HOLOMORPHIC CORRESPONDENCES BETWEEN CR MANIFOLDS.

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1. Introduction.

One of the interesting phenomena in several complex variables is the analytic continuation of a germ of a biholomorphic map $f: M \to M'$, defined at a point $p \in M$, where $M, M' \subset \mathbb{C}^n$ are real-analytic hypersurfaces. Already Poincaré [Po] observed that a biholomorphic map sending an open piece of a unit sphere in $\mathbb{C}^2$ to another such open piece must be an automorphism of the unit ball. This was proved for $\mathbb{C}^n$ in [Ta] and [Al]. Clearly such an extension is possible only for $n > 1$, thus showing the very special nature of CR maps between real-analytic CR manifolds.

Pinchuk [Pi1],[Pi3] proved that the germ of a non-degenerate holomorphic map from a real-analytic strictly pseudoconvex hypersurface M in $\mathbb{C}^n$ to a unit sphere in $\mathbb{C}^N$, $N \geq n$, extends as a holomorphic map along any path on M. Webster [We] proved that the germ of a biholomorphic map between real-algebraic Levi non-degenerate hypersurfaces in $\mathbb{C}^n$ extends as an algebraic map, and also gave sufficient conditions for the map to be rational. In that paper he studied the geometric properties of Segre varieties, which were originally introduced in [Se].

Much attention was devoted to the generalization of Webster's theorem to the case of different dimensions and higher codimensions. In this situation both M and M' are assumed to be real algebraic submanifolds or sets; that is, defined by the zero locus of a system of real polynomials. Under certain conditions, it then turns out that a locally defined holomorphic map between such objects must necessarily be algebraic. See Baouendi, Ebenfelt and Rothschild [BER1], Huang[Hu], Sharipov and Sukhov [SS], Baouendi, Huang and Rothschild [BHR], Coupet, Meylan and Sukhov [CMS], Zaitsev [Za], Merker [Me1] and many additional references contained therein. The special case of hyperquadrics was also considered in Tumanov [Tu], Forstnerič [Fo], Sukhov [Su] and other papers.

On the other hand, much less is known if at least one of the submanifolds is not assumed to be real-algebraic. In this case the map need not be algebraic; however, analytic continuation along a real-analytic hypersurface is also possible. Pinchuk [Pi2] proved that a germ of a biholomorphic map f from a strictly pseudoconvex, real-analytic, non-spherical hypersurface M to a compact, strictly pseudoconvex, real-analytic hypersurface $M' \subset \mathbb{C}^n$ extends holomorphically along any path on M. A similar result was shown in [Sh1] for the case when M is essentially finite, smooth, real-analytic and $M' \subset \mathbb{C}^n$ is compact, real-algebraic and strictly pseudoconvex. Levi non-degeneracy of the target hypersurface ensures that the extended map is single valued. If the target hypersurface is just assumed to be compact and smooth real-algebraic, the extension in general will be multiple-valued as was proved in [Sh2]. This naturally leads to consideration of holomorphic correspondences, a multiple-valued generalization of holomorphic mappings.

In this paper we study the analytic continuation of germs of holomorphic mappings from smooth real-analytic CR submanifolds of arbitrary codimension to compact smooth real-algebraic generic submanifolds in $\mathbb{P}^N$ of general codimension. The continuation that we obtain is a holomorphic correspondence from a neighborhood of the submanifold in the pre-image to $\mathbb{P}^N$. This is analogous

Date: July 30, 2003.

to the algebraicity of the map asserted in the case when both submanifolds are real-algebraic. We also study some applications to maps between pseudoconcave CR submanifolds in $\mathbb{P}^n$. It is rather surprising that under certain conditions, a local CR map between such objects turns out to be the restriction of a rational, or even a linear map in $\mathbb{P}^n$ without the assumption of algebraicity of the submanifold in the pre-image.

We remark that the proofs of the main results of this paper differ significantly from those utilized in [Sh2], where the main construction essentially uses the fact that the Segre varieties have codimension one.

In the next section we present the main results. In Section 3 we give some background on CR manifolds, Segre varieties and holomorphic correspondences. Section 4 contains the proof of the local extension of a holomorphic map as a holomorphic correspondence. In Section 5 we prove the global extension. The last section contains applications of the main theorem to pseudoconcave CR submanifolds in $\mathbb{P}^n$.

The second author would like to acknowledge the support of the Max-Planck-Institute for mathematics in Bonn during the preparation of this paper. We would also like to thank S. Popescu and S. Lu for a number of very helpful discussions.

2. Statement of Results.

2.1. **Analytic Continuation.** By a holomorphic correspondence we mean a complex analytic subset $A \subset X \times X'$, where X and X' are complex manifolds, such that (i) $\dim A \equiv \dim X$, and (ii) the projection $\pi : A \to X$ is proper. It is natural to define a multiple valued map $F = \pi' \circ \pi^{-1} : X \to X'$ associated with A. A holomorphic correspondence F is called a *finite* correspondence, if $F(p)$ and $F^{-1}(p')$ are finite sets for any $p \in X$ and $p' \in X'$. We say that F *splits* at a point $q \in X$ if there exist a neighborhood U_q of q and an integer k such that $F|_{U_q}$ is just the union of k holomorphic maps $F^j : U_q \to X'$, $j = 1, \ldots, k$.

Recall that if M is an abstract paracompact real analytic CR manifold of type (m, d), then by [AF] there exists a complex manifold X of dimension $n = m + d$ such that M can be generically embedded into X as a CR submanifold.

Consider the following situation: M is a smooth real-analytic minimal CR manifold of type (m, d), $m, d > 0$; $M' \subset \mathbb{P}^N$ is a compact smooth real-algebraic essentially finite generic submanifold of type (m, d'), $d' > 0$. The main results of the paper are the following.

Theorem 2.1. *Suppose that $\omega \subset M$ is a relatively compact connected open subset, and $f : \omega \to M'$ is a real-analytic CR map such that $df|_{H_p} : H_pM \to H_{f(p)}M'$ is an isomorphism between the holomorphic tangent spaces for almost all $p \in \omega$. Let $q \in \partial\omega$. Then there exists a neighborhood $U_q \subset X$ of q such that f extends to U_q as a holomorphic correspondence $F : U_q \to \mathbb{P}^N$ with $F(U_q \cap M) \subset M'$.*

We remark that in Theorem 2.1 we may assume that $f : \omega \to M'$ is just a smooth CR mapping, since by the result in [Da], f is in fact real-analytic. Also note that we do not claim in the above theorem that the extended correspondence near q is finite.

Theorem 2.2. *Assume in addition that M is essentially finite and $d' \geq d$. Let $p \in M$, $U_p \subset X$ be a neighborhood of p, and let $f : U_p \to \mathbb{P}^N$ be a holomorphic mapping of maximal rank such that $f(U_p \cap M) \subset M'$. Suppose that M_1 is a relatively compact simply-connected open subset of M containing p. Then*

1. *There exists a neighborhood $U \subset X$ of M_1 such that f extends as a finite holomorphic correspondence $F : U \to \mathbb{P}^N$ with $F(U \cap M_1) \subset M'$.*

(2) *F splits into holomorphic mappings at every point $q \in M_1 \setminus F^{-1}(\Sigma')$, where Σ' is the set of points on M' near which the Segre map is not locally injective.*

To illustrate the conclusions of Theorem 2.2 we consider a simple example. Let p and q be coprime positive integers, and M and M' be the CR hypersurfaces in $\mathbb{P}^n$ given in homogeneous coordinates by

$$M = \{z \in \mathbb{P}^n : |z_0|^{2p} + \cdots + |z_k|^{2p} - |z_{k+1}|^{2p} - \cdots - |z_n|^{2p} = 0\}, \tag{1}$$

$$M' = \{z' \in \mathbb{P}^n : |z'_0|^{2q} + \cdots + |z'_k|^{2q} - |z'_{k+1}|^{2q} - \cdots - |z'_n|^{2q} = 0\}, \tag{2}$$

with $k < n$. The finite holomorphic correspondence $F(z) = (z_0^{p/q}, \ldots, z_n^{p/q})$ maps M to M', is $p^n : q^n$ valued, and F splits into q^n holomorphic mappings outside the branching locus of F. If we choose a point $p \in M$ outside the branching locus, and if at p we choose the germ of one of the q^n branches of F, then Theorem 2.2 reproduces the entire correspondence F.

Furthermore, we obtain the following.

Theorem 2.3. *Let M be a simply-connected compact smooth real-analytic Levi non-degenerate CR manifold of type $(n-1, 1)$, $n > 1$. Let M' be the hyperquadric in $\mathbb{P}^n$ given in homogeneous coordinates by*

$$|z_0|^2 + |z_1|^2 + \cdots + |z_k|^2 - |z_{k+1}|^2 - \cdots - |z_n|^2 = 0. \tag{3}$$

Suppose that ω is a connected open set in M, and $f : \omega \to M'$ is a CR map that is a local homeomorphism. Then M and M' are globally CR equivalent; hence M has a CR embedding as a hypersurface in $\mathbb{P}^n$. In the special case where $0 < k < n-1$, and M is apriori a CR submanifold of $\mathbb{P}^n$, then f is the restriction to M of a linear automorphism of $\mathbb{P}^n$.

Note that Theorem 2.3 includes both the case when M' is a sphere in $\mathbb{C}^n$, which was proved before in [Pi1], and the case when M' is a compact pseudoconcave hyperquadric. If M is not assumed to be simply-connected, then f in general may not extend to a global map from M to M' as examples in [BS] show for the case when M' is a sphere.

Theorems 2.1, 2.2 and 2.3 can be considered as a generalization of the results on algebraicity of a local CR map between real-algebraic submanifolds. Non-algebraic holomorphic correspondences can be easily constructed from the examples considered in [BS].

Our proof of Theorems 2.1 and 2.2 is based on the technique of Segre varieties. As it was mentioned in the introduction, Webster was the the first to use Segre varieties in the context of holomorphic mappings. His ideas were further developed in [DW], [DF2], [DFY], [DP1] and other papers. Our main construction is based on the technique of these papers. A somewhat different approach was developed in [BJT], [BR] and [BER1]. We rely on the characterization of minimality in terms of Segre sets proved in [BER1].

Let us briefly explain the idea of the main construction. Let $p = 0 \in M$, and let U_0 be a small neighborhood of the origin, where the map f is defined. The first step is to show that f extends as a holomorphic correspondence F_1 to a neighborhood U_1 of Q_0, the Segre variety of the origin. This can be understood as follows. If $w \in Q_0$, then $0 \in Q_w$, and $f(U_0 \cap Q_w)$ is a complex subvariety in the target space which passes through $f(0)$. If $f(U_0 \cap Q_w) \subset Q'_{w'}$ for some w', then we set $F_1(w) = w'$. For w close to the origin, $f(Q_w \cap U_0) \subset Q'_{f(w)}$ by the invariance property of Segre varieties, and thus the extension defined in this way agrees with f on U_0. For essentially finite manifolds, if w' is sufficiently close to M', then there exists a finite number of points which have the same Segre variety as w'. Therefore in general F_1 may not be single valued. The graph of F_1 can be described as

$$A_1 = \{(w, w') \in U_1 \times \mathbb{P}^N : f(Q_w \cap U_0) \subset Q'_{w'}\}. \tag{4}$$

The next step of the proof is to define inductively the analytic continuation of f to bigger sets. This can be achieved using Segre sets (see the next section for definitions). Briefly, once the analytic continuation of f as a correspondence F_1 is established, we can use a similar construction to define the extension of F_1 along the Segre varieties of points on Q_0. Let

$$A_2 = \{(w, w') \in U_2 \times \mathbb{P}^N : F_1(Q_w \cap U_1) \subset Q'_{w'}\}, \tag{5}$$

where U_2 is a neighborhood of $Q_0^2 = \cup_{w \in Q_0} Q_w$. We note that for w' close to M' the condition for (w, w') in (5) roughly speaking means that all branches of F_1 map Q_w into $Q'_{w'}$. We can repeat this process for A_3, using the extension A_2, etc.

Finally, the crucial observation is that after a certain number of iterations, we obtain an extension of f to a neighborhood of the origin, which is independent of the choice of U_0, where f was originally defined. This follows from the minimality of M and the property of Segre sets proved in [BER1]. Furthermore, for compact minimal manifolds such neighborhoods can be chosen of uniform size for all points on M, which allows us to extend f to any simply-connected relatively compact subset of M.

2.2. **Pseudoconcave CR manifolds.** Our main application of Theorem 2.2 concerns pseudoconcave CR submanifolds embedded into projective spaces. We recall that a CR manifold is called pseudoconcave, if at each point its Levi form has at least one positive and at least one negative eigenvalue in every characteristic conormal direction.

Theorem 2.4. *Consider a connected C^∞-smooth compact pseudoconcave CR submanifold M of $\mathbb{P}^n$, having type (m, d) with $m, d > 0$.*

(a) *Let $f : M \to \mathbb{P}^N$ be a continuous CR map. Then f is the restriction to M of a rational map $F : \mathbb{P}^n \to \mathbb{P}^N$.*
(b) *Assume $n = m + d$, so that M is generic in $\mathbb{P}^n$. Let $f : M \to \mathbb{P}^n$ be a CR map which is a local diffeomorphism onto $f(M)$. Then f is the restriction of a linear automorphism of $\mathbb{P}^n$.*

For a locally biholomorphic map from a compact smooth pseudoconcave hypersurface in $\mathbb{P}^n$ to $\mathbb{P}^n$ part (b) was first shown in [Iv]. Our proof of Theorem 2.4 is based on the result of [HN5], where it is shown that any CR meromorphic function on M is necessarily rational.

Combining Theorem 2.4 with Theorem 2.2, we obtain the following results.

Theorem 2.5. *Let $M \subset \mathbb{P}^n$ (resp. $M' \subset \mathbb{P}^N$) be a compact smooth real-analytic pseudoconcave essentially finite CR submanifold of type (m, d) (resp. (m, d')), $m, d, d' > 0$ and $d' \geq d$. Let M be simply connected and let $M' \subset \mathbb{P}^N$ be generic real-algebraic, and such that the Segre map associated with M' is locally injective. Let $p \in M$, U_p be a neighborhood of p in $\mathbb{P}^n$, and let $f : U_p \cap M \to M'$ be a germ of a smooth CR map of maximal rank. Then*

(a) *f is the restriction of a rational map $F : \mathbb{P}^n \to \mathbb{P}^N$;*
(b) *If $n = N = m + d$ and the Segre map associated with M is locally injective, then f is the restriction of a linear automorphism of $\mathbb{P}^n$.*

If $M, M' \subset \mathbb{P}^n$ are both of hypersurface type (i.e. $d = d' = 1$), then f may be assumed to be just a local CR homeomorphism, since in this case f extends smoothly to a neighborhood of p, and by [HN2], the Jacobian of f does not vanish at p.

Corollary 2.6. *If a real-analytic submanifold M, satisfying the conditions of Theorem 2.5, is locally CR equivalent to a real-algebraic CR submanifold M' satisfying Theorem 2.5, then M is necessarily real-algebraic.*

Remark. The above results concerning pseudoconcave CR manifolds hold also under a weaker assumption on M and M', namely, instead of pseudoconcavity it is enough to assume that M and M' satisfy the so-called Property E. For a generic M this means that for each $p \in M$ every local CR function defined near p extends to a holomorphic function in a full neighborhood of p in the ambient space. For details see [HN5].

3. CR manifolds, Segre varieties and holomorphic correspondences.

An abstract smooth CR manifold of type (m, d) consists of a connected smooth paracompact manifold M of dimension $2m + d$, a smooth subbundle HM of TM of rank $2m$, which is called the holomorphic tangent space of M, and a smooth complex structure J on the fibers of HM. Let $T^{0,1}M$ be the complex subbundle of the complexification $\mathbb{C}HM$ of HM, which corresponds to the $-i$ eigenspace of J:

$$T^{0,1}M = \{Y + iJY | Y \in HM\}. \tag{6}$$

We also require that the formal integrability condition

$$[C^\infty(M, T^{0,1}M), C^\infty(M, T^{0,1}M)] \subset C^\infty(M, T^{0,1}M) \tag{7}$$

holds. We call m the CR dimension of M and d the CR codimension. M is called minimal at $p \in M$, if there exists no local CR manifold $N \subset M$ passing through p having CR dimension m, but strictly smaller real dimension.

The characteristic bundle H^0M is defined to be the annihilator of HM in T^*M. Its purpose is to parametrize the Levi form, which for every $p \in M$ is defined for $v \in H^0_pM$ and $Y \in H_pM$ by

$$\mathcal{L}(v; Y) = d\tilde{v}(Y, JY) = \langle v, [J\tilde{Y}, \tilde{Y}] \rangle, \tag{8}$$

where $\tilde{v} \in C^\infty(M, H^0M)$ and $\tilde{Y} \in (M, HM)$ are smooth extensions of v and Y. For each fixed v it is a Hermitian quadratic form for the complex structure J_p on H_pM. A CR manifold M is said to be pseudoconcave if the Levi from $\mathcal{L}(v, \cdot)$ has at least one negative and one positive eigenvalue for every $p \in M$ and every nonzero $v \in H^0_pM$.

Let M and M' be two abstract smooth CR manifolds, with holomorphic tangent spaces HM and HM'. A smooth map $f : M \to M'$ is CR if $f_*(HM) \subset HM'$, and $f_*(Jw) = J'f_*(w)$ for every $w \in HM$. A CR embedding of an abstract CR manifold M into a complex manifold X is a CR map which is an embedding. We say that the embedding is *generic* if the complex dimension of X is $(m + d)$.

If M is a real-analytic CR manifold of type (m, d), then by [AH], M is locally CR embeddable. Furthermore, by [AF] there exists a complex manifold X such that M can be globally generically embedded into X. Consider a connected open set ω on M. When M is real-analytic, and f is a real-analytic CR function in ω, then there is a connected open set Ω_f in X, with $\omega = M \cap \Omega_f$, and a holomorphic extension $\tilde{f}$ of f to Ω_f. When M is a generic C^∞-smooth pseudoconcave CR submanifold of X, then there exists a connected open set Ω in X, with $\omega = M \cap \Omega$, such that any CR distribution in ω has a unique holomorphic extension to Ω. See [BP], [NV], [HN3], [HN4].

Most of our considerations of real-analytic CR manifolds will be local, and therefore by the above mentioned results we can assume without loss of generality that M is a generically embedded CR submanifold of some open set in $\mathbb{C}^n$, where $n = m+d$. Note here that a compact pseudoconcave CR manifold cannot be embedded (even non-generically) into any Stein manifold (see e.g. [HN1]). Therefore, for application purposes the main theorem is formulated for the target submanifold embedded into $\mathbb{P}^N$.

Let M be a generic smooth real-analytic submanifold of $\mathbb{C}^n$ and let $p \in M$. Then in a sufficiently small neighborhood U of p, M is given by

$$M = \{z \in U : \rho_j(z, \overline{z}) = 0, \quad j = 1, \dots d\}, \tag{9}$$

where each ρ_j is a real-valued real-analytic function and

$$\overline{\partial}\rho_1 \wedge \cdots \wedge \overline{\partial}\rho_d \neq 0 \text{ on } M \cap U. \tag{10}$$

We set $\rho = (\rho_1, \rho_2, ..., \rho_d)$. There exists a biholomorphic change of coordinates near p, $z = (\xi, \zeta) \in \mathbb{C}^m \times \mathbb{C}^d = \mathbb{C}^n$ such that in the new coordinates, $p = 0$, and M is given by

$$\text{Im } \zeta = \phi(\xi, \overline{\xi}, \text{Re } \zeta), \tag{11}$$

where ϕ is a vector-valued real-analytic function with $\phi(0) = 0$ and $d\phi(0) = 0$.

If U is sufficiently small, to every point $w \in U$ we can associate to M its so-called Segre variety in U defined as

$$Q_w = \{z \in U : \rho_j(z, \overline{w}) = 0, \; j = 1, \dots, d\}, \tag{12}$$

where $\rho_j(z, \overline{w})$ is the complexification of the defining functions of M. Another important variety associated with the submanifold M and the neighborhood U is

$$I_w = \{z \in U : Q_w = Q_z\}. \tag{13}$$

From the reality condition on the defining functions the following simple but important properties of Segre varieties follow:

$$z \in Q_w \;\Leftrightarrow\; w \in Q_z, \tag{14}$$

$$z \in Q_z \;\Leftrightarrow\; z \in M, \tag{15}$$

$$w \in M \Leftrightarrow I_w \subset M. \tag{16}$$

If $0 \in M$, then from (10) and the implicit mapping theorem, there exist a local change of coordinates near the origin, and a pair of small neighborhoods U and U_0 of the origin, $U \Subset U_0$, where U_0 is given in the product form

$$U_0 = {}'U_0 \times \tilde{U}_0, \quad {}'U_0 \subset \mathbb{C}^m, \quad \tilde{U}_0 \subset \mathbb{C}^d, \tag{17}$$

such that for every $w \in U$, the set $Q_w \cap U_0$ can be represented as the graph of a holomorphic mapping. That is

$$Q_w \cap U_0 = \{z = (\xi, \zeta) \in {}'U_0 \times \tilde{U}_0 : \; \zeta = h(\xi, \overline{w})\}, \tag{18}$$

where $h(\xi, \overline{w})$ is holomorphic in ξ and $\overline{w}$. Thus Q_w is a complex submanifold of U of complex codimension d.

The main use of Segre varieties comes from the fact that they are invariant under biholomorphic mappings. More precisely, given a holomorphic map $f : U \to U'$, sending a generic smooth real-analytic submanifold M to another such submanifold M', $f(p) = p'$ implies $f(Q_p \cap U) \subset Q'_{p'}$ for p sufficiently close to M. An analogous property holds also for holomorphic correspondences.

The proof of the basic properties of Segre varieties in higher codimensions is similar to the hypersurface case and can be found in [BER2] or [Me2].

A real analytic submanifold M is called essentially finite at $p \in M$, if $I_p = \{p\}$ in a small neighborhood of p. The *Segre map* is defined by $\lambda : w \to Q_w$. A manifold M being essentially finite now means that the Segre map is finite near M. It can be shown (see e.g. [Me2]) that any generic Levi non-degenerate CR submanifold of $\mathbb{C}^n$ is essentially finite. Moreover, if M is a compact generic submanifold of $\mathbb{C}^n$, then it is automatically essentially finite, since by [DF1], any compact real-analytic subset of $\mathbb{C}^n$ does not contain any non-trivial germs of complex-analytic varieties. See also [BER2].

In [BER1] the authors introduced so-called Segre sets. We briefly recall this construction here. Let M be a generic smooth real-analytic submanifold of $\mathbb{C}^n$, $0 \in M$ and let $Q_0 = Q_0^1$ be the usual Segre variety of 0 as defined in (12). Define

$$Q_0^j = \bigcup_{z \in Q_0^{j-1}} Q_z, \quad j > 1. \tag{19}$$

Then

$$Q_0^1 \subset Q_0^2 \subset \cdots \subset Q_0^j \subset \cdots . \tag{20}$$

Indeed, let k be the smallest integer such that $Q_0^k \not\subset Q_0^{k+1}$. Clearly, $k \geq 2$. If $z \in Q_0^k \setminus Q_0^{k+1}$, then there exists $w \in Q_0^{k-1}$ such that $z \in Q_w$. By assumption, $Q_0^{k-1} \subset Q_0^k$. Therefore $w \in Q_0^k$, and $Q_w \subset Q_0^{k+1}$; in particular $z \in Q_0^{k+1}$, which is a contradiction.

According to [BER1] (see also [BER2]), there exists an integer j_0, $0 < j_0 < \infty$, such that $\cup_{j \leq j_0} Q_0^j$ contains a neighborhood of the origin in $\mathbb{C}^n$, provided that M is minimal at 0. We define

$$\Omega_0 = \{z : z \in Q_0^j,\ j \leq j_0\}. \tag{21}$$

Moreover, if M is compact, or is a relatively compact open set of a bigger minimal submanifold, then there exists $\epsilon > 0$ such that for any point $p \in M$, the neighborhood Ω_p, defined as in (21), contains a ball of radius ϵ centered at p.

Suppose now that the manifold $M \subset \mathbb{P}^n$ is connected and defined by real polynomials. Then the Segre varieties associated with M can be defined globally as projective algebraic varieties in $\mathbb{P}^n$. Indeed, let $M \cap \mathbb{C}^n$ be given as a connected component of the set defined by

$$\{z \in \mathbb{C}^n : \rho_j(z, \overline{z}) = 0,\ j = 1, \dots, d\}, \tag{22}$$

where ρ_j are real polynomials. We can projectivize each ρ_j to define M in $\mathbb{P}^n$ in homogeneous coordinates

$$\hat{z} = [\hat{z}_0 : \hat{z}_1 : \cdots : \hat{z}_n], \quad z_k = \frac{\hat{z}_k}{\hat{z}_0}, \quad k = 1, \dots, n, \tag{23}$$

as a connected component of the set defined by

$$\{\hat{z} \in \mathbb{P}^n : \hat{\rho}_j(\hat{z}, \overline{\hat{z}}) = 0\}. \tag{24}$$

We may define now the *polar* of M as

$$\hat{M}^c = \{(\hat{z}, \hat{\zeta}) \in \mathbb{P}^n \times \mathbb{P}^n : \hat{\rho}_j(\hat{z}, \hat{\zeta}) = 0, \quad j = 1, \dots, d\}. \tag{25}$$

Then $\hat{M}^c$ is a complex algebraic variety in $\mathbb{P}^n \times \mathbb{P}^n$. Given $\tau \in \mathbb{P}^n$, we set

$$\hat{Q}_\tau = \hat{M}^c \cap \{(\hat{z}, \hat{\zeta}) \in \mathbb{P}^n \times \mathbb{P}^n : \hat{\zeta} = \overline{\tau}\}. \tag{26}$$

We define the projection of $\hat{Q}_\tau$ to the first coordinate to be the Segre variety of τ.

Alternatively, given (22) one can define the polar as

$$M^c = \{(z, \zeta) \in \mathbb{C}^n \times \mathbb{C}^n : \rho_j(z, \zeta) = 0, \quad j = 1, \dots, d\}. \tag{27}$$

The submanifold $M \cap \mathbb{C}^n$ can be recovered by intersecting M^c with the totally real subspace $\mathcal{T} = \{(z, \zeta) \in \mathbb{C}^{2n} : \zeta = \overline{z}\}$ and taking an appropriate connected component. Given $w \in \mathbb{C}^n$, we define

$$Q_w^c = M^c \cap \{(z, \zeta) \in \mathbb{C}^{2n} : \zeta = \overline{w}\}. \tag{28}$$

The standard Segre variety Q_w can now be recovered by projecting Q_w^c to $\mathbb{C}_z^n$. The algebraic varieties M^c and Q_w^c can be projectivized which gives us objects geometrically equivalent to (25) and (26). Note that the closure $\overline{\mathcal{T}} \subset \mathbb{P}^{2n}$ of the set $\mathcal{T}$ is a smooth submanifold of $\mathbb{P}^{2n}$, and thus M can be identified with a connected component of $\overline{M^c} \cap \overline{\mathcal{T}}$.

We note that condition (10) implies that for w close to M, the Segre variety Q_w contains a connected component $\tilde{Q}_w$ of dimension m. However in general, Q_w may have other components, which apriori may even have different dimension (higher than m). For $w \in M$, $w \in \tilde{Q}_w$, and for w close to M, $\tilde{Q}_w$ is the component which near w is given by (18).

We will understand essential finiteness of a real algebraic submanifold $M \subset \mathbb{P}^n$ in the sense that for every $w \in M$, the set

$$\tilde{I}_w := \{z \in M : \tilde{Q}_z = \tilde{Q}_w\} \tag{29}$$

is finite. In this case we can further show that generically the various $\tilde{I}_w$ have the same number of points. Note that the set $\tilde{I}_w$ is a globally defined object in $\mathbb{P}^n$.

Lemma 3.1. *Let M be a compact smooth real-algebraic essentially finite generic CR submanifold in $\mathbb{P}^n$. Then there exist a neighborhood $U \subset \mathbb{P}^n$ of M and an integer $R \geq 1$ such that*

$$\#\tilde{I}_w = \#\{z \in U : \tilde{Q}_z = \tilde{Q}_w\} = R \tag{30}$$

for almost all $w \in U$.

Proof. For each $p \in M$ choose a product neighborhood U_p as in (17). Let $U = \bigcup_{p \in M} U_p$. We define the map λ from U to the set of all algebraic varieties in $\mathbb{P}^n$ of dimension m by letting $\lambda(w) := \tilde{Q}_w$. Then $\tilde{Q}_w$ depends anti-holomorphically on w. Locally, near almost every $w \in U$, the varieties $\tilde{Q}_w$ have the same algebraic degree. Since algebraic varieties of positive codimension do not divide U, the $\tilde{Q}_w$ have fixed degree for almost all w in U. Algebraic varieties of fixed dimension and degree are known to be parametrized by the so-called Chow variety (see e.g. [Ha]), and the parametrization λ is an algebraic map between algebraic varieties.

Let $Y = \lambda(U)$. Then since M is essentially finite, $\dim Y = n$. It follows (see e.g. [Mu]) that there exists an algebraic variety $Z \subset Y$ such that for any $q \in Y \setminus Z$,

$$\#\lambda^{-1}(q) = \deg(\lambda) := R, \tag{31}$$

where R is a positive integer. From (31) the assertion follows. $\square$

Let X and X' be complex manifolds and $D \subset X$, $D' \subset X'$ be open sets. Recall that if $A \subset D \times D'$ is a holomorphic correspondence, then $\pi : A \to D$ is proper. A is called a proper holomorphic correspondence, if $\pi' : A \to D'$ is also proper. Throughout the paper we will identify the multiple valued map $F := \pi' \circ \pi^{-1}$ with its graph A, so given a point $p \in D$,

$$F(p) = \{p' \in D : \pi' \circ \pi^{-1}(p)\} \tag{32}$$

is a compact subset of D'. Given a complex analytic subset $G' \subset D'$, $F^{-1}(G')$ is a complex analytic subset of D. Indeed ${\pi'}^{-1}(G')$ is clearly analytic, and since π is proper, it follows from Remmert's theorem that $\pi({\pi'}^{-1}(G))$ is analytic in D. F is called a *finite-valued* holomorphic correspondence if $F(p)$ is a finite set for any $p \in D$, and a *finite* correspondence if in addition $F^{-1}(p')$ is finite for any $p' \in X'$. If X' is a Stein manifold, then any proper holomorphic correspondence F is automatically finite-valued. To see this observe that by the proper embedding theorem, X' can be viewed as a submanifold of $\mathbb{C}^{n'}$ for some $n' > 1$, and a compact complex analytic set $F(p)$ must be discrete. Similarly, if X and X' are both Stein, then any proper holomorphic correspondence is finite.

Given a finite-valued holomorphic correspondence $A \subset D\times D'$, there exists a complex subvariety $S \subset D$ (possibly empty) such that for any point $p \in D\setminus S$, there exists a neighborhood $U_p \subset D\setminus S$, such that F *splits* into k holomorphic maps $F^j : U_p \to D'$, $j = 1,\dots,k$, that represent F. The integer k is independent of p, and the F^j are called the *branches* of F.

Given a locally complex analytic set A in X of pure dimension p, we say that A *extends analytically* to an open set $U \subset X$, $A\cap U \neq \varnothing$, if there exists a (closed) complex-analytic set A^* in U such that (i) $\dim A^* \equiv p$, (ii) $A \cap U \subset A^*$ and (iii) every irreducible component of A^* has a nonempty intersection with A of dimension p. By the uniqueness theorem for analytic sets such analytic continuation of A is uniquely defined. From this we define the analytic continuation of holomorphic correspondences:

Definition 3.2. *Let $D \subset X$ and $D' \subset X'$ be open sets and let $A \subset D \times D'$ be a holomorphic correspondence. We say that A extends as a holomorphic correspondence to an open set $U \subset X$, $D\cap U \neq \varnothing$, if there exists an open set $U' \subset X'$ such that $A\cap(U\times U') \neq \varnothing$, A extends analytically to a set $A^* \subset U \times U'$, and $\pi : A^* \to U$ is proper.*

If A and A^* are both finite-valued, then A^* may have more branches in $D \cap U$ than A. The following lemma gives a simple criterion for the extension to have the same number of branches.

Lemma 3.3. *Let $A^* \subset U\times X'$ be a finite-valued holomorphic correspondence which is an analytic extension of a finite correspondence $A \subset D \times D'$. Suppose that for any $z \in (D \cap U)$,*

$$\#\{\pi^{-1}(z)\} = \#\{\pi^{*-1}(z)\}, \tag{33}$$

where $\pi : A \to D$ and $\pi^* : A^* \to U$ are the projections. Then $A\cup A^*$ is a holomorphic correspondence in $(D \cup U) \times X'$.

The proof is the same as in [Sh2], Lemma 2.

4. Local continuation as a correspondence.

In this section we prove Theorem 2.1. We may assume the following situation: $\Omega \subset \mathbb{C}^n$ is a connected open set, $\Omega\cap M = \omega$, $f : \Omega \to \mathbb{P}^N$ is a holomorphic map, and $f(\Omega \cap M) \subset M'$. Let $q \in \partial\Omega \cap M$ and let U be a neighborhood of q such that for any $z \in U$, the Segre variety Q_z is defined in a strictly larger set, and can be represented as in (18). We show that f extends to some neighborhood of q as a holomorphic correspondence. We choose a point $a \in \omega$ such that $df|_{H_aM}$ is an isomorphism, and so close to q that $q \in \Omega_a$, where Ω_a is defined as in (21). Fix some neighborhood $U_a \subset \Omega$ of a. We will show that $f|_{U_a}$ extends as a holomorphic correspondence to Ω_a, in particular to a neighborhood of q.

Assume for simplicity that $a = 0$. Choose a small neighborhood U_0 of the origin and shrink U_a in such a way, that for any $w \in U_0$, the set $Q_w \cap U_a$ is non-empty and connected. Let $U' \subset \mathbb{P}^N$ be the neighborhood of M' as in Lemma 3.1. Define

$$A_0 = \{(w,w') \in U_0 \times U' : f(Q_w \cap U_a) \subset Q'_{w'}\}. \tag{34}$$

Then A_0 is a complex-analytic subset of $U_0 \times U'$. In the case when M and M' are hypersurfaces, this was shown in [Sh2], the proof in the general case is analogous (see also similar constructions later in this section). Note that in (34),

$$\dim Q_w = \dim Q'_{w'} = \dim f(Q_w \cap U_a) = m. \tag{35}$$

Indeed, since $df|_{H_0M}$ is an isomorphism, $\dim Q_0 = \dim f(Q_0 \cap U_0)$. Without loss of generality we may assume that M near 0 and M' near $f(0)$ are chosen as in (11). Let $J_f(z)$ be the Jacobian matrix associated with the map f. Then in the chosen coordinate systems, $df|_{H_0M}$ being an isomorphism means that the principal minor of $J_f(0)$ of size $m \times m$ has a non-zero determinant. The same property also holds for points sufficiently close to the origin. Therefore, after shrinking U_0, (35) holds for all w in U_0. After further shrinking U_0 if necessary, we may assume that if $(w, w') \in A_0$, then $w' \in \tilde{I}'_{f(w)}$, which is a finite set by the assumption on M'. It follows that $\dim A_0 \equiv n$. Finally, (16) implies that if U_0 is sufficiently small, then A_0 has no limit points on $U_0 \times \partial U'$, and therefore the natural projection from A_0 to the first coordinate is proper. This shows that A_0 defines a holomorphic correspondence. Denote the corresponding multiple valued map by F_0.

We shrink the neighborhood U_0 of the origin, where F_0 is defined, and choose a 'thin' neighborhood U_1 of $Q_0 \cap U$ such that for any $w \in U_1$, the set $Q_w \cap U_0$ is non-empty and connected. Note that from (14), $0 \in Q_w$ for any $w \in Q_0$. Denote now by A_0 the graph of F_0 in $U_0 \times \mathbb{P}^N$. We define the set A_1 as follows:

$$A_1 = \left\{(w, w') \in U_1 \times \mathbb{P}^N : F_0(Q_w \cap U_0) \subset Q'_{w'}\right\}. \tag{36}$$

Then A_1 is a complex-analytic subset of $U_1 \times \mathbb{P}^N$. To verify this assertion we prove the following:

1. $A_1 \neq \varnothing$. Indeed, by the invariance property of Segre varieties, $f(Q_w \cap U_0) \subset Q'_{f(w)}$ for w sufficiently close to the origin. Then
$$F_0(Q_w \cap U_0) \subset Q'_{w'}, \tag{37}$$
where $w' \in F_0(w)$. To see this, suppose that $z \in Q_w \cap U_0$ and $z' \in F_0(z)$. By construction, $F_0(Q_z \cap U_0) \subset Q'_{z'}$. If w is sufficiently close to the origin, $w \in Q_z \cap U_0$, and therefore $w' \in F_0(w) \subset Q'_{z'}$. This implies $z' \in Q'_{w'}$. And since $z \in Q_w \cap U_0$ was arbitrary, (37) holds. Moreover, $F_0(w) = \tilde{I}'_{w'}$, and thus
$$A_0|_{(U_1 \cap U_0) \times \mathbb{P}^N} \subset A_1, \tag{38}$$
in particular A_1 is non-empty.
2. A_1 is a complex-analytic set in a neighborhood of any of its points. Indeed, let $(w_0, {w'}_0) \in A_1$, $z_0 \in Q_{w_0} \cap U_0$ be an arbitrary point, and let $U_{z_0} \subset U_0$ be a small neighborhood of z_0. Since $Q_w \cap U_0$ is connected for all $w \in U_1$, $F_0(Q_w \cap U_{z_0}) \subset Q'_{w'}$ implies $F_0(Q_w \cap U_0) \subset Q'_{w'}$. Therefore in (36) U_0 can be replaced with U_{z_0}. Choose U_{w_0} so small that $Q_w \cap U_{z_0} \neq \varnothing$ for all $w \in U_{w_0}$. Since the pre-image of an analytic set under a holomorphic correspondence is an analytic set, $S_{w'} := F_0^{-1}(Q'_{w'})$ is an analytic subset of U_0. Let $U'_{w'_0}$ be so small that $S_{w'} \cap U_{z_0} \neq \varnothing$ for all $w' \in U'_{w'_0}$. Let $S_{w'}$ near z_0 be given by
$$S_{w'} = \{z \in U_{z_0} : \phi_j(z, \overline{w}') = 0,\ j = 1, \dots, \tilde{j}\}, \tag{39}$$
where the ϕ_j depend holomorphically on $\overline{w}'$. We may assume that U_{z_0} is chosen as in (17), and therefore $z \in Q_w$ simply means $z = (\xi, \zeta)$ and $\zeta = h(\xi, \overline{w})$. Then the condition $F_0(Q_w \cap U_{z_0}) \subset Q'_{w'}$ is equivalent to
$$\phi_j\left((\xi, h(\xi, \overline{w})), \overline{w}'\right) = 0, \quad \xi \in {}'U_{z_0},\ j = 1, \dots, \tilde{j}. \tag{40}$$
This is an infinite system of holomorphic equations (after conjugation) which defines A_1 as an analytic set in $U_{w_0} \times U'_{w'_0}$.
3. A_1 is closed in $U_1 \times \mathbb{P}^N$. Indeed, suppose that $(w^j, {w'}^j) \to (w^0, {w'}^0)$, as $j \to \infty$, where $(w^j, {w'}^j) \in A_1$ and $(w^0, {w'}^0) \in U_1 \times \mathbb{P}^N$. Since $Q_{w^j} \to Q_{w^0}$, and $Q'_{w'^j} \to Q'_{w'^0}$ as $j \to \infty$, by analyticity $F_0(Q_{w^0} \cap U_0) \subset Q'_{w'^0}$, which implies that $(w^0, {w'}^0) \in A_1$ and thus A_1 is a closed set.

It follows from 1–3 that A_1 is a complex-analytic subset of $U_1 \times \mathbb{P}^N$. Let $\pi_1 : A_1 \to U_1$ and $\pi_1' : A_1 \to \mathbb{P}^N$ be the coordinate projections. Since $\mathbb{P}^N$ is compact, π_1 is proper. We consider only the irreducible components of A_1 of dimension n which contain A_0. The union of all such components we denote again by A_1. Thus A_1 is an analytic continuation of A_0 as a holomorphic correspondence. Denoting $U_0 \cap U_1$ again by U_0, we may assume from the uniqueness theorem for analytic sets that

$$A_1|_{U_0 \times \mathbb{P}^N} = A_0. \tag{41}$$

We set $F_1 := \pi_1' \circ \pi_1^{-1} : U_1 \to \mathbb{P}^N$. Note that in general, F_1 may not be finite-valued. However, by the Cartan-Remmert theorem (see e.g. [Lo]) combined with Remmert's proper mapping theorem, the set of points

$$\Sigma_1 = \{z \in U_1 : \dim \pi_1^{-1}(z) > 0\} \tag{42}$$

is a complex subvariety of U_1. Since $\dim A_1 \equiv n$, we have $\dim \Sigma_1 < n$, in particular, Σ_1 does not divide U_1, and $A_1|_{(U_1 \setminus \Sigma_1) \times \mathbb{P}^N}$ is a finite-valued holomorphic correspondence.

By construction, if $(w, w') \in A_1$, then for any $(z, z') \in A_1$ such that $z \in Q_w \cap U_0$, we necessarily have $z' \in Q'_{w'}$. We wish to conclude the same for all z in $Q_w \cap U_1$. The difficulty is that in general, $Q_w \cap U_1$ may have more than one connected component. To prove the assertion we argue by contradiction, and assume that there exists a point $(w^0, w'^0) \in A_1$ and $(z, z') \in A_1$, such that $z \in Q_{w^0} \cap U_1$, but $z' \notin Q'_{w'^0}$. Connect w^0 and the origin with a smooth path γ contained in U_1. We may assume that w^0 is the first point on γ for which the desired property does not hold. Without loss of generality we may also assume that for all $p \in \gamma$ between the origin and w^0 (excluding w^0) there exists a small neighborhood U_p such that whenever $z \in U_p$, all components of $Q_z \cap U_1$ are mapped by F_1 into the same Segre variety. For each point $p \in \gamma$ between the origin and w^0, we construct the following set:

$$A_p = \left\{(w, w') \in U(Q_p) \times \mathbb{P}^N : F_1(Q_w \cap U_p) \subset Q'_{w'}\right\}, \tag{43}$$

where $U_p \subset U_1$ is a neighborhood of p and $U(Q_p)$ is a neighborhood of $Q_p \cap U$, which are chosen in such a way that U_p satisfies the property described above and that for any $w \in U(Q_p)$, the set $Q_w \cap U_p$ is connected. Repeating the argument that was used for A_1, one can prove that each A_p is a complex analytic set, which defines a holomorphic correspondence. For $p = 0$, this is just the set A_1. Moreover, for any p between the origin and w^0 we have

$$A_1|_{(U(Q_p) \cap U_1) \times \mathbb{P}^N} \subset A_p|_{(U(Q_p) \cap U_1) \times \mathbb{P}^N}. \tag{44}$$

Indeed, suppose that $w \in U(Q_p) \cap U_1$ and $(w, w') \in A_1$. Let $z \in Q_w \cap U_p$ be an arbitrary point, and $z' \in F_1(z)$. Then $F_0(Q_z \cap U_0) \subset Q'_{z'}$. From (41) we have $F_1(Q_z \cap U_0) \subset Q'_{z'}$. By the assumption on U_p, $F_1(Q_z \cap U_1) \subset Q'_{z'}$, in particular $F_1(w) \subset Q'_{z'}$. Therefore, $w' \in Q'_{z'}$, and $z' \in Q'_{w'}$. Since $z \in Q_w \cap U_p$ was arbitrary, it follows that $F_1(Q_w \cap U_p) \subset Q'_{w'}$. But this means that $(w, w') \in A_p$, and thus (44) holds.

For any p, $Q_p \cap U$ is a connected set in $U(Q_p)$ and therefore is mapped by A_p into the same Segre variety. By continuity and from (44) we conclude that $F_1(Q_{w_0} \cap U_1) \subset Q'_{w'_0}$, which contradicts the assumption. Thus for any $(w, w') \in A_1$, if $(z, z') \in A_1$ and $z \in Q_w \cap U_1$, then $z' \in Q'_{w'}$.

We now define recursively for $j > 1$ the following sets:

$$A_j = \left\{(w, w') \in U_j \times \mathbb{P}^N : F_{j-1}(Q_w \cap U_{j-1}) \subset Q'_{w'}\right\}. \tag{45}$$

Here the open set U_j is defined as follows. Suppose that the set $A_{j-1} \subset U_{j-1} \times \mathbb{P}^N$ is already defined and $U_{j-1} \subset U$ is some connected open set. We let U_j be the set of points w in U such that $Q_w \cap U_{j-1} \neq \varnothing$. Furthermore, after shrinking at each step, if necessary, the sets U_k for $k < j$, we may assume that $U_{k-1} \subset U_k$ for $1 \le k \le j$. Note that it follows from the construction that $Q_0^k \subset U_k$ for $1 \le k \le j$.

We claim that for all $j > 0$, A_j is a complex-analytic subset of $U_j \times \mathbb{P}^N$, which satisfies the following properties:

(i) $A_j|_{U_{j-1}\times\mathbb{P}^N} = A_{j-1}$;
(ii) A_j defines a holomorphic correspondence $F_j : U_j \to \mathbb{P}^N$;
(iii) for any $(w, w') \in A_j$, if $(z, z') \in A_j$ and $z \in Q_w \cap U_j$, then $z' \in Q'_{w'}$.

Condition (iii) can be understood in the sense that the map F_j, associated with A_j, sends all connected components of $Q_w \cap U_j$ into $Q'_{w'}$ provided that $(w, w') \in A_j$.

The proof is by induction. The case $j = 1$ is already proved. Suppose that A_{j-1} is as claimed. We show that the set defined by (45) is also a holomorphic correspondence satisfying properties (i)–(iii).

(i) Let $w \in U_{j-1}$, and $(w, w') \in A_{j-1}$. Then by definition, $F_{j-2}(Q_w \cap U_{j-2}) \subset Q'_{w'}$. From property (i), which by the induction hypothesis holds for F_{j-1}, the correspondences F_{j-2} and F_{j-1} agree in U_{j-2}, and therefore we have

$$F_{j-1}(Q_w \cap U_{j-2}) \subset Q'_{w'}. \tag{46}$$

From (iii), F_{j-1} maps all components of $Q_w \cap U_{j-1}$ into the same Segre variety. Therefore (46) implies $F_{j-1}(Q_w \cap U_{j-1}) \subset Q'_{w'}$, which by definition means that $(w, w') \in A_j$. In particular, the set A_j is non-empty. Condition (i) for A_j will be completely proved, once we know that A_j is a complex-analytic set, $\dim A_j \equiv n$, and select only the irreducible components of A_j which have intersection with A_{j-1} of dimension n.

Proof of (ii). Let $(w^0, w'^0) \in A_j$. If $Q_w \cap U_{j-1}$ is connected for all w sufficiently close to w^0, then the proof of the fact that A_j is complex-analytic near $(w^0, w'^0) \in A_j$ is the same as for A_1 in Step 2. Let $\tilde U$ be the largest connected relatively open subset of U_j such that $0 \in \tilde U$ and for all $z \in \tilde U$, F_{j-1} maps $Q_z \cap U_{j-1}$ into the same Segre variety. From property (iii) for A_{j-1} we have $U_{j-1} \subset \tilde U$. Then (45) defines a holomorphic correspondence $\tilde A \subset \tilde U \times \mathbb{P}^N$. The proof is the same as in Step 2 for A_1. Denote by $\tilde F$ the multiple valued map associated with $\tilde A$. By repeating the argument used for A_1, we can show that for any $w \in \tilde U$, $\tilde F$ maps all connected components of $Q_w \cap \tilde U$ into the same Segre variety.

For each point $p \in \tilde U$ we may define now the following set

$$A_p = \{(w, w') \in U(Q_p) \times \mathbb{P}^N : \tilde F(Q_w \cap U_p) \subset Q'_{w'}\}, \tag{47}$$

where the neighborhoods $U(Q_p)$ of Q_p and U_p of p are chosen as in the construction of the set defined by (43). Let F_p be the map associated with A_p. Clearly, F_p coincides with F_{j-1} for p sufficiently close to the origin. We claim that the map F_p agrees with F_{j-1} in $U(Q_p) \cap U_{j-1}$ for all $p \in \tilde U$. Indeed, suppose that $w \in U(Q_p) \cap U_{j-1}$ and $(w, w') \in A_{j-1}$. Then $F_{j-2}(Q_w \cap U_{j-2}) \subset Q'_{w'}$. To prove the assertion we need to show that

$$\tilde F(Q_w \cap U_p) \subset Q'_{w'}. \tag{48}$$

Let $z \in Q_w \cap U_p$ be an arbitrary point, and $z' \in \tilde F(z)$. Then $F_{j-1}(Q_z \cap U_{j-1}) \subset Q'_{z'}$. Since $\tilde F$ and F_{j-1} agree in U_{j-1}, it follows that $\tilde F(Q_z \cap U_{j-1}) \subset Q'_{z'}$. For $z \in \tilde U$, $\tilde F$ maps different components of $Q_z \cap \tilde U$ into the same Segre varieties, and therefore we have $\tilde F(Q_z \cap \tilde U) \subset Q'_{z'}$. In particular, $w' \in Q'_{z'}$, which implies $z' \in Q'_{w'}$. Since z was arbitrary, (48) holds.

By the construction, F_p maps Q_p into the same Segre variety for all $p \in \tilde{U}$. By analyticity this means that for any point w in $\partial\tilde{U} \cap U_j$, F_{j-1} maps $Q_w \cap U_{j-1}$ into the same Segre variety. Therefore, $\tilde{U} = U_j$. We choose only irreducible components of A_j of dimension n which contain A_{j-1}. This proves (ii) and also completes the proof of (i).

Finally, property (iii) can be shown the same way as it was done for A_1.

By the construction, from minimality of M and from [BER1], for some $j_0 > 1$, the set A_{j_0} defines a holomorphic correspondence F_{j_0} in a neighborhood $\Omega_0 \subset U_{j_0}$ of the origin. Note that the size of this neighborhood depends only on the geometry of M and is independent of U_0, where f was originally defined. It remains to show now that F_{j_0} satisfies

$$F_{j_0}(M \cap \Omega_0) \subset M'. \tag{49}$$

If $(z, z') \in A_{j_0}$, then $F_{j_0-1}(Q_z \cap U_{j_0-1}) \subset Q'_{z'}$. From property (i), $F_{j_0}(Q_z \cap U_{j_0-1}) \subset Q'_{z'}$, and from (iii) it follows that

$$F_{j_0}(Q_z \cap U_{j_0}) \subset Q'_{z'}. \tag{50}$$

Suppose now that for some $z^0 \in M \cap \Omega_0$, $F_{j_0}(z^0) \not\subset M'$. Then there exists $z' \in F_{j_0}(z^0) \setminus M'$. Note that $F_{j_0}(M \cap U_0) \subset M'$, and therefore by continuity we may find z^0 and z' such that z' is close to M'. From (50) we have $F_{j_0}(Q_{z^0} \cap U_{j_0}) \subset Q'_{z'}$. Since $z^0 \in Q_{z^0}$, we have $F_{j_0}(z^0) \subset Q'_{z'}$, in particular, $z' \in Q'_{z'}$. But from (15), $z' \notin Q'_{z'}$, since $z' \notin M'$. This contradiction proves (49).

Theorem 2.1 is proved.

5. Extension as a finite correspondence along M.

In this section we give the proof of Theorem 2.2. One difficulty in the proof of analytic continuation of correspondences lies in the fact that the continued correspondence may acquire additional branches. To deal with this we define the notion of a *complete* correspondence as follows.

Definition 5.1. *Let $M \subset \mathbb{C}^n$ be a smooth real-analytic essentially finite generic CR submanifold, and let $M' \subset \mathbb{P}^N$ be a smooth compact real-algebraic essentially finite generic submanifold. Let $F : U \to \mathbb{P}^N$ be a holomorphic correspondence such that $F(U \cap M) \subset M'$. Then F is called complete if for every $z \in M$, we have $F(z) = \tilde{I}'_{z'}$ for some (and therefore for any) $z' \in F(z)$. Here $\tilde{I}'_{z'}$ is defined as in* (29).

Note that since M' is essentially finite, a complete correspondence is finite-valued near M, but in general it may be reducible, even if defined on all of M.

Assume that M is locally, near $p \in M$, generically embedded into an open set in $\mathbb{C}^n$, so $n = m + d$. Let f be a holomorphic map defined in a neighborhood $U_p \subset \mathbb{C}^n$ of $p \in M$, of maximal rank and such that $f(U_p \cap M) \subset M'$. We replace f with a complete correspondence. For that we choose a small neighborhood U_0 of p and shrink U_p in such a way, that for any $w \in U_0$, the set $Q_w \cap U_p$ is non-empty and connected. We define A_0 as in (34). Denote the corresponding multiple valued map by F_0. Then F_0 is a complete holomorphic correspondence. Indeed, since f is of maximal rank and $d' \geq d$, for $z \in U_0$, $\dim f(Q_z \cap U_p) = \dim Q_z = m$. For z sufficiently close to p, $f(z)$ is close to M', and since M' is essentially finite, the only points whose Segre varieties can contain $f(Q_z \cap U_p)$ are in $I'_{f(z)}$. Furthermore, F_0 is a finite correspondence. To see this we observe that if $E \subset U_0$ is a positive dimensional set such that $F_0(E)$ is discrete, then by the construction $f(Q_z \cap U_p) \subset Q'_{z'}$ for all $z \in E$. But since $\bigcup_{z \in E} Q_z \cap U_p$ has dimension bigger than m, this contradicts the fact that f is of maximal rank in U_p. Therefore the pre-image of any point under F_0 is finite, and thus F_0 is a finite correspondence.

We now show that F_0 extends as a finite correspondence along any path on M_1. Our construction of the analytic continuation of F_0 will preserve completeness, and therefore from Lemmas 3.1 and 3.3 and we conclude that such analytic continuation will have the same number of branches near M_1. The problem can be localized as follows. Let $\gamma : [0,1] \to M_1$ be the given path, $\gamma(0) = p$, and assume that for $t_0 \leq 1$, $q = \gamma(t_0)$ is the first point on γ to which F_0 does not extend as a finite correspondence. We choose $t_1 \in [0, t_0)$ so close to t_0, that for $a = \gamma(t_1)$ we have $q \in \Omega_a$, where Ω_a is defined as in (21). Then by Theorem 2.1, F_0 extends as a holomorphic correspondence $A \subset \Omega_a \times \mathbb{P}^n$. Thus we only need to show that after possibly shrinking the neighborhood Ω_a, the set A is a finite correspondence. Let $\pi : A \to \Omega_a$ and $\pi' : A \to \mathbb{P}^N$ be the natural projections, and set $F := \pi' \circ \pi^{-1}$.

From (49) there exists a neighborhood $\tilde{U} \subset \Omega_a$ of $M \cap \Omega_a$ such that $F(\tilde{U}) \subset U'$, where $U' \subset \mathbb{P}^N$ is a neighborhood of M' as in Lemma 3.1. We now repeat the argument of analytic continuation of A_0 along γ by constructing the sets A_j^*, $j = 1, 2, \dots$, with the only difference that the standard neighborhoods of the form (17) are chosen so small that they are contained in $\tilde{U}$. Since the new set Ω_a may be smaller than $\tilde{U}$, this continuation may require more than one step. More precisely, we choose a sequence of points $\{a_\nu\}_{\nu=0}^{l}$ such that $a_\nu \in \gamma$, $a_0 = a$, $a_l = q$ and $a_\nu \in \Omega_{a_{\nu-1}}$ for $0 < \nu \leq l$. For each a_ν starting with a_0 we use Theorem 2.1 to extend a finite correspondence F_ν defined in a neighborhood of a_ν to a holomorphic correspondence $F_{\nu+1}$ defined in $\Omega_{a_\nu} \subset \tilde{U}$. This time we show in addition that at each step the extension is a finite correspondence. Since $a_{\nu+1} \in \Omega_{a_\nu}$, the process can be continued until we reach the point q.

Suppose that $F_{\nu+1} : \Omega_{a_\nu} \to \mathbb{P}^N$ is a holomorphic correspondence, which is obtained from the finite correspondence F_ν defined in a small neighborhood of the point a_ν for some ν by the inductive construction of the sets A_j^*; that is

$$A_j^* = \left\{ (w, w') \in U_j^* \times \mathbb{P}^N : F_{j-1}^*(Q_w \cap U_{j-1}^*) \subset Q'_{w'} \right\}, \tag{51}$$

where the Segre set $Q_{a_\nu}^j$ is contained in U_j^*, the map F_ν is associated with the set A_0^*, and $F_{\nu+1}$ corresponds to the set $A_{j_0}^*$ with $U_{j_0}^* = \Omega_{a_\nu}$. Let F_j^* be the map associated with A_j^*.

Clearly, the sets A_j^* are contained in the set A, which defines a correspondence $F : \tilde{U} \to U'$. We claim that $F_{\nu+1}$ is a finite correspondence. To prove this assertion we let k be the smallest integer such that F_k^* is not a finite correspondence. By assumption, $k > 0$. Suppose that there exists a point $w' \in U'$ such that the analytic set ${F_k^*}^{-1}(w') \subset U_k^*$ has positive dimension. By the construction we have

$$F_{k-1}^*(Q_z \cap U_{k-1}^*) \subset Q'_{w'}, \quad \text{for all} \quad z \in {F_k^*}^{-1}(w'). \tag{52}$$

Since M is essentially finite, there exists a set $E \subset {F_k^*}^{-1}(w')$ such that $\dim \cup_{z \in E} Q_z = m + 1$. It follows from (52) that $\dim F_{k-1}^*(\cup_{z \in E} Q_z) \leq m$. But this contradicts the induction hypothesis that F_{k-1}^* is finite.

Suppose now that there exists a point $w \in U_k^*$ such that $F_k^*(w)$ is not discrete. By the construction this means that $F_{k-1}^*(Q_w \cap U_{k-1}^*) \subset Q'_{z'}$, where z' belongs to the non-discrete set. Since F_{k-1}^* is finite, $F_{k-1}^*(Q_w \cap U_{k-1}^*)$ has dimension m. But then there can be only finitely many z' whose Segre varieties can contain $F_{k-1}^*(Q_w \cap U_{k-1}^*)$. Therefore F_k^* is also finite for any $k \geq 0$.

Hence, the obtained extension $F_{\nu+1}$ is a finite correspondence. We can repeat the same argument for extending F_0 along $\gamma \cap \tilde{U}$ until we reach the point q.

Thus we have proved that F_0 extends along any path on M_1 as a finite correspondence F. It follows from the construction that the number of branches of F coincides with the number R defined in Lemma 3.1 for almost all points on M_1. It remains to observe now that from the simple connectivity of M_1 and the monodromy theorem, the extension of F_0 along homotopically equivalent paths gives the same result. The version of the monodromy theorem for finite-valued correspondences can be found in [Sh2].

Part (2) of the statement of the theorem follows from the construction of the extension F. The proof of Theorem 2.2 is now complete.

6. Pseudoconcave submanifolds in $\mathbb{P}^n$.

In this section we prove the rest of the results stated in Section 2.

Proof of Theorem 2.4.

(a) According to Thm 5.2 of [HN5], there exists an $m+d$ dimensional irreducible algebraic subvariety Y of $\mathbb{P}^n$ such that M is a generic CR submanifold of the regular part of Y, reg Y. Because of the pseudoconcavity of M (or because of Property E), the continuous CR map f is smooth and has a unique holomorphic extension to an open neighborhood Ω of M in reg Y. Thus f can be regarded as a holomorphic map from Ω to $\mathbb{P}^N$. This means that f may be given by N meromorphic functions $f_1, f_2, \dots f_N$ in Ω. To see this, we choose homogeneous coordinates $[z_0 : z_1 : \dots : z_N]$ in $\mathbb{P}^N$ such that the hyperplane $\{z_0 = 0\}$ is in general position with respect to $f(\Omega)$, and set $\Omega_j = f^{-1}(V_j)$, where $V_j = \mathbb{P}^N \setminus \{z_j = 0\}$, $j = 0, 1, \dots, N$. The Ω_j give an open cover of Ω, in each V_j we have the inhomogeneous coordinates $(w_{1j}, w_{2j}, \dots, w_{Nj})$, where $w_{ij} = z_i/z_j$, and $f|_{\Omega_j}$ is given by holomorphic functions

$$\Omega_j \ni t \to (w_{1j}(t), w_{2j}(t), \dots, w_{Nj}(t)). \tag{53}$$

We define the meromorphic functions $f_1, f_2, \dots, f_N$ by $f_k(t) = w_{k0}(t)$ in Ω_0, and by $f_k(t) = w_{kj}(t)/w_{0j}(t)$ in Ω_j for $j, k = 1, 2, \dots, N$. Note that these definitions are consistent on the overlaps.

By Thm 5.2 of [HN5], each f_k is the restriction to M of a rational function on Y, and hence can be regarded as a rational function on $\mathbb{P}^n$. This gives the desired rational map from $\mathbb{P}^n$ to $\mathbb{P}^N$.

(b) Since M is generic in $\mathbb{P}^n$, and since f is a local CR diffeomorphism, the Jacobian $\det J_f$ of the extension of f to $\mathbb{P}^n$ is not identically zero. Hence, the set $\Sigma = \{z \in \mathbb{P}^n : \det J_f(z) = 0\}$ if non-empty, is a subvariety of $\mathbb{P}^n \setminus \Lambda$ of complex codimension one, where Λ is the indeterminacy locus of f. Suppose $\Sigma \neq \varnothing$. Then, since f is locally biholomorphic near any point on M, $M \cap \Sigma = \varnothing$. On the other hand, by the Remmert-Stein theorem, the closure of Σ is a subvariety of $\mathbb{P}^n$ of codimension one. It is well known that its complement in $\mathbb{P}^n$ is therefore a Stein manifold. But a pseudoconcave M (or an M satisfying Property E) has no CR embedding into a Stein manifold (see [HN1] and [HN5]). Thus $\Sigma = \varnothing$.

Let $F : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ be a polynomial map such that $f \circ \pi = \pi \circ F$, where $\pi : \mathbb{C}^{n+1} \to \mathbb{P}^n$ is the canonical projection. Without loss of generality assume that the components of F are homogeneous polynomials of degree k without common factors. We claim that $\det J_F(z) \neq 0$ for any point $z \in \mathbb{C}^{n+1}$. Indeed, suppose on the contrary that

$$E = \{z \in \mathbb{C}^{n+1} : \det J_F(z) = 0\} \tag{54}$$

is a non-empty subvariety of complex codimension one. Then $F(E) \neq \{0\}$, and therefore there exists a point $p \in E$ such that $F(p) \neq 0$. For $z \in \mathbb{C}^{n+1}$, let $L_z := \{\lambda z : \lambda \in \mathbb{C}\}$ be the complex line passing through the point z and the origin. Since the Jacobian of f

does not vanish outside the indeterminacy locus, and $F(L_p) \neq \{0\}$, there exists a small neighborhood U of p such that for all z and w in U, $z \neq w$,

$$F(L_z) \cap F(L_w) = \{0\}. \tag{55}$$

Furthermore, $F|_{U \cap L_z} = \lambda^k z$, and after shrinking U if necessary, we may assume that $F|_{U \cap L_z}$ is an injective function for all $z \in U$. From this and (55) we conclude that F is injective in U, which contradicts the assumption that $p \in E$. Thus $\det J_F \neq 0$ and therefore is a constant.

Finally, observe that $\det J_F(z)$ is a homogeneous polynomial of degree $(k-1)(n+1)$, and being constant means that $k = 1$, i.e. F is a linear automorphism.

□

Proof of Theorem 2.5 and Corollary 2.6. We may regard M as being a generic CR submanifold of a complex manifold X. Note that the pseudoconcavity of M (or Property E) implies that M is minimal, because minimality is well-known to be equivalent to wedge extendability. There is a neighborhood V_p of p in X such that f extends to a holomorphic mapping $f : V_p \to \mathbb{P}^N$. It is easy to check that, by possibly shrinking V_p, the extended map f has maximal rank in V_p. Hence we may conclude from Theorem 2.2 that f extends to a finite holomorphic correspondence $F : V \to \mathbb{P}^N$, where $V \subset X$ is some neighborhood of M.

Since the Segre map associated with M' is injective, F splits at every point of M, and every map F^j of the splitting is a CR map from M to M'. One of them, say F^1 is the extension of f. By Theorem 2.4(a), F^1 extends to a rational map from $\mathbb{P}^n$ to $\mathbb{P}^N$. Moreover, if $n = N$ and M is generic in $\mathbb{P}^n$, then F^1 is locally biholomorphic, because the Segre map associated with M is injective. Thus by Theorem 2.4(b) F^1 extends to a linear automorphism of $\mathbb{P}^n$. Rationality of F^1 implies that M must also be algebraic, which proves Corollary 2.6. □

Proof of Theorem 2.3. Since M and M' are Levi non-degenerate, the associated Segre maps are locally injective, and M and M' satisfy the conditions of Theorem 2.2. If M and M' are strictly pseudoconvex then f extends as a locally biholomorphic map to a neighborhood of p by [PT]. If M is pseudoconcave, the result follows from [HN2]. Therefore the map f defined near p also satisfies the conditions of Theorem 2.2. Thus f extends as a finite correspondence along M. Since the set Σ', where the Segre map associated with M' branches, is empty, the extended correspondence is single-valued. Since the Segre map associated with M is injective, the extension $f : M \to M'$ is a locally biholomorphic map.

We now show that the extension is globally biholomorphic in a neighborhood of M. For that we note that M' is simply connected. Indeed, if $k = n-1$ or $k = 0$, then $M' = S^{2n-1}$ which is simply connected. If $0 < k < n-1$, then we choose an affine patch V'_n in $\mathbb{P}^n$ where $z'_n \neq 0$. Then

$$M' \cap V'_n = \{|w'_0|^2 + \cdots + |w'_k|^2 - |w'_{k+1}|^2 - \cdots - |w'_{n-1}|^2 = 1\}, \tag{56}$$

where $w'_j = z'_j / z'_n$. Let π be the projection from M' to the coordinates $(w'_{k+1}, \ldots, w'_{n-1})$. Then π is onto, and for any $w = (w'_{k+1}, \ldots, w'_{n-1})$, $\pi^{-1}(w) \cong S^{2k+1}$, which is simply connected. Therefore, M' is also simply connected. Because of that the germ of a map f^{-1} extends holomorphically along any path on M' to a holomorphic map $f^{-1} : M' \to M$. Thus $f|_M$ maps M one-to-one and onto M', and f is globally biholomorphic.

If M' is pseudoconcave, then by the invariance of the Levi form, so is M. Thus by Theorem 2.4(b) f extends to a linear automorphism of $\mathbb{P}^n$. □

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