Quantum integrability of the Beltrami-Laplace operator for geodesically equivalent metrics.

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Abstract

In this paper we show that if there exist a Riemannian metric $\bar{g}$ on M^n, geodesically equivalent to the metric g, then it is possible to construct self-adjoint operators $\mathcal{I}_0, \mathcal{I}_1, \dots, \mathcal{I}_{n-1} : L_2 \to L_2$ such that the operators $\mathcal{I}_k$ commute with the operator Δ and pairwise commute. If the metrics $g, \bar{g}$ are strictly non-proportional at least at one point then the operators $\mathcal{I}_0, \mathcal{I}_1, \dots, \mathcal{I}_{n-1}$ are linearly independent. If in addition the manifold is closed then the operators are self-adjoint and we have complete quantum integrability of the Beltrami-Laplace operator. We also formulate (without proofs) when the Liouville integrability of a geodesic flow implies the existence a geodesically equivalent metric.

Let g, $\bar{g}$ be two Riemannian metrics on the same manifold M^n. They are geodesically equivalent, if they have the same geodesics, considered as unparameterized curves.

Consider the Beltrami-Laplace operator $\Delta := g^{ij}\nabla_i\nabla_j$, where ∇ denotes the covariant derivative, corresponding to the metric g. The Beltrami-Laplace operator acts on the space L_2 of measurable functions $\psi : M^n \to R$ such that the Lebesgue integral

$$\int_{M^n} \psi^2 \sqrt{det(g)} dx^1 dx^2 ... dx^n$$

is finite.

Denote by G the linear operator $g^{-1}\bar{g} = (g^{i\alpha}\bar{g}_{\alpha j})$. Consider the characteristic polynomial $det(G - \mu E) = c_0\mu^n + c_1\mu^{n-1} + ... + c^n$. The coefficients $c_1, .., c_n$ are smooth functions on the manifold M^n, and $c_0 \equiv (-1)^n$. Consider the linear operators

$$S_0, S_1, ..., S_{n-1} : TM^n \to TM^n$$

given by the general formula

$$S_k \stackrel{\text{def}}{=} \left(\frac{det(g)}{det(\bar{g})}\right)^{\frac{k+2}{n+1}} \sum_{i=0}^{k} c_i G^{k-i+1}.$$

Consider the operators

$$\mathcal{I}_0,\ \mathcal{I}_1, ..., \mathcal{I}_{n-1} : L_2 \to L_2 \tag{1}$$

given by the general formula

$$\mathcal{I}_k := g^{ij}\nabla_i (S_k)^p_j \nabla_p. \tag{2}$$

Remark 1 . *The operator $\mathcal{I}_{n-1}$ is exactly the operator $-\Delta$.*

Theorem 1 . *If the metrics g and $\bar{g}$ on M^n are geodesically equivalent then the operators $\mathcal{I}_k$ pairwise commute. In particular they commute with the Beltrami-Laplace operator Δ.*

Theorem 1 is a "quantum" analog of the following theorem.

Consider the functions $I_k : TM^n \to R$, $k = 0, ..., n-1$, given by formulae

$$I_k(x,\xi) = g(S_k\xi, \xi), \tag{3}$$

where $g(\nu, \xi)$ denotes the dot product of the tangent vectors $\nu, \xi \in T_{x_0}M^n$ in the metrics g.

Theorem 2 . *If the metrics g and $\bar{g}$ on M^n are geodesically equivalent then the functions I_k are integrals of the geodesic flow of the metric g and pairwise commute (if we canonically identify by the metric g the tangent and the cotangent bundles of M^n and consider the standard symplectic form on the cotangent bundle) .*

Theorem 2 was proved in [19]. For closed two-dimensional manifolds Theorem 1 was essentially proved in [8].

Remark 2 ([19]) . *Let metrics g, $\bar{g}$ be geodesically equivalent. If at a point $x_0 \in M^n$ the number of different roots of the polynomial $det(G - \mu E)$ is equal to n_1 then at almost all points of $T_{x_0}M^n$ the dimension of the linear space generated by $dI_0, dI_1, ..., dI_{n-1}$ is greater or equal than n_1.*

In particular, if the characteristic polynomial $det(G - \mu E)$ has no multiple roots at a point $x_0 \in M^n$ then the functions $I_0, ..., I_{n-1}$ are functionally independent almost everywhere in $TU(x_0)$, where $U(x_0)$ is a sufficiently small neighborhood of x_0.

If at every point of a neighborhood $U(x_0)$ of a point $x_0 \in M^n$ the number of different eigenvalues of the polynomial $det(G - \mu E)$ is less or equal than n_1 then at all points of $TU(x_0)$ the dimension of the linear space generated by $dI_0, dI_1, ..., dI_{n-1}$ is less or equal than n_1.

The metrics $g, \bar{g}$ are *strictly non-proportional* at a point $x_0 \in M^n$ if the characteristic polynomial $det(G - \mu E)$ has no multiple roots at the point $x_0 \in M^n$.

Corollary 1 . *Suppose M^n is connected. Let metrics g, $\bar{g}$ on M^n be geodesically equivalent and strictly non-proportional at least at one point of M^n. Then the metrics are strictly non-proportional almost everywhere.*

More generally, suppose M^n is connected and the metrics $g,\ \bar{g}$ on M^n are geodesically equivalent. If at every point of a neighborhood $U \subset M^n$ the number of different eigenvalues of the polynomial $det(G-\mu E)$ is less or equal than n_1 then at every point of M^n the number of different eigenvalues of the polynomial $det(G-\mu E)$ is less or equal than n_1.

Corollary 2 . *Let metrics $g,\ \bar{g}$ on a connected manifold M^n be geodesically equivalent. The are strictly non proportional at least at one point of M^n if and only if the operators $\mathcal{I}_0, ..., \mathcal{I}_{n-1}$ are linearly independent.*

Corollaries 1,2 easily follows from the following observation. If we have an integrable Hamiltonian systems then the dimension of the linear space generated by differentials of the integrals is constant along each orbit.

Proof of Corollary 1. Take a geodesic $\gamma : R \to M^n$ and consider the points $x_0 = \gamma(0),\ x_1 = \gamma(1) \in M^n$. Suppose that the metric $\bar{g}$ is geodesically equivalent to the metric $\bar{g}$ and strictly non-proportional with g at the point x_0. Let us prove that in each neighborhood of the point x_1 there exist a point x suck that the metrics are strictly non-proportional at x.

The geodesic orbit of the geodesic γ is the curve $(\gamma, \dot{\gamma}) : R \to TM^n$ (assuming $\dot{\gamma} = \dot{\gamma}(t) \in T_{\gamma(t)}M^n$ is the velocity vector of γ.) Take a small neighborhood $O^{2n} \subset TM^n$ of the point $(x_0, \dot{\gamma}(0)) \in TM^n$. Consider the union of all points of all geodesic orbits with start from the points of the neighborhood. Denote it by U. Since the metrics are strictly non-proportional at x_0, the differentials $dI_0, dI_1, ..., dI_{n-1}$ of the integrals $I_1, I_2, ..., I_{n-1}$ are linear independent at almost all points of U. Evidently the set U contains a neighborhood of the point $(x_1, \dot{\gamma}(1))$. Then the integrals $I_0, I_1, ..., I_{n-1}$ are functionally independent at almost all points of a neighborhood of the point $(x_1, \dot{\gamma}(1))$. Using Remark 2 we have that in each neighborhood of x_1 there exist a point where the metrics are strictly non-proportional.

Now suppose M^n is closed (= compact + without boundary) and connected. Let metrics $g,\ \bar{g}$ on M^n be geodesically equivalent. Suppose the metrics are strictly non-proportional at least at one point of the manifold. Then we have complete Liouville integrability for the geodesic flow of the metric g and complete quantum integrability for the Beltrami-Laplace operator of the metric g. Liouville integrability means in particular that almost all connected components of level submanifolds $\{I_0 = C_0, I_1 = C_1, ..., I_{n-1} = C_{n-1}\}$, where $C_1, C_2, ..., C_{n-1}$ are constants, are n-dimensional tori, and orbits on such tori are windings.

Quantum integrability means that there exists a countable basis

$$\Phi = \{\phi_1, \phi_2, ..., \phi_m, ...\}$$

of the space L_2 such that each ϕ_m is an eigenfunction of each operator $\mathcal{I}_k$.

Moreover, in our case the variables can be separated. More precisely, take any function ϕ from the basis Φ. Since ϕ is an eigenfunction of each operator $\mathcal{I}_k$ we have that ϕ is a solution of the system of n partial differential equations

$$\mathcal{I}_k \phi = \lambda_k \phi, \quad k = 0, 1, ..., n-1. \tag{4}$$

The separation of variables means that in a neighborhood of almost any point there exist coordinates $(x^1, x^2, ..., x^n)$ such that in this coordinates the system (4) is equivalent to the system

$$\frac{\partial^2}{(\partial x^k)^2}\phi = F_k(x^k, \lambda_0, \lambda_1, ..., \lambda_{n-1})\phi, \quad k = 0, 1, ..., n-1, \tag{5}$$

where the function F_k depends on the variable x^k and on the parameters $\lambda_0, \lambda_1, ..., \lambda_{n-1}$. Then ϕ is the product

$$X_1(x^1)X_2(x^2)...X_n(x^n),$$

and each X_k is a solution of (5) so that we reduced the system of partial differential equations (4) to the system of ordinary differential equations

$$\frac{\partial^2}{(\partial x^k)^2}X_k(x^k) = F_k(x^k, \lambda_0, \lambda_1, ..., \lambda_{n-1})X_k(x^k), \quad k = 0, 1, ..., n-1. \tag{6}$$

Does the Liouville integrability of the geodesic flow implies the existence of a geodesically equivalent metric? Let $g, \bar{g}$ be two metrics on M^n. Consider the functions $I_k : TM^n \to R$, $k = 0, ..., n-1$, given by (3). With the help of g identify the tangent and the cotangent bundles of M^n and consider the standard symplectic structure on T^*M^n.

Theorem 3 . *Let the functions I_k commute and let they be functionally independent almost everywhere. Then the metrics g, $\bar{g}$ are geodesically equivalent.*

Consider the operators $\mathcal{I}_k : L_2 \to L_2$, $k = 0, ..., n-1$, given by 1.

Corollary 3 . *Suppose the manifold M^n is connected. Let the operators $\mathcal{I}_k$ commute and let they be linear independent. Then the metrics $g, \bar{g}$ are geodesically equivalent.*

Let the function I_k commute. If they are not functionally independent then the metrics can be not geodesically equivalent.

Consider the tangent bundle TM^n. Consider the operator $G = g^{-1}\bar{g}$. The operator G is evidently self-adjoint (with respect to both metrics). Then all eigenvalues of the operator G are real and we have a decomposition of any tangent space T_xM^n to the direct sum $V_x^1 \oplus V_x^2 \oplus ... \oplus V_x^{n_1}$ of eigenspaces.

Recall that the distribution

$$V_x^1, V_x^2, ..., V_x^{n_1} \subset T_xM^n, \quad V_x^1 \oplus V_x^2 \oplus ... \oplus V_x^{n_1} = T_xM^n$$

is integrable if at a neighborhood of almost every points of M^n there exist a coordinate system $x^1, x^2, ..., x^n$ such that the vectors tangent to the coordinate lines lie in the elements of the distribution.

Theorem 4 . *Let the functions I_k commute and let the distribution of the eigenspaces of the operator G is integrable. Then the metrics $g, \bar{g}$ are geodesically equivalent.*

Are there many interesting examples of geodesically equivalent metrics on closed manifolds? The following theorem (essentially due to N. S. Sinjukov [15]) gives us a construction that, given a pair of geodesically equivalent metrics, produces another pair of geodesically equivalent metrics. Starting from the metric of constant curvature on the sphere, we obtain the metric of the ellipsoid and the metric of the Poisson sphere.

Let $g, \bar g$ be Riemannian metrics on the manifold M^n. Consider the linear operator $B : TM^n \to TM^n$ given by $B^i_j = \left(\frac{det(\bar g)}{det(g)}\right)^{\frac{1}{n+1}} \bar g^{ip} g_{pj}$. By definition put metric gB equal $g_{ip}B^p_j$. By definition put metric $\bar g B$ equal $\bar g_{ip}B^p_j$. In other words, the dot product $gB(\xi, \nu)$ of arbitrary vectors $\xi, \nu \in T_{x_0}M^n$ is equal to $g(B\xi, \nu)$ and the dot product $\bar g B(\xi, \nu)$ is equal to $\bar g(B\xi, \nu)$. Evidently the linear operator B is self-adjoint with respect to the metrics g and $\bar g$ and the metrics gB, $\bar g B$ are well-definite.

Theorem 5 . *The metrics g and $\bar g$ are geodesically equivalent if and only if the metrics gB and $\bar g B$ are geodesically equivalent.*

Evidently the metrics g and $\bar g$ are strictly non-proportional at a point $x \in M^n$ if and only if the metrics gB and $\bar g B$ are strictly non-proportional at the point x.

Thus if we have a pair of geodesically equivalent metrics g, $\bar g$ then we can construct the other pair of geodesically equivalent metrics gB, $\bar g B$. We can apply the construction once more, the result is another pair of geodesically equivalent metrics. It is natural to denote it by gB^2, $\bar g B^2$ since this pair is given by

$$gB^2(\xi, \nu) = g(B^2\xi, \nu), \ \ \bar g B^2(\xi, \nu) = \bar g(B^2\xi, \nu).$$

We can go in other direction and consider the metrics gB^{-1}, $\bar g B^{-1}$ given by

$$gB^{-1}(\xi, \nu) = g(B^{-1}\xi, \nu), \ \ \bar g B^{-1}(\xi, \nu) = \bar g(B^{-1}\xi, \nu).$$

They are geodesically equivalent also.

To start the process, we need an initial pair of geodesically equivalent metrics g, $\bar g$. We take the following one (obtained by E. Beltrami [1], [2]). The metric $\bar g$ is the restriction of the Euclidean metric

$$(dx^1)^2 + (dx^2)^2 + ... + (dx^{n+1})^2$$

to the standard sphere

$$S^n = \{(x^1, x^2, ..., x^{n+1}) \in R^{n+1} : (x^1)^2 + (x^2)^2 + ... + (x^{n+1})^2 = 1\}.$$

The metric g is the pull-back $l^*\bar g$, where the diffeomorphism $l : S^n \to S^n$ is given by

$$l(x) \stackrel{\text{def}}{=} \frac{Ax}{\|Ax\|},$$

where A is an arbitrary non-degenerate linear transformation of R^{n+1}, Ax means "the transformation A applied to the vector $x = (x^1, x^2, ..., x^{n+1})$" and $\|x\|$ is the standard norm $\sqrt{(x^1)^2 + (x^2)^2 + ... + (x^{n+1})^2}$.

Since the mapping l evidently takes the geodesics of the metric g to the geodesics of the metric g, the metrics g, $\bar g$ are geodesically equivalent. For this choice of metrics g and $\bar g$ the metric gB^{-1} is the metric of an ellipsoid and the metric $\bar g B^2$ is the metric of a Poisson sphere. Varying the linear operator L, we can obtain metrics of all possible ellipsoids and all possible Poisson spheres.

The metric of the ellipsoid

$$E \stackrel{\text{def}}{=} \left\{ (x^1, ..., x^{n+1}) \in R^{n+1} : \frac{(x^1)^2}{a_1} + \frac{(x^2)^2}{a_2} + ... + \frac{(x^{n+1})^2}{a_{n+1}} = 1 \right\}$$

is the restriction of the metric

$$(dx^1)^2 + (dx^2)^2 + ... + (dx^{n+1})^2$$

to the ellipsoid E. By the metric of the Poisson sphere we, follow [5], mean the restriction of the metric

$$\frac{1}{\frac{(x^1)^2}{a_1^2} + \frac{(x^2)^2}{a_2^2} + ... + \frac{(x^{n+1})^2}{a_{n+1}^2}} \left((dx^1)^2 + ... + (dx^{n+1})^2 \right)$$

to the ellipsoid E.

The metric of Poisson sphere has the following mechanical sense. Consider the free motion of (n+1)-dimension rigid body in (n+1)-dimensional space around a fixed point. The configuration space of the correspondent Hamiltonian system is $SO(n+1)$ and the corresponding Hamiltonian is left-invariant. Consider the embedding $SO(n) \longmapsto SO(n+1)$ as the stabilizer of a vector $v \in R^{n+1}$. Consider the action of the group $SO(n)$ on $SO(n+1)$ by left translations. The Hamiltonian of the motion is evidently invariant modulo this action, and the reduced system on $TSO(n+1)/SO(n) \cong TS^n$ is the geodesic flow of the (appropriate) Poisson metric.

Theorem 6 ([19], independently obtained by S. Tabachnikov [16]). *The restriction of the Euclidean metric*

$$(dx^1)^2 + ... + (dx^{n+1})^2$$

to the standard ellipsoid

$$E^n = \{(x^1, x^2, ..., x^{n+1}) \in R^{n+1} : \frac{(x^1)^2}{a_1} + ... + \frac{(x^{n+1})^2}{a_{n+1}} = 1\}$$

is geodesically equivalent to the restriction of the metric

$$\frac{1}{\left(\frac{x^1}{a_1}\right)^2 + ... + \left(\frac{x^{n+1}}{a_{n+1}}\right)^2} \left(\frac{(dx^1)^2}{a_1} + ... + \frac{(dx^{n+1})^2}{a_{n+1}} \right) \tag{7}$$

to the same ellipsoid.

Theorem 7 (Topalov, [20]). *The restriction of the metric*

$$\frac{1}{\frac{(x^1)^2}{a_1^2}+...+\frac{(x^{n+1})^2}{a_{n+1}^2}}\left((dx^1)^2+...+(dx^{n+1})^2\right) \tag{8}$$

to the ellipsoid

$$E^n=\{(x^1,x^2,...,x^{n+1})\in R^{n+1}:\frac{(x^1)^2}{a_1}+...+\frac{(x^{n+1})^2}{a_{n+1}}=1\}$$

is geodesically equivalent to the restriction of the metric

$$a_1(dx^1)^2+...+a_{n+1}(dx^{n+1})^2-(x^1dx^1+x^2dx^2+...+x^{n+1}dx^{n+1})^2 \tag{9}$$

to the same ellipsoid.

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Proof of Theorem 1.

Let G be the diagonal matrix $diag(\rho_1,\rho_2,...,\rho_n)$, where

$$\rho_i=\frac{1}{\phi_i(\phi_1\phi_2\phi_3...\phi_n)}.$$

Consider the characteristic polynomial $det(G-\mu E)=c_0\mu^n+c_1\mu^{n-1}+...+c^n$ of the matrix G. . Consider the matrixes $S_0,S_1,...,S_{n-1}$ given by the general formula

$$S_k\stackrel{\text{def}}{=}\left(\frac{det(g)}{det(\bar g)}\right)^{\frac{k+2}{n+1}}\sum_{i=0}^{k}c_iG^{k-i+1}.$$

Denote by σ_k the elementary symmetric polynomial of degree k of n variables variables $\phi_1,\phi_2,...,\phi_n$. Denote by $\sigma_k(\hat{\phi_i})$ the elementary symmetric polynomial of degree k of $n-1$ variables $\phi_1,\phi_2,...,\phi_{i-1},\phi_{i+1},\phi_n$.

Lemma 1 .

$$S_k=(-1)^{n-k}diag(\sigma_{n-k-1}(\hat\phi_1),\sigma_{n-k-1}(\hat\phi_2),...,\sigma_{n-k-1}(\hat\phi_n)).$$

Proof. Easy to see that the coefficients of the characteristic polynomial are given by

$$c_k=(-1)^{n-k}\frac{\sigma_{n-k}}{(\phi_1\phi_2...\phi_n)^{n+1}}.$$

In particular,

$$det(G)=\frac{1}{(\phi_1\phi_2...\phi_n)^{n+1}}.$$

Then for $k=0$ we have

$$\begin{aligned}
S_0 &= \left(\frac{1}{det(G)}\right)^{\frac{2}{n+1}} c_0 G \\
&= (-1)^n(\phi_1,\phi_2,...,\phi_n)^2 diag(\frac{1}{\phi_1(\phi_1\phi_2...\phi_n)},\frac{1}{\phi_2(\phi_1\phi_2...\phi_n)},...,\frac{1}{\phi_n(\phi_1\phi_2...\phi_n)}) \\
&= (-1)^n diag(\phi_2\phi_3...\phi_n,\phi_1\phi_3...\phi_n,...,\phi_1\phi_2...\phi_{n-3}\phi_{n-2},\phi_1\phi_2...\phi_{n-2}\phi_{n-1})).
\end{aligned}$$

Suppose that the lemma is true for $k=l-1$. Then for S_l we have

$$\begin{aligned}
S_l &= \left(\frac{1}{det(G)}\right)^{\frac{1}{n+1}} G(S_{l-1}+\left(\frac{1}{det(G)}\right)^{\frac{k+1}{n+1}} c_l E) \\
&= diag(\frac{1}{\phi_1},\frac{1}{\phi_2},...,\frac{1}{\phi_n})diag((-1)^{n-l}\sigma_{n-l}(\hat{\phi}_1)+(-1)^{n-l+1}\sigma_{n-l}, \\
&\quad (-1)^{n-l}\sigma_{n-l}(\hat{\phi}_2)+(-1)^{n-l+1}\sigma_{n-l},...,(-1)^{n-l}\sigma_{n-l}(\hat{\phi}_n)+(-1)^{n-l+1}\sigma_{n-l}) \\
&= (-1)^{n-l-1}diag(\sigma_{n-l-1}(\hat{\phi}_1),\sigma_{n-l-1}(\hat{\phi}_2),...,\sigma_{n-l-1}(\hat{\phi}_n)).
\end{aligned}$$

Q. E. D.

For simplicity we assume that the manifold is connected and that the metrics are strictly non-proportional at least at one point of the manifold. Then they are strictly non-proportional at almost all points of the manifold and it is sufficient to check the statement of the theorem in such points.

If geodesically equivalent metrics are strictly non-proportionally at a point x_0 then in a neighborhood of x_0 there exist coordinates $x^1,x^2,...,x^n$ such that the metrics are given by the formulae (see [7]

$$\Pi_1(dx^1)^2+\ \Pi_2(dx^2)^2+\cdots+\Pi_n(dx^n)^2 \tag{10}$$
$$\rho^1\Pi_1(dx^1)^2+\rho^2\Pi_2(dx^2)^2+\cdots+\rho^n\Pi_n(dx^n)^2, \tag{11}$$

respectively, where

$$\begin{aligned}
\Pi_i &\stackrel{\text{def}}{=} (\phi_i-\phi_1)\cdots(\phi_i-\phi_{i-1})(\phi_{i+1}-\phi_i)\cdots(\phi_m-\phi_i), nonumber \quad (12) \\
\rho^i &= \frac{1}{\phi_1...\phi_m}\frac{1}{\phi_i}
\end{aligned}$$

and $\phi_1,\phi_2,...,\phi_n$, $0<\phi_1<\phi_2<...<\phi_n$, are smooth functions such that the function ϕ_i depends on x^i only.

First we show that in this coordinates the operators $\mathcal{I}_k$ have the form

$$
\begin{aligned}
(-1)^n \mathcal{I}_{n-1} &= \Pi_1 \frac{\partial^2}{(\partial x^1)^2} + \cdots + \Pi_n \frac{\partial^2}{(\partial x^n)^2} \\
(-1)^{n-2} \mathcal{I}_{n-2} &= \sigma_1(\phi_2, ..., \phi_m) \Pi_1 \frac{\partial^2}{(\partial x^1)^2} + \cdots + \sigma_1(\phi_1, ..., \phi_{m-1}) \Pi_n \frac{\partial^2}{(\partial x^n)^2}, \\
(-1)^{n-3} \mathcal{I}_{n-3} &= \sigma_2(\phi_2, ..., \phi_m) \Pi_1 \frac{\partial^2}{(\partial x^1)^2} + \cdots + \sigma_2(\phi_1, ..., \phi_{m-1}) \Pi_n \frac{\partial^2}{(\partial x^n)^2}, \\
&\vdots \\
-\mathcal{I}_0 &= (\phi_2 ... \phi_m) \Pi_1 \frac{\partial^2}{(\partial x^1)^2} + \cdots + (\phi_1 ... \phi_{m-1}) \Pi_n \frac{\partial^2}{(\partial x^n)^2},
\end{aligned}
$$

where σ_k denotes the elementary symmetric polynomial of degree k. Then we prove that the operators of this form commute.

Recall that for arbitrary function $F : M^n \to R$ and for arbitrary covector $T = T_i$ covariant derivatives $\nabla_j F$ and $\nabla_j T_i$ are given by

$$
\begin{aligned}
\nabla_j F &= \frac{\partial F}{\partial x^j} && (13) \\
\nabla_j T_i &= \frac{\partial T_i}{\partial x^j} - \sum_{h=1}^{n} \Gamma^h_{ij} T_h, && (14)
\end{aligned}
$$

where the Christoffel symbols Γ^i_{jk} are given by the formula

$$
\Gamma^i_{jk} = \frac{1}{2} \sum_{h=1}^{n} \left[g^{ih} \left(\frac{\partial}{\partial x^j} g_{hk} + \frac{\partial}{\partial x^k} g_{jh} - \frac{\partial}{\partial x^h} g_{jk} \right) \right].
$$

In the Levi-Civita coordinates the matrix of the metric g is the diagonal matrix $diag(\Pi_1, \Pi_2, ..., \Pi_n)$, so we do not need Γ^i_{jk} for $j \neq k$ to calculate the operator $\mathcal{I}_k := g^{ij} \nabla_i (S_k)^p_j \nabla_p$.

If $i \neq j$ we have

$$
\begin{aligned}
\Gamma^i_{jj} &= \frac{1}{2} \left[g^{ii} \left(\frac{\partial}{\partial x^j} g_{ij} + \frac{\partial}{\partial x^j} g_{ji} - \frac{\partial}{\partial x^i} g_{jj} \right) \right] \\
&= -\frac{1}{2} g^{ii} \frac{\partial}{\partial x^i} g_{jj} \\
&= \frac{1}{2} \frac{\phi_i{}'}{\Pi_i} \left[\frac{1}{\phi_i - \phi_j} \right]. && (15)
\end{aligned}
$$

For $i = j$ we have

$$
\begin{aligned}
\Gamma^i_{ii} &= \frac{1}{2} \left[g^{ii} \left(\frac{\partial}{\partial x^i} g_{ii} + \frac{\partial}{\partial x^i} g_{ii} - \frac{\partial}{\partial x^i} g_{ii} \right) \right] \\
&= \frac{1}{2} g^{ii} \frac{\partial}{\partial x^i} g_{ii} \\
&= \frac{1}{2} \frac{\phi_i{}'}{\Pi_i} \left[\frac{1}{\phi_i - \phi_1} + ... + \frac{1}{\phi_i - \phi_{i-1}} + \frac{1}{\phi_i - \phi_{i+1}} + ... + \frac{1}{\phi_i - \phi_n} \right]. && (16)
\end{aligned}
$$

In particular, for each i we have

$$\sum_{j=1}^{n} \Gamma^i_{jj} = 0. \tag{17}$$

Let us calculate $\mathcal{I}_k := \sum_{i=1}^n \sum_{j=1}^n g^{ij} \nabla_i (S_k)^p_j \nabla_p$. In view Lemma 1 the operator S_k is diagonal with the i-th element on the diagonal given by

$$(S_k)^i_i = \sigma_{n-k-1}(\phi_1, \phi_2, ..., \phi_{i-1}, \phi_{i+1}, ..., \phi_n).$$

Then

$$T_j = \sum_{p=1}^{n} \left[(S_k)^p_j \nabla_p F = \sigma_{n-k-1}(\phi_1, \phi_2, ..., \phi_{j-1}, \phi_{j+1}, ..., \phi_n) \frac{\partial F}{\partial x^j} \right]$$

and

$$\begin{aligned}
\mathcal{I}_k(F) &= \sum_{i,j=1}^{n} \left[g^{ij} \nabla_i \sum_{p=1}^{n} [(S_k)^p_j \nabla_p F] \right] \\
&= \sum_{i,j=1}^{n} [g^{ij} \nabla_i T_j] \\
&= g^{ii} \sigma_{n-k-1}(\phi_1, \phi_2, ..., \phi_{i-1}, \phi_{i+1}, ..., \phi_n) \frac{\partial^2 F}{(\partial x^i)^2} \\
&- \sum_{i=1}^{n} \sum_{h=1}^{n} \left[\Gamma^h_{ii} \frac{\partial F}{(\partial x^h)} \right] \\
&= \sum_{i=1}^{n} \left[\frac{\sigma_{n-k-1}(\phi_1, \phi_2, ..., \phi_{i-1}, \phi_{i+1}, ..., \phi_n)}{\Pi_i} \frac{\partial^2 F}{(\partial x^i)^2} \right] \\
&- \sum_{i=1}^{n} \sum_{h=1}^{n} \left[\Gamma^h_{ii} \frac{\partial F}{(\partial x^h)} \right].
\end{aligned}$$

Since in view of (17) $\sum_{i=1}^n \sum_{h=1}^n \left[\Gamma^h_{ii} \frac{\partial F}{\partial x^h} \right]$ is zero we have

$$\mathcal{I}_k = \sum_{i=1}^{n} \left[\frac{\sigma_{n-k-1}(\phi_1, \phi_2, ..., \phi_{i-1}, \phi_{i+1}, ..., \phi_n)}{\Pi_i} \frac{\partial^2}{(\partial x^i)^2} \right].$$

Q.E.D.

Now let us prove that the operators $\mathcal{I}_k$ commute. Consider the operator

$$[\mathcal{I}_{l+1-n}, \mathcal{I}_{m+1-n}] = \mathcal{I}_{l+1-n}\mathcal{I}_{m+1-n} - \mathcal{I}_{m+1-n}\mathcal{I}_{l+1-n}.$$

Since each $\mathcal{I}_k$ is a differential operator of order 2, the operator $[\mathcal{I}_{l+1-n}, \mathcal{I}_{m+1-n}]$ is a differential operator of order no greater than 4.

It is easy to see that the components of order 4, 3, 1, 0 of the operator $[\mathcal{I}_{l+1-n}, \mathcal{I}_{m+1-n}]$ are zero. More precisely, the component of order 4 vanishes since for any i, j the operators $\frac{\partial^2}{(\partial x^i)^2}$ and $\frac{\partial^2}{(\partial x^j)^2}$ commute. The component of order 3 vanishes since the functions I_l and I_m are in involution. The components of order 1 and 0 vanish since the operators $\mathcal{I}_l$ and $\mathcal{I}_m$ have no component of order 1 and 0.

Thus we have to verify that the component of order 2 vanishes also. Evidently the coefficient by $\frac{\partial^2}{\partial x^j \partial x^i}$ is zero for $i \neq j$. The coefficient by $\frac{\partial^2}{(\partial x^j)^2}$ is given by

$$\sum_{i=1}^{n} \left[\frac{\sigma_l(\phi_1, \phi_2, ..., \phi_{i-1}, \phi_{i+1}, ..., \phi_n)}{\Pi_i} \frac{\partial^2}{(\partial x^i)^2} \frac{\sigma_m(\phi_1, \phi_2, ..., \phi_{j-1}, \phi_{j+1}, ..., \phi_n)}{\Pi_j} \right.$$

$$\left. - \frac{\sigma_m(\phi_1, \phi_2, ..., \phi_{i-1}, \phi_{i+1}, ..., \phi_n)}{\Pi_i} \frac{\partial^2}{(\partial x^i)^2} \frac{\sigma_l(\phi_1, \phi_2, ..., \phi_{j-1}, \phi_{j+1}, ..., \phi_n)}{\Pi_j} \right].$$

So it is sufficient to prove that (for every i, j)

$$\frac{\sigma_l(\phi_1, \phi_2, ..., \phi_{i-1}, \phi_{i+1}, ..., \phi_n)}{\Pi_i} \frac{\partial^2}{(\partial x^i)^2} \frac{\sigma_m(\phi_1, \phi_2, ..., \phi_{j-1}, \phi_{j+1}, ..., \phi_n)}{\Pi_j}$$

$$- \frac{\sigma_m(\phi_1, \phi_2, ..., \phi_{i-1}, \phi_{i+1}, ..., \phi_n)}{\Pi_i} \frac{\partial^2}{(\partial x^i)^2} \frac{\sigma_l(\phi_1, \phi_2, ..., \phi_{j-1}, \phi_{j+1}, ..., \phi_n)}{\Pi_j}.$$

is zero. Since $\sigma_l(\phi_1, \phi_2, ..., \phi_{i-1}, \phi_{i+1}, ..., \phi_n)$ does not depend on x^i, it is sufficient to prove that

$$\frac{1}{\Pi_j}[\sigma_l(\phi_1, \phi_2 ..., \phi_{i-1}, \phi_{i+1}, ..., \phi_n)\sigma_m(\phi_1, \phi_2 ..., \phi_{j-1}, \phi_{j+1}, ..., \phi_n)$$

$$-\sigma_m(\phi_1, \phi_2 ..., \phi_{i-1}, \phi_{i+1}, ..., \phi_n)\sigma_l(\phi_1, \phi_2 ..., \phi_{j-1}, \phi_{j+1}, ..., \phi_n)] \qquad (18)$$

does not depend on x^i.

The last statement is trivial for $i = j$. Assume $i < j$. Denote by $\bar{\sigma}_k$ the elementary symmetric polynomial

$$\sigma_k(\phi_1, \phi_2, ..., \phi_{i-1}, \phi_{i+1}, ..., \phi_{j-1}\phi_{j+1}, ..., \phi_n).$$

Substitute the trivial identities

$$\begin{aligned}
\sigma_l(\phi_1, \phi_2, ..., \phi_{i-1}, \phi_{i+1}, ..., \phi_n) &= \bar{\sigma}_{l-1}\phi_i + \bar{\sigma}_l \\
\sigma_m(\phi_1, \phi_2, ..., \phi_{i-1}, \phi_{i+1}, ..., \phi_n) &= \bar{\sigma}_{m-1}\phi_i + \bar{\sigma}_m \\
\sigma_l(\phi_1, \phi_2, ..., \phi_{j-1}, \phi_{j+1}, ..., \phi_n) &= \bar{\sigma}_{l-1}\phi_i + \bar{\sigma}_l \\
\sigma_m(\phi_1, \phi_2, ..., \phi_{j-1}, \phi_{j+1}, ..., \phi_n) &= \bar{\sigma}_{m-1}\phi_i + \bar{\sigma}_m
\end{aligned}$$

in (18). After simple calculation we obtain

$$\frac{(\bar{\sigma}_l \bar{\sigma}_m - \bar{\sigma}_m \bar{\sigma}_l)(\phi_i - \phi_j)}{\Pi_j}$$

which does not depend on x^i. The statement is proved for $i < j$, the prove for $i > j$ is essentially the same. Q.E.D.

If the manifold is closed then the operators $\mathcal{I}_k$ are self-adjoint.

In this section we assume that the manifold M^n is closed. Let g and $\bar{g}$ be geodesically equivalent metrics on M^n. Consider the operators $\mathcal{I}_k$ given by (2). Recall that L_2 is a Hilbert space with the following dot product:

$$< \phi, \psi >= \int_{M^n} \phi\psi dV,$$

where dV is the standard volume form $dV = \sqrt{det(g)}dx^1dx^2...dx^n$ on M^n.

Lemma 2 . *The operators $\mathcal{I}_k$ are self-adjoint.*

In other words for any $\phi, \psi \in L_2$

$$< \phi, \mathcal{I}_k(\psi) >=< \mathcal{I}_k(\phi), \psi >$$

for any $\phi, \psi \in L_2$.

Proof. Recall that for each covector field T_i on the manifold and for each function ϕ we have

$$\int_{M^n} g^{ij}\nabla_i T_j dV = 0 \quad \text{(Divergence theorem).} \tag{19}$$

$$\phi\nabla_i T_j = \nabla_i(\phi T_j) - T_j\nabla_i\phi \quad \text{(Leibnitz rule).} \tag{20}$$

Then

$$\begin{aligned} < \phi, \mathcal{I}_k(\psi) > &= \int_{M^n} \phi\mathcal{I}_k(\psi)dV \\ &= \int_{M^n} \phi g^{ij}\nabla_i(S_k)^p_j\nabla_p\psi dV \\ &= \int_{M^n} g^{ij}\nabla_i((S_k)^p_j\phi\nabla_p\psi)dV - \int_{M^n} g^{ij}(\nabla_i\phi)((S_k)^p_j\nabla_p\psi)dV \\ &= -\int_{M^n} g^{ip}(S_k)^j_p(\nabla_i\phi)(\nabla_j\psi). \end{aligned} \tag{21}$$

Since the operator S_k is self-adjoint with respect to the metrics g, the two-form $g^{ip}(S_k)^j_p$ is symmetric and therefore the operator $\mathcal{I}_k$ is self-adjoint. Q. E. D.

Proof of Theorems 6,7.

Consider the space R^{n+1} with the standard coordinates $x = (x^1, x^2, ..., x^{n+1})$ and the sphere

$$S^n \subset R^{n+1}, \quad S^n = \{x \in R^{n+1} : (x^1)^2 + (x^2)^2 + ... + (x^{n+1})^2 = 1\}.$$

Let A be the diagonal matrix $diag(\frac{1}{\sqrt{a_1}}, \frac{1}{\sqrt{a_2}}, ..., \frac{1}{\sqrt{a_{n+1}}})$, where $0 < a_1 \leq a_2 \leq ... \leq a_{n+1}$.

In order to prove Theorems 6,7 we consider the linear transformation $x \longmapsto Ax$ of the space R^{n+1} and construct the corresponding projective transformation l of the sphere S^n and the metric $\bar{g} = l^* g$ on the sphere (where g is the restriction of the Euclidean metric to the sphere). Then we construct the operator B and show that the metrics gB, $\bar{g}B$ are essentially the metrics from Theorem 6 and the metrics gB^2, $\bar{g}B^2$ are essentially the metrics from Theorem 7 so that Theorems 6,7 follow from Theorem 5.

Take an arbitrary point $x = (x^1, x^2, ..., x^{n+1}) \in S^n$. Denote by $T_x R^{n+1}$ and $T_x S^n$ the tangent spaces at the point x to R^{n+1} and S^n, respectively. Identify the tangent space $T_x R^{n+1}$ with R^{n+1} by moving the origin of the coordinate system to the point x. Consider the matrix

$$\bar{A} = \frac{1}{\sqrt{\frac{(x^1)^2}{a_1} + \frac{(x^2)^2}{a_2} + ... + \frac{(x^{n+1})^2}{a_{n+1}}}} A$$

and the matrix P with the general element $P_{ij} = x^i x^j$. The matrixes $\bar{A}$, P depend on the choice of the point x. Easy to see that for each vector $v \in T_x R^{n+1}$ the vector Pv is the orthogonal projection of the vector v to the vector normal to the sphere. Then

$$PP = P, \tag{22}$$

since the square of the projection is the projection and

$$P\bar{A}^2 P = P, \tag{23}$$

since in view of (22) the matrix $P\bar{A}^2P$ commute with P, takes $T_x S^n$ to zero and therefore is proportional to the projection P. Easy to se that the coefficient of proportionality is 1, since for the normal vector $\eta = (x^1, x^2, ..., x^{n+1})$ to the sphere we have $P\bar{A}^2 P\eta = \eta$.

Consider the mapping

$$L : R^{n+1} \to R^{n+1}, \ L(x) = \frac{\sqrt{(x^1)^2 + (x^2)^2 + ... + (x^{n+1})^2}}{\sqrt{\frac{(x^1)^2}{a_1} + \frac{(x^2)^2}{a_2} + ... + \frac{(x^{n+1})^2}{a_{n+1}}}} Ax.$$

The mapping L is a diffeomorphism on $R^{n+1} \setminus 0$, takes the sphere S^n to itself and coincide with the mapping $l : x \longmapsto \frac{Ax}{|Ax|}$ on the sphere.

The Jacobian J of the mapping L at the point $x \in S^n$ is equal to

$$\bar{A} - \bar{A}P\bar{A}^2 + \bar{A}P.$$

Then the pull-back of the Euclidean metric in R^{n+1} is given by

$$\begin{array}{rcl} J^{\mathrm{T}}J &=& (\bar{A}-\bar{A}^2P\bar{A}+P\bar{A})(\bar{A}-\bar{A}P\bar{A}^2+\bar{A}P) \\ &=& \bar{A}^2-\bar{A}^2P\bar{A}^2+\bar{A}^2P \\ &-& \bar{A}^2P\bar{A}^2+\bar{A}^2P\bar{A}^2P\bar{A}^2-\bar{A}^2P\bar{A}^2P \\ &+& P\bar{A}^2-P\bar{A}^2P\bar{A}^2+P\bar{A}^2P \\ &=& \bar{A}^2-\bar{A}^2P\bar{A}^2+P. \end{array}$$

Let us show that the matrix

$$G^{-1}=\bar{A}^{-2}-\bar{A}^{-2}P-P\bar{A}^{-2}+P\bar{A}^{-2}P+P$$

is inverse to the matrix $G=\bar{A}^2-\bar{A}^2P\bar{A}^2+P$.

Denote by $\mathbf{1}$ the identity $(n+1)\times(n+1)$ matrix $diag(1,1,...,1)$. We have

$$\begin{array}{rcl} G^{-1}G &=& (\bar{A}^{-2}-\bar{A}^{-2}P-P\bar{A}^{-2}+P\bar{A}^{-2}P+P)(\bar{A}^2-\bar{A}^2P\bar{A}^2+P) \\ &=& \mathbf{1}-\bar{A}^{-2}P\bar{A}^2-P+P\bar{A}^{-2}P\bar{A}^2+P\bar{A}^2 \\ &-& P\bar{A}^2+\bar{A}^{-2}P\bar{A}^2P\bar{A}^2+PP\bar{A}^2-P\bar{A}^{-2}P\bar{A}^2P\bar{A}^2 \\ &+& \bar{A}^{-2}P-\bar{A}{-2}PP-P\bar{A}^{-2}P+P\bar{A}^{-2}PP+PP \\ &=& \mathbf{1}. \end{array}$$

Since the Euclidean metric g_{euclid} is given by the identity matrix $\mathbf{1}$, then the matrix of the operator $(\bar{l}^*g_{euclid})^{-1}g_{euclid}$ is the matrix G^{-1}. Let us show that G^{-1} takes tangent vectors to sphere to tangent vectors to sphere and therefore the restriction of the operator $(\bar{l}^*g_{euclid})^{-1}g_{euclid}$ to T_xS^n coincide with the operator $\bar{g}^{-1}g$. Evidently T_xS^n is the eigenspace of the matrix P (with the eigenvalue 0). Then it is sufficient to show that the matrix P commute with G^{-1}. We have

$$\begin{array}{rcl} G^{-1}P &=& (\bar{A}^{-2}-\bar{A}^{-2}P-P\bar{A}^{-2}+P\bar{A}^{-2}P+P)P \\ &=& \bar{A}^{-2}P-\bar{A}^{-2}PP-P\bar{A}^{-2}P+P\bar{A}^{-2}PP+PP \\ &=& \bar{A}^{-2}P-\bar{A}^{-2}P-P\bar{A}^{-2}P+P\bar{A}^{-2}P+P \\ &=& \bar{P}A^{-2}-P\bar{A}^{-2}-P\bar{A}^{-2}P+P\bar{A}^{-2}P+P \\ &=& P(\bar{A}^{-2}-\bar{A}^{-2}P-P\bar{A}^{-2}+P\bar{A}^{-2}P+P). \end{array}$$

Thus the operator B at the tangent space T_xS^n is proportional to the restriction of the operator given by the matrix G^{-1} to the tangent space T_xS^n. Let us find the coefficient of proportionality. By definition it is equal to $\left(\frac{det(\bar{g})}{det(g)}\right)^{\frac{1}{n+1}}$. It is easy to see that

$$\frac{det(\bar{g})}{det(g)}=det(G).$$

More precisely, since the matrix P commute with the matrix G^{-1}, the tangent space T_xS^n is invariant under multiplication by the matrix G^{-1}. Since on T_xS^n

the operator corresponding to the matrix G^{-1} coincide with the operator $g^{-1}\bar{g}$ then all eigenvalues and eigenvectors of the operator $g^{-1}\bar{g}$ are eigenvalues and eigenvectors of the matrix G and n eigenvectors of the matrix G lie in T_xS^n. The only eigenvector which does not lie in T_xS^n is the normal vector η to the sphere. The corresponding eigenvalue is evidently equal to 1. Using the fact that the determinant is equal to the product of all eigenvalues we obtain

$$\frac{det(\bar{g})}{det(g)} = \frac{1}{det(\bar{g}^{-1}g)} = \frac{1}{det(G^{-1})} = det(G).$$

Let us find the determinant of the matrix G. We have

$$\begin{aligned} det(G) &= det^2(J) \\ &= det^2(\bar{A} - \bar{A}P\bar{A}^2 + \bar{A}P) \\ &= \frac{1}{det^2(A)} det^2(\bar{A}^2 - \bar{A}^2P\bar{A}^2 + \bar{A}^2P) \end{aligned}$$

Let us show that

$$det(\bar{A}^2 - \bar{A}^2P\bar{A}^2 + \bar{A}^2P) = det(\bar{A}^2 - \bar{A}^2P\bar{A}^2 + P).$$

Easy to check that

$$(\bar{A}^2 - \bar{A}^2P\bar{A}^2 + \bar{A}^2P)(1-P) = (\bar{A}^2 - \bar{A}^2P\bar{A}^2 + P)(1-P)$$

so that the operators $(\bar{A}^2 - \bar{A}^2P\bar{A}^2 + \bar{A}^2P)$ and $(\bar{A}^2 - \bar{A}^2P\bar{A}^2 + P)$ coincide on TS^n. Then there exist an eigenvector $v \notin TS^n$ of the operator $(\bar{A}^2 - \bar{A}^2P\bar{A}^2 + \bar{A}^2P)$. The corresponding eigenvalue is equal to 1 since $P(\bar{A}^2 - \bar{A}^2P\bar{A}^2 + \bar{A}^2P) = P$. Since the determinant is the product of all eigenvalues we have

$$det(\bar{A}^2 - \bar{A}^2P\bar{A}^2 + \bar{A}^2P) = det(\bar{A}^2 - \bar{A}^2P\bar{A}^2 + P).$$

Thus

$$\frac{1}{det^2(A)} det^2(G) = det(G)$$

and

$$det(G) = \left(\frac{1}{\frac{(x^1)^2}{a_1} + \frac{(x^2)^2}{a_2} + ... + \frac{(x^{n+1})^2}{a_{n+1}}} \right)^{n+1}.$$

Finally the operator B is the restriction of the operator given by the matrix

$$\left(\frac{1}{\frac{(x^1)^2}{a_1} + \frac{(x^2)^2}{a_2} + ... + \frac{(x^{n+1})^2}{a_{n+1}}} \right) G^{-1}$$

to TS^n. Then the metrics gB, $\bar{g}B$ are the restrictions to TS^n of the metrics given by the matrixes

$$\left(\frac{1}{\frac{(x^1)^2}{a_1} + \frac{(x^2)^2}{a_2} + ... + \frac{(x^{n+1})^2}{a_{n+1}}} \right) G^{-1}, \quad \left(\frac{1}{\frac{(x^1)^2}{a_1} + \frac{(x^2)^2}{a_2} + ... + \frac{(x^{n+1})^2}{a_{n+1}}} \right) \mathbf{1},$$

respectively.

The restriction to the sphere of the metric with matrix

$$\left(\frac{1}{\frac{(x^1)^2}{a_1}+\frac{(x^2)^2}{a_2}+...+\frac{(x^{n+1})^2}{a_{n+1}}}\right)G^{-1}$$

coincides with the restriction to the sphere of the metric with the matrix

$$\left(\frac{1}{\frac{(x^1)^2}{a_1}+\frac{(x^2)^2}{a_2}+...+\frac{(x^{n+1})^2}{a_{n+1}}}\right)\bar{A}^{-2}=A^{-2}$$

To show this it is sufficient to check that

$$(\mathbf{1}-P)G^{-1}(\mathbf{1}-P)=(\mathbf{1}-P)\bar{A}(\mathbf{1}-P)$$

which is trivial.

Let us show that the restriction of the metric with the matrix A to the sphere is the metric of the ellipsoid. Consider the ellipsoid

$$\left\{\frac{(x^1)^2}{a_1}+\frac{(x^2)^2}{a_2}+...+\frac{(x^{n+1})^2}{a_{n+1}}=1\right\}.$$

Consider the mapping

$$m:R^{n+1}\to R^{n+1},\quad m(x^1,x^2,...,x^n)=(\frac{x^1}{a_1},\frac{x^2}{a_2},...,\frac{x^{n+1}}{a_{n+1}}).$$

This mapping takes the ellipsoid to the sphere.

The Jacobian of this mapping is the A. Then the pull-back metric of the metric with the matrix A^{-2} is the metric $A^TA^{-2}A=\mathbf{1}$.

Analogically the pull-back of the metric $\bar{g}B$ to the sphere gives us the metric (7). More precisely, the matrix of the pull-back of the metric with the matrix

$$\left(\frac{1}{\frac{(x^1)^2}{a_1}+\frac{(x^2)^2}{a_2}+...+\frac{(x^{n+1})^2}{a_{n+1}}}\right)\mathbf{1}$$

is given by the matrix

$$\frac{1}{\frac{(x^1)^2}{a_1}+\frac{(x^2)^2}{a_2}+...+\frac{(x^{n+1})^2}{a_{n+1}}}A^2.$$

Theorem 6 is proved.

The metric gB^2, $\bar{g}B^2$ are the restriction to the sphere of the metrics with matrixes

$$\left(\frac{1}{\frac{(x^1)^2}{a_1}+\frac{(x^2)^2}{a_2}+...+\frac{(x^{n+1})^2}{a_{n+1}}}\right)^2(G^{-1})^2,\quad \left(\frac{1}{\frac{(x^1)^2}{a_1}+\frac{(x^2)^2}{a_2}+...+\frac{(x^{n+1})^2}{a_{n+1}}}\right)^2(G^{-1}),$$

respectively. Easy to check that the matrix $(1-P)(G^{-1})^2(1-P)$ is the matrix $\bar{A}^{-4}-\bar{A}^{-2}P\bar{A}^{-2}$. Then the pull-back of the corresponding metrics gives us the metrics with the matrix

$$A^{-2}-P, \qquad \left(\frac{1}{\frac{(x^1)^2}{a_1}+\frac{(x^2)^2}{a_2}+\ldots+\frac{(x^{n+1})^2}{a_{n+1}}}\right)\mathbf{1},$$

which give us the metrics (8), (9). Q. E. D.

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