\psi)=\arccos\left(\frac{1+r\cos(\psi)}{\sqrt{1+2r\cos(\psi)+r^2}}\right)$$

where $\psi\in[0,\pi]$. If $\psi\in[-\pi,0]$, then we let $\varphi_1(r,\psi)=-\varphi_1(r,-\psi)$. Clearly,

$$\cos(\varphi_1(r,\psi)))=\frac{1+r\cos(\psi)}{\sqrt{1+2r\cos(\psi)+r^2}}, \tag{43}$$

$$\sin(\varphi_1(r,\psi)))=\frac{r\sin(\psi)}{\sqrt{1+2r\cos(\psi)+r^2}}, \tag{44}$$

If we consider the circumference with radius equal to $r$ and center in the point $(1,0)$, and consider the triangle with the apixes $1+r\exp(i\psi)$, 0 and 1 then we find easily that the value $0 < \varphi_1(r,\psi) < \psi$, if $\psi \in (0,\,\pi)$, and the value $\varphi_1(r,\psi)$ increases in $(0,\,\pi)$ with increasing of $\psi$ in $(0,\,\pi)$. Let

$$(45)\qquad \varphi_2(r,\psi) = \frac{\psi + \varphi_1(r,\psi)}{2},\ \varphi_3(r,\psi) = \frac{\psi - \varphi_1(r,\psi)}{2} < \frac{\pi}{4}.$$

Then, clearly,

$$(46)\qquad 0 \le \varphi_2(r,\psi) \le \psi,\ 0 \le \varphi_3(r,\psi) < \pi/2,\ \text{ if } \psi \in [0,\,\pi],$$

the value $\varphi_2(r,\psi)$ increases in $(0,\,\pi)$ with increasing of $\psi$ in $(0,\,\pi)$,

$$\varphi_2(r,0) = \varphi_3(r,0) = \varphi_3(r,\pi) = 0,\ \varphi_2(r,\pi) = \pi.$$

Clearly,

$$\cos(2\varphi_3(r,\psi)) = \cos(\psi - var\phi_1(r,\psi))) =$$
$$\frac{r + \cos(\psi)}{\sqrt{1 + 2r\cos(\psi) + r^2}} > 0.$$

Therefore

$$(47)\qquad 2\phi_3(r,\psi) < \frac{\pi}{2} \text{ and } \varphi_3(r,\psi) < \frac{\pi}{4}.$$

Clearly, the roots $\eta$ of the polynomial $D_1^{\vee}(r,\psi;\eta)$ are

$$(48)\qquad \eta = \eta_k(r,\psi) = -1 - 2r\exp i\psi) -$$
$$(-1)^k 2\sqrt{r}\exp(i(\varphi_2(\psi))(1 + r^2 + 2rcos(\psi))^{1/4},$$

where $\psi \in [-\pi,\pi]$, $k = 0,\,1$. Therefore

$$(49)\qquad \eta_k(r,-\psi) = \overline{\eta_k(r,\psi)} \text{ for } \psi \in [-\pi,\pi].$$

In view of (48),

$$(50)\qquad |\eta_k(r,\psi)|^2 = 1 + 4r^2 + 4r(1 + r^2 + 2rcos(\psi))^{1/2} +$$
$$4r\cos(\psi) + (-1)^k 4\sqrt{r}(1 + r^2 + 2r\cos(\psi))^{1/4}\cos(\varphi_2(\psi)) +$$
$$(-1)^k 8r^{3/2}(1 + r^2 + 2r\cos(\psi))^{1/4}\cos(\phi_3(\psi)) =$$
$$(2r-1)^2 + 8r\cos^2(\psi/2) + 4r((r-1)^2 + 4r\cos^2(\psi/2))^{1/2} +$$
$$(-1)^k 4\sqrt{r}((r-1)^2 + 4r\cos^2(\psi/2))^{1/4})\times$$
$$(2r\cos(\varphi_3(\psi)) + \cos(\varphi_2(\psi))$$

where $\psi \in [-\pi,\pi]$, $k = 0,\,1$. In view of (47),

$$(51)\qquad \Delta(r,\psi) :=$$
$$4\sqrt{r}(1 + r^2 + 2r\cos(\psi))^{1/4}\cos(\varphi_2(\psi)) +$$
$$8r^{3/2}(1 + r^2 + 2r\cos(\psi))^{1/4}\cos(\varphi_3(\psi)) =$$
$$4r^{1/2}(1 + r^2 + 2r\cos(\psi))^{1/4})(2r\cos(\varphi_3(\psi) + \cos(\varphi_2(\psi)) \ge$$
$$4r^{1/2}(1 + r^2 + 2r\cos(\psi))^{1/4})(r\sqrt{2} + \cos(\varphi_2(\psi)) \ge 0,$$

if $r \geq 1$, $\psi \in [-pi/2, \pi/2]$. It follows from (50) that

$$1 = |\eta_1(r,\psi)|^2|\eta_0(r,\psi)|^2 = \tag{52}$$

$$(1+4r^2+4r(1+r^2+2r\cos(\psi))^{1/2}+4r\cos(\psi))^2-(\Delta(r,\psi))^2.$$

Since

$$(2r-1)^2+8r\cos^2(\psi/2)+4r((r-1)^2+4r\cos^2(\psi/2))^{1/2}$$

decreases together with increasing $\psi \in [0,\pi]$, when $r$ is fixed in $[1,+\infty)$, it follows from (52), (51), that $\Delta(r,\psi)$ decreases with increasing $\psi \in [0,\pi]$. Therefore, in view of (50), if $r$ is fixed in $[1,+\infty))$, then the value $|\eta_0(r,\psi)|^2$ decreases with increasing of $\psi \in (0,\pi)$, and $|\eta_1(r,\psi)|^2 = 1/|\eta_0(r,\psi)|^2$ increases with increasing of $\psi \in [0,+\pi]$. Since

$$(2r-1)^2+8r\cos^2(\psi/2)+4r((r-1)^2+4r\cos^2(\psi/2))^{1/2}$$

increases with increasing $r \in [0,+\infty)$, when $\psi$ is fixed in $[0,+\pi]$, it follows from (52), (51), that $\Delta(r,\psi)$ decreases with increasing $r \in [1,+\infty)$. Therefore, in view of (50), if $\psi$ is fixed in $[0,\pi]$, then $|\eta_0(r,\psi)|^2$ decreases with increasing of $r \in [1,+\infty)$, and $|\eta_1(r,\psi)|^2 = 1/|\eta_0(r,\psi)|^2$ increases with increasing of $r \in (1,+\infty)$. Therefore

(53)

$$1 \leq |\eta_0(r,\pi)|^2 = (2r-1)^2+4r(r-1)+4\sqrt{r(r-1)}(2r-1) <$$

$$|\eta_0(r,\psi)|^2 < |\eta_0(r,0)|^2 =$$

$$(2r+1)^2+4r(r+1)+4(2r+1)\sqrt{r(r+1)}$$

and

$$\eta_1(r,0) = \frac{1}{\eta_0(r,0)} < |\eta_1(r,\psi)| = \frac{1}{|\eta_0(r,\psi)|} < \tag{54}$$

$$(2r-1)^2+4r(r-1)-4\sqrt{r(r-1)}(2r-1) = \eta_1(r,\pi) = \frac{1}{\eta_0(r,\pi)} \leq 1,$$

if $\psi \in [-\pi,0)\cup(0,\pi]$, $r \geq 1$.

## §8.2. Comparison of the functions $f_{0,2}(z;\nu)$ and $f_{0,3}(z;\nu)$.

**Lemma 8.1.1.** *Let $z \geq 1, \lambda_1 \in (0,1/2)$ Then*

$$f_{0,3}(z;\nu) = f_{0,2}(z,\nu)\nu^{-\lambda_1}O(1), \tag{55}$$

*where $\nu \in \mathbb{N}$, and $O(1)$ depends only from $z$.*

**Proof.** Since $|z| = z$ now it follows that $r = \sqrt{z} \geq 1$. Let

$$f_0 = f_0(r,\eta) = D_1^\vee(r,\pi,\eta) = (\eta+1)^2 - 4r\eta,$$

$$\eta_k = \eta_k(r) = \eta_k(r,\pi) = 2r-1+(-1)^k 2\sqrt{r^2-r} \text{ where } k=0,\,1.$$

Then

$$f_0(r,\eta_1(r)) = 0, \frac{\partial f_0}{\partial \eta}(r,\eta_1(r)) = -4\sqrt{r^2-r}, \tag{56}$$

(57) $$\frac{\partial^2 f_0}{\partial \eta^2}(r,\eta) = 2,$$

(58) $$0 < \eta_1(r) = 1/\eta_2(r) < 1,\ \frac{\partial f_0}{\partial \eta}(r,\eta_1(r)) < 0,$$

if $r > 1$,

$$\eta_1(1) = \eta_2(1) = 1,\ \frac{\partial f_0}{\partial \eta}(1,1) = 0.$$

Let

(59) $$\tau = (1+\eta)/(1-\eta) \text{ where } 0 \le \eta < 1,$$

In view of (58), let

(60) $$\tau_1 = \tau_1(r) = (1+\eta_1(r))/(1-\eta_1(r)) \text{ where } r > 1,$$

Clearly, $\tau_1(r) \in (1,+\infty)$, if $r > 1$. In view of (232) in [63], let

$$g(z,\tau) = \tau^4/((\tau^2-1)^2 z) - 1,\ \text{where } z \ge 1, 1 < \tau$$

(61) $$h(z,\eta) = (D^\wedge(z;\eta))/(16z^2\eta^2) =$$

$$f_0(\sqrt{z},\eta)((\eta+1)^2 + 4\sqrt{z}\eta)/(16z\eta^2),$$

where $z \ge 1, 0 < \eta \le 1$. Clearly,

(62) $$h(z,\eta) = g(z,(1+\eta)/(1-\eta)) \text{ where } z \ge 1,\ 0 < \eta < 1.$$

Let further

(63) $$u(z,\tau) = 2\ln((\tau-1)/\tau+1)) + \tau\ln(1+g(z,\tau)),$$

where $z \ge 1, \tau > 1$,

(64) $$w(z,\eta) = u(z,(1+\eta)/(1-\eta)) = 2\ln(\eta)) +$$

$$(\ln(1+h(z,\eta))(1+\eta)/(1-\eta),\ \text{where } z \ge 1, 0 < \eta < 1,$$

(65) $$w_1(z) = w(z,\eta_1(\sqrt{z})) = 2\ln(\eta_1(\sqrt{z}) +$$

$$(\ln(1+h(z,\eta_1(\sqrt{z}))(1+\eta_1(z))/(1-\eta_1(z)),$$

where $z > 1$. Since $\eta_0(\sqrt{z}) \ge 1$, it follows that

(66) $$h(z,\eta_1(\sqrt{z})) = 0,\ h(z,\eta)(\eta - \eta_1(\sqrt{z})) < 0,$$

if $z \ge 1$, $0 < \eta < 1$, $\eta \ne \eta_1(\sqrt{z})$. Therefore in view of (59), (62),

(67) $$g(z,\tau_1(\sqrt{z})) = 0,\ g(z,\tau)(\tau - \tau_1(\sqrt{z})) < 0,$$

if $z \geq 1$, $1 < \tau$, $\tau \neq \tau_1(\sqrt{z})$. In view of (56) and (61),

(68)
$$\frac{\partial h}{\partial \eta}(z, \eta_1(\sqrt{z})) =$$
$$-\frac{(\eta_1(\sqrt{z})+1)^2 + 4\sqrt{z}\eta_1(\sqrt{z})}{4z\eta_1(\sqrt{z})}\sqrt{z-\sqrt{z}},$$

where $z \geq 1$,

(69)
$$\varepsilon_0(z) := -\frac{1}{2}\frac{\partial h}{\partial \eta}(z, \eta_1(\sqrt{z})) > 0, \text{ if } z > 1.$$

In view of (68), $\frac{\partial h}{\partial \eta}(1,1) = 0$; therefore, in view of (56), (57) and (61),

$$\frac{\partial^2 h}{\partial \eta^2}(1,1) = 1.$$

In view of (63), (64),

(70)
$$\frac{\partial u}{\partial \tau}(z,\tau) = 4/(\tau^2-1) + 4 - 4\tau^2/(\tau^2-1) +$$
$$\ln(1+g(z,\tau)) = \ln(1+g(z,\tau)),$$

(71)
$$\frac{\partial w}{\partial \eta}(z,\eta) =$$
$$(2/(1-\eta)^2)\frac{\partial u}{\partial \tau}(z,(1+\eta)/(1-\eta)) =$$
$$(2/(1-\eta)^2)\ln(1+h(z,\eta)),$$

where $z \geq 1$, $0 < \eta < 1$,

(72)
$$\frac{\partial^2 w}{\partial \eta^2}(z,\eta) =$$
$$(2/(1-\eta)^2)\frac{\partial h}{\partial \eta}(z,\eta)/(1+h(z,\eta)+$$
$$(4/(1-\eta)^3)\ln(1+h(z,\eta)),$$

where $z \geq 1$, $0 < \eta < 1$. Therefore, if $z > 1$, then, in view of (64) – (72),

(73)
$$\frac{\partial w}{\partial \eta}(z, \eta_1(\sqrt{(z)})) = 0,\ -\varepsilon_1(z) = -\frac{1}{2}f\frac{\partial^2 w}{\partial \eta^2}(z, \eta_1(\sqrt{z})) =$$
$$\varepsilon_0(z)(2/(1-\eta_1(\sqrt{z}))^2) > 0.$$

If $z = 1$, then, in view of (65), (61), (66), (64) – (72),

(74)
$$w_1(1) := \lim_{z\to 1+0} w_1(z) =$$
$$\lim_{z\to 1+0}(\ln(1+h(z,\eta_1(\sqrt{z}))/h(z,\eta_1(\sqrt{z}))\times$$

$$\lim_{z\to 1+0}(h(z,\eta_1(\sqrt{z})(1+\eta_1(z))/(1-\eta_1(z)) = 2\times$$

$$\lim_{z\to 1+0}(h(z,\eta_1(\sqrt{z}))/(1-\eta_1(z)) = 0,$$

$$w(1,1) := \lim_{\eta\to 1-0}(w(1,\eta)) = \lim_{\eta\to 1-0}(2\ln(\eta)) + \tag{75}$$

$$\lim_{\eta\to 1-0}(\ln(1+h(1,\eta)/h(1,\eta)\times$$

$$\lim_{\eta\to 1-0}(h(1,\eta)(1+\eta)/(1-\eta) = 2\times$$

$$\lim_{\eta\to 1-0}(h(1,\eta))/(1-\eta) = 2\times$$

$$(\lim_{\eta\to 1-0}((\eta+1)^2+4\sqrt{z}\eta)/(16z\eta^2))\times$$

$$\lim_{\eta\to 1-0}((\eta-1)^2/(1-\eta)) = 0$$

So, $w_1(1) = w(1,\eta_1(1)) = w(1,1) = 0$. Further we have

$$\frac{\partial w}{\partial \eta}(1,1) = \lim_{\eta\to 1-0}(w(1,\eta)/(\eta-1)) = \tag{76}$$

$$\lim_{\eta\to 1-0}((2\ln(\eta))/(\eta-1)) - 2\times$$

$$\lim_{\eta\to 1-0}(\ln(1+h(1,\eta)/h(1,\eta)\times$$

$$\lim_{\eta\to 1-0}(h(1,\eta)/(\eta-1)^2 =$$

$$2-2(\lim_{\eta\to 1-0}((\eta+1)^2+4\sqrt{z}\eta)/(16z\eta^2)) = 2-1 = 1.$$

In view of (61), (71), (66),

$$\lim_{\eta\to 1-0}\left(\frac{\partial w}{\partial \eta}(1,\eta)\right) = \tag{77}$$

$$\left(\lim_{\eta\to 1-0}\left(\frac{\ln(1+h(1,\eta)}{h(1,\eta)}\right)\right)\times$$

$$\left(\lim_{\eta\to 1-0}\left(\frac{2h(1,\eta)}{(\eta-1)^2}\right)\right) = 1.$$

I use below the results of [39] with

$$d^{\vee} = d^{\wedge} = 1,\ m = n = 1.$$

We represent $(R(t,\nu))^2$, where $R(t,\nu)$ is defined in (2) in the form

$$(R(t,\nu))^2 = R_1(t,\nu)R_2(t,\nu), \tag{78}$$

where

$$R_1(t,\nu) = \frac{\prod_{j=1}^{\nu}(t-j)}{\prod_{j=0}^{\nu-1}(t+j)} = \tag{79}$$

$$\prod_{j=0}^{\nu-1}\frac{t-1-j}{t+j},\ R_2(t,\nu)=(t+\nu)^{-2}.$$

It follows from (79) that

(80) $$R_1(t,\nu)z^{-t}=\frac{(\Gamma(t))^4}{(\Gamma(t-\nu)^2(\Gamma(t+\nu))^2}z^{-t}.$$

In view of (3), (6), we can take $t\geq\nu+1$ in further calculations and use Stirlings formula in the form

(81) $$\ln\Gamma(x)=(x-\frac{1}{2})logx-x+O(1).$$

with $x\geq 1$ and $O(1)=\theta(x)C$, where $|\theta(x)|\leq 1$ and $C$ is appropriate absolute constant. Below $O(1)$ will be depend only from $z\geq 1$. We put $t=\nu\tau$ now. Then

(82) $$(R_1(t+1,\nu))z^{-t-1}=(R_1(t,\nu))z^{-t}=t^4/z(t^2-\nu^2)^2=$$
$$(R_1(t,\nu))(1+g(z,\tau))=(R_1(t,\nu))(1+h(z,\eta)),$$

where

$$t\in\mathbb{N},\ \nu\in\mathbb{N},\ \tau=t/nu,\ eta=(\tau-1)/(\tau-1).$$

In view of (80), (81),

(83) $$\ln(R_1(t,\nu)z^{-t})=4(t-1/2)\ln(t)-4t-$$
$$2(t-\nu-1/2)\ln(t-\nu)+2t-2\nu-$$
$$2(t+\nu-1/2)\ln(t+\nu)+2t+2\nu-tln(z)+O(1)=$$
$$\nu\tau\ln(1+g(1+g(z,\tau))+2\nu\ln((\tau-1)/(\tau+1))-$$
$$\frac{1}{2}\ln(\tau^4/(\tau^2-1)^2)+O(1)=$$
$$u(z,\tau)\nu-\frac{1}{2}\ln(\tau^4/(\tau^2-1)^2)+O(1)=$$
$$w(z,\eta)\nu-\frac{1}{2}\ln(\tau^4/(\tau^2-1)^2)+O(1),$$

where $\nu\in\mathbb{N}$, $t\in[\nu+1,+\infty]\cap\mathbb{N}$. Therefore

(84) $$\ln(R_1(\nu\tau,\nu)z^{-\nu\tau})-\ln(R_1(\nu\tau_1(\sqrt{z}),\nu)z^{-\nu\tau_1(\sqrt{z})})=$$
$$\nu(u(z,\tau)-u(z,\tau_1(\sqrt{z})-\frac{1}{2}\ln(\tau^4/(\tau^2-1)^2)+$$
$$\frac{1}{2}\ln((\tau_1(\sqrt{z}))^4/((\tau_1(\sqrt{z}))^2-1)^2)-$$
$$\nu(\tau-\tau_1(\sqrt{z})\ln(z)+O(1).$$

In view of (67), (83), (64),

(85) $$\max_{t\in[\nu+1,+\infty)\cap\mathbb{Z}}(R_1(t,\nu)z^{-t})=$$

$$R_1(\nu(\tau_1(\sqrt{z})+\theta(z;\nu)/\nu);\nu)z^{-(\nu\tau_1(\sqrt{z})+\theta(z;\nu))} =$$
$$R_1(\nu\tau,\nu)z^{-\nu\tau}$$

with $\tau = \tau_1(\sqrt{z})+\theta(z;\nu)/\nu$, where $0 \le \theta(z;\nu) < 1$. In view (84) – (85), (64), (66), (83),

$$\max_{t\in[\nu+1,+\infty)\cap\mathbb{Z}} (R_1(t,\nu)z^{-t}) = (\eta_1(\sqrt{z}))^{2\nu}e^{O(1)}, \tag{86}$$

where $z > 1$. If $z > 1$, then all summands in (3) are positive, and its sum, which is equal to $f_2(z;\nu)$, is bigger, than a single summands with $t = \nu(\tau_1(\sqrt{z})+\theta(z;\nu)/nu)$; therefore, in view of (78), (79), (86),

$$\nu^{-2}(\eta_1(\sqrt{z})^{2\nu}\exp(O(1)) \le f_{0,2}(z;\nu). \tag{87}$$

On the other hand, if $z > 1$, then, in view of (78), (79), (86),

$$f_{0,2}(z;\nu) \le \left(\max_{t\in[\nu+1,+\infty)\cap\mathbb{Z}} (R_1(t,\nu)z^{-t})\right)\sum_{t=1+\nu}^{+\infty} R_2(t,\nu) = \tag{88}$$
$$\nu^{-1}(\eta_1(\sqrt{z}))^{2\nu}\exp(O(1)).$$

Since $(t-1-k)/(t+k)$ increases, when $k$ is fixed in $[0,\nu-1]$ and $t$ increases in $(\nu+1,\infty)$, it follows from (79) and (74) that

$$\sup_{t\in[\nu+1,+\infty)\cap\mathbb{Z}} (R_1(t,\nu)z^{-t}) = 1 = e^{w_1(1)}. \tag{89}$$

In view of (78), (79), (83), (89),

$$\nu^{-4}e^{O(1)} \le R^2(2\nu^2-\nu,\nu) \le f^2(1;\nu) \le \tag{90}$$
$$\left(\sup_{t\in\nu+\mathbb{N}} (R_1(t,\nu))\right)\sum_{t=1+\nu}^{+\infty} R_2(t;\nu) = \nu^{-1}e^{O(1)}.$$

In view of (71), (72), (76), (77), there exists $\delta_1(z) \in (0,\eta_1(\sqrt{z}))$ such that

$$\left|\frac{\partial w}{\partial\eta}(z,\eta) - \frac{\partial w}{\partial\eta}(z,\eta_1(\sqrt{z}))\right| \le 1/2 \tag{91}$$

for $z \ge 1$, $|\eta-\eta_1(\sqrt{z})| < \delta_1(z)$, $0 < \eta < 1$,

$$\left|\frac{\partial h}{\partial\eta}(z,\eta) - \frac{\partial h}{\partial\eta}(z,\eta_1(\sqrt{z}))\right| \le \varepsilon_0(z) \tag{92}$$

for $z > 1$, $|\eta-\eta_1(\sqrt{z})| < \delta_1(z)$, $0 < \eta < 1$,

$$\left|\frac{\partial^2 w}{\partial\eta^2}(z,\eta) - \frac{\partial w}{\partial\eta}(z,\eta_1(\sqrt{z}))\right| \le \varepsilon_1(z) \tag{93}$$

for $z > 1$, $|\eta-\eta_1(\sqrt{z})| < \delta_1(z)$, $0 < \eta < 1$. In view of (73) and (93),

$$\frac{\partial^2 w}{\partial\eta^2}(z,\eta) \le -\varepsilon_0(z) \tag{94}$$

for $z>1$, $|\eta-\eta_1(\sqrt{z})|<\delta_1(z)$, $\eta<1$. In view of (69) and (92),

$$\frac{\partial h}{\partial \eta}(z,\eta)\leq -\varepsilon_0(z) \tag{95}$$

for $z>1$, $|\eta-\eta_1(\sqrt{z})|<\delta_1(z)$, $\eta<1$. In view of (76) and (91),

$$\frac{\partial w}{\partial \eta}(1,\eta)\geq \frac{1}{2} \text{ for } 1-\delta_1(1)<\eta<1. \tag{96}$$

If $\eta\geq\eta_1(\sqrt{z}-\delta_1(z)$ then

$$\tau\geq\frac{1+\eta_1(\sqrt{z}-\delta_1(z)}{1-\eta_1(\sqrt{z}+\delta_1(z)},$$

and $\frac{1}{2}\ln(\tau^4/(\tau^2-1)^2)=O(1)$; therefore, in view of (83), (94, (73), (96), if

$$z>1,\ 0<\eta=(t-\nu)/(t+\nu)<1,\ |\eta-\eta_1(\sqrt{z})|<\delta_1(z),$$

then

$$\begin{gathered}\ln(R_1(t,\nu)z^{-t})=w_(z,\eta)\nu+O(1)=\\ w(z,\eta_1(\sqrt{z})\nu+(w(z,\eta)-w(z,\eta_1(\sqrt{z}))\nu+O(1)\leq\\ w_1(z)\nu-\frac{1}{2}\varepsilon_1(\eta-\eta_1(\sqrt{z})^2\nu+O(1),\end{gathered} \tag{97}$$

and, if $0<\eta=(t-\nu)/(t+\nu)<1$, $1-\delta_1(1)<\eta<1$, then

$$\begin{gathered}\ln(R_1(t,\nu))=w(1,\eta)\nu+O(1)=\\ w_(1,1)\nu+(w_(1,\eta)-w_(1,1)\nu+O(1)\leq-(1-\eta)\nu/2+O(1).\end{gathered} \tag{98}$$

We fix $\lambda_1\in(0,1/2)$, and let $\lambda_2=2\lambda_1$. Clearly, if $z>1$, then

$$\begin{gathered}\tau_1(\sqrt{z})-\frac{1+\eta_1(\sqrt{z})-\delta_1(z)}{1-\eta_1(\sqrt{z}+\delta_1(z))}=\\ \frac{2\delta_1(z)}{(1-\eta_1(\sqrt{z})+\delta_1(z)))(1-\eta_1(\sqrt{z})}>2\delta_1(z).\end{gathered} \tag{99}$$

Let $\nu_1(z)$ is fixed in $((3/\delta_1(z))^{1/\lambda_1},+\infty)\cap\mathbb{Z}$ for $z\geq1$. If $z>1$, then each of the sets

$$\mathfrak{M}_1(z;\nu)=(\nu\tau_1(\sqrt{z})-\nu2\delta_1(z),\nu\tau_1(\sqrt{z})-\nu\delta_1(z))\cap\mathbb{Z},$$

$$\mathfrak{M}_2(z;\nu)=(\nu\tau_1(\sqrt{z})+\nu\delta_1(z),\nu\tau_1(\sqrt{z})+2\nu\delta_1(z))\cap\mathbb{Z}$$

is not empty for $\nu\in[\nu_1(z),+\infty)\cap\mathbb{Z}$. Clearly, the set

$$\mathfrak{M}_3(\nu)=(\nu(\nu^{2\lambda_1}-1),\nu(2\nu^{2\lambda_1}-1),\cap\mathbb{Z},$$

is not empty for $\nu\in[\nu_1(1),+\infty)\cap\mathbb{Z}$. Let $t_k(z;\nu)$ is fixed in $\mathfrak{M}_k(z,\nu)$ for $z>1$, $k=1$, 2, $\nu\in[\nu_1(z),+\infty)\cap\mathbb{Z}$, and let $t_3(\nu)$ is fixed in $\mathfrak{M}_3(\nu)$ for $\nu\in[\nu_1(z),+\infty)\cap\mathbb{Z}$. Then

$$\nu<\nu\frac{1+\eta_1(\sqrt{z})-\delta_1(z)}{1-\eta_1(\sqrt{z})+\delta_1(z)}< \tag{100}$$

$$\nu\tau_1(\sqrt{z}) - \nu 2\delta_1(z) < t_1(z;\nu) <$$
$$\nu\tau_1(\sqrt{z}) - 2\nu^{1-\lambda_1} <$$
$$\nu\tau_1(\sqrt{z}) + \nu^{1-\lambda_1} < t_2(z;\nu) <$$
$$\nu\tau_1(\sqrt{z}) + 2\nu^{1-\lambda_1} < \nu(\tau_1(\sqrt{z}) + \delta_1(z)),$$

where $z > 1$, $\nu \in \nu \in [\nu_1(z), +\infty) \cap \mathbb{Z}$,

(101) $$\nu + 1 < 4\nu < \nu(\nu^{2\lambda_1} - 1) < t_3(\nu) < (2\nu^{2\lambda_1} - 1),$$

where $nu \in \nu \in [\nu_1(1), +\infty) \cap \mathbb{Z}$. In view of (100),

(102) $$\nu^{-\lambda_1}(\tau_1(\sqrt{z}) + \delta_1(z) + 1)^{-2} <$$
$$\frac{|t_k(z;\nu)/\nu - \tau_1(\sqrt{z})|}{(t_2(z;\nu)/\nu + 1)^2} <$$
$$< \left| \frac{t_k(z;\nu)/\nu - 1}{t_k(z;\nu)/\nu + 1} - \eta_1(\sqrt{z}) \right| =$$
$$\left| \frac{t_k(z;\nu)/\nu - 1}{t_k(z;\nu)/\nu + 1} - \frac{\tau_1(\sqrt{z}) - 1}{\tau_1(\sqrt{z}) + 1} \right| <$$
$$|t_k(z;\nu)/\nu - \tau_1(\sqrt{z}| < 2\nu^{-\lambda_1} < \delta_1(z)$$

where $z > 1$, $\nu \in [\nu_1(z), +\infty) \cap \mathbb{Z}$. In view of (101),

(103) $$1 - 2\nu^{-2\lambda_1} < \frac{t_3(\nu) - \nu}{t_3(\nu) + \nu} < 1 - \nu^{-2\lambda_1}$$

where $\nu \in \nu \in [\nu_1(1), +\infty) \cap \mathbb{Z}$. In view of (97), (102), if

$$z > 1,\ \nu \in [\nu_1(z), +\infty) \cap \mathbb{Z} \text{ and } t \in [\nu + 1,\ t_1(z;\nu)] \cup [t_2(z;\nu),\ +\infty),$$

then

(104) $$0 < R_1(t,\nu)z^{-t} \le \max_{k=1,2} R_1(t_k(z,\nu),\nu)z^{-t_k(z,\nu)} =$$
$$\exp(w_1(z)\nu - \varepsilon_2(z)\nu^{1-2\lambda_1})O(1),$$

where
$$\varepsilon_2(z) = \frac{1}{2}\varepsilon_1(z)(\tau_1(\sqrt{z}) + \delta_1(z) + 1)^{-4}.$$

In view of (103),

$$\nu^{-2\lambda_1} < 1 - \frac{t_3(\nu)/\nu - 1}{t_3(\nu)/\nu + 1} < 2\nu^{-2\lambda_1}.$$

Therefore, if $t \in [\nu + 1,\ t_3(\nu)]$, then

(105)
$$0 < R_1(t,\nu) \le R_1(t_3(\nu),\nu) \le \exp(-\nu^{1-2\lambda_1}/2 + O(1)) =$$
$$\exp(w_1(1)\nu - \nu^{1-2\lambda_1}/2)O(1).$$

In view of (2),

$$(R(t,\nu))^{-2}\frac{\partial(R(t,\nu))^2}{\partial t}= \tag{106}$$

$$2\ln(t^2/(t^2-\nu^2))+O(1/(t-\nu)=O(\ln(\nu)),$$

where $\nu\in[2,\,+\infty)\cap\mathbb{Z}$, $t\in[\nu+1,\,+\infty)\cap\mathbb{Z}$,

$$((R(t,\nu))^2z^{-t})^{-1}\frac{\partial(R(t,\nu))^2z^{-t}}{\partial t}= \tag{107}$$

$$-\ln(z)+(R(t,\nu))^{-2}\frac{\partial(R(t,\nu))^2}{\partial t}=$$

$$-\ln(z)+2\ln(t^2/(t^2-\nu^2))+O(1/(t-\nu)=\ln(1+g(z,\tau))+O(1/(t-\nu)=$$

$$\ln(1+h(z,\eta))+O(1/(t-\nu)=O(\ln(\nu)),$$

where

$$\nu\in[2,\,+\infty)\cap\mathbb{Z},\, t\in[\nu+1,\,+\infty)\cap\mathbb{Z},$$

$$\tau=t/\nu,\,\eta=\frac{\tau-1}{\tau+1},\, z\geq 1.$$

In view of (107), (104), (65), (66), (87),

$$\left(\sum_{t=\nu+1}^{t_1(z,\nu)}\left|R(t,\nu))^2z^{-t}\ln(z)-\frac{\partial((R(t,\nu))^2)}{\partial t}z^{-t}\right|\right)+ \tag{108}$$

$$\sum_{t=t_2(z,\nu)}^{+\infty}\left|R(t,\nu))^2z^{-t}\ln(z)-\frac{\partial((R(t,\nu))^2)}{\partial t}z^{-t}\right|=$$

$$\left(\sum_{t=\nu+1}^{t_1(z,\nu)}\left|-\frac{\partial((R(t,\nu))^2z^{-t})}{\partial t}\right|\right)+$$

$$\sum_{t=t_2(z,\nu)}^{+\infty}\left|-\frac{\partial((R(t,\nu))^2z^{-t})}{\partial t}\right|=$$

$$(O(\ln(\nu)))\left(\sum_{t=\nu+1}^{t_1(z,\nu)}(R(t,\nu))^2z^{-t}\right)+$$

$$(O(\ln(\nu)))\sum_{t=t_2(z,\nu)}^{+\infty}(R(t,\nu))^2z^{-t}=$$

$$\exp(w_1(z)\nu-\varepsilon_2(z)\nu^{1-2\lambda_1})(O(\ln(\nu)))\times$$

$$\left(\sum_{t=\nu+1}^{t_1(z,\nu)}R_2(t,\nu)\right)+$$

$$\exp(w_1(z)\nu-\varepsilon_2(z)\nu^{1-2\lambda_1})(O(\ln(\nu)))\times$$

$$\sum_{t=t_2(z,\nu)}^{+\infty} R_2(t,\nu)) =$$

$$\exp(w_1(z)\nu - \varepsilon_2(z)\nu^{1-2\lambda_1}) \left(O\left(\frac{\ln(\nu)}{\nu}\right)\right) =$$

$$(\eta_1(\sqrt{z})^{2\nu} \exp(-\varepsilon_2(z)\nu^{1-2\lambda_1}) \left(O\left(\frac{\ln(\nu)}{\nu}\right)\right) =$$

$$f_2(z;\nu)\exp(-\varepsilon_2(z)\nu^{1-2\lambda_1})(O(\ln(\nu))\nu)$$

for $z > 1$, $\nu \in [\nu_1(z), +\infty) \cap \mathbb{Z}$. In view of (107), (105), (66), (90),

$$\left(\sum_{t=\nu+1}^{t_3(z,\nu)} \left|-\frac{\partial(R(t,\nu))^2}{\partial t}\right|\right) = \tag{109}$$

$$(O(\ln(\nu))) \sum_{t=\nu+1}^{t_3(z,\nu)} (R(t,\nu))^2 =$$

$$(O(\ln(\nu)))\exp(-\nu^{1-\lambda_2}/2 + O(1)) \sum_{t=\nu+1}^{t_3(z,\nu)} R_2(t,\nu) =$$

$$\exp(-\nu^{1-\lambda_2}/2)\left(O\left(\frac{\ln(\nu)}{\nu}\right)\right) =$$

$$f_2(1;\nu)\exp(-\varepsilon_2(z)\nu^{1-2\lambda_1})(O(\ln(\nu))\nu^3).$$

If $z > 1$, $z > 1$, $\nu \in [\nu_1(z), +\infty) \cap \mathbb{Z}$, and $t_1(z;\nu) \le t \le t_2(z;\nu)$, then, in view of (102),

$$-\delta_1(z) < \frac{t_1(z;\nu)-\nu}{t_1(z;\nu)+\nu} - \eta_1(\sqrt{z}) \le \frac{t-\nu}{t+\nu} - \eta_1(\sqrt{z} \le$$

$$\frac{t_2(z;\nu)-\nu}{t_2(z;\nu)+\nu} - \eta_1(\sqrt{z}) < \delta_1(z),$$

i.e, for $\eta = (\tau-1)/(\tau+1) = (t-\nu)/(t+\nu)$ the inequality $|\eta - \eta_1(\sqrt{z})| < \delta_1(z)$ holds; therefore, in view of (95)

$$\frac{\partial \ln(1+h(z,\eta))}{\partial \eta} \le -\frac{1}{2}\varepsilon_0(z) \tag{110}$$

and, in view of (107),

$$((R(t,\nu))^2 z^{-t})^{-1} \frac{\partial (R(t,\nu))^2 z^{-t}}{\partial t}(t,\nu) = O(\nu^{-\lambda_1}). \tag{111}$$

If $t_3(z;\nu) \le t$ then, in view of (103),

$$-\delta_1(1) < 2\nu^{-2\lambda_1} < \frac{t_3(\nu)-\nu}{t_3(\nu)+\nu} - \eta_1(1) =$$

$$\frac{t_3(\nu)-\nu}{t_3(\nu)+\nu} - 1 \le \frac{t-\nu}{t+\nu} - 1 << \frac{t_1(z;\nu)-\nu}{t_1(z;\nu)+\nu} - 1 < -\nu^{-2\lambda_1} < 0,$$

in view of (95),

$$\frac{\partial \ln(1+h(1,\eta)}{\partial \eta} \le -\frac{1}{2}\varepsilon_0(z), \tag{112}$$

and, in view of (107),

$$((R(t,\nu))^2)^{-1}\frac{\partial (R(t,\nu))^2}{\partial t} = O(\nu^{-\lambda_1}). \tag{113}$$

In of view of (111),

$$\sum_{t\in(t_1(z,\nu),t_2(z,\nu)\cap\mathbb{Z}} -\frac{\partial (R(t,\nu))^2}{\partial t} z^{-t} + \ln(z)(R(t,\nu))^2 z^{-t} = \tag{114}$$

$$O(\nu^{-\lambda_1}) \sum_{t\in(t_1(z,\nu),t_2(z,\nu)\cap\mathbb{Z}} (R(t,\nu))^2 z^{-t}.$$

In of view of (113),

$$\sum_{t\in(t_3(\nu),+\infty)\cap\mathbb{Z}} -\frac{\partial (R(t,\nu))^2}{\partial t} = \tag{115}$$

$$O(\nu^{-\lambda_1}) \sum_{t\in(t_3(\nu),+\infty)\cap\mathbb{Z}} (R(t,\nu))^2 z^{-t}.$$

The equality (55) follows from (108), (114), (109), (115), (4), (5), (6). ■

## §8.3 To what absolute values of roots of charactristic polynomial (38) correspond the obtained solutions of the equation (37).

According to well-known classical Perron's theorem, if $y(\nu)$ is non-zero solution of difference equation of Poincar'e type, then the following equality holds

$$\limsup_{\nu\to+\infty} |y(\nu)|^{1/\nu} = \rho, \tag{116}$$

where $\rho = |\eta|$, and $\eta$ is a root of characterisic polynomial of this equation. If (116) holds, then we will say that the solution $y(\nu)$ corresponds to $\rho$. It follows from (87) ,(88) and (90) that solution $y(\nu) = f_{0,2}(z;\nu)$ corresponds to $(\tilde{\eta}_1(\sqrt{z}))^2 = (\eta_1(\sqrt{z},\pi))^2$, if $z \ge 1$.

**Lemma 8.3.1.** *Let $s \in \mathbb{N}_0, n \in \mathbb{N}$,*

$$a_i^{\sim} \in \mathbb{C},\ a_i(\nu) \in \mathbb{C},$$

$$a_n(\nu) = 1,\ a_i(\nu) - a_i^{\sim} = O(1/(\nu+1)) \tag{117}$$

*for $\nu \in \mathbb{N}_0$ and $i = 0, \ldots, n$. Let us consider the following difference equation*

$$\sum_{k=0}^{n} a_k(\nu) y(\nu+k) = 0, \tag{118}$$

*where* $\nu \in \mathbb{N}_0$*. For any* $m^* \in \mathbb{N}_0$ *let* $V_{m^*}$ *denotes the linear over* $\mathbb{C}$ *space of solutions* $y = y(\nu)$ *of the equation*

$$\sum_{k=0}^{n} a_k(\nu)y(\nu + k) = 0, \tag{119}$$

*where* $\nu \in [m^*, +\infty) \cap \mathbb{Z}$*. Let the absolute values of all the roots of the characteristical polynomial*

$$T(z) = \sum_{k=0}^{n} a_k^{\sim} z^k \tag{120}$$

*are among the numbers* $\{\rho_i \colon 1 \leq i \leq 1 + s\}$ *such that* $\rho_{s+1} = 0$ *and* $\rho_j < \rho_i$ *for* $1 \leq i < j \leq s + 1$*. Let* $e_i$ *and* $k_i$ *denote respectively the sum and the maximum of the multiplicities of those roots, whose absolute value is equal to the number* $\rho_i$*, where* $i = 1, \ldots, s + 1$*, and let* $k^* = k_{s+1}$*. We suppose that, if* $s > 0$*, then*

$$e_i > 0 \tag{121}$$

*for* $i = 1 \ldots, s$*. For given* $y = y(\nu)$ *in* $\mathbb{C}^{[m^*,+\infty)\cap\mathbb{Z}}$*, let*

$$\omega_{n,y}(\nu) = \max(|y(\nu)|, \ldots, |y(\nu + n - 1)|).$$

*a) Then there exist* $A > 0$*,* $m^* \in \mathbb{N}$*,* $\alpha^\wedge(\nu) > 0$ *with* $\nu \in [m^*, +\infty) \cap \mathbb{Z}$ *and the subspaces* $V^\vee_{m^*,1}, \ldots, V^\vee_{m^*,s+1}$ *such that*

$$V_{m^*} = V^\vee_{m^*,1} \oplus \ldots, \oplus V^\vee_{m^*,s+1}, \ \dim_{\mathbb{C}}(V^\vee_{m^*,i}) = e_i, \ 1 \leq i \leq s + 1,$$

*and, if* $y \in V^\vee_{m^*,\theta}$ *for some* $\theta \in \{1, ..., s\}$*, then*

$$\exp(-A(ln(\nu) + \nu^{1-1/k_\theta}))(\rho_\theta)^\nu \omega_n(y)(m^*) \leq \omega_{n,y}(\nu) \tag{122}$$

*for* $\nu \in [m^*, +\infty) \cap \mathbb{Z}$*; moreover, the spaces*

$$V^\wedge_{m^*,j} = V^\vee_{m^ast,j} \oplus \ldots \oplus V^\vee_{m^*,s+1},$$

*where* $j = 1 \ldots, s + 1$*, have the following properties:*

*if* $y \in V^\wedge_{m^*,\theta}$ *for some* $\theta \in \{1, ..., s\}$*, then*

$$\omega_{n,y}(\nu) \leq \exp(A(\ln(\nu) + \nu^{1-1/k_\theta}))(\rho_\theta)^\nu \omega_{n,y}(m^*); \tag{123}$$

*if*

$$k^* > 0, \tag{124}$$

*and* $y \in V^\vee_{m^*,s+1}\,(= V^\wedge_{m^*,s+1})$*, then*

$$|y(\nu)| \leq (A/\nu)^{\nu/k^*} \omega_{n,y}(m^*), \tag{125}$$

*where* $\nu \in m + \mathbb{N} - 1$*.*

*b) If $V$ be an arbitrary linear subspace of linear space$V_{m^*}$ such that*

$$V \cap V_{m^*,\theta+1} = \{0\},$$

*where $\theta \in \{1\ \dots,\ s\}$, then for this $V$ there exists a constant $A^* = A^*(V) > 0$ such that*

$$\exp(-A^*(ln(\nu) + \nu^{1-1/k}))(\rho_\theta)^\nu \omega_n(y)(m^*) \le \omega_{n,y}(\nu) \tag{126}$$

*where $y \in V$, $k = \max(k_1,\ \dots\ k_s)$ and $\nu \in [m^*, +\infty) \cap \mathbb{Z}$.*

**Proof.** The proof can be found in [49] – [53].

**Remark 1.** It follows from the Lemma 8.3.1 that the linear space $V^\wedge_{m^*,\theta}$, where $\theta = 1,\ \dots,\ s+1$, does not depend from the construction and is defined uniquelly by means of the equality

$$V^\wedge_{m^*,\theta} = \{y \in V_m\colon\ \limsup |y(\nu)|^{1/\nu} \le \rho_\theta\}.$$

**Lemma 8.3.2.** *Let $V$ be a $r$-dimensional linear subspace of the linear space $V^*_m$, let $r \ge 1$ and let $V \cap V^\vee_{m^*,s+1} = \{0\}$. Let further $\{y_1(\nu),\ \dots,\ y_r(\nu)\}$ is a basis of the space $V$. Let*

$$k_3(V) = \max\{k \in \mathbb{Z}\colon 1 \le k \le s,\ V \subset V^\wedge_{m^*,k}\},$$

*and*

$$k_4(V) = \min\{k \in \mathbb{Z}\colon 1 \le k \le s,\ V \cap V^\wedge_{m^*,k+1} = \{0\}\}.$$

*For $X \in \mathbb{C}^r$,*

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix}$$

*let*

$$q_\infty(X) = max\{|x_1|, \dots, |x_r|\}, \tag{127}$$

$$y = y^\vee(X,\nu) = x_1 y^\vee_1(\nu)\ + \dots +\ x_r y^\vee_r(\nu). \tag{128}$$

*Then for every $\varepsilon \in (0,1)$ there exist $C_3(\varepsilon) > 0$ and $C_4(\varepsilon) > 0$ such that*

$$C_4(\varepsilon)(\rho_{k_4}(1-\varepsilon))^\nu q_\infty(X) \le \omega_{n,y}(\nu) \le \tag{129}$$

$$C_3(\varepsilon)(\rho_{k_3}+\varepsilon)^\nu q_\infty(X).$$

**Proof.** Any $y \in V$ has an unique representaition in the form (128) with column $X = X(y)$. Let $q_\infty(y) := q_\infty(X(y))$. Then $q_\infty(y)$ and $\omega_{m^*}(y)$ are two norms on finite-dimensional linear over $\mathbb{C}$ space $V$. Therefore there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$C_2 q_\infty(y) \le \omega_{m^*}(y) \le C_1 q_\infty(y).$$

Hence, according to the Lemma 8.3.1, for every $\varepsilon \in (0,1)$ there exist constants $C_5(\varepsilon) > 0$ and $C_6(\varepsilon) > 0$ such that

$$C_6(\varepsilon)(\rho_{k_4}(1-\varepsilon))^\nu \omega_{m^*,y} \le \omega_{n,y}(\nu) \le C_5(\varepsilon)(\rho_{k_3}+\varepsilon)^\nu \omega_{m^*,y}.$$

Then (129) holds with $C_3(\varepsilon) = C_1C_5(\varepsilon)$ and $C_4(\varepsilon) = C_2C_5(\varepsilon)$. ■

We apply the Lemma 8.3.1 to our case $l = 0$. We have $n = 4$ for the equation (37). Let $z > 1$. Then it follows from (53) – (54) that

$$1 < \rho_2 = |\eta_0(r,\pi)|^2 = (2r-1)^2 + 4r(r-1) + 4\sqrt{r(r-1)}(2r-1) <$$

$$\rho_1 = |\eta_0(r,0)|^2 = (2r+1)^2 + 4r((r+1) + 4\sqrt{r(r+1)}(2r+1),$$

$$\rho_4 = 1/\rho_1 < \rho_3 = 1/\rho_2 < 1,$$

$$s = 4,\ e_1 = e_2 = e_3 = e_4 = k_1 = k_2 = k_3 = k_4 = 1.$$

We note that, in view of (26),

$$(\tilde{\eta}_i(z))^2 =$$

$$(2r + (-1)^i + 2\sqrt{r^2 + (-1)^i r})^2 =$$

$$(2r + (-1)^i)^2 + 4r(r + (-1)^i) + 4(2r + (-1)^i)\sqrt{r(r + (-1)^i)} = \rho_{i+1}$$

for $i = 0, 1$. Let is fixed the number $m^*$, which is specified in the Lemma 8.3.1. Then $V_{m^*,5} = \{0\}$, and $V^{\wedge}_{m^*,4} = V^{\vee}_{m^*,4}$. Since the solution $y(\nu) = f_{0,2}(z;\nu)$ corresponds to $\rho_3$, it belongs to $V^{\wedge}_{m^*,3} \diagdown V^{\wedge}_{m^*,4}$. Let $y_4(z;\nu)$ is non-zero solution in $V^{\vee}_{m^*,4}$. Then $y(\nu) = f_{0,2}(z;\nu)$ and $y_4(z;\nu)$ compose the basis of $V^{\wedge}_{m^*,3}$. In view of (55), the solution $y(\nu) = f_{0,3}(z;\nu)$ belongs to $V^{\wedge}_{m^*,3}$. Hence,

$$f_{0,3}(z;\nu) = \alpha f_{0,2}(z;\nu) + \beta y_4(z;\nu),$$

where $\alpha \in \mathbb{C},\ \beta \in \mathbb{C}$. In view of (87) and (123),

$$f_{0,3}(z;\nu) = f_{0,2}(z;\nu)(\alpha + \beta y_4(z;\nu)/f_{0,2}(z;\nu)) =$$

$$f_{0,2}(z;\nu)(\alpha + O(1)\nu^{O(1)}(\rho_4/\rho_3)^\nu).$$

Hence, in view of (55), $\alpha = 0$ and $f_{0,3}(z;\nu)$ belongs to $V^{\wedge}_{m^*,4}$ and, if it is non-zero solution of the equation (37, then it corresponds to $\rho_4$ in this case.

Let $z = 1$. Then it follows from (53) – (54) that

$$s = 3,\ e_1 = e_3 = k_1 = k_3 = 1, e_2 = k_2 = 2$$

$$\rho_1 = |\eta_0(r,0)|^2 = 17 + 12\sqrt{2} > \rho_2 = |\eta_0(r,\pi)|^2 =$$

$$|\eta_1(r,\pi)|^2 = 1 > \rho_3 = |\eta_1(r,0)|^2 = 1/\rho_1.$$

Let is fixed $m^*$, which is specified in the Lemma 8.3.1. Then $V_{m^*,5} = \{0\}$, and $V^{\wedge}_{m^*,4} = V^{\vee}_{m^*,4}$. The solution $y(\nu) = f_{0,2}(1;\nu)$ corresponds to $\rho_2 = 1$ in this case. It is proved in the §7.4 of [63], that our difference equation has also solution $y(\nu) = 1$, which, clearly, corresponds to $\rho_2 = 1$; it is proved there also that the solutions $y(\nu) = f_{0,2}(1;\nu)$ and $y(\nu) = 1$, compose a linearly independent system over $\mathbb{C}$; since each of these solutios correspond to $\rho 2$, it follows that they are contained in $V^{\wedge}_{m^*,2} \diagdown V^{\wedge}_{m^*,3}$. Let $y_3(\nu)$ is non-zero solution in $V^{\vee}_{m^*,3}$. Let

$$0 = \alpha f_{0,2}(1;\nu) + \beta y_(\nu) + \gamma,$$

where $\alpha \in \mathbb{C},\ \beta \in \mathbb{C}$ and $\nu \in [m, +\infty) \cap \mathbb{Z}$ Then, in view of 123 for $y = y_3(\nu)$ and (90), $\gamma = 0$. Then, in view of 123 for $y = y_3(\nu)$ and (90),

$$0 = f_{0,2}(1;\nu)(\alpha + \beta y_3(1;\nu)/f_{0,2}(1;\nu)) =$$

$$\alpha f_{0,2}(1;\nu)(\alpha+O(1)\nu^{O(1)}(\rho_3/\rho_2)^\nu),$$

and therefore $\alpha=\beta=0$. Then $y(\nu)=f_{0,2}(1;\nu),\ y(\nu)=1$ and $y_3(\nu)$ compose the linearly independent system over $\mathbb{C}$; according to the assertions of the Lemma 8.3.1, $\dim_{\mathbb{C}}(V^{\wedge}_{m^*,2})=3$; hence, $y(\nu)=f_{0,2}(1;\nu),\ y(\nu)=1$ and $y_3(\nu)$ compose the basis of $V^{\wedge}_{m^*,2}$. In view of (55), the solution $y(\nu)=f_{0,3}(1;\nu)$ belongs to $V^{\wedge}_{m^*,2}$. Therefore

$$f_{0,3}(1;\nu)=\alpha f_{0,2}(1;\nu)+\beta y_(\nu)+\gamma,$$

where $\alpha\in\mathbb{C},\ \beta\in\mathbb{C}$ and $\gamma\in\mathbb{C}$. It follows from (87) ,(88), (90) and (55), that $\gamma=0$; therefore

$$f_{0,3}(1;\nu)=f_{0,2}(1;\nu)(\alpha+\beta y_3(1;\nu)/f_{0,2}(1;\nu))=$$

$$f_{0,2}(1;\nu)(\alpha+O(1)\nu^{O(1)}(\rho_3/\rho_2)^\nu.$$

Hence, in view of (55), $\alpha=0$ and $f_{0,3}(1;\nu)$ belongs to $V^{\wedge}_{m^*,3}$ and, if it is non-zero solution of the equation (37, then it corresponds to $\rho_3$ in this case.

Let, finally, $z\le -1$. Then $\phi=\pi/2$ in (40), and, as it has been mentioned already in §8.1, we must consider for $\psi$ in (42) two values $\psi=-\frac{\pi}{2}$ and $\psi=\frac{\pi}{2}$. In view of (49), (53), (54),

$$s=2,\ e_1=e_2=k_1=k_2=2,$$

(130) $$V^{\wedge}_{m^*,2}=V^{\vee}_{m^*,2},\ V^{\wedge}_{m^*,3}=V^{\vee}_{m^*,3}=\{0\},\ \dim_{\mathbb{C}}V^{\vee}_{m^*,2}=2,$$

(131) $$\rho_1=|\eta_0(r,\pi/2)|^2=|\eta_0(r,-\pi/2)|^2>$$

$$|\eta_0(r,\pi)|^2\ge|\eta_1(r,\pi)|^2>$$

$$\rho_2=1/\rho_1=|\eta_1(r,\pi/2)|^2,$$

where $r=\sqrt{-z}\ge 1$. Clearly,

$$\cos(\varphi_1(\pi/2))=\sin(2\varphi_3(\pi/2))=1/\sqrt{r^2+1},$$

$$\cos(2\varphi_3(\pi/2))=\sin(\varphi_1(\pi/2))=r/\sqrt{r^2+1},$$

$$\cos(2\varphi_2(\pi/2))=-\sin(\varphi_1(\pi/2))=-r/\sqrt{r^2+1},$$

$$\cos(\varphi_2(\pi/2))=\sqrt{(1-r/\sqrt{r^2+1})/2},$$

$$\cos(\varphi_3(\pi/2))=\sqrt{(1+r/\sqrt{r^2+1})/2}.$$

Therefore, in view of (50),

$$\rho_k=|\eta_{k-1}(r,\pi/2)|^2=1+4r^2+4r\sqrt{1+r^2}+$$

$$(-1)^k2\sqrt{2r}\left(2r\sqrt{\sqrt{r^2+1}+r}+\sqrt{\sqrt{r^2+1}-r}\right)=$$

$$1+4r^2+4r\sqrt{1+r^2}+$$

$$(-1)^{1-k}2\sqrt{2r}\sqrt{(4r^2+1)\sqrt{r^2+1}+r(4r^2+3},$$

where $k = 1, 2,\ r = \sqrt{-z} \geq 1$. We note that, in view of (27),

$$(\tilde{\eta}_2(z))^2 = 4r^2 + 1 + 4r^2\sqrt{r^2+1} + \tag{132}$$

$$2\sqrt{2r}\sqrt{r + \sqrt{r^2+1})^3} =$$

$$4r^2 + 1 + 4r^2\sqrt{r^2+1} +$$

$$2\sqrt{2r}\sqrt{r(4r^2+3) + (4r^2+1)\sqrt{r^2+1}} = \rho_1,$$

in this case. Since $|f_{0,2}(z;\nu)| \leq f_{0,2}(|z|;\nu)$, and the solution $y(\nu) = f_{0,2}(|z|;\nu)$ correspond to $|\eta_1(r,\pi)|^2$, it follows from (38) that $f_{0,2}(z;\nu)$ cannot correspond to $\rho_1$ and, hence, if it is non-zero solution of the equation (37), then it corresponds to $\rho_2$. If $t \geq \nu + 1$ then, in view of (2),

$$((R^{2+l})(t,\nu))^{-1}\left(\frac{\partial}{\partial t}(R^{2+l})\right)(t,\nu),$$

where $l = 0,\ 1,\ 2,\ |z| \geq 1$. is sum of $(2\nu+1)$ summands, which are $O(1)$, where $O(1)$ depends only from $z$. Therefore in view of (6),

$$f_{l,4}(z,\nu) = f_{l,2}(|z|,\nu)(2\nu+1)O(1),$$

where $l = 0,\ 1,\ 2,\ |z| \geq 1$ and $O(1)$, depends only from $z$. Consequently, when $y(\nu) = f_{0,4}(z;\nu)$ is non-zero solution of the equation (37), then it corresponds to $\rho_2$. In view of (5), if $y(\nu) = f_{0,3}(z;\nu)$ is non-zero solution of the equation (37), then it corresponds to $\rho_2$. Moreover, if

$$x_k \in \mathbb{R},\ \text{for } k = 1,\ 2,$$

$$y(x_1, x_2, z, \nu) = \sum_{k=1}^{2} x_k f_{2k}(z;\nu)$$

and $y(x_1, x_2, z, \nu)$ is non-zero solution of the equation (37), then it corresponds to $\rho_2$.

## §8.4. Properties of some sequences.

Here we prove, as a generalization of the Lemma 3.2.1 in [43], the following
**Lemma 8.4.1.** *Let*

$$z \in \mathbb{Q},\ |z| \geq 1,\ z \neq 1,\ b \in \mathbb{N}, a = bz \in \mathbb{Z},$$

*Then the four sequences*

$$\{\alpha^*_{0,i}(z;\kappa+k)\}_{k=1}^{+\infty}),\ \{\beta^*_{0,j}(z;\kappa+k)\}_{k=1}^{+\infty}), \tag{133}$$

*where $i = 1, 2,\ j = 0, 1,$ compose a linearly independent system over $\mathbb{C}$ for any $\kappa \in \mathbb{N}$.*

**Proof.** The proof for $|z| > 1$ can be found in [39] (Lemma 14) in more general situation. The proof for $z = -1$ can be found in [43] (Lemma 3.2.1). Making use of the simplicity of the situation, which we consider now, I give here more short proof. Let $\mathfrak{O}_p = \{u \in \mathbb{Q}\colon ord_p(u) \geq 0\}$. According to (101) and (102) in [62] (see §8.6 below), the polynomials (133) have a form

$$\alpha^*_{0,i}(z;\nu) = \sum_{k=0}^{\nu} \alpha_{0,i,k,\nu} z^k, \tag{134}$$

with $\alpha_{0,i,k,\nu} \in \mathbb{Q}$ for $i = 1,\, 2,\, k \in [0,\nu] \cap \mathbb{Z},\, \nu \in [0,+\infty) \cap \mathbb{Z}$,

$$\beta^*_{0,j}(z;\nu) = \tag{135}$$

$$\sum_{i=1}^{2}\left(\sum_{k=0}^{\nu} \alpha_{0,i,k,\nu}\left(\sum_{t=1}^{k} z^{k-t}\binom{i+j-1}{j}(t)^{-i-j}\right)\right) =$$

$$\sum_{i=1}^{2}\left(\sum_{t=1}^{\nu}\binom{i+j-1}{j}(t)^{-i-j}\left(\sum_{k=t}^{\nu} \alpha_{0,i,k,\nu} z^{k-t}\right)\right)$$

for $j = 0,\, 1,\, \nu \in [0,+\infty) \cap \mathbb{Z}$. Let $p$ be an arbitrary prime greater than 3. In view of (70) – (80) in (23),

$$\alpha_{0,i,k,2p} \in p^i \mathfrak{O}_p \tag{136}$$

for $i = 1,\, 2$ $k \in [1, 2p-1] \cap \mathbb{Z} \diagdown \{p\}$,

$$\alpha_{0,2,0,2p} = 1,\ \alpha_{0,1,0,2p} \in -6/p + \mathfrak{O}_p \tag{137}$$

$$\alpha_{0,2,p,2p} \in 36 + p\mathfrak{O}_p,\ \alpha_{0,2,2p,2p} \in 36 + p\mathfrak{O}_p, \tag{138}$$

$$\alpha_{0,1,p,2p} \in -60/p + \mathfrak{O}_p,\ \alpha_{0,1,2p,2p} \in 66/p) + \mathfrak{O}_p. \tag{139}$$

Therefore if the prime $p$ isn't a divisor of $2ab$ then

$$\alpha^*_{0,2}(z;2p) \in 36(z + z^2) + 1) + p^2\mathfrak{O}_p, \tag{140}$$

$$ord_p(\alpha^*_{0,1}(z;2p) \in 66z^2/p - 60z/p - 6/p + \mathfrak{O}_p. \tag{141}$$

In view of (135) – (141),

$$\beta^*_{0,j}(z;2p) \in \tag{142}$$

$$\sum_{i=1}^{2}\left(\sum_{\tau=1}^{2}\binom{i+j-1}{j}(p\tau)^{-i-j}\left(\sum_{k=p\tau}^{2p} \alpha_{0,i,k,2p} z^{k-p\tau}\right)\right) +$$

$$p^{-1}\mathfrak{O}_p \subset$$

$$\sum_{i=1}^{2}\left(\left(\sum_{\tau=1}^{2}\binom{i+j-1}{j}(p\tau)^{-i-j}\left(\sum_{\kappa=\tau}^{2} \alpha_{0,i,p\kappa,2p} z^{p(\kappa-\tau)}\right)\right) + p^i\mathfrak{O}_p\right) + p^{-1}\mathfrak{O}_p =$$

$$\sum_{i=1}^{2}(\left(\sum_{\tau=1}^{2}\binom{i+j-1}{j}(p\tau)^{-i-j}\left(\sum_{\kappa=\tau}^{2} \alpha_{0,i,p\kappa,2p} z^{p(\kappa-\tau)}\right)\right) + p^{-1}\mathfrak{O}_p \subset$$

$$\binom{1+j-1}{j}(p\tau)^{-1-j}(-60/p+\mathfrak{O}_p)+\binom{2+j-1}{j}(p\tau)^{-2-j}(36+p\mathfrak{O}_p)+$$
$$\binom{1+j-1}{j}(p\tau)^{-1-j}(66/p+\mathfrak{O}_p)(z+p\mathfrak{O}_p)+$$
$$\binom{2+j-1}{j}(p\tau)^{-2-j}(36+p\mathfrak{O}_p)(z+p\mathfrak{O}_p)+$$
$$\binom{1+j-1}{j}(p2)^{-1-j}(66/p+\mathfrak{O}_p)+$$
$$\binom{2+j-1}{j}(p2)^{-2-j}(36+p\mathfrak{O}_p)+p^{-1}\mathfrak{O}_p=$$
$$6(p2)^{-2-j}\times$$
$$(11-2^{1+j}10+(1+j)9(1+2^{2+j})+2^{1+j}11+(1+j)9(2^{2+j})z)+$$
$$p^{-1-j}\mathfrak{O}_p$$

where $j=0,1$. Let

$$F=6ab(36a^2+36ab+b^2)(66a^2-60ab-6b^2)(58a+36b)(188a+133b).$$

Clearly, if $a\in\mathbb{Z}$, $b\in\mathbb{N}$, $|a|\geq b$, $a\neq b$, then $F\neq 0$. Therefore, if $p>|F|$, then

(143) $$ord_p(\alpha^*_{0,i}(z;2p))=-2+i,$$

where $i=1,\,2$, and

(144) $$ord_p(\beta^*_{0,j}(z;2p))=-2-j,$$

where $j=0,1$. As it was mentioned in §7.4 of [63], $z-1$ is a divisor of the polynomial $\alpha^*_{0,1}(z;\nu)$; let

(145) $$P^*_0(z;\nu):=\frac{\alpha^*_{0,1}(z;\nu)}{z-1}\in\mathbb{Q}[z,\nu];$$

then

(146) $$P^*_0(1;\nu)=\frac{d\alpha^*_{0,1}}{dz}(1;\nu).$$

Let

(147) $$P^*_1(z;\nu):=\alpha^*_{0,2}(z;\nu),P^*_2(z;\nu):=\alpha^*_{0,1}(z;\nu),$$

(148) $$P^*_{3+j}(z;\nu):=\beta^*_{0,j}(z;\nu)\,\text{for}\,j=0,1.$$

Then, in view of (143) – (144)

(149) $$ord_p(P^*_{0,i}(z;2p)=1-i$$

where $i=1,\,2,\,3,\,4$. We must prove that for any $\kappa\in\mathbb{N}$ four sequences

(150) $$\{P^*_i(z;\kappa+k)\}_{k=1}^{+\infty},$$

where $i = 1, 2, 3, 4$, compose a linearly independent system over $\mathbb{C}$.

First we prove that for each $\kappa \in \mathbb{N}$ four sequences (150) compose a linearly independent system over $\mathbb{Q}$. Suppose the contrary. Then there exist $\kappa \in \mathbb{N}$ and $a_i \in \mathbb{Z}$, where $i = 1, ..., 4$, such that

$$\sum_{i=1}^{4} |a_i| > 0$$

and

$$\sigma := \sum_{i=1}^{4} a_i P_i^*(z; \nu) = 0, \tag{151}$$

where $\nu \in \mathbb{N}$, $\nu > \kappa$. Let $k = \max\{i \in \{1, 2, 3, 4\} \colon a_i \neq 0\}$ Let $p$ be any prime such that $p > F + \sum_{i=1}^{4} |a_i|$. Then $ord_p(\sigma) = 1 - k$, an we obtain a contradiction. So four sequenses (150) compose a linearly independent system over $\mathbb{Q}$. Hence, the composed by these sequences infinite $4 \times \mathbb{N}$-matrix contain an invertible $4 \times 4$-submatrix. ■

**Lemma 8.4.2.** *Let*

$$z \in \mathbb{Q},\ |z| \geq 1,\ b \in \mathbb{N}, a = bz \in \mathbb{Z},$$

*Then the four sequences*

$$\{P_i(z; \kappa + k)^*\}_{k=1}^{+\infty}, \tag{152}$$

*where* $i = 0, 1, 3, 4$ *compose a linearly independent system over* $\mathbb{C}$ *for any number* $\kappa \in \mathbb{N}$.

**Proof.** If $z \neq 1$ the assertion of the Lemma is direct Corollary of the Lemma 8.4.1. If $z = 1$ the assertion of the Lemma is Corollary of the Lemma 7.4.1 in [63]. ■

If $\nu \in [2, +\infty) \cap \mathbb{Z}$, we let $D_\nu$ denote the smallest number in $\mathbb{N}$ with property that the following inequality holds for every $k = 1, ..., 2\nu$ and for every prime $p, 1 \leq p \leq \nu :$

$$ord_p(k^{-1} D_\nu) \geq 0.$$

It is clear that for any $\varepsilon > 0$

$$D_\nu = \prod_{p \leq \nu} p^{(\ln(2nu))/\ln(p)} = \tag{153}$$

$$\exp((ln(2\nu))(\nu/ln(\nu) + O(\nu/(ln(\nu))^2) = \exp(\nu(1 + O(1)/\ln(\nu))).$$

**Lemma 8.4.3.** *Let*

$$P_i^\wedge(z; \nu) = P_i^*(z; \nu)(D_\nu)^3 \tag{154}$$

*where* $i = 1, 2, 3, 4$. *Then*

$$P_i^\wedge(z; \nu) \in \mathbb{Z}[z] \tag{155}$$

*for* $i = 0,\ 1,\ 2,\ 3,\ 4$ *and* $\nu \in [2, +\infty) \cap \mathbb{Z}$.

**Proof.** For $i = 1,\ 2,\ 3,\ 4$ the proof can be found in [23], page 48. Since assertion of the Lemma holds for $i = 1$, it follows from (145) and Horner rule that it holds for $i = 0$ also. ■

**Lemma 8.4.4.** *For any* $\kappa \in \mathbb{N}$ *four sequences*

$$\{P_i^\wedge(z; \kappa + k)\}_{k=1}^{+\infty}, \tag{156}$$

*where* $i = 0,\ 1,\ 3,\ 4$ *compose a linearly independent system over* $\mathbb{C}$.

**Proof.** The infinite $4 \times \mathbb{N}$-matrix produced by the sequences (152), contains 4 columns, which composed an invertible $4 \times 4$-matrix $M^*$; we suppose that $k_1,\ k_2,\ k_3,\ k_4$ are the numbers of these columns. Let further $M^\wedge$ be the corresponding matrix, composed by the columns with numbers $k_1,\ k_2,\ k_3,\ k_4$ in the $4 \times \mathbb{N}$-matrix, produced by the sequenses (156). Then

$$\det(M^\wedge) = \det(M^*) \prod_{i=1}^{4} (D_{\kappa+k_i})^3 \neq 0.$$

■

## §8.5. Proof of the Theorem 2.

Let $\{m,\ n\} \subset \mathbb{N}$,

$$a_{i,k} \in \mathbb{R}$$

for $i = 1, \dots, m,\ k = 1, \dots, n$,

$$\alpha_j^\wedge(\nu) \in \mathbb{Z}$$

where $j = 1, \dots, m+n$ and $\nu \in \mathbb{N}$. Let there are $\gamma_0^\wedge, r_1^\wedge \geq 1, \dots, r_m^\wedge \geq 1$ such that

$$|\alpha_i(\nu)| < \gamma_0^\wedge (r_i^\wedge)^\nu \tag{157}$$

where $i = 1, \dots, m$ and $\nu \in \mathbb{N}$. Let $y_k(\nu) = -\alpha_{m+k}^\wedge(\nu) + \sum\limits_{i=1}^{m} a_{i,k}\alpha_i^\wedge(\nu)$, where $k = 1, \dots, n$ and $\nu \in \mathbb{N}$. If

$$X = \begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix} \in \mathbb{R}^n, \tag{158}$$

then let

$$q_\infty(X) = \max(|Z_1|, \dots, |Z_n|), \tag{159}$$

$$y^\wedge(X) = y^\wedge(X, \nu) = \sum_{k=1}^{n} y_k^\wedge(\nu) Z_k$$

for $\nu \in \mathbb{N}$, let

$$\phi_i^*(X) = \sum_{k=1}^{n} a_{i,k} Z_k$$

for $i = 1, \dots, m$, and let

$$\alpha_0^\wedge(X,\nu) = \sum_{k=1}^{n} \alpha_{m+k}^\wedge(\nu) Z_k$$

for $\nu \in \mathbb{N}$. Clearly,

$$y^\wedge(X,\nu) = -\alpha_0^\wedge(X,\nu) + \sum_{i=1}^{m} \alpha_i^\wedge(\nu)\phi_i(X)$$

for $X \in \mathfrak{R}^n$ and $\nu \in \mathbb{N}$,

$$\alpha_0^\wedge(X,\nu) \in \mathbb{Z}$$

for $X \in \mathbb{Z}^n$ and $\nu \in \mathbb{N}$.

**Lemma 8.5.1.** *Let* $\{l^\vee, n\} \subset \mathbb{N}$, $\gamma_1^\wedge > 0$, $\gamma_2^\wedge > \frac{1}{2}$, $R_1 \geq R_2 > 1$,

(160) $$\alpha_i = (\log(r_i^\wedge R_1/R_2))/\log(R_2),$$

*where* $i = 1, \dots, m$, *let* $X \in \mathbb{Z}^n \backslash \{0\}$,

$$\gamma_3^\wedge = \gamma_1^\wedge (R_1)^{(-\log(2\gamma_2 R_2))/\log(R_2)}, \gamma_4^\wedge = \gamma_3^\wedge \left( \sum_{i=1}^{m} \gamma_0 (r_i^\wedge)^{(log(2\gamma_2^\wedge))/\log(R_2)+l^\vee} \right)^{-1}$$

*and let for each* $\nu \in \mathbb{N} - 1$ *hold the inequalities*

(161) $$\gamma_1^\wedge (R_1)^{-\nu} q_\infty(X) \leq \sup\{|y^\wedge(X,\kappa)| \colon \kappa = \nu, \dots, \nu + l^\vee - 1\},$$

(162) $$|y^\wedge(X,\nu)| \leq \gamma_2^\wedge (R_2)^{-\nu} q_\infty(X)$$

*Then*

(163) $$\sup\{\|\phi_i^*(X)\|(q_\infty(X))^{\alpha_i} \colon i = 1, \dots, m\} \geq \gamma_4^\wedge.$$

**Proof.** Proof may be found in [42], Theorem 2.3.1. ■

Let now $z = a/b \geq 1$, where $a \in \mathbb{N}$, $b \in \mathbb{N}$. In fiew of (5), (145) – (148), (32), (33) above and (99) in [62] (see §8.6 below),

(164) $$f_{l,3}^\vee(z,\nu) = f_{l,3}(z,\nu) = (\ln(z)) f_{l,2}(z,\nu) + f_{l,4}(z,\nu) =$$

$$P_0^*(z;\nu)((\ln(z))L_{1,1}(1/z) + 1L_{1,2}(1/z)) +$$
$$P_1^*(z;\nu)((\ln(z))L_{0,2}(1/z) + 2L_{0,3}(1/z)) -$$
$$P_3^*(z;\nu)(\ln(z)) - P_4^*(z;\nu) =$$
$$P_0^*(z;\nu)\tilde{\varphi}_1(z,\ln(z),1) +$$
$$P_1^*(z;\nu)\tilde{\varphi}_2(z,\ln(z),1) -$$
$$P_3^*(z;\nu)\tilde{\varphi}_3(z,\ln(z),1) - P_4^*(z;\nu)$$

According to the Lemma 8.4.2, $y(\nu) = f_3(z;\nu)$ iz non-zero solution of the equation (37), and, according to results of the §8.4, if $r = \sqrt{z} > 1$, then it corresponds to

$$\rho_4 = |\eta_0(r,0)|^{-2} =$$

$$1/((2r+1)^2+4r(r+1)+4(2r+1)\sqrt{r(r+1)}),$$

and, if $r=\sqrt{z}=1$, then it corresponds to

$$\rho_3=|\eta_0(1,0)|^{-2}=1/(17+12\sqrt{2}).$$

So, if $z\geq 1$, then $y(\nu)=f_3(z;\nu)$ corresponds to $|\eta_0(\sqrt{z},0)|^{-2}$.

We take in the Lemma 8.5.1 $n=1,\ m=3$,

$$a_{1,1}=(\ln(z))L_{1,1}(1/z)+L_{1,2}(1/z),$$

$$a_{2,1}=(\ln(z))L_{0,2}(1/z)+2L_{0,3}(1/z),\ a_{3,1}=\ln(z),$$
$$\alpha_1^{\wedge}(\nu)=b^{\nu}(D_\nu)^3P_0^*(z;\nu),\alpha_2^{\wedge}(\nu)=b^{\nu}(D_\nu)^3P_1^*(z;\nu),$$
$$\alpha_3^{\wedge}(\nu)=-b^{\nu}(D_\nu)^3P_3^*(z;\nu),\alpha_4^{\wedge}(\nu)=b^{\nu}(D_\nu)^3P_4^*(z;\nu).$$

For any $k=0,\ 1,\ 3,\ 4$ the solution $y(\nu)=P_k^*(z;\nu)$, of the equation (37) corresponds to

$$\rho_k^{\vee}\leq\rho_1=|\eta_0(r,0)|^2=$$
$$(-1-2r-2\sqrt{r(r+1)})^2,$$

where $r=\sqrt{z}$. Therfore, in view of (153), for any $\varepsilon_1\in(0,1)$ there exists a constant $\gamma_0^{\wedge}=\gamma_0^{\wedge}(\varepsilon_1)$ such that with

(165) $$r_i=(\rho_1 be^3)^{1+\varepsilon_1}=(|\eta_0(r,0)|be^3)^{2(1+\varepsilon_1},$$

where $i=1,\ 2,\ 3,\ 4$, the following inequality holds:

(166) $$|\alpha_i^{\wedge}(\nu)|<\gamma_0^{\wedge}r_i^{\nu}=\gamma_0^{\wedge}(\rho_1 be^3)^{(1+\varepsilon_1)\nu},$$

where $i=1,\ 2,\ 3$ and $\nu\in\mathbb{N}$. Since $n=1$ now, it follows that

(167) $$y_1(\nu)=f_3(z;\nu),\ X=(q)\in\mathbb{R}^1,\ q_\infty(X)=|q|,$$

$$y^{\wedge}(X)=y^{\wedge}(X,\nu)=qf_3(z;\nu),$$
$$\varphi_1^*(X)=q\tilde{\varphi}_1(z,\ln(z),1)$$
$$\varphi_2(X)=q\tilde{\varphi}_2(z,\ln(z),1)$$
$$\varphi_3(X)=q\tilde{\varphi}_3(z,\ln(z),1).$$

Since the solution $y(\nu)=f_3(z;\nu)$ corresponds to $|\eta_0(\sqrt{z},0)|^{-2}$, it follows from the Lemma 8.3.1 that there exist constants

$$\gamma_1=\gamma_1(z,\ \varepsilon_1)>0,\ \gamma_2=\gamma_2(z,\ \varepsilon_1)>1/2$$

such that

(168) $$\gamma_1(R_1)^{-\nu}|q\leq\sup\{|qf_3(z;\nu+\kappa)|:\kappa=0,\ ...,\ 3\},$$

(169) $$\{|qf_3(z;\nu+\kappa)|\leq|q\gamma_2(R_2)^{-\nu}$$

if

(170) $$R_1=(\rho_1/be^3)^{1+\varepsilon_1}=(|\eta_0(r,0)/|be^3)^{2(1+\varepsilon_1}\geq$$

$$R_2 = (\rho_1/be^3)^{1-\varepsilon_1}(|\eta_0(r,0)/|be^3)^{2(1-\varepsilon_1} > 1,\ \nu \in \mathbb{N}.$$

The condition $R_2 > 1$ will be fulfilled, if

(171) $$\rho_1/(be^3) = (|\eta_0(r,0)|^2)/(be^3) =$$

$$(-1-2r-2\sqrt{r(r+1)})^2/(be^3) > 1.$$

The condition (171) is equivalent to the condition

(172) $$\sqrt{be^3} < 1 + 2r + 2\sqrt{r(r+1)}.$$

Since $(\sqrt{be^3} > 1 > 1+2r-2\sqrt{r(r+1)} = 1/(1+2r+2\sqrt{r(r+1)})$, it follows that the condition (171) is equivalent to the condition

$$(\sqrt{be^3}-1)^2 - 4\sqrt{z}\sqrt{be^3} < 0.$$

The last inequality is equivalent to the condition

$$z = (-1)^{[k/2]}z > (\sqrt{be^3}-1)^4/(16be^3) =$$

$$(\sqrt{be^3}-(-1)^k)^4\times$$

$$(e^{3/2}b^{1/2}+(-1)^k)^4/(e^3b+1))^{[k/2]}/(16e^3b)$$

with $k=0$, i.e. to the condition $z \in D_0(b)$.

So, if $z \in D_0(b)$, then in view of (163),

(173) $$q^{-\alpha}\gamma_4^{\wedge} \leq$$

$$\max(\|q\tilde{\varphi}_1(z,\ln(z),1)\|, \|q\tilde{\varphi}_2(z,\ln(z),1)\|, \|q\tilde{\varphi}_3(z,\ln(z),1)\|),$$

where $\gamma_4^{\wedge}$ is a positive constant, which depends from $z$ and $\varepsilon_1$, and where

$$\alpha = \alpha(\varepsilon_1) =$$

$$\frac{(1+\varepsilon_1)\ln(\rho_1(be^3)) + 2\varepsilon_1)\ln(\rho_1/(be^3))}{(1-\varepsilon_1)\ln(\rho_1/(be^3))} =$$

$$\frac{(1+\varepsilon_1)\ln((\tilde{\eta}_0(z))^2(be^3)++2\varepsilon_1)\ln((\tilde{\eta}_0(z))^2/(be^3)}{(1-\varepsilon_1)\ln((\tilde{\eta}_0(z))^2/(be^3))}.$$

Since $\alpha(0) = \beta_0(z)$, where the value $\beta_0(z)$ is specified in (28), it follows that for any $\varepsilon > 0$ the inequality $\alpha(\varepsilon_1) < \beta_0(z) + \varepsilon$ holds for sufficiently small $\varepsilon_1$ and, when $z \in D_0(b)$, then, according to (29), (26), the inequality (34) holds with $\gamma_0(z,\varepsilon)$ equal to $\gamma_4^{\wedge}$ in (173). Let

$$x_k \in \mathbb{R},\ \text{for } k = 1,\ 2, \sum_{k=1}^{2}|x_k| > 0,$$

(174) $$y(x_1,x_2,z,\nu) = \sum_{k=1}^{2} x_k f_{2k}(z;\nu) =$$

$$P_0^*(z;\nu)(x_1L_{1,1}(1/z) + x_2L_{1,2}(1/z)) +$$

$$P_1^*(z;\nu)(x_1L_{0,2}(1/z)+2x_2L_{0,3}(1/z))-$$
$$P_3^*(z;\nu)x_1-P_4^*(z;\nu)=$$
$$P_0^*(z;\nu)\tilde{\varphi}_1(z,x_1,x_2)+$$
$$P_1^*(z;\nu)\tilde{\varphi}_2(z,x_1,x_2)-$$
$$P_3^*(z;\nu)x_1-P_4^*(z;\nu)x_2.$$

If $a\in\mathbb{N}$, $b\in\mathbb{N}$, $a\geq b$, $z=-a/b$, then, as it follows from the assertion of the Lemma 8.4.2, $y(x_1,x_2,z,\nu)$ is non-zero solution of the equation (37); in view of (132) and according to results of the §8.3, it corresponds to $(\tilde{\eta}_2(z))^{-2}$.

If $a\in\mathbb{N}$, $b\in\mathbb{N}$, $a>b$, $z=a/b>1$, then, in view of (5),

$$y(x_1,x_2,z,\nu)=x_1f_2(z;\nu)+x_2(f_3(z;\nu)-(\ln(z))f_2(z;\nu)=$$
$$(x_1-x_2\ln(z))f_2(z;\nu)+x_2f_3(z;\nu);$$

according to the results of the §8.3, if $x_1\neq x_2\ln(z)$, then $y(x_1,x_2,z,\nu)$ corresponds to $|\eta_0(r,\pi)|^{-2}$ and, if $x_1=x_2\ln(z)$, then $y(\nu)=y(x_1,x_2,z,\nu)$ corresponds to $|\eta_0(r,\,0)|^{-2}$, where $r=\sqrt{|z|}$. We want to consider first the case, when $x_1\in\mathbb{R}$, $x_2\in\mathbb{Z}\diagdown\{0\}$ and $x_1\neq x_2\ln(z)$ now.

We apply Lemma 8.5.1 with $n=1,\ m=3$,

$$a_{1,1}=x_1L_{1,1}(1/z)+x_2L_{1,2}(1/z))=\tilde{\varphi}_1(z,x_1,x_2),$$
$$a_{2,1}=x_1L_{0,2}(1/z)+2x_2L_{0,3}(1/z)=\tilde{\varphi}_1(z,x_1,x_2),$$
$$a_{3,1}=x_1=\varphi_3(z,x_1,x_2),$$
$$\alpha_1^\wedge(\nu)=b^\nu(D_\nu)^3P_0^*(z;\nu),\alpha_2^\wedge(\nu)=b^\nu(D_\nu)^3P_1^*(z;\nu),$$
$$\alpha_3^\wedge(\nu)=-b^\nu(D_\nu)^3P_3^*(z;\nu),\alpha_4^\wedge(\nu)=x_2b^\nu(D_\nu)^3P_4^*(z;\nu).$$

As above, for any $\varepsilon_1\in(0,1)$ there exists a constant $\gamma_0^\wedge=\gamma_0^\wedge(\varepsilon_1)$ such that with $r_i$ from (165) the inequality (166) holds. Since $n=1$ in Lemma 8.5.1 now, it follows that

(175) $$y_1(\nu)=y(x_1,x_2,z,\nu),\ X=(q)\in\mathbb{R}^1,\ q_\infty(X)=$$
$$|q|,\ y^\wedge(X)=y^\wedge(X,\nu)=qy(x_1,x_2,z;\nu),$$
$$\varphi_1^*(X)=q\tilde{\varphi}_1(z,x_1,x2)$$
$$\varphi_2(X)=q\tilde{\varphi}_2(z,x_1,x_2)$$
$$\varphi_3(X)=q\tilde{\varphi}_3(z,x_1,x_2)=x_1.$$

Since the solution $y_1(\nu)=y(x_1,x_2,z,\nu)$ corresponds to $|\eta_0(\sqrt{z},\pi)|^{-2}$, it follows from Lemma 8.3.1 that there exist constants

$$\gamma_1=\gamma_1(z,\,\varepsilon_1)>0,\ \gamma_2=\gamma_2(z,\,\varepsilon_1)>1/2$$

such that

(176) $$\gamma_1(R_1)^{-\nu}|q\leq\sup\{|qy(x_1,x_2,z,\nu+\kappa)|:\kappa=0,\,...,\,3\},$$

(177) $$\{|qy(x_1,x_2,z,\nu)| \le |q\gamma_2(R_2)^{-\nu},$$

if

(178) $$R_1 = (|\eta_0(\sqrt{z},\pi)|^2/be^3)^{1+\varepsilon_1} \ge$$

$$R_2 = (|\eta_0(\sqrt{z},\pi)|^{-2}/be^3)^{1-\varepsilon_1} > 1,\ \nu \in \mathbb{N}.$$

The condition $R_2 > 1$ will be fulfilled, if

(179) $$(|\eta_0(r,0)|^2)/(be^3) =$$

$$(2r-1+2\sqrt{r(r-1)})^2/(be^3) > 1.$$

The condition (179) is equivalent to the condition

(180) $$\sqrt{be^3} < 2r-1+2\sqrt{r(r-1)}.$$

Since $(\sqrt{be^3} > 1 \ge 2r-1-2\sqrt{r(r-1)} = 1/(2r-1+2\sqrt{r(r-1)})$, it follows that the condition (179) is equivalent to the condition

$$(\sqrt{be^3}+1)^2 - 4\sqrt{z}\sqrt{be^3} < 0.$$

The last inequality is equivalent to the condition

$$z = (-1)^{[k/2]}z > (\sqrt{be^3}+1)^4/(16be^3) =$$

$$(\sqrt{be^3}-(-1)^k)^4 \times$$
$$(e^{3/2}b^{1/2}+(-1)^k)^4/(e^3b+1))^{[k/2]}/(16e^3b)$$

with $k=1$, i.e. to the condition $z \in D_1(b)$.

So, if $z \in D_1(b)$, then in view of (163),

(181) $$q^{-\alpha}\gamma_4^{\wedge} \le$$

$$\max(\|q\tilde{\varphi}_1(z,x_1,x_2)\|, \|q\tilde{\varphi}_2(z,x_1,x_2)\|, \|q\tilde{\varphi}_3(z,x_1,x_2)\|),$$

where $\gamma_4^{\wedge}$ is a positive constant, which depends from $z$ and $\varepsilon_1$, and where

$$\alpha = \alpha(\varepsilon_1) =$$

$$\frac{(1+\varepsilon_1)\ln((2r-1+2\sqrt{r(r-1)})^2)(be^3) + +2\varepsilon_1)\ln\left(\frac{(2r-1+2\sqrt{r(r-1)})^2}{be^3}\right)}{(1-\varepsilon_1)\ln((2r-1+2\sqrt{r(r-1)})^2/(be^3))} =$$
$$\frac{(1+\varepsilon_1)\ln((\tilde{\eta}_1(z))^2(be^3) + +2\varepsilon_1)\ln((\tilde{\eta}_1(z))^2/(be^3)}{(1-\varepsilon_1)\ln((\tilde{\eta}_1(z))^2/(be^3))}.$$

Since $\alpha_0 = \beta_1(z)$, where the value $\beta_1(z)$ is specified in (28), it follows that for any $\varepsilon > 0$ the inequality $\alpha(\varepsilon_1) < \beta_1(z) + \varepsilon$ holds for sufficiently small $\varepsilon_1$ and, when $z \in D_1(b)$, then, according to (29), (26) the inequality (35) holds with $k=1$, $\gamma_1^*(z,x_1,x_2,\varepsilon)$ equal to $\gamma_4^{\wedge}$ in (181). Since

$$(\tilde{\eta}_1(z))^2 = (|\eta_0(\sqrt{z},\pi)|^2) < (|\eta_0(\sqrt{z},0)|^2) = (\tilde{\eta}_0(z))^2,$$

where $z \geq 1$, it follows from (28), (29), that $\alpha_0(z) = \beta_0(z) < \beta_1(z)$; on the other hand, in view of (30), $D_1(b) \subset D_0(b)$. Consequently, (35) is a corollary of (34), if $x_1 = x_2 \ln(z), x_2 \in \mathbb{Z} \diagdown \{0\}$.

Let $a \in \mathbb{N}$, $b \in \mathbb{N}$, $z = -a/b = -r^2$, $r \geq 1$. In view of (43) – (45)

$$\cos(\varphi_1(r, \pi/2)) = \sin(2\varphi_2(r, \pi/2)) = \tag{182}$$

$$\sin(2\varphi_3(r, \pi/2)) = \frac{1}{\sqrt{1+r^2}},$$

$$\sin(\varphi_1(r, \pi/2))) = -\cos(2\varphi_2(r, \pi/2)) = \tag{183}$$

$$\cos(2\varphi_3(r, \pi/2)) = \frac{r}{\sqrt{1+r^2}}.$$

Therefore, in view (50),

$$|\eta_k(r, \pi/2)|^2 = 4r^2 + 1 + 4r(r^2+1)^{1/2} + \tag{184}$$

$$(-1)^k 4\sqrt{r}\left(2r\sqrt{(\sqrt{1+r^2}+r)/2} + \sqrt{(\sqrt{1+r^2}-r)/2}\right),$$

where $r \geq 1$, $k = 0, 1$. First I want to check this equality directly. In view of (42),

$$D_1^{\vee}(r, \pi/2; \eta) = (\eta+1)^2 + 4r \exp i\pi/\eta = \tag{185}$$

$$\eta^2 + 2(1+2ri)\eta + 1 = (\eta + 1 + 2ri)^2 + 4r(r-i).$$

Therefore the roots $\eta$ of this polynomial are

$$-1 - 2ri - 2\sqrt{r}\varepsilon\left(\sqrt{\sqrt{r^2+1}-r)/2} + i\sqrt{\sqrt{r^2+1}+r)/2}\right),$$

where $\varepsilon^2 = 1$, and the squares of their absolute values are

$$1 + 4r^2 + 4r\sqrt{r^2+1} +$$

$$\varepsilon 4\sqrt{r}((\sqrt{\sqrt{r^2+1}-r)/2} + 2r\sqrt{\sqrt{r^2+1}+r)/2}).$$

Since $|\eta_1(r, \pi/2)|^2 < |\eta_0(r, \pi/2)|^2$ it follows that $\varepsilon$ for $\eta_k(r, \pi/2)|^2$ is equal to $(-1)^k$. So, (184) is checked. In view of (27),

$$(\tilde{\eta}_2(z))^2 = 2r^2 + 1 + 2r\sqrt{r^2+1} + 2r^2 + 2r\sqrt{r^2+1} + \tag{186}$$

$$2(r + \sqrt{r^2+1})\sqrt{2(r\sqrt{r^2+1} + r^2)} =$$

$$4r^2 + 1 + 4r\sqrt{r^2+1} + 2(r + \sqrt{r^2+1})\sqrt{2(r\sqrt{r^2+1} + r^2)},$$

$$\left(2(r + \sqrt{r^2+1})\sqrt{2(r\sqrt{r^2+1} + r^2)}\right)^2 = \tag{187}$$

$$8r(r + \sqrt{r^2+1})^3 = 8r(4r^3 + 3r^2 + (4r^2+1)\sqrt{r^2+1}),$$

(188) $$\left(4\sqrt{r}\left(2r\sqrt{\frac{\sqrt{1+r^2}+r}{2}}+\sqrt{\frac{\sqrt{1+r^2}-r}{2}}\right)\right)^2 =$$

$$16r(2r^2\sqrt{1+r^2}+2r^3+(\sqrt{1+r^2})/2-r/2+2r =$$
$$8r(4r^3+3r+(4r^2+1)\sqrt{1+r^2}.$$

In view of (187) , (188), (186) and (184),

(189) $$(\tilde{\eta}_2(z))^2 = |\eta_0(r,\pi/2)|^2,$$

where $z = -r^2$, $r \geq 1$. The function $\rho_0(r) = |\eta_(r,\pi/2)|^2$. is a continuous increasing function which maps $[0,+\infty)$ onto $[1,+\infty)$. We want to find the inverse map $r = r_0(\rho)$ of $[1,+\infty)$ onto $[0,+\infty)$. In view of (184),

$$|\eta_0(r,\pi/2)|^2 \geq 1 \geq \frac{1}{|\eta_0(r,\pi/2)|^2} = |\eta_1(r,\pi/2)|^2,$$

and

$$|\eta_k(r,\pi/2)|^2,\ k=0,1$$

are roots $\rho$ of the trinomial

$$\rho^2 - 2(4r^2+1+4r(r^2+1)^{1/2})\rho + 1,$$

moreover $\rho = \rho_0(r) \geq 1$. Hence, for $r = r_0(\rho)$ we have

$$r^2+\sqrt{r^4+r^2} = \frac{1}{2}(\rho+1/rho)-1 = \frac{(\rho-1)^2}{8\rho},$$

$$r^4+r^2 = r^4+\frac{(\rho-1)^4}{64\rho^2} - r^2\frac{(\rho-1)^2}{4\rho},$$

$$(r^2 = \frac{(\rho-1)^4/(64\rho^2)}{1+(\rho-1)^2/(4\rho)} =$$
$$(\rho-1)^4/(16\rho(\rho+1)^2),$$

and, finally,

(190) $$r_0(\rho) = (\rho-1)^2/(4(\rho+1)\sqrt{\rho}).$$

We apply the Lemma 8.5.1 to the function $y_1(\nu) = y(x_1,x_2,z,\nu)$ again, but now for $z=-r^2$ with $r \geq 1$. The inequality (166) holds with

(191) $$r_i = |\eta_0(r,\pi/2)|^{2(1+\varepsilon_1)},$$

where $i = 1,\ 2,\ 3,\ 4$. Since $y(x_1,x_2,z,\nu$ corresponds to $|\eta_0(\sqrt{-z},pi/2)|^{-2}$, it follows from Lemma 8.3.1 that there exist constants

$$\gamma_1 = \gamma_1(z,\ \varepsilon_1) > 0,\ \gamma_2 = \gamma_2(z,\ \varepsilon_1) > 1/2$$

such that (176) – (177) hold with

(192) $$R_1 = (|\eta_0(\sqrt{-z},\pi/2)|^2/be^3)^{1+\varepsilon_1} \geq$$

$$R_2 = (|\eta_0(\sqrt{-z}, \pi/2)|^2/be^3)^{1-\varepsilon_1} > 1,\ \nu \in \mathbb{N}.$$

In view of (190) with $\rho = e^3 b$, the condition $R_2 > 1$ will be fulfilled, if

(193) $$-z = (-1)^{[k/2]} z = r^2 >$$

$$(r_0(e^3 b))^2 = (e^3 b - 1)^4/(16 e^3 b(e^3 b + 1)^2 =$$
$$(e^{3/2} b^{1/2} - (-1)^k)^4 \times$$
$$(e^{3/2} b^{1/2} + (-1)^k)^4/(e^3 b + 1))^{[k/2]}/(16 e^3 b),$$

where $k = 2$, i.e. if $z \in D_2(b)$. So, if $z \in D_2(b)$, then in view of (163),

(194) $$q^{-\alpha} \gamma_4^{\wedge} \leq$$

$$\max(\|q\tilde{\varphi}_1(z, x_1, x_2)\|, \|q\tilde{\varphi}_2(z, x_1, x_2)\|, \|q\tilde{\varphi}_3(z, x_1, x_2)\|),$$

where $\gamma_4^{\wedge}$ is a positive constant, which depends from $z$ and $\varepsilon_1$, and where, in view of (189),

(195) $$\alpha = \alpha(\varepsilon_1) =$$

$$\frac{(1+\varepsilon_1)\ln((\tilde{\eta}_2(z))^2(be^3) + +2\varepsilon_1)\ln((\tilde{\eta}_2(z))^2/(be^3))}{(1-\varepsilon_1)\ln((\tilde{\eta}_2(z))^2/(be^3))}.$$

Since $\alpha(0) = \beta_2(z)$, where the value $beta_2(z)$ is specified in (28), it follows that for any $\varepsilon > 0$ the inequality $\alpha(\varepsilon_1) < \beta_2(z) + \varepsilon$ holds for sufficiently small $\varepsilon_1$ and, when $z \in D_2(b)$, then, according to (29), (26) the inequality (35) holds with $k = 2$, $\gamma_2^*(z, x_1, x_2, \varepsilon)$ equal to $\gamma_4^{\wedge}$ in (181).

In previous results $x_1$ and $x_2$ were fixed. Let we consider the case when $x_1$ and $x_2$ change. Let $a \in \mathbb{N}$, $b \in \mathbb{N}$, $z = -a/b = -r^2$, $r \geq 1$, and let $z \in D_2(b)$. We apply Lemma 8.5.1 with $n = 2,\ m = 2$,

$$a_{1,1} = L_{1,1}(1/z),\ a_{1,2} = L_{1,2}(1/z)),$$

$$a_{2,1} = L_{0,2}(1/z),\ a_{2,2} = 2L_{0,3}(1/z),$$
$$\alpha_1^{\wedge}(\nu) = b^{\nu}(D_{\nu})^3 P_0^*(z;\nu), \alpha_2^{\wedge}(\nu) = b^{\nu}(D_{\nu})^3 P_1^*(z;\nu),$$
$$\alpha_3^{\wedge}(\nu) = b^{\nu}(D_{\nu})^3 P_3^*(z;\nu), \alpha_4^{\wedge}(\nu) = x_2 b^{\nu}(D_{\nu})^3 P_4^*(z;\nu),$$
$$y_1(\nu) = b^{\nu}(D_{\nu})^3 f_{0,2}^{\vee}(z,\nu) = b^{\nu}(D_{\nu})^3 P_0^*(z;\nu) L_{1,1}(1/z) +$$
$$b^{\nu}(D_{\nu})^3 P_1^*(z;\nu) L_{0,2}(1/z) - b^{\nu}(D_{\nu})^3 P_3^*(z;\nu) =$$
$$\alpha_1^{\wedge}(\nu) a_{1,1} + \alpha_2^{\wedge}(\nu) a_{2,1} - \alpha_3^{\wedge}(\nu),$$
$$y_2(\nu) = b^{\nu}(D_{\nu})^3 f_{0,4}^{\vee}(z,\nu) = b^{\nu}(D_{\nu})^3 P_0^*(z;\nu) L_{1,2}(1/z) +$$
$$b^{\nu}(D_{\nu})^3 P_1^*(z;\nu) 2L_{0,3}(1/z) - b^{\nu}(D_{\nu})^3 P_4^*(z;\nu) =$$
$$\alpha_1^{\wedge}(\nu) a_{1,2} + \alpha_2^{\wedge}(\nu) a_{2,2} - \alpha_4^{\wedge}(\nu),$$

(196) $$X = \begin{pmatrix} x_1 \\ x_2 ) \end{pmatrix} \in \mathbb{R}^2,$$

(197) $$\frac{1}{2}(|x_1| + |x_2|) \leq q_\infty(X) := \max(|x_1|,\ |x_2|) \leq |x_1| + |x_2|$$

(198) $$y^\wedge(X) = y^\wedge(X, z, \nu) =$$

$$y^\wedge(x_1, x_2, z, \nu) = x_1 y_1(\nu) + x_2 y_2(\nu)$$

with $z \in D_2(b)$, $\nu \in \mathbb{N}$,

(199) $$\varphi_i(X) = a_{i,1}x_1 + a_{i,2}x_2,$$

where $i = 1, 2$,

(200) $$\alpha_0(X, \nu) = \alpha_3^\wedge(\nu)x_1 + \alpha_4^\wedge(\nu)x_2.$$

According to the Lemma 8.4.2, $y(\nu) = \{P_i^*(z; nu)^*\}$, where $i = 0, 1, 3, 4$ is non-zero solution of the equation (37); hence, in view of (184) it correspond to some $\rho_i^{**} \leq (\tilde{\eta}_2(z))^2 = |\eta_0(r, \pi/2)|^2$. Therefore the inequality (166) holds with $r_i$ specified in (191).

In view of (31), if $z \leq -1$, $k = 0, 1$, $s > 0$ then

(201) $$(-1)^k L_{k,s}(1/z) > 0 = (-kz + 1) \sum_{n=1}^{+\infty} (1/z)^n / n^s < 0.$$

Therefore, according to the Lemma 8.4.2,

(202) $$f_{0,2}^\vee(z, \nu) = P_0^*(z; \nu) L_{1,1}(1/z) +$$

$$P_1^*(z; \nu) L_{0,2}(1/z) - P_3^*(z; \nu),$$

(203) $$f_{0,4}^\vee(z, \nu) = P_0^*(z; \nu) L_{1,2}(1/z) +$$

$$P_1^*(z; \nu) 2 L_{0,3}(1/z) - P_4^*(z; \nu)$$

compose the basis of the space $V = V_{m^*,2}^\vee = V_{m^*,2}^\wedge$ from the Lemma 8.3.1. Let

(204) $$y^*(X) = y^*(X, z, \nu) = y(x_1, x_2, z, \nu) = x_1 f_2(z, \nu) + x_2 f_4(z, \nu)$$

with $z \in D_2(b)$, $\nu \in \mathbb{N}$ and $X$ in (196). We apply Lemma 8.3.2 now. Then we have $r = 2$, $k_3(V) = k_4(V) = 2$. Therefore, according to the Lemma 8.3.2, for any $\varepsilon_1 \in (0, 1)$ there exist $C_7 = C_7(z, \varepsilon_1) > 0$ and $C_8 = C_8(z, \varepsilon_1) > 0$ such that

(205) $$C_8(R_1 b e^3)^{-\nu} q_\infty(X) \leq$$

$$\sup\{|y^*(X, z, \nu + \kappa)| : \kappa = 0, ..., 3\},$$

(206) $$\{|y^*(X, z, \nu)| \leq |q_\infty(X) C_7 (R_2 b e^3)^{-\nu}$$

with $R_1$ and $R_2$ in (192). In view of (198), and (204),

$$y^\wedge(X, z, \nu) = (D_\nu)^3 y^*(X, z, \nu).$$

Therefore, in view of (153), there exist constants

$$\gamma_1 = \gamma_1(z, \varepsilon_1) > 0, \ \gamma_2 = \gamma_2(z, \varepsilon_1) > 1/2$$

such that

(207)
$$\gamma_1(R_1be^3)^{-\nu}q_\infty(X)\le\sup\{|y^\wedge(X,z,\nu+\kappa)|,\ :\kappa=0,\ ...,\ 3\},$$

(208)
$$\{verty^*(X,z,\nu)|\le|q_\infty(X)(R_2be^3)^{-\nu}$$

with $R_1$ and $R_2$ in (192). So, if $z\in D_2(b)$, then in view of (163),

(209)
$$\max(\|\tilde\varphi_1(z,x_1,x_2)\|,\|\tilde\varphi_2(z,x_1,x_2)\|)(|x_1|+|x_1|)^\alpha\ge$$
$$\max(\|\tilde\varphi_1(z,x_1,x_2)\|,\|\tilde\varphi_2(z,x_1,x_2)\|)q_\infty(X)\ge\gamma_4^\wedge$$

where $\gamma_4^\wedge$ is a positive constant, which depends from $z$ and $\varepsilon_1$, and $\alpha=\alpha(z,\varepsilon_1$ is specified in (195). In view of (28) and (29), $\alpha(0)=\beta_2(z)=\alpha_2(z)$; where $beta_2(z)$ is specified in (28); therefore it follows that for any $\varepsilon>0$ the inequality $\alpha(\varepsilon_1)<\alpha_2(z)+\varepsilon$ holds for sufficiently small $\varepsilon_1$ and, if $z\in D_2(b)$, then (36) holds with $k=2$, $\gamma_0^*(z,\varepsilon)$ equal to $\gamma_4^\wedge$ in (207).

Let, finally, Let $a\in\mathbb N$, $b\in\mathbb N$, $z=a/b=r^2$, $r\ge1$, and let $z\in D_2(b)$. We apply Lemma 8.5.1 with $n=2$, $m=2$ again. Then the inequality (166) holds with $r_i$ in (165). If $z>1$, then $f_{0,2}^\vee(z,\nu)$ and $f_{0,4}^\vee(z,\nu)$ compose the basis of the space $V=V_{m^*,3}^\wedge=V_{m^*,3}^\vee\oplus V_{m^*,4}^\vee$ from the Lemma 8.3.1; $\dim_{\mathbb C}(V_{m^*,k}^\vee)=1$ for $k=1,\ 2$, $k_3(V)=3$, $k_4(V)=4$ If $z=1$, then $f_{0,2}^\vee(z,\nu)$ and $f_{0,4}^\vee(z,\nu)$ compose the basis of the subspace $V$ of $V_{m^*,2}^\wedge=V_{m^*,2}^\vee\oplus V_{m^*,3}^\vee$ from the Lemma 8.3.1; $\dim_{\mathbb C}(V_{m^*,2}^\vee)=2$, $\dim_{\mathbb C}(V_{m^*,3}^\vee)=1$, $k_3(V)=2$, $k_4(V)=3$. In both cases
$$\rho_{k_3(V)}=(\eta_0(\sqrt z,\pi)^{-2},\ \rho_{k_4(v)}=(\eta_0(\sqrt z,0)^{-2},$$
the inequalities (205) and (207) hold with $R_1$ in (170) and $R_2$ in (178). Hence, if $z\in D_1(b)$, then (209) holds with a positive constant $\gamma_4^\wedge$, which depends from $z$ and $\varepsilon_1$, and with

(210)
$$\alpha=\alpha(\varepsilon_1)=$$
$$\frac{(1+\varepsilon_1)\ln((\tilde\eta_0(z))^2(be^3)}{+}(1-\varepsilon_1)\ln((\tilde\eta_1(z))^2/(be^3))+$$
$$\frac{(1+\varepsilon_1)\ln((\tilde\eta_0(z))^2/(be^3)+-(1-\varepsilon_1)\ln((\tilde\eta_1(z))^2/(be^3))}{(1-\varepsilon_1)\ln((\tilde\eta_1(z))^2/(be^3))}.$$

In view of (29), we have the equality $\alpha(0)=\alpha_1(z)$; therefore for any $\varepsilon>0$ the inequality $\alpha(\varepsilon_1)<\alpha_1(z)+\varepsilon$ holds for sufficiently small $\varepsilon_1$ and, if $z\in D_1(b)$, then (36) holds with $k=1$, $\gamma_0^*(z,\varepsilon)$ equal to $\gamma_4^\wedge$ in (207). ■

The Theorem 2 is proved.

## §8.6. Corrections in the previous my papers.

The last equation in §6.5 of Part 6 must have the form
$$(\nu+1)^3y(1;\nu+1)+\nu^3y(\nu-1)=(17\nu^3+51\nu^2+27\nu+5)y(\nu).$$
instead of
$$(\nu+1)^3y(1;\nu+1)+\nu^3y\nu-1)=(17\nu^3+51\nu^2+27\nu+5)y(\nu)(34\nu^3+85\nu^2.$$

On the page 6 in [63] must stand

$$\tilde{\eta}_2(z) = \sqrt{|z|} + \sqrt{|z|+1} + \sum_{k=0}^{1} \sqrt{\sqrt{|z|^2+|z|} + (-1)^k \sqrt{|z|}} =$$

$$\sqrt{|z|} + \sqrt{|z|+1} + \sqrt{2(\sqrt{|z|^2+|z|} + |z|)},$$

instead of what is written there. On the page 8 in [63] must stand

$$\beta_2 = \alpha_2 = 1+$$

$$\frac{6}{2\ln\left(1+\sqrt{2}+\sqrt{\sqrt{2}+1}+\sqrt{\sqrt{2}-1}\right)-3} = 106,00187...$$

instead of what is written there. On the page 6 in [63] must stand

$$\tilde{\eta}_i(z) = \left(\sum_{k=0}^{1} \sqrt{\sqrt{|z|} + k(-1)^i}\right)^2 =$$

$$2\sqrt{|z|} + (-1)^i + 2\sqrt{|z| + (-1)^i\sqrt{|z|}}$$

for $i = 0, 1$, instead of what is written there.

In the formulation of the the Theorem 2 in the [63] must stand $a = bz \in \mathbb{Z}$ instead of $bz \in \mathbb{Z}$.

The equality (99) in [62] must have the form

$$f_{l,2+2j}(z,\nu) =$$

$$\sum_{i=1}^{2+l}\left(\sum_{t=1}^{\infty}\left(\sum_{k=0}^{\nu}\alpha_{l,i,k,\nu}z^k z^{-t-k}\binom{i+j-1}{j}(t+k)^{-i-j}\right)\right) =$$

$$\sum_{i=1}^{2+l}\left(\sum_{k=0}^{\nu}\alpha_{l,i,k,\nu}z^k\left(\sum_{t=1}^{\infty}z^{-t-k}\binom{i+j-1}{j}(t+k)^{-i-j}\right)\right) =$$

$$\sum_{i=1}^{2+l}\left(\sum_{k=0}^{\nu}\alpha_{l,i,k,\nu}z^k\left(\left(\binom{i+j-1}{j}L_{i+j}(1/z) - \sum_{t=1}^{k}z^{-t}\binom{i+j-1}{j}(t)^{-i-j}\right)\right)\right) =$$

$$\left(\sum_{i=1}^{2+l}\alpha^*_{l,i}(z;\nu)\binom{i+j-1}{j}L_{i+j}(1/z)\right) - \beta^*_{l,j}(z;\nu) =$$

$$\left(\sum_{i=-\infty}^{\infty}\alpha^*_{l,i}(z;\nu)\binom{i+j-1}{j}L_{i+j}(1/z)\right) - \beta^*_{l,j}(z;\nu),$$

instead of what is written there. The equality (102) in [62] must have the form

$$\beta^*_{l,j}(z;\nu) =$$

$$\sum_{i=1}^{2+l}\left(\sum_{k=0}^{\nu}\alpha_{l,i,k,\nu}\left(\sum_{t=1}^{k}\binom{i+j-1}{j}z^{k-t}(t)^{-i-j}\right)\right)$$

The expression for $\alpha_k$ on the page 6 in [63] must have a form

$$\alpha_k = \beta_k + \frac{(1-(-1)^k)(\ln(\tilde{\eta}_0(z)/\tilde{\eta}_1(z))}{ln((\tilde{\eta}_1(z))^2/e^3 b)},$$

instead of what is written there.

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