Eta Invariants and Manifolds with Boundary

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0. Introduction

Let M be a compact oriented Riemannian manifold of dimension n. Let

$$D : C^\infty(M,S) \to C^\infty(M,S)$$

be a first order elliptic differential operator on M which is formally self–adjoint (with respect to some Hermitian fibre metric in S). For the moment suppose that M has no boundary. Then D is essentially self–adjoint in $L^2(M,S)$ and the eta invariant is a non–local spectral invariant of D. It was introduced by Atiyah, Patodi and Singer [APS1]. We recall its definition. Let λ_j run over the eigenvalues of D. Then the eta function of D is defined as

$$\eta(s,D) = \sum_{\lambda_j \neq 0} \frac{\operatorname{sign}\lambda_j}{|\lambda_j|^s}, \quad \operatorname{Re}(s) > n. \tag{0.1}$$

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The series is absolutely converging in the half-plane $\text{Re}(s) > n$ and admits a meromorphic continuation to the whole complex plane. The analytic continuation is based on the following alternative expression for the eta function

$$\eta(s, D) = \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^\infty t^{(s-1)/2} \operatorname{Tr}\left(De^{-tD^2}\right) dt. \tag{0.2}$$

It is a nontrivial result that $\eta(s, D)$ is regular at $s = 0$ [APS3], [Gi2]. Then the eta invariant is defined to be $\eta(0, D)$. The eta invariant is a measure of the spectral asymmetry of D. It arises naturally as the boundary correction term in the index theorem for manifolds with boundary proved by Atiyah, Patodi and Singer [APS1]. We note that this index theorem can be recovered in many different ways. For example, one may glue a half-cylinder or a cone to the boundary of the manifold in question and work in the L^2-setting [Ch1], [Me], [Mü]. This means that the spectral boundary conditions used in [APS1] are replaced by the L^2-conditions. It turns out that the L^2-index of the naturally extended operator is closely related to the index of the original boundary value problem.

In this paper we shall study eta invariants for manifolds with boundary. Thus, we assume that M has a nonempty boundary Y. There are various posibilities to define eta invariants for manifolds with boundary. One way is to introduce boundary conditions. In [GS], Gilkey and Smith have studied eta invariants for a certain restricted class of elliptic boundary value problems. The associated closed extensions are, in general, non-self-adjoint. For first order operators, however, there exists a natural choice of boundary conditions which gives rise to a self-adjoint extension. These are the spectral boundary conditions of Atiyah, Patodi and Singer [APS1]. For compatible Dirac type operators this approach was used in [DW].

Instead of imposing boundary conditions one may, for example, glue a cone or a half-cylinder to the boundary of M and consider the corresponding eta invariant in the L^2-setting. This may be also viewed as a global boundary condition. Eta invariants for manifolds with conical singularities were studied by Cheeger [Ch1], [Ch2] for the operator associated to the signature operator and by Bismut and Cheeger [BC] for Dirac operators. In this paper, we shall consider the case were a half-cylinder is attached to the boundary.

We suppose that the Riemannian metric of M is a product in a neighborhood $I \times Y$ of the boundary. Furthermore, we assume that, on this neighborhood, D takes the form

$$D = \gamma\left(\frac{\partial}{\partial u} + A\right) \tag{0.3}$$

where γ and A satisfy the conditions (1.2), (1.3). In particular, A is symmetric. Then we introduce spectral boundary conditions as in [APS1]. We use the negative spectral projection Π_- of A. If $\operatorname{Ker} A \neq \{0\}$, the corresponding extension of D is not self-adjoint. In this case we proceed as in [DW,pp.162] and pick a unitary involution $\sigma : \operatorname{Ker} A \to \operatorname{Ker} A$ such that $\sigma\gamma = -\gamma\sigma$. Under the given assumptions, such an involution always exists. Let P_- denote the orthogonal projection onto $\operatorname{Ker}(\sigma + \operatorname{Id})$.The boundary conditions are then defined by $(\Pi_- + P_-)(\varphi|Y) = 0$, $\varphi \in C^\infty(M, S)$. The associated closed extension D_σ is self-adjoint and has pure point spectrum. A similar phenomena occurs also in the case of conical singularities [Ch1], [Ch2]. One has to impose ideal boundary conditions which corresponds exactly to the choice of a Lagrangian subspace of $\operatorname{Ker} A$. In this context, Cheeger was the first to consider this type of boundary conditions.

In §1 we study the spectrum of D_σ more closely. It has essentially the same formal properties as the spectrum of D on a closed manifold. In particular, Weyl's law holds for the counting function of the eigenvalues λ_j of D_σ, that is,

$$\#\{\lambda_j \,|\, |\lambda_j| \leq \lambda\} \sim \frac{\operatorname{Vol}(M)}{(4\pi)^{n/2}\,\Gamma(n/2+1)}\,\lambda^n$$

as $\lambda \to \infty$ (Corollary 1.22). This enables us to introduce the eta function $\eta(s, D_\sigma)$ by the same formula (0.1). The study of the heat equation implies in the same way as in the closed case that $\eta(s, D_\sigma)$ has a meromorphic continuation to the whole complex plane. The case of a compatible Dirac type operator (cf. §1 for the definition) was treated in [DW]. In thiscase $\eta(s, D_\sigma)$ is regular in the half-plane $\operatorname{Re}(s) > -1$. In particular, the eta invariant of D_σ is given by

$$\eta(0, D_\sigma) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} \operatorname{Tr}\big(D_\sigma e^{-tD_\sigma^2}\big)\, dt.$$

The question of regularity of $\eta(s, D_\sigma)$ at $s = 0$ is not completely answered in this paper. In §2 we study the behaviour of the eta invariant under variations which stay constant near the boundary. It follows that, for such variations, the residue is a homotopy invariant. This implies, in particular, that $\eta(s, D_\sigma)$ is regular at $s = 0$ for all Dirac type operators. We also investigate the dependence of the eta invariant on the choice of the unitary involution σ. If σ_0, σ_1 are two unitary involutions of $\operatorname{Ker} A$ anticommuting with γ, then we show in Theorem 2.21 that

$$\eta(0, D_{\sigma_1}) \;-\; \eta(0, D_{\sigma_0}) \equiv -\frac{1}{\pi i} \log\, \det(\sigma_0\sigma_1 | \operatorname{Ker}(\gamma - i)) \mod \mathbf{Z}.$$

This result was proved independently by Lesch and Wojciechowski [LW].

In anology with the closed case one may expect that eta invariants for manifolds with boundary shall arise as boundary correction terms in an index theorem for manifolds with corners. We do not know yet if there exists an appropriate boundary value problem for a manifold with corners generalizing the APS boundary conditions in the case of a smooth boundary. One may, however, use the L^2-approach to derive such an index formula. For this purpose we need to study eta invariants within the L^2-framework. This means that we enlarge M by gluing the half-cylinder $\mathbf{R}^+ \times Y$ to the boundary Y of M. If we equip $\mathbf{R}^+ \times Y$ with the product metric, then the resulting manifold Z becomes a complete Riemannian manifold. The operator D has a natural extension to Z and its closure in L^2 will be denoted by $\mathcal{D}$. It is easy to see that $\mathcal{D}$ is self-adjoint. Since $\mathcal{D}$ has a nontrivial continuous spectrum, the eta invariant of $\mathcal{D}$ can not be defined in the same way as for D_σ. Instead we consider the kernel $E(x, y, t)$ of $\mathcal{D} \exp -t\mathcal{D}^2$. In §3 we study this kernel and prove that $\operatorname{tr} E(x,x,t)$ is absolutely integrable on Z. The integral $\int_Z \operatorname{tr} E(x,x,t)\,dx$ will be the substitute for $\operatorname{Tr}(De^{-tD^2})$ in (0.2). It has also an interpretation as relative trace. Namely, consider $D_0 = \gamma(\partial/\partial u + A)$ as operator in $C^\infty(\mathbf{R}^+ \times Y, S)$. We impose spectral boundary conditions at the bottom of the cylinder. The corresponding closure $\mathcal{D}_0$ is self–adjoint. Moreover, for $t > 0$, $\mathcal{D}\exp -t\mathcal{D}^2 - \mathcal{D}_0 \exp -t\mathcal{D}_0^2$ is of the trace class and the following relative trace formula holds

$$\operatorname{Tr}(\mathcal{D}e^{-t\mathcal{D}^2} - \mathcal{D}_0 e^{-t\mathcal{D}_0^2}) = \int_Z \operatorname{tr} E(x,x,t)\,dx. \tag{0.4}$$

In order to be able to define the eta function of $\mathcal{D}$ using (0.4), we have to study the asymptotic behaviour of (0.4) as $t \to 0$ and $t \to \infty$. The small time asymptotic follows essentially from the corresponding local heat expansion on a closed manifold and the explicit description of the heat kernel on the cylinder. To obtain the large time asymptotic we need some results about the spectral decomposition of $\mathcal{D}$ which we recall in §4. To study the continuous spectrum we may regard $\mathcal{D}$ as a perturbation of $\mathcal{D}_0$ and apply standard techniques of scattering theory. It follows that the wave operators $W_\pm(\mathcal{D},\mathcal{D}_0)$ (cf. (4.8) for their definition) exist and are complete. Thus, the absolutely continuous part of $\mathcal{D}$ is unitarily equivalent to $\mathcal{D}_0$. Moreover, the scattering operator $C = W_+^* \circ W_-$ is well–defined. Let $C(\lambda)$, $\lambda \in \mathbf{R}$, be the corresponding scattering matrix determined by the spectral decomposition of C with respect to the spectral measure of $\mathcal{D}_0$. Let μ_j run over the eigenvalues of A and denote by $\mathcal{E}(\mu_j)$ the μ_j–eigenspace of A. For $\lambda \in \mathbf{R}$, $C(\lambda)$ is a unitary operator in $\oplus_{\mu_j^2<\lambda^2}\mathcal{E}(\mu_j)$. Let $\mu_1 > 0$ be the smallest positive eigenvalue of A. If $|\lambda| < \mu_1$, then $C(\lambda)$ acts in $\operatorname{Ker} A$. It admits an analytic continuation to a meromorphic function of $\lambda \in \Sigma_1 = \mathbf{C} - ((-\infty,-\mu_1] \cup [\mu_1,\infty))$ with values in the linear operators in $\operatorname{Ker} A$. Moreover, $C(\lambda)$ satisfies the following functional equation

$$C(-\lambda)\,C(\lambda) = \mathrm{Id}, \quad \gamma C(\lambda) = -C(\lambda)\gamma, \quad \gamma \in \Sigma_1. \tag{0.5}$$

In §5 we determine the large time asymptotic of (0.4). The main result is Corollary 5.16 which states that

$$\int_Z \operatorname{tr} E(x,x,t)\,dx = -\frac{1}{2\pi}\int_0^{\mu_1} \lambda e^{-t\lambda^2}\operatorname{Tr}(\gamma C(-\lambda)\,C'(\lambda))\,d\lambda \; + \; O(e^{-ct}) \tag{0.6}$$

for $t \geq 1$. Here $C'(z) = (\partial/\partial z)C(z)$. In fact, we expect a more general formula to be true. Observe that the scattering matrix $C(\lambda)$ is real analytic at all real points λ which do not belong to $\operatorname{Spec}(A)$. Denote by $C'(\lambda)$ the derivative of $C(\lambda)$ at $\lambda \notin \operatorname{Spec}(A)$. We claim that the following relative trace formula holds

$$\operatorname{Tr}(\mathcal{D}e^{-t\mathcal{D}^2} - \mathcal{D}_0 e^{-t\mathcal{D}_0^2}) = \sum_{\lambda_j} \lambda_j e^{-t\lambda_j^2} - \frac{1}{2\pi}\int_0^\infty \lambda e^{-t\lambda^2}\operatorname{Tr}(\gamma C(-\lambda)C'(\lambda))\,d\lambda.$$

Here the $\lambda_j's$ are running over the eigenvalues of $\mathcal{D}$. Formula (0.6) would then be an immediate consequence of this trace formula.

Since $C(\lambda)$ is analytic, this formula leads to an asymptotic expansion of $\int_Z \operatorname{tr} E(x,x,t)\,dx$ as $t \to \infty$. The coefficients of this expansion are determined by the scattering matrix. They are nonlocal in contrast to the coefficients occurring in the asymptotic expansion for $t \to 0$.

Based on these results, we introduce in §6 the eta function $\eta(s,\mathcal{D})$. If D is a compatible Dirac type operator, then $\eta(s,\mathcal{D})$ is regular at $s=0$ and the eta invariant is given by

$$\eta(0,\mathcal{D}) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} \int_Z \operatorname{tr} E(x,x,t)\,dx\,dt. \tag{0.7}$$

One of our main goals is to compare the two types of eta invariants studied in this paper. First note that, by (0.5), $\tau = C(0)$ is a unitary involution of $\operatorname{Ker} A$ which anticommutes with γ. In particular, we may use τ to define the boundary conditions for D. There is also an equivalent description in terms of Lagrangian subspaces of $\operatorname{Ker} A$. Observe that $\operatorname{Ker} A$ has a natural symplectic structure defined by $\Phi(x,y) = \langle \gamma x, y\rangle$ where $\langle x,y\rangle$ denotes the L^2 inner product of $x, y \in \operatorname{Ker} A$. Then $L = \operatorname{Ker}(C(0) - \operatorname{Id})$ is a Lagrangian subspace, that is, it satisfies $L \oplus \gamma L = \operatorname{Ker} A$ and $\Phi(L,L) = 0$. Furthermore, given $\phi \in \operatorname{Ker} A$, there is associated a generalized eigensection $E(\phi,\lambda)$ of D (cf. §4). If $\phi \in L$ then $\varphi = \frac{1}{2}E(\phi,0)$ satisfies $D\varphi = 0$ and, on $\mathbf{R}^+ \times Y$, it has the form $\phi + \psi$ where ψ is square integrable. In particular, $\varphi \neq 0$. In other words, ϕ is the limiting value of an extended L^2–solution of $D\varphi = 0$ in the sense of [APS1]. It follows from Lemma 8.5 that L is precisely the subspace of all limiting values of extended solutions. Thus, the continuous spectrum of $\mathcal{D}$ gives rise to a distinguished choice of an involution σ of $\operatorname{Ker} A$ – the on–shell scattering matrix $C(0)$ – or, equivalently, to a distinguished Lagrangian subspace of $\operatorname{Ker} A$. Our main result can then be stated as follows

Theorem 0.1. *Let $D : C^\infty(M,S) \to C^\infty(M,S)$ be a compatible Dirac type operator which, on a neighborhood $I \times Y$ of Y, takes the form (0.3). Let $C(\lambda) : \operatorname{Ker} A \to \operatorname{Ker} A$ be the associated scattering matrix in the range $|\lambda| < \mu_1$ and put $\tau = C(0)$. Then we have*

$$\eta(0, D_\tau) = \eta(0,\mathcal{D}).$$

In part II we shall employ this formula to prove a splitting formula for eta invariants.

To prove Theorem 0.1, we pick $a > 0$ and consider the manifold $M_a = M \cup ([0,a] \times Y)$. The operator D has a natural extension $D(a)$ to a compatible Dirac type operator on M_a. It follows from the variational formulas of §2 that $\eta(0, D(a)_\tau)$ is independent of a. Therefore, it is sufficient to show that $\lim_{a\to\infty} \eta(0, D(a)_\tau) = \eta(0,\mathcal{D})$. To establish this result, we follow partially the approach used by Douglas and Wojciechowski [DW]. Namely, we start out with formula (0.2) and split the integral as follows

$$\int_0^{\sqrt{a}} + \int_{\sqrt{a}}^\infty .$$

In §7 we prove that, as $a \to \infty$, the first integral converges to $\eta(0,\mathcal{D})$. To deal with the second integral, we write $\operatorname{Tr}\big(D(a)_\tau e^{-tD(a)_\tau^2}\big)$ as $S_1(a,t) + S_2(a,t)$ where S_1 is the contribution to the trace given by all eigenvalues $\lambda(a)$ satisfying $|\lambda(a)| > a^{-\kappa}$ for some $0 < \kappa < 1$. Then it is easy to see that $\int_{\sqrt{a}}^{\infty} S_1(a,t)\,dt$ tends to zero as $a \to \infty$. It remains to study the behaviour of $\int_{\sqrt{a}}^{\infty} S_2(a,t)\,dt$ as $a \to \infty$. This is done in §8. If $\operatorname{Ker} A = \{0\}$, then the continuous spectrum of $\mathcal{D}$ has a gap at 0 which implies that the nonzero eigenvalues of $D(a)_{\Pi_-}$ stay bounded away from zero and the proof is finished. This case was studied in [DW]. The difficult part is the case when $\operatorname{Ker} A \neq \{0\}$. Then the continuous spectrum of $\mathcal{D}$ has no gap at zero and eigenvalues of $D(a)_\tau$ will cluster at zero if $a \to \infty$. The crux of the argument is to show that the nonzero spectrum of $D(a)_\tau$ becomes asymptotically symmetric near zero and therefore, cancels out in the limit $a \to \infty$. Let $\varphi \neq 0$ be an eigensection of $D(a)_\tau$ with eigenvalue λ. On $[0,a] \times Y$, φ takes the following form

$$\varphi = e^{-i\lambda u}\psi_+ \ + \ e^{i\lambda u}\psi_- \ + \ \varphi_1$$

where $\psi_\pm \in \operatorname{Ker} A$, $\gamma\psi_\pm = \pm i\,\psi_\pm$ and $\varphi_1(u,\cdot)$ is orthogonal to $\operatorname{Ker} A$ for each $u \in [0,a]$. We call

$$\varphi_0 = e^{-i\lambda u}\psi_+ \ + \ e^{i\lambda u}\psi_-$$

the constant term of φ. In Proposition 8.14 we show that there exist $a_0, \delta > 0$ such that, for $a \geq a_0$ and $0 < |\lambda| < \delta$, the constant term of φ is nonzero. Thus, the eigensections of $D(a)_\tau$ with sufficiently small nonzero eigenvalues are determined by their constant terms. We continue by investigating the properties of the constant terms. Write ψ_+ as $\psi_+ = \phi - i\gamma\phi$ where $\phi \in \operatorname{Ker}(C(0) - \operatorname{Id})$. Associated to ϕ there is a generalized eigensection $E(\phi, z)$ of $\mathcal{D}$ with eigenvalue $z \in \mathbf{R}$. The main observation is that the constant term of φ differs from the constant term of $E(\phi,\lambda)$ by a term whose norm is exponentially small as $a \to \infty$. The constant term of $E(\phi,\lambda)$ has the form

$$e^{-i\lambda u}\psi_+ \ + \ e^{i\lambda u}\, C(\lambda)\psi_+.$$

Therefore, the constant term of φ satisfies

$$\| \psi_- - C(\lambda)\psi_+ \| \leq e^{-ca}, \quad a \geq a_0. \tag{0.8}$$

Let $L_- = \operatorname{Ker}(C(0) + \operatorname{Id})$ and denote by P_- the orthogonal projection of $\operatorname{Ker} A$ onto L_-. Let $I : L_- \to \operatorname{Ker}(\gamma - i)$ be defined by $I(\phi) = \phi - i\gamma\phi$. Then we consider the linear operator

$$S(\lambda) = P_- \circ C(\lambda) \circ I$$

acting in L_-. It follows from (0.8) that the function of z, $\det(e^{2iza}S(z) + \operatorname{Id})$, has a real zero ρ such that $|\rho - \lambda| < e^{-ca}$. Moreover, the multiplicity of the eigenvalue λ can be estimated by the multiplicity of ρ. Then we study more closely the real zeros of $\det(e^{2iza}S(z) + \operatorname{Id})$ near $z = 0$. The final result, Theorem 8.32, shows that, up to exponentially small terms, we may replace the small eigenvalues by the real zeros of $\det(e^{2iza}S(z) + \operatorname{Id})$ near $z = 0$. Since $S(\lambda)$ satisfies

$$S(-\lambda)\,S(\lambda) = \operatorname{Id} + O(\lambda^2), \quad |\lambda| < \varepsilon,$$

it follows then that the nonzero spectrum of $D(a)_\tau$ is indeed asymptotically symmetric near zero.

1. Eta Invariants for Manifolds with Boundary

Let M be a compact oriented C^∞ Riemannanian manifold of dimension n with smooth boundary $\partial M = Y$. We shall assume that the Riemannian metric of M is a product near the boundary.

Let $S \to M$ be a complex vector bundle over M equipped with a Hermitian fibre metric which is also a product near the boundary. Let $C^\infty(M,S)$ denote the space of smooth sections of S and $C_0^\infty(M,S)$ the subspace of $C^\infty(M,S)$ consisting of all sections with support contained in the interior of M. Given $s,s' \in C^\infty(M,S)$, let $\langle s,s'\rangle$ denote the inner product of s,s' defined by the fibre metric of S and the Riemannian metric of M. By $L^2(M,S)$ we shall denote the completion of $C_0^\infty(M,S)$ with respect to this inner product. Let

$$D : C^\infty(M,S) \to C^\infty(M,S)$$

be a linear first order differential operator on M which is formally self-adjoint, that is, D satisfies $\langle Ds,s'\rangle = \langle s,Ds'\rangle$ for all $s,s' \in C_0^\infty(M,S)$. We assume that, in a collar neighborhood $(-1,0] \times Y$ of the boundary, D takes the form

$$D = \gamma\Big(\frac{\partial}{\partial u} + A\Big) \tag{1.1}$$

where $\gamma : S|Y \to S|Y$ is a bundle isomorphism and

$$A : C^\infty(Y,S|Y) \to C^\infty(Y,S|Y)$$

is an elliptic operator on Y satisfying

$$\gamma^2 = -\mathrm{Id}, \quad \gamma^* = -\gamma \tag{1.2}$$

and

$$A\gamma = -\gamma A, \quad A^* = A. \tag{1.3}$$

Here A^* means the formal adjoint of A. Thus, A is symmetric. Examples of such operators are Dirac type operators.

Since Y is closed, A is essentially self-adjoint and has pure point spectrum. Let ϕ be an eigensection of A with eigenvalue μ. By (1.3), $\gamma\phi$ is also an eigensection of A with eigenvalue $-\mu$. Thus, the non-zero spectrum of A is symmetric.

If we regard D as an unbounded operator in $L^2(M,S)$ with domain $C_0^\infty(M,S)$, then D is symmetric. To obtain a self-adjoint extension of $D : C_0^\infty(M,S) \to L^2(M,S)$ one has to introduce boundary conditions. Appropriate boundary conditions are the spectral boundary conditions introduced by Atiyah, Patodi and Singer [APS1]. Let $\tilde{\Pi}_+$ (resp. $\tilde{\Pi}_-$) denote the orthogonal projection of $L^2(Y,S|Y)$ onto the subspace spanned by the eigensections of A with positive (resp. negative) eigenvalues. Note that the following equality holds:

$$\gamma\,\tilde{\Pi}_+ = \tilde{\Pi}_-\,\gamma. \tag{1.4}$$

If $\operatorname{Ker} A \neq \{0\}$, then the boundary conditions defined by $\tilde{\Pi}_\pm$ are not self–adjoint. In this case we proceed as in [DW,pp.162]. By (1.3), γ induces a map of $\operatorname{Ker} A$ into itself which we also denote by γ. We make the following

Assumption. *There exists a unitary involution*

$$\sigma : \operatorname{Ker} A \to \operatorname{Ker} A \quad \textit{with} \quad \sigma\gamma = -\gamma\sigma. \tag{1.5}$$

As we shall see in Proposition 4.26, this assumption is always satisfied. Let $L_\pm$ denote the ± 1–eigenspaces of σ. Then we have an orthogonal splitting

$$\operatorname{Ker} A = L_+ \oplus L_- \tag{1.6}$$

with

$$\gamma(L_\pm) = L_\mp. \tag{1.7}$$

In particular, $\operatorname{Ker} A$ is even–dimensional. We consider a special case. Let $S|Y = S^+ \oplus S^-$ be the splitting of $S|Y$ into the $\pm i$-eigenspaces of γ. In view of (1.3), we obtain operators

$$A_\pm : C^\infty(Y, S^\pm) \to C^\infty(Y, S^\mp) \quad \text{with} \quad A_+^* = A_-.$$

If D is a Dirac type operator, it follows from Theorem 3 of [Pa, Chap. XVII] that $\operatorname{Ind} A_+ = 0$. Thus, we get an orthogonal splitting

$$\operatorname{Ker} A = \operatorname{Ker} A_+ \oplus \operatorname{Ker} A_-$$

and $\dim \operatorname{Ker} A_+ = \dim \operatorname{Ker} A_-$. Using this spitting one may construct involutions σ as in (1.5).

Let σ be such an involution and let $P_\pm^\sigma$ denote the orthogonal projection of $L^2(Y, S|Y)$ onto $L_\pm$. Put

$$\Pi_\pm^\sigma = \tilde{\Pi}_\pm + P_\pm^\sigma. \tag{1.8}$$

Note that the following equality holds

$$-\gamma\Pi_+^\sigma\gamma = \operatorname{Id} - \Pi_+^\sigma = \Pi_-^\sigma. \tag{1.9}$$

Let $H^1(M, S)$ denote the first Sobolev space. Put

$$\operatorname{dom}(D_\sigma) = \{\varphi \in H^1(M, S) \mid \Pi_-^\sigma(\varphi|Y) = 0\} \tag{1.10}$$

and define $D_\sigma : \mathrm{dom}(D_\sigma) \to L^2(M,S)$ by $D_\sigma \varphi = D\varphi$ where, on the right hand side, derivations are taken in the sense of distributions. If $\operatorname{Ker} A = \{0\}$, there is only one involution. In this case we shall write D_{Π_-} in place of D_σ.

Lemma 1.11. *The operator D_σ is essentially self-adjoint.*

Proof. Let

$$C^\infty(M,S;\Pi_-^\sigma) = \{\varphi \in C^\infty(M,S) \mid \Pi_-^\sigma(\varphi|Y) = 0\}. \tag{1.12}$$

Then we may construct a two-sided parametrix

$$R : C^\infty(M,S) \to C^\infty(M,S;\Pi_-^\sigma)$$

for D_σ in the same way as in [APS1,p.54]. Thus $DR - \mathrm{Id}$ and $RD - \mathrm{Id}$ are smoothing operators and the lemma follows by standard arguments. Q.E.D.

Now we shall study the heat operator $\exp -tD_\sigma^2$. For this purpose we first consider the heat equation on the half-cylinder $X = \mathbf{R}^+ \times Y$. Let $\pi : X \to Y$ be the canonical projection and $S_X = \pi^*(S|Y)$. Let $D^X : C^\infty(S_X) \to C^\infty(S_X)$ be defined by $D^X = \gamma(\partial/\partial u + A)$. Then $D^X : C_0^\infty(S_X) \to L^2(S_X)$ is symmetric and, if we impose boundary conditions by $\Pi_-^\sigma(\varphi(0,\cdot)) = 0$, we obtain a self-adjpoint extension D_σ^X. Let $e_{1,\sigma}$ be the kernel of the heat operator $\exp -t(D_\sigma^X)^2$. Then $e_{1,\sigma}$ is a smooth kernel which satisfies

$$\Big(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial u^2} + A_x^2\Big) e_{1,\sigma}((u,x),(v,y),t) = 0, \quad \lim_{t\to 0} e_{1,\sigma}(z,z',t) = \delta_{z,z'}$$

$$\Pi_-^\sigma\big(e_{1,\sigma}((0,\cdot),z,t)\big) = 0, \quad \Pi_+^\sigma\Big(\frac{\partial}{\partial u} e_{1,\sigma}((0,\cdot),z,t)|_{u=0}\Big) = 0.$$

It can be given by an explicit formula. Let ϕ_j, $j \in \mathbf{N}$, be an orthonormal basis for $\mathrm{Ran}(\Pi_+^\sigma)$ consisting of eigensections of A with eigenvalues $0 \le \mu_1 \le \mu_2 \le \cdots$. Then we have

$$\begin{aligned} e_{1,\sigma}((u,x),(v,y),t) = \sum_{j=1}^\infty \Big\{ \frac{e^{-\mu_j^2 t}}{\sqrt{4\pi t}} \Big(\exp\Big\{-\frac{(u-v)^2}{4t}\Big\} + \exp\Big\{-\frac{(u+v)^2}{4t}\Big\}\Big) \\ - \mu_j e^{\mu_j(u+v)} \operatorname{erfc}\Big(\frac{u+v}{2\sqrt{t}} + \mu_j\sqrt{t}\Big)\Big\} \phi_j(x) \otimes \overline{\phi_j(y)} \\ + \sum_{j=1}^\infty \Big\{ \frac{e^{-\mu_j^2 t}}{\sqrt{4\pi t}} \Big(\exp\Big\{-\frac{(u-v)^2}{4t}\Big\} - \exp\Big\{-\frac{(u+v)^2}{4t}\Big\}\Big) \gamma\phi_j(x) \otimes \overline{\gamma\phi_j(y)} \end{aligned} \tag{1.13}$$

where erfc is the complementary error function defined by

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2}\, du.$$

Let $\hat{M} = M \cup -M$ be the double of M. Then S extends to a bundle $\hat{S}$ over $\hat{M}$. Because of (1.1), D has a natural extension to an elliptic operator $\hat{D} : C^\infty(\hat{S}) \to C^\infty(\hat{S})$. Let e_2 denote the restriction to M of the fundamental solution of $\partial/\partial t + \hat{D}^2$. Then a parametrix e_σ for the kernel K_σ of $\exp -tD_\sigma^2$ is obtained by patching together $e_{1,\sigma}$ and e_2 as in [APS1, p.55]. More precisely, let $\rho(a,b)$ denote an increasing C^∞ function of the real variable u, such that $\rho = 0$ for $u \le a$ and $\rho = 1$ for $u \ge b$. Suppose the metric of M is a product on the collar neighborhood $(-1,0] \times Y$ of Y. We define four C^∞ functions $\phi_1, \phi_2, \psi_1, \psi_2$ by

$$\begin{aligned} \phi_1 &= \rho(-1,-5/6), \quad && \psi_1 = \rho(-4/6,-3/6) \\ \phi_2 &= 1 - \rho(-2/6,-1/6), \quad && \psi_2 = 1 - \psi_1. \end{aligned} \tag{1.14}$$

We regard these functions of u as functions on the cylinder $[-1,0] \times Y$ and then extend them to M in the obvious way. Then we put

$$e_\sigma = \phi_1 e_{1,\sigma} \psi_1 \; + \; \phi_2 e_2 \psi_2. \tag{1.15}$$

This is a parametrix for the heat kernel K_σ and K_σ is obtained from e_σ as usually by a convergent series of the form

$$K_\sigma = e_\sigma + \sum_{m=1}^{\infty} (-1)^m c_m * e_\sigma, \tag{1.16}$$

where $*$ denotes convolution of kernels, $c_1 = (\partial/\partial t + D^2)e_\sigma$ and $c_m = c_{m-1} * c_1$, $m \ge 2$. It follows from (1.16) that, for $t > 0$, K_σ is a C^∞ kernel which differs from e_σ by an exponentially small term as $t \to 0$.

Lemma 1.17. (i) *The operators* $\exp -tD_\sigma^2$ *and* $D_\sigma \exp -tD_\sigma^2$ *are of the trace class for* $t > 0$.

(ii) *As* $t \to 0$, *there exist asymptotic expansions*

$$\operatorname{Tr}\left(e^{-tD_\sigma^2}\right) \sim \sum_{j=0}^{\infty} a_j(D_\sigma)\, t^{(j-n)/2} \tag{1.18}$$

and

$$\operatorname{Tr}\left(D_\sigma e^{-tD_\sigma^2}\right) \sim \sum_{j=0}^{\infty} b_j(D_\sigma)\, t^{(j-n-1)/2}. \tag{1.19}$$

(iii) *There exist local densities* $a_j(D_\sigma)(x)$ *and* $b_j(D_\sigma)(x)$ *such that*

$$a_j(D_\sigma) = \int_M a_j(D_\sigma)(x) \quad \text{and} \quad b_j(D_\sigma) = \int_M b_j(D_\sigma)(x).$$

The local densities $a_j(D_\sigma)(x)$, $b_j(D_\sigma)(x)$ are polynomials in the jets of the total symbol of D_σ with coefficients which are smooth functions of the leading symbol. Moreover, $b_j(D_\sigma) = 0$ if j is even.

Proof. Since, for $t > 0$, $K_\sigma(x, y, t)$ is a smooth kernel, it follows that $\exp -tD_\sigma^2$ and $D_\sigma \exp -tD_\sigma^2$ are Hilbert–Schmidt operators. Employing the semi-group property, we get (i). Furthermore, we have

$$\mathrm{Tr}\big(e^{-tD_\sigma^2}\big) = \int_M \mathrm{tr}\, K_\sigma(x, x, t)\, dx \tag{1.20}$$

and

$$\mathrm{Tr}\big(D_\sigma e^{-tD_\sigma^2}\big) = \int_M \mathrm{tr}\,\big(D_x K_\sigma(x, y, t)\big|_{x=y}\big)\, dx. \tag{1.21}$$

For the asymptotic expansion, we may replace K_σ by its parametrix e_σ. The asymptotic behaviour of $\int_{[-1,0]\times Y} \mathrm{tr}\, e_1(x, x, t)\, dx$ can be studied explicitely by using (1.13). For the interior parametrix we use the local heat expansion. This implies (1.18). Furthermore, (1.15) implies that

$$\int_Y \mathrm{tr}\big(\gamma(\frac{\partial}{\partial u} + A)\, e_1\big((u, y), (v, y), t\big)\big|_{u=v}\big)\, dy = 0$$

and, by Lemma 1.7.7. of [Gi1], there exists a local expansion of the form

$$\mathrm{tr}\big(D_x e_2(x, y, t)\big|_{x=y}\big) \sim \sum_{j=0}^{\infty} c_j(x)\, t^{(j-n-1)/2}$$

as $t \to 0$. This proves (1.19). Q.E.D.

By Lemma 1.17, (i), D_σ has pure point spectrum. Let $\cdots \le \lambda_j \le \lambda_{j+1} \le \cdots$ be the eigenvalues of D_σ where each eigenvalue is repeated according to its multiplicity. Consider the counting function

$$N(\lambda) = \#\{\lambda_j \mid |\lambda_j| \le \lambda\}, \quad \lambda \ge 0.$$

Applying a standard Tauberian theorem to (1.18), we get

Corollary 1.22. *As $\lambda \to \infty$, one has*

$$N(\lambda) = \frac{\mathrm{Vol}(M)}{(4\pi)^{n/2}\,\Gamma(\frac{n}{2}+1)}\,\lambda^n + o(\lambda^n).$$

Therefore, we can introduce the corresponding zeta and eta function. Let

$$\zeta(s, D_\sigma) = \sum_{\lambda_j \neq 0} |\lambda_j|^{-s}, \tag{1.23}$$

and

$$\eta(s, D_\sigma) = \sum_{\lambda_j \neq 0} \operatorname{sign} \lambda_j \, |\lambda_j|^{-s}. \tag{1.24}$$

By Corollary 1.22, both sides are absolutely converging in the half–plane $\mathrm{Re}(s) > n$. Let $h = \dim \mathrm{Ker}(D_\sigma)$. Then, by Mellin transform, we obtain

$$\zeta(s, D_\sigma) = \frac{1}{\Gamma\left(\frac{s}{2}\right)} \int_0^\infty t^{s/2-1} \left(\mathrm{Tr}(e^{-tD_\sigma^2}) - h\right) dt \tag{1.25}$$

and

$$\eta(s, D_\sigma) = \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^\infty t^{(s-1)/2} \mathrm{Tr}\left(D_\sigma e^{-tD_\sigma^2}\right) dt. \tag{1.26}$$

By Lemma 1.17, these integrals are absolutely convergent for $\mathrm{Re}(s) > n$ and admit meromorphic continuations to $\mathbf{C}$. For compatible Dirac type operators (see below) this was established in [DW]. Thus, $\zeta(s, D_\sigma)$ and $\eta(s, D_\sigma)$ are meromorphic functions of $s \in \mathbf{C}$. The poles can be determined from the corresponding asymptotic expansions (1.18) and (1.19). Of particular interest is the behaviour at $s = 0$. The zeta function $\zeta(s, D_\sigma)$ is always regular at $s = 0$ and $\zeta(0, D_\sigma) = a_n(D_\sigma) - h$. The eta function $\eta(s, D_\sigma)$ has a simple pole at $s = 0$ with

$$\operatorname*{Res}_{s=0} \eta(s, D_\sigma) = \frac{2}{\sqrt{\pi}} b_n(D_\sigma). \tag{1.27}$$

By Lemma 1.17, (iii), the residue is zero for n even. Now suppose that n is odd. We shall not study the behaviour of the residue in general, but only discuss this question for the case of an operator of Dirac type. We briefly recall the definition of such an operator (cf. [GL], [BG]).

Let $\mathrm{Clif}(M) = \mathrm{Clif}(TM)$ be the complexified Clifford algebra bundle over M. The Riemannian metric and connection of TM can be naturally extended to $\mathrm{Clif}(M)$. Let S be a complex vector bundle over M. A $\mathrm{Clif}(M)$ module structure on S is a unital algebra morphism

$$\nu : \mathrm{Clif}(M) \to \mathrm{End}(S).$$

A vector bundle S with a $\mathrm{Clif}(M)$ module structure is called a *Clifford bundle* over M if it is equipped with a Hermitian fibre metric and a unitary connection ∇ such that

(i) For each unit vector $e \in T_xM$, the module multiplication $e : S_x \to S_x$ is an isometry.

(ii) $\nabla \nu = 0$.

A connection on S which satisfies (ii) is called *compatible.* Note that ∇ is compatible iff for all $\phi \in C^\infty(\mathrm{Clif}(M))$ and $\psi \in C^\infty(S)$ the following relation holds

$$\nabla(\phi\psi) = \phi\nabla(\psi) + (\nabla\phi)\psi.$$

We shall assume that the fibre metric and the connection of S are also products near the boundary.

If S is a Clifford bundle there is a natural first order elliptic differential operator $D : C^\infty(S) \to C^\infty(S)$ associated to S which is defined as the composition

$$C^\infty(S) \xrightarrow{\nabla} C^\infty(S \otimes T^*M) \to C^\infty(S \otimes TM) \to C^\infty(S).$$

Here the second arrow is defined by the Riemannian metric of M and the third arrow by the $\mathrm{Clif}(M)$ module structure of S. This is the Dirac operator attached to S and, following [BG], we call D a compatible Dirac type operator. Let $X_1, ..., X_n$ denote a local orthonormal frame field. Then D can be written as

$$D = \sum_{k=1}^{n} X_k \cdot \nabla_{X_k}.$$

Let $\psi \in C^\infty(\mathrm{End}(S))$. Then we call

$$D^\psi = D + \psi$$

an operator of Dirac type. First consider a compatible operator D of Dirac type. Recall that the coefficients of the asymptotic expansion (1.19) are completely determined by the interior parametrix e_2. Therefore, we can apply Theorem 3.4 of [BG] and get

Proposition 1.28. *Let D be a compatible operator of Dirac type.*
(a) *If j is even, then $b_j(D_\sigma) = 0$.*
(b) *If n is even, then $b_j(D_\sigma) = 0$ for all j.*
(c) *If $j \leq n$, then $b_j(D_\sigma) = 0$.*

By (1.26), this implies

Corollary 1.29. *Let D be a compatible operator of Dirac type. Then $\eta(s, D_\sigma)$ is holomorphic in the half-plane $\mathrm{Re}(s) > -2$. Moreover, the eta invariant $\eta(0, D_\sigma)$ is given by*

$$\eta(0, D_\sigma) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} \,\mathrm{Tr}\big(D_\sigma e^{-tD_\sigma^2}\big)\, dt. \tag{1.30}$$

This result was also proved in [DW]. In the next section we shall continue with the investigation of the residues of the eta function for general Dirac type operators.

Suppose that $n = 2k$, $k \in \mathbf{N}$, and D is a compatible Dirac type operator. Consider the standard involution $\tau : S \to S$ defined by

$$\tau = i^k e_1 \cdots e_{2k}$$

where $e_1, ..., e_{2k}$ is a local tangent frame field. Then we have

$$\tau D = -D\tau \quad \text{and} \quad \tau A = A\tau. \tag{1.31}$$

Hence, τ commutes with the spectral projections $\tilde{\Pi}_{\pm}$ and induces a map $\tau : \operatorname{Ker} A \to \operatorname{Ker} A$. Suppose that the involution (1.5) satisfies $\tau\sigma = \sigma\tau$. Then τ also commutes with $\Pi^{\sigma}_{\pm}$. Therefore, by (1.31), we obtain $\tau D_{\sigma} = -D_{\sigma}\tau$. This implies that the spectrum of D_{σ} is symmetric and, hence, the eta function vanishes identically. In particular, this is the case if $\operatorname{Ker} A = \{0\}$. Thus, the interesting case is the odd-dimensional one.

2. Variation of Eta Invariants

In this section we shall study the behaviour of the eta invariant under variation of the operator and the boundary conditions. We first study the case where the boundary conditions are held fixed. This means that the operator D remains constant near the boundary and the involution σ of $\operatorname{Ker} A$ is not varied. As above, we assume that all metrics and connections are products near the boundary.

Proposition 2.1. *Let D_v be a C^∞ one-parameter family of formally self-adjoint elliptic first order differential operators on M. Suppose that, on a collar neighborhood $(-1,0]\times Y$, D_v is given by*

$$D_v = \gamma(\frac{\partial}{\partial u} + A)$$

with γ and A independent of v and satisfying (1.2), (1.3). Let σ be a unitary involution of $\operatorname{Ker} A$ as in (1.5). Let $B_v = (D_v)_\sigma$ be the self-adjoint extension of D_v defined by σ and put $\dot{B}_v = (d/dv)B_v$. Then

$$\frac{\partial}{\partial v}\operatorname{Tr}(B_v e^{-tB_v^2}) = (1 + 2t\frac{\partial}{\partial t})\operatorname{Tr}(\dot{B}_v e^{-tB_v^2}).$$

Proof. The operators D_v act on smooth sections of a fixed vector bundle S. However, the fibre metric of S and the Riemannian metric of M may depend on v and, therefore, the inner product in $C^\infty(M,S)$ may depent on v. In any case, the corresponding Hilbert spaces $L^2(M,S)_v$ have equivalent norms. Hence, the trace functional is independent of v [La, p.161]. Moreover, by our assumptions, the domains of the operators B_v agree as topological vector spaces. Hence, we may regard B_v as a one-parameter family of linear operators in a fixed Hilbert space $L^2(M,S)_0$ with domain independent of v. Thus, $\dot{B}_v = dB_v/dv$ is well-defined and

$$\frac{\partial}{\partial v}\operatorname{Tr}(B_v e^{-tB_v^2}) = \operatorname{Tr}(\frac{\partial}{\partial v}(B_v e^{-tB_v^2})) = \operatorname{Tr}(\dot{B}_v e^{-tB_v^2}) + \operatorname{Tr}(B_v \frac{\partial}{\partial v} e^{-tB_v^2}). \tag{2.2}$$

To determine the derivative of the heat operator with respect to the parameter v, we proceed as in [Me]. We use the identity

$$\left(\frac{\partial}{\partial t} + B_v^2\right)\frac{\partial}{\partial v}e^{-tB_v^2} = -(\dot{B}_v B_v + B_v \dot{B}_v)\, e^{-tB_v^2}. \tag{2.3}$$

Since the initial condition is independent of v, we can use Duhamel's principle to solve (2.3). This leads to

$$\frac{\partial}{\partial v}e^{-tB_v^2} = -\int_0^t e^{-(t-r)B_v^2}(\dot{B}_v B_v + B_v \dot{B}_v)\, e^{-rB_v^2}\, dr. \tag{2.4}$$

Using (2.4) and the trace identities, we get

$$\operatorname{Tr}\big(B_v \frac{\partial}{\partial v} e^{-tB_v^2}\big) = -2t \operatorname{Tr}\big(\dot{B}_v B_v^2 e^{-tB_v^2}\big) = 2t \frac{\partial}{\partial t} \operatorname{Tr}\big(\dot{B}_v e^{-tB_v^2}\big).$$

Q.E.D.

Let $K_v(x,y,t)$ be the kernel of $\exp -tB_v^2$. Then it follows in the same way as in the proof of Lemma 1.17 that

$$\operatorname{Tr}\big(\dot{B}_v e^{-tB_v^2}\big) = \int_M \operatorname{tr}\big((\dot{D}_v)_x K_v(x,y,t)\big|_{x=y}\big)\, dx$$

where $\dot{D}_v = (d/dv)D_v$ is a first order differential operator. If we employ Lemma 1.7.7 of [Gil], it follows that, as $t \to 0$, there exists an asymptotic expansion of the form

$$\operatorname{Tr}\big(\dot{B}_v e^{-tB_v^2}\big) \sim \sum_{j=0}^{\infty} c_j(v)\, t^{(j-n-1)/2}. \tag{2.5}$$

The coefficients $c_j(v)$ are again local in the sense that there exist densities $c_j(v,x)$ such that $c_j(v) = \int_M c_j(v,x)$.

Proposition 2.6. *Let the assumptions be the same as in Proposition 2.1. Moreover, suppose that* $\dim \operatorname{Ker}(B_v)$ *is constant. Then, for* $\operatorname{Re}(s) > n$, *we have*

$$\frac{\partial}{\partial v}\eta(s,B_v) = -\frac{s}{\Gamma\big(\frac{s+1}{2}\big)} \int_0^{\infty} t^{(s-1)/2} \operatorname{Tr}\big(\dot{B}_v e^{-tB_v^2}\big)\, dv. \tag{2.7}$$

The integral is absolutely converging.

Proof. We follow the proof of Proposition 8.39 in [Me]. Let $\operatorname{Re}(s) > n$ and $T > 0$. Using Proposition 2.1, (2.5) and integration by parts, we obtain

$$\begin{aligned}\frac{\partial}{\partial v}\int_0^T t^{(s-1)/2} \operatorname{Tr}\big(B_v e^{-tB_v^2}\big)\, dt &= \int_0^T t^{(s-1)/2}\big(1 + 2t\frac{\partial}{\partial t}\big) \operatorname{Tr}\big(\dot{B}_v e^{-tB_v^2}\big)\, dt \\ &= 2T^{(s+1)/2} \operatorname{Tr}\big(\dot{B}_v e^{-TB_v^2}\big) \;-\; s\int_0^T t^{(s-1)/2} \operatorname{Tr}\big(\dot{B}_v e^{-tB_v^2}\big)\, dt.\end{aligned} \tag{2.8}$$

Let H_v be the orthogonal projection of $L^2(M,S)_v$ onto $\operatorname{Ker} B_v$. Since $\dim \operatorname{Ker}(B_v)$ is constant, H_v depends smoothly on v. By the self-adjointness of B_v, we have $B_v H_v = H_v B_v = 0$ and, therefore,

$$B_v = (\mathrm{Id} - H_v) B_v (\mathrm{Id} - H_v).$$

This implies

$$\dot{B}_v = -\dot{H}_v B_v (\mathrm{Id} - H_v) \;+\; (\mathrm{Id} - H_v)\dot{B}_v(\mathrm{Id} - H_v) \;-\; (\mathrm{Id} - H_v) B_v \dot{H}_v.$$

Since $\| (\mathrm{Id} - H_v) \exp -tB_v^2 \| \leq e^{-tc}$ for some $c = c(v) > 0$, it follows that

$$\left|\mathrm{Tr}\big(\dot{B}_v e^{-tB_v^2}\big)\right| \leq C_1\, e^{-tc_1}.$$

If we pass to the limit $T \to \infty$, the first term on the right hand side of (2.8) vanishes and the proposition follows. Q.E.D.

By (2.5), the integral on the right hand side of (2.7) admits a meromorphic continuation to $\mathbf{C}$. At $s = 0$ it has a simple pole with residue equal to $2c_n(v)$. This implies

Corollary 2.9. *Let the assumptions be as in Proposition* 2.6. *Then* $(\partial/\partial v)\eta(s, B_v)$ *is holomorphic at* $s = 0$ *with*

$$\frac{\partial}{\partial v}\, \eta(s, B_v)\big|_{s=0} = -\frac{2}{\sqrt{\pi}}\, c_n(v)$$

where $c_n(v)$ *is the* n*–th coefficient in the asymptotic expansion (2.5).*

Now observe that the poles of $\eta(s, B_v)$ are located at $s = n - j$, $j \in \mathbf{N}$. In particular, poles stay separated during a deformation. Since $(\partial/\partial v)\eta(s, B_v)$ is holomorphic near $s = 0$, it follows that $\mathrm{Res}_{s=0}\eta(s, B_v)$ is independent of v. We shall now extend this result to the case when $\dim \mathrm{Ker}(B_v)$ is not necessarily constant.

To study $\eta(s, B_v)$ near $v = 0$ we pick $c \in \mathbf{R}$ not an eigenvalue of $\pm B_0$. By continuity it is not an eigenvalue of any $\pm B_v$ for $|v| < \varepsilon$. Let P_c denote the orthogonal projection of $L^2(M, S)_v$ onto the subspace spanned by all eigensections with eigenvalue λ satisfying $|\lambda| < c$. Put

$$B'_v = B_v(\mathrm{Id} - P_c) \; + \; P_c. \tag{2.10}$$

Then, for $|v| < \varepsilon$, B'_v is invertible and depends smoothly on v. Since P_c has finite rank, the eta function is also defined for B'_v and

$$\eta(s, B_v) = \eta(s, B'_v) \; + \sum_{|\lambda_j| < c} \mathrm{sign}\lambda_j \, |\lambda_j|^{-s} \; - \; \mathrm{Tr}(P_c).$$

Thus $\eta(s, B_v)$ and $\eta(s, B'_v)$ differ by an entire function. In particular, $\eta(s, B_v)$ and $\eta(s, B'_v)$ have the same residue at $s = 0$. Furthermore, the proofs of Propositions 2.1 and 2.6 work for B'_v as well. In fact, the proof of (2.7) simplifies because B'_v is invertible. Thus

$$\frac{\partial}{\partial v}\eta(s, B'_v) = -\frac{s}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^\infty t^{(s-1)/2}\, \mathrm{Tr}\big(\dot{B}'_v e^{-t(B'_v)^2}\big)\, dv \tag{2.11}$$

for $\mathrm{Re}(s) > n$. Since P_v is a finite rank operator, it is easy to see that

$$\mathrm{Tr}\big(\dot{B}'_v e^{-t(B'_v)^2}\big) = \mathrm{Tr}\big(\dot{B}_v e^{-tB_v^2}\big) \; + \; O(1)$$

as $t \to 0$. Together with (2.5) it follows that the integral on the right hand side of (2.11) admits a meromorphic continuation to $\mathrm{Re}(s) > -1$. Moreover, it has a simple pole at $s = 0$ with residue $2c_n(v)$ where $c_n(v)$ is the corresponding coefficient in (2.5). Therefore, $(\partial/\partial v)\eta(s, B'_v)$ is holomorphic at $s = 0$ and

$$\frac{\partial}{\partial v}\eta(s, B'_v)\big|_{s=0} = -\frac{2}{\sqrt{\pi}}\, c_n(v).$$

This implies

Corollary 2.12. *Let the assumptions be the same as in Proposition 2.1. Then the residue of $\eta(s, B_v)$ at $s = 0$ does not depend on v.*

Proof. As explained above, we have

$$\operatorname*{Res}_{s=0}\, \eta(s, B_v) = \operatorname*{Res}_{s=0}\, \eta(s, B'_v).$$

Moreover, poles of $\eta(s, B'_v)$ may only occur at $s = n - j$, $j \in \mathbf{N}$. Let $\gamma \subset \mathbf{C}$ be the circle of radius $1/2$ with center at 0. Then $(\partial/\partial v)\eta(s, B'_v)$ is holomorphic in the interior of γ and, therefore,

$$\frac{\partial}{\partial v} \operatorname*{Res}_{s=0}\, \eta(s, B'_v) = \frac{1}{2\pi i} \int_\gamma \frac{\partial}{\partial v}\eta(s, B'_v)\, ds = 0.$$

Q.E.D.

Thus $\mathrm{Res}_{s=0}\, \eta(s, D_\sigma)$ is a homotopy invariant of D_σ.

As an application we consider a compatible Dirac type operator $D : C^\infty(M, S) \to C^\infty(M, S)$ which, on $(-1, 0] \times Y$, takes the form (1.1). Let $\psi \in C^\infty(\mathrm{End}(S))$ be such that $\psi^* = \psi$. Moreover suppose that, on $(-1, 0] \times Y$, ψ satisfies $(\partial/\partial u)\psi(u, y) = 0$ and $\gamma\psi = -\psi\gamma$. Put $D^\psi = D + \psi$. Then D^ψ is formally self-adjoint and, near Y, it takes the form (1.1). Let $\chi \in C^\infty(\mathbf{R})$ be such that $\chi(u) = 0$ for $u \le -1$ and $\chi(u) = 1$ for $u \ge -1/2$. We regard χ as a function on $(-1, 0] \times Y$ in the obvious way and then extend it by zero to a smooth function on M. For $v \in \mathbf{R}$, put

$$D^\psi_v = D + v(1 - \chi)\psi + \chi\psi.$$

Then D^ψ_v is a one–parameter family of Dirac type operators which satisfy the assumptions of Proposition 2.1. Let σ be a unitary involution of $\operatorname{Ker} A$ as in (1.5). In view of Corollary 2.12, the residue at $s = 0$ of $\eta(s, (D^\psi)_\sigma)$ equals the residue at $s = 0$ of $\eta(s, (D^\psi_0)_\sigma)$ which is determined by the coefficient $b_n((D^\psi_0)_\sigma)$ of the asymptotic expansion (1.19). Since D is a compatible Dirac type operator the corresponding local density $b_n(x, (D^\psi_0)_\sigma)$ has support in $(-1, 0] \times Y$. Therefore, in order to determine b_n, we may replace M by the half–cylinder $\mathbf{R}^- \times Y$. Let $\hat{S}$ be the pullback of $S|Y$ to $\mathbf{R}^- \times Y$ and let $\hat{D} = \gamma(\partial/\partial u + A) + \chi\psi$ regarded as operator in $C^\infty(\mathbf{R}^- \times Y, \hat{S})$. Here $\gamma(\partial/\partial u + A)$ is the expression for D on $(-1, 0] \times Y$. Let

$\hat{\psi} \in C^\infty(\mathrm{End}(\hat{S}))$ be defined by $\hat{\psi}(u,y) = \psi(0,y)$, $y \in Y$. Note that $\hat{\psi}$ satisfies $\gamma\hat{\psi} = -\hat{\psi}\gamma$. For $v \in \mathbf{R}$, put

$$\hat{D}_v = \hat{D} + v(1-\chi)\hat{\psi}.$$

Thus $\hat{D}_0 = \hat{D}$. Moreover, on $(-1,0] \times Y$, we have $\hat{D}_v = \gamma(\partial/\partial u + \hat{A})$. We use Π^σ_-, defined with respect to $\hat{A}$, to introduce spectral boundary conditions. Let $(\hat{D}_v)_\sigma$ be the corresponding self-adjoint extension in L^2. Now we observe that Lemma 3.9, Proposition 3.11 and 3.12 apply in the present case as well. This implies that the integral

$$\int_Z \mathrm{tr}\big((\hat{D}_v)_\sigma e^{-t(\hat{D}_v)^2_\sigma}(x,x)\big)\,dx$$

is absolutely convergent and has an asymptotic expansion as $t \to 0$. For $v = 0$, the coefficient of $t^{-1/2}$ equals our b_n above. Furthermore, if we proceed as in the proof of Proposition 2.1, it follows that

$$\frac{\partial}{\partial u}\int_Z \mathrm{tr}\big((\hat{D}_v)_\sigma e^{-t(\hat{D}_v)^2_\sigma}(x,x)\big)\,dx = \big(1+2t\frac{\partial}{\partial t}\big)\int_Z \mathrm{tr}\big((1-\chi(x))\hat{\psi}(x)\,e^{-t(\hat{D}_v)^2_\sigma}(x,x)\big)\,dx.$$

Since γ_x anticommutes with $\hat{\psi}(x)$ and $\gamma_x \circ \exp -t(\hat{D}_v)^2_\sigma(x,x) = \exp -t(\hat{D}_v)^2_\sigma(x,x) \circ \gamma_x$ it follows that the right hand side vanishes. This implies that $(\partial/\partial v)b_n(v) = 0$. But $b_n(1) = 0$. Thus $b_n \equiv 0$ and we proved

Proposition 2.13. *Let $D : C^\infty(M,S) \to C^\infty(M,S)$ be any Dirac type operator which satisfies* (1.1). *Let D_σ be a self-adjoint extension defined by some unitary involution* (1.5). *Then $\eta(s,D_\sigma)$ is regular at $s=0$.*

Let D_v be a smooth one-parameter family of Dirac type operators such that, on $(-1,0] \times Y$, $D_v = \gamma(\partial/\partial u + A)$ with γ, A independent of v and satisfying (1.2), (1.3). Let σ be any unitary involution of $\mathrm{Ker}\,A$ as in (1.5). Put $B_v = (D_v)_\sigma$. Then $\eta(s,B_v)$ is holomorphic at $s=0$. However, if eigenvalues cross zero, $\eta(0,B_v)$ is not smooth in v, but has integer jumps. Let

$$\overline{\eta}(0,B_v) = \eta(0,B_v) \mod \mathbf{Z} \tag{2.14}$$

be the reduced eta invariant which takes values in $\mathbf{R}/\mathbf{Z}$. If B'_v is defined as in (2.10), it is clear that $\overline{\eta}(0,B_v) = \overline{\eta}(0,B'_v)$. Using our results above, we get

Proposition 2.15. (i) *The reduced eta invariant $\overline{\eta}(0,B_v)$ is a smooth function of v and*

$$\frac{d}{dv}\overline{\eta}(0,B_v) = -\frac{2}{\sqrt{\pi}}\,c_n(v).$$

(ii) *If* $\dim \mathrm{Ker}(B_v)$ *is constant, then $\eta(0,B_v)$ is smooth and*

$$\frac{d}{dv}\eta(0,B_v) = -\frac{2}{\sqrt{\pi}}\,c_n(v).$$

Here $c_n(v)$ is determined by the asymptotic expansion (2.5). Moreover, there exists a density $c_n(x;v)$ which is locally computable from the jets of the complete symbole of D_v such that $c_n(v) = \int_M c_n(x;v)$.

We shall now discuss two applications of our variational formulas. Let D be a Dirac type operator on M which satisfies (1.1)–(1.3). Let $a \geq 0$ and set

$$M_a = M \cup ([0,a] \times Y).$$

The bundle S can be extended in the obvious way to a vector bundle S_a over M_a and D has a natural extension to a Dirac type operator $D(a)$ acting in $C^\infty(M_a, S_a)$ which has the same properties as $D = D(0)$. Let σ be a unitary involution of $\operatorname{Ker} A$ as in (1.5). Let $D(a)_\sigma$ be the self–adjoint extension of $D(a) : C_0^\infty(M_a, S_a) \to L^2(M_a, S_a)$ defined above.

Proposition 2.16. *The eta invariant $\eta(0, D(a)_\sigma)$ is independent of a.*

Proof. First we shall show that $\dim \operatorname{Ker} D(a)_\sigma$ is independent of a. Let $\varphi \in \operatorname{Ker} D(a)_\sigma$. This is equivalent to say that $\varphi \in C^\infty(S_a)$ satisfies

$$D(a)\varphi = 0 \quad \text{and} \quad \Pi_-^\sigma(\varphi|(\{a\} \times Y)) = 0. \tag{2.17}$$

Let ϕ_j, $j \in \mathbf{N}$, be an orthonormal basis for $\operatorname{Ran}(\Pi_+^\sigma)$ consisting of eigensections of A with eigenvalues $0 \leq \mu_1 \leq \mu_2 \leq \cdots$. In view of (2.17), we may expand $\varphi|([0,a] \times Y)$ in terms of the ϕ_j:

$$\varphi(u,y) = \sum_{j=1}^{\infty} e^{-\mu_j u}\, \phi_j(y).$$

Let $a' > a$. Then φ can be extended in the obvious way to $\tilde\varphi \in \operatorname{Ker} D(a')_\sigma$ and the map $\varphi \mapsto \tilde\varphi$ defines an isomorphism of $\operatorname{Ker} D(a)_\sigma$ onto $\operatorname{Ker} D(a')_\sigma$. Next, observe that there exists a smooth family of diffeomorphisms $f_a : (-1,0] \to (-1,a]$ which satisfies the following properties

$$f_a(u) = u \quad \text{for} \quad u \in (-1,-2/3) \quad \text{and} \quad f_a(u) = u + a \quad \text{for} \quad u \in (-1/3, 0].$$

Let $\psi_a : (-1,0] \times Y \to (-1,a] \times Y$ be defined by $\psi_a(u,y) = (f_a(u), y)$ and extend ψ_a to a diffeomorphism $\psi_a : M \to M_a$ in the canonical way, i.e., ψ_a is the identity on $M - ((-1,0] \times Y)$. There is also a bundle isomorphism $\tilde\psi_a : S \to S_a$ which covers ψ_a. This induces an isomorphism $\psi_a^* : C^\infty(M_a, S_a) \to C^\infty(M,S)$. Let $\tilde D(a) = \psi_a^* \circ D(a) \circ (\psi_a^*)^{-1}$. Then $\tilde D(a)$ is a family of Dirac type operators on M and $\tilde D(a) = \gamma(\partial/\partial u + A)$ near Y. Furthermore, $\tilde D(a)_\sigma = \psi_a^* \circ D(a)_\sigma \circ (\psi_a^*)^{-1}$. Hence

$$\eta(s, D(a)_\sigma) = \eta(s, \tilde D(a)_\sigma) \quad \text{and} \quad \psi_a^*(\operatorname{Ker} D(a)_\sigma) = \operatorname{Ker} \tilde D(a)_\sigma.$$

In particular, $\dim \operatorname{Ker} \tilde D(a)_\sigma$ is constant and we can apply Proposition 2.15, (ii), which gives

$$\frac{d}{da}\eta(0, D(a)_\sigma) = -\frac{2}{\sqrt{\pi}}\, c_n(a).$$

Now let $\mathbf{S}_a^1$ be the circle of radius $2a$, $\pi : \mathbf{S}_a^1 \times Y \to Y$ the natural projection and $\hat{S}_a = \pi^*(S|Y)$. We define $\hat{D}_a : C^\infty(\hat{S}_a) \to C^\infty(\hat{S}_a)$ by $\hat{D}_a = \gamma(\partial/\partial u + A)$. Since $c_n(a)$ is locally computable, it follows in the same way as above that

$$\frac{d}{da}\,\eta(0, \hat{D}_a) = -\frac{2}{\sqrt{\pi}}\, c_n(a).$$

But a direct computation shows that the spectrum of $\hat{D}_a$ is symmetric. Hence $\eta(s, \hat{D}_a) = 0$ and, therefore, $c_n(a) = 0$. Q.E.D.

Next we shall study the dependence of the eta invariant $\eta(0, D_\sigma)$ on the choice of σ. This question was independently settled by Lesch and Wojciechowski [LW]. Following [LW], we pick a self–adjoint endomorphism T of $\operatorname{Ker}(\gamma - \mathrm{Id})$ such that $e^{2\pi i T} = \sigma_0\sigma_1|\operatorname{Ker}(\gamma - \mathrm{Id})$ and $-\pi < T \leq \pi$, i.e., $T = \frac{1}{2\pi i}\log(\sigma_0\sigma_1|\operatorname{Ker}(\gamma - \mathrm{Id}))$. We extend T to $\operatorname{Ker} A$ by putting $T = 0$ on $\operatorname{Ker}(\gamma + \mathrm{Id})$. Let $\rho_v = e^{2\pi i v T}$ and put

$$\sigma_v = \rho_v^* \sigma_0 \rho_v, \quad 0 \leq v \leq 1.$$

This is a one–parameter family of unitary involutions of $\operatorname{Ker} A$ which anticommute with γ and connects σ_0 to σ_1. In order to study the variation of the eta invariant of D_{σ_v} we have to transform the family D_{σ_v} into one with fixed domain. This can be done as follows. Let $f \in C^\infty(\mathbf{R})$ be such that $f(u) = 1$ for $-1/3 < u$ and $f(u) = 0$ for $u < -2/3$. Note that, by Fubini's theorem, we may identify $L^2([-1, 0] \times Y, S)$ with $L^2([-1, 0]; L^2(S|Y))$. Therefore, we may regard $L^2([-1, 0]; \operatorname{Ker} A)$ as a closed subspace of $L^2(M, S)$. With respect to this identification, we define a one–parameter family U_v, $0 \leq v \leq 1$, of unitary operators in $L^2(M, S)$ as follows: Set $U_v = \mathrm{Id}$ on $L^2([0, 1]; \operatorname{Ker} A)^\perp$ and

$$(U_v\varphi)(u) = e^{2\pi i v f(u) T}(\varphi(u)), \quad \varphi \in L^2([-1, 0]; \operatorname{Ker} A).$$

Let $\Pi_\pm^v$ be the orthogonal projection (1.8) defined with respect to σ_v, $0 \leq v \leq 1$. Then, by definition, we have

$$U_v \circ \Pi_\pm^v = \Pi_\pm^0, \quad 0 \leq v \leq 1. \tag{2.18}$$

Put

$$D'_{\sigma_v} = U_v D_{\sigma_v} U_v^*, \quad 0 \leq v \leq 1. \tag{2.19}$$

By (2.18), we get

$$\operatorname{dom} D'_{\sigma_v} = \operatorname{dom} D_{\sigma_0}.$$

Hence D'_{σ_v}, $0 \leq v \leq 1$, is a smooth family of self–adjoint operators in $L^2(M, S)$ with fixed domain. Moreover, it follows from the definition of U_v that $U_v(C_0^\infty(M, S)) = C_0^\infty(M, S)$. Put $D'_v = U_v D U_v^*$. Then $D'_v : C_0^\infty(M, S) \to L^2(M, S)$ is symmetric and D'_{σ_v} is the self–adjoint extension of D'_v defined by the boundary conditions $\Pi_-^0(\varphi|\partial M) = 0$. This implies

$$D'_{\sigma_v} = D_{\sigma_0} - 2\pi i v f' \gamma T, \quad 0 \leq v \leq 1. \tag{2.20}$$

By (2.18), D_{σ_v} and D'_{σ_v} have the same spectrum. Hence, the eta function $\eta(s, D'_{\sigma_v})$ is well-defined and equals $\eta(s, D_{\sigma_v})$. Note that D'_v is not a differential operator, but our results above can be easily extended to D'_{σ_v}. In particular, this applies to Proposition 2.15. Thus

$$\frac{d}{dv}\,\overline{\eta}(0, D_{\sigma_v}) = -\frac{2}{\sqrt{\pi}}\, c_n(v)$$

where $c_n(v)$ is the coefficient of $t^{-1/2}$ in the asymptotic expansion of $\mathrm{Tr}\big(\dot{D}'_{\sigma_v} \exp -t(D'_{\sigma_v})^2\big)$. By (2.19) and (2.20), the trace equals

$$\mathrm{Tr}\big(\dot{D}'_{\sigma_v} U_v e^{-tD^2_{\sigma_v}} U_v^*\big) = \mathrm{Tr}\big(U_v^* \dot{D}'_{\sigma_v} U_v e^{-tD^2_{\sigma_v}}\big) = -2\pi i \mathrm{Tr}(f'\gamma T\, e^{-tD^2_{\sigma_v}}).$$

Since the support of f' is contained in $(-1,0)$, we may replace $\exp -tD^2_{\sigma_v}$ by its parametrix on $[-1,0] \times Y$ which can be taken to be

$$\frac{1}{\sqrt{4\pi t}} \exp\Big\{-\frac{(u-u')^2}{4t}\Big\}\, e^{-tA^2}(x,y).$$

This shows that

$$\mathrm{Tr}\big(f'\gamma T e^{-tD^2_{\sigma_v}}\big) = \frac{1}{\sqrt{4\pi t}}\,\mathrm{Tr}(\gamma T) \;+\; O(e^{-c/t})$$

as $t \to 0$ and, therefore,

$$c_n(v) = \frac{2\pi}{\sqrt{4\pi}}\,\mathrm{Tr}(T) = \frac{1}{2\sqrt{\pi}\, i}\, \log\det(\sigma_0\sigma_1 | \mathrm{Ker}(\gamma - \mathrm{Id})).$$

Thus we have proved

Theorem 2.21. *Let $D : C^\infty(M,S) \to C^\infty(M,S)$ be a Dirac type operator which, on $(-1,0] \times Y$, takes the form $D = \gamma(\partial/\partial u + A)$ with conditions (1.2), (1.3) satisfied. Let σ_0, σ_1 be two unitary involutions of $\mathrm{Ker}\, A$ such that $\sigma_i\gamma = -\gamma\sigma_i$, $i = 0,1$. Then*

$$\eta(0, D_{\sigma_1}) \;-\; \eta(0, D_{\sigma_0}) \equiv -\frac{1}{\pi i}\, \log\det\big(\sigma_0\sigma_1 | \mathrm{Ker}(\gamma - i)\big) \mod \mathbf{Z}.$$

This result was proved independently by Lesch and Wojciechowski [LW].

3. Heat Kernels on Manifolds with Cylindrical Ends

Let the setting be the same as in section 1. We introduce the non–compact manifold

$$Z = M \cup (\mathbf{R}^+ \times Y)$$

by gluing the half–cylinder $\mathbf{R}^+ \times Y$ to the boundary Y of M. We equip $\mathbf{R}^+ \times Y$ with the canonical product metric. Together with the given metric on M we get a smooth metric on Z. Then Z becomes a complete Riemannian manifold of infinite volume. We extend the bundle S with its fibre metric and the operator D to Z in the obvious way. The extended bundle and operator will be also denoted by S and D, respectively. Thus, on $\mathbf{R}^+ \times Y$,

$$D = \gamma\Big(\frac{\partial}{\partial u} + A\Big)$$

where γ, A satisfy (1.2), (1.3).

Let $C_0^\infty(Z,S)$ be the space of compactly supported smooth sections of S over Z and $L^2(Z,S)$ the completion of $C_0^\infty(Z,S)$ with respect to the natural inner product defined by the fibre metric of S and the metric of Z. Then

$$D : C_0^\infty(Z,S) \to L^2(Z,S) \tag{3.1}$$

is symmetric.

Lemma 3.1. *The operator* (3.1) *is essentially self–adjoint.*

Proof. It suffices to show that $(D \pm i)C_0^\infty(Z,S)$ is dense in $L^2(Z,S)$. Suppose that $\psi \in L^2(Z,S)$ is orthogonal to $(D \pm i)C_0^\infty(Z,S)$. By elliptic regularity, ψ is smooth and satisfies $D\psi = \mp i\psi$. If we expand ψ on $\mathbf{R}^+ \times Y$ in terms of the eigensections of γA, it follows that ψ satisfies an estimate of the form

$$\| \psi(u,y) \| \le C\, e^{-cu}, \quad (u,y) \in \mathbf{R}^+ \times Y,$$

for some constants $C, c > 0$. Applying Green's formula, we get $\langle D\psi, \psi\rangle = \langle \psi, D\psi\rangle$ and, therefore, $\psi = 0$. Q.E.D.

Let $\mathcal{D}$ denote the unique self–adjoint extension of D. In this section we shall investigate the kernel $K(x,y,t)$ of the heat operator $\exp -t\mathcal{D}^2$. We construct a parametrix for K as follows. Let Q_2 be the restriction to M of the fundamental solution of $\partial/\partial t + \hat{D}^2$ on the double $\hat{M}$ of M, i.e., $Q_2 = e_2$ in the notation of (1.15). Furthermore, let Q_1 be the fundamental solution of $\partial/\partial t - \partial^2/\partial u^2 + A^2$ on $\mathbf{R} \times Y$. Thus

$$Q_1\big((u,x),(v,y),t\big) = \frac{1}{\sqrt{4\pi t}} \exp\Big\{-\frac{(u-v)^2}{4t}\Big\}\, e^{-tA^2}(x,y)$$

where $e^{-tA^2}(x,y)$ is the kernel of $\exp -tA^2$. Let the functions $\phi_1, \phi_2, \psi_1, \psi_2$ be defined by (1.14) and put

$$Q = \phi_1 Q_1 \psi_1 \; + \; \phi_2 Q_2 \psi_2. \tag{3.3}$$

Then Q is a parametrix for K and K is obtained by a convergent series similar to (1.16).

$$K = Q \; + \; \sum_{m=1}^{\infty} (-1)^m \, Q_m * Q \tag{3.4}$$

where $Q_1 = (\partial/\partial t + D^2)Q$, $Q_m = Q_{m-1} * Q_1$ for $m \geq 2$ and $*$ denotes convolution of kernels. For $t > 0$, K is a C^∞ kernel which represents $\exp -t\mathcal{D}^2$. In particular, it satisfies $(\partial/\partial t + D_x^2)K(x,y,t) = 0$. Moreover, for each $x_0 \in Z$ and $m \in \mathbf{N}$, there exist constants $C, c > 0$ such that

$$\| D_x^k D_y^l \big(K(x,y,t) \; - \; Q(x,y,t)\big) \| \leq C \exp\Big(-c\big(d(x,x_0)^2 + d(y,x_0)^2 + 1\big)/t\Big) \, e^{ct} \tag{3.5}$$

for all $x, y \in Z$, $k, l \leq m$ and $t > 0$.

Let $D_0 = \gamma(\partial/\partial u + A)$ regarded as operator in $C^\infty(\mathbf{R}^+ \times Y, S)$. Suppose that there exists a unitary involution σ of $\operatorname{Ker} A$ such that $\gamma\sigma = -\sigma\gamma$. Let Π_+^σ be the orthogonal projection (1.8) with respect to σ and put

$$C^\infty(\mathbf{R}^+ \times Y, S; \Pi_+^\sigma) = \{\varphi \in C^\infty(\mathbf{R}^+ \times Y, S) \mid \Pi_+^\sigma(\varphi(0,\cdot)) = 0\}.$$

Denote by $C_0^\infty(\mathbf{R}^+ \times Y, S; \Pi_+^\sigma)$ the subspace of $C^\infty(\mathbf{R}^+ \times Y, S; \Pi_+^\sigma)$ consisting of sections which vanish for $u \gg 0$. Then $D_0 : C_0^\infty(\mathbf{R}^+ \times Y, S; \Pi_+^\sigma) \to L^2(\mathbf{R}^+ \times Y, S)$ is essentially self-adjoint. Let $\mathcal{D}_0$ be the unique self-adjoint extension. We observe that the kernel K_0 of $\exp -t\mathcal{D}_0^2$ is given by formula (1.13) with the roles of ϕ_j and $\gamma\phi_j$ switched. From this formula for K_0 follows immediately that, for each $m \in \mathbf{N}$, there exist $C_1, c_1 > 0$ such that

$$\| \frac{\partial^k}{\partial u^k} \frac{\partial^l}{\partial v^l} A_y^p A_{y'}^q \Big(K_0((u,y),(v,y'),t) \; - \; Q_1((u,y),(v,y'),t)\Big) \| \leq C_1 \exp(-c_1(u^2+v^2)/t) \tag{3.6}$$

for $y, y' \in Y$, $u, v \geq 1$ and $k, l, p, q \leq m$. We extend $\exp -t\mathcal{D}_0^2$ by zero to an operator in $L^2(Z,S)$.

Theorem 3.7. *For $t > 0$, the operators*

$$\exp -t\mathcal{D}^2 \; - \; \exp -t\mathcal{D}_0^2 \quad \textit{and} \quad \mathcal{D}\exp -t\mathcal{D}^2 \; - \; \mathcal{D}_0 \exp -t\mathcal{D}_0^2$$

are of the trace class.

Proof. Pick $\chi \in C^\infty(Z)$ such that $0 < \chi \leq 1$, $\chi(z) = 1$ for $z \in M$ and $\chi(u,y) = (1+u^2)^{-1}$ for $(u,y) \in [1,\infty) \times Y$. Denote by U_χ the operator in $L^2(Z,S)$ defined by multiplication by χ. Then we may write

$$\begin{aligned} \exp -t\mathcal{D}^2 \; - \; \exp -t\mathcal{D}_0^2 = &\big(\exp -\tfrac{t}{2}\mathcal{D}^2 \; - \; \exp -\tfrac{t}{2}\mathcal{D}_0^2\big) \circ U_\chi^{-1} \circ U_\chi \circ \exp -\tfrac{t}{2}\mathcal{D}^2 \\ &+ \exp -\tfrac{t}{2}\mathcal{D}_0^2 \circ U_\chi \circ U_\chi^{-1} \circ \big(\exp -\tfrac{t}{2}\mathcal{D}^2 \; - \; \exp -\tfrac{t}{2}\mathcal{D}_0^2\big). \end{aligned}$$

It follows from (3.5) that $\left(\exp -\frac{t}{2}\mathcal{D}^2 - \exp -\frac{t}{2}\mathcal{D}_0^2\right) \circ U_\chi^{-1}$ and $U_\chi^{-1} \circ \left(\exp -\frac{t}{2}\mathcal{D}^2 - \exp -\frac{t}{2}\mathcal{D}_0^2\right)$ are Hilbert–Schmidt operators. Furthermore, the function

$$(z, z') \in (\mathbf{R}^+ \times Y) \times (\mathbf{R}^+ \times Y) \mapsto \chi(z') \parallel Q_1(z, z', t) \parallel$$

belongs to $L^2\left((\mathbf{R}^+ \times Y) \times (\mathbf{R}^+ \times Y)\right)$. Together with (3.6) this shows that $\exp -t\mathcal{D}_0^2 \circ U_\chi$ is Hilbert–Schmidt. By (3.5), it also follows that $U_\chi \circ \exp -t\mathcal{D}^2$ is a Hilbert–Schmidt operator. Thus $\exp -t\mathcal{D}^2 - \exp -t\mathcal{D}_0^2$ can be written as a product of Hilbert–Schmidt operators and, therefore, is of the trace class. The remaining case is similar. Q.E.D.

Put

$$E(x, y, t) = D_x K(x, y, t). \tag{3.8}$$

This is the kernel of $\mathcal{D} \exp -t\mathcal{D}^2$.

Lemma 3.9. *For each $t > 0$, the function $x \mapsto \operatorname{tr} E(x, x, t)$ is absolutely integrable on Z.*

Proof. It follows from (3.5) that

$$\operatorname{tr}\left\{D_x\left(K(x, y, t) - Q(x, y, t)\right)\big|_{x=y}\right\}$$

is absolutely integrable on Z and the integrated absolute value is $O(e^{-c/t})$ as $t \to 0$. Furthermore, by definition of Q_1,

$$\gamma\left(\frac{\partial}{\partial u} + A_w\right)Q_1\left((u, w), (v, w'), t\right)\Big|_{\substack{u=v \\ w=w'}} = \frac{1}{\sqrt{4\pi t}}\gamma A_w e^{-tA^2}(w, w')\big|_{w=w'}.$$

Since $\gamma A = -A\gamma$ and γ acts fibrewise, it follows that

$$\operatorname{tr}\left(D_x Q_1(x, y, t)\big|_{x=y}\right) = 0.$$

Thus

$$\operatorname{tr}\left(D_x Q(x, y, t)\big|_{x=y}\right) = \operatorname{tr}\left(D_x(\phi_2(x)\, Q_2(x, y, t)\big|_{x=y}\right). \tag{3.10}$$

The right hand side has compact support which implies the lemma. Q.E.D.

Proposition 3.11. *For $t > 0$, we have*

$$\operatorname{Tr}\left(\mathcal{D}e^{-t\mathcal{D}^2} - \mathcal{D}_0 e^{-t\mathcal{D}_0^2}\right) = \int_Z \operatorname{tr} E(z, z, t)\, dz.$$

Proof. Let $E_0(z,z',t)$ be the kernel of $\mathcal{D}_0 \exp -t\mathcal{D}_0^2$. Then $E_0(z,z',t) = (D_0)_z K_0(z,z',t)$. Using the explicit description of K_0 similar to (1.13), we get

$$\begin{aligned}
&\operatorname{tr} E_0\big((u,y),(u,y),t\big) \\
&= \sum_{j=1}^{\infty} \frac{e^{-\mu_j^2 t}}{\sqrt{4\pi t}} \left\{\mu_j(1-e^{-u^2/t}) + \frac{u}{t} e^{-u^2/t}\right\} \big(\langle\gamma\phi_j(y),\phi_j(y)\rangle + \langle\phi_j(y),\gamma\phi_j(y)\rangle\big) = 0.
\end{aligned}$$

The last equality follows because $\gamma_y^* = -\gamma_y$, $y \in Y$. Since $E - E_0$ is the kernel of $\mathcal{D}\exp -t\mathcal{D}^2 - \mathcal{D}_0 \exp -t\mathcal{D}_0^2$, the Proposition follows from Lemma 3.9 by standard arguments. Q.E.D.

Proposition 3.12. (a) *As $t \to 0$, there exists an asymptotic expansion of the form*

$$\int_Z \operatorname{tr} E(z,z,t)\,dz \sim \sum_{j=0}^{\infty} a_j(D)\, t^{(j-n-1)/2}.$$

Moreover, there exist local densities $a_j(D)(x)$ with support contained in M such that $a_j(D) = \int_Z a_j(D)(x)$.
(b) *If D is a compatible Dirac type operator, then $a_j(D) = 0$ for $j \le n$ and $a_k(D) = 0$ for k even.*

Proof. It follows from (3.5) and (3.10) that

$$\int_Z \operatorname{tr} E(z,z,t)\,dz = \int_Z \operatorname{tr}\big(D_x(\phi_2(z)Q_2(z,z',t)\big|_{z=z'}\big)\,dz \;+\; O(e^{-c/t}).$$

The integral on the right hand side equals

$$(3.13)\quad \int_Z \phi_2(z)\operatorname{tr}\big(D_z Q_2(z,z',t)\big|_{z=z'}\big)\,dz \;+\; \int_{-1}^{0} \phi_2'(u) \int_Y \operatorname{tr}\big(\gamma Q_2((u,y),(u,y),t)\big)\,dy\,du.$$

If we employ Theorem 0.2 of [BG], we obtain an asymptotic expansion of the first integral. This expansion has the properties claimed by the proposition. To deal with the second integral we may replace Q_2 on $[-1,0] \times Y$ by an appropriate parametrix, say $(4\pi t)^{-1/2}\exp(-(u-v)^2/4t)\exp -tA^2$. Hence, up to an exponentially small term, the second integral equals

$$\frac{1}{\sqrt{4\pi t}} \operatorname{Tr}\big(\gamma e^{-tA^2}\big).$$

Let $S|Y = S_+ \oplus S_-$ be the splitting into the $\pm i$–eigenspaces of γ and $A_\pm$ the restriction of A to $C^\infty(S_\pm)$. Then

$$\operatorname{Tr}\big(\gamma e^{-tA^2}\big) = i\Big\{\operatorname{Tr}\big(e^{-tA_-A_+}\big) - \operatorname{Tr}\big(e^{-tA_+A_-}\big)\Big\} = i\operatorname{Ind} A_+.$$

this proves (a). If D is a compatible Dirac type operator, then $\operatorname{Ind} A_+ = 0$ by Theorem 3 of [Pa, Ch. XVII]. Moreover, by Theorem 3.4 of [BG], the coefficients b_j in the asymptotic expansion of the first integral of (3.13) vanish if either $j \le n$ or $j = 2k$, $k \in \mathbf{N}$. Q.E.D.

4. The Spectral Decomposition

In this section we summarize some results about the spectral decomposition of the self-adjoint operators $\mathcal{D}$ introduced in the previous section.

Theorem 4.1. *The point spectrum of $\mathcal{D}$ consists of a sequence $\cdots \le \lambda_j \le \lambda_{j+1} \le \cdots$ of eigenvalues of finite multiplicity with $\pm\infty$ as the only possible points of accumulation. There exists $C > 0$ such that*

$$\#\{\lambda_j \mid |\lambda_j| \le \lambda\} \le C(1 + \lambda^{2n}), \quad \lambda \ge 0.$$

Proof. It is sufficient to prove that the spectrum of $\mathcal{D}^2$ consists of eigenvalues $0 \le \tilde{\lambda}_1 \le \tilde{\lambda}_2 \le \cdots$ of finite multiplicity and

$$\#\{\tilde{\lambda}_j \mid \tilde{\lambda}_j \le \lambda\} \le C(1 + \lambda^n), \quad \lambda \ge 0,$$

for some constant $C > 0$. If $\mathcal{D}^2$ is the Laplacian of Z acting on functions, then this has been proved by Donnelly [Do]. His method extends without difficulties to the present case. Q.E.D.

Let $L^2_d(Z, S)$ be the subspace of $L^2(Z, S)$ spanned by all eigensections of $\mathcal{D}$. This is also the discrete subspace for $\mathcal{D}^2$. Let $\mathcal{D}_d$ denote the restriction of $\mathcal{D}$ to $L^2_d(Z, S)$.

Corollary 4.2. *For $t > 0$, $\exp -t\mathcal{D}^2_d$ is of the trace class and we have*

$$\operatorname{Tr}(\exp -t\mathcal{D}^2_d) = \sum_j e^{-\lambda_j t}.$$

The proof can be derived from Theorem 4.1 by standard arguments.

Next we study the behaviour of the eigensections of $\mathcal{D}$ at infinity. Let ϕ_j, $j \in \mathbf{N}$, be an orthonormal basis of $\operatorname{Ran}(\Pi^\sigma_+)$ consisting of eigensections of A with eigenvalues $0 \le \mu_1 \le \mu_2 \le \cdots$. Then $\gamma\phi_j$, $j \in \mathbf{N}$, is an orthonormal basis for $\operatorname{Ran}(\Pi^\sigma_-)$ with eigenvalues $-\mu_j$. Since $L^2(Z, S)$ is the direct sum of $\operatorname{Ran}(\Pi^\sigma_+)$ and $\operatorname{Ran}(\Pi^\sigma_-)$, we get in this way an orthonormal basis for $L^2(Z, S)$. Put

$$\psi^\pm_j = \frac{1}{\sqrt{2}}(\phi_j \pm \gamma\phi_j), \quad j \in \mathbf{N}. \tag{4.3}$$

Then ψ^+_j and ψ^-_j are eigensections of γA with eigenvalues μ_j and $-\mu_j$, respectively. Moreover, we have

$$\psi^-_j = -\gamma\psi^+_j \tag{4.4}$$

and $\{\psi_j^+, \psi_j^-\}$ is an orthonormal basis of eigensections of γA. Suppose that $\varphi \in L^2(Z,S)$ satisfies $D\varphi = \lambda\varphi$, $\lambda \in \mathbf{R}$. Then, on $\mathbf{R}^+ \times Y$, we may expand φ in terms of the basis just constructed:

$$\varphi(u,y) = \sum_{j=1}^{\infty} \{f_j(u)\,\psi_j^+(y) \;+\; g_j(u)\,\psi_j^-(y)\}.$$

The coefficients f_j, g_j satisfy

$$\begin{pmatrix} \mu_j & \partial/\partial u \\ -\partial/\partial u & -\mu_j \end{pmatrix} \begin{pmatrix} f_j \\ g_j \end{pmatrix} = \lambda \begin{pmatrix} f_j \\ g_j \end{pmatrix}.$$

Using the square integrability of φ, we obtain

$$\varphi(u,y) = \sum_{\mu_j > |\lambda|} a_j \left\{ e^{-\sqrt{\mu_j^2 - \lambda^2}\, u}\, \psi_j^+(y) \;+\; \frac{\mu_j - \lambda}{\sqrt{\mu_j^2 - \lambda^2}}\, e^{-\sqrt{\mu_j^2 - \lambda^2}\, u}\, \psi_j^-(y) \right\}. \tag{4.5}$$

In particular, if $\lambda = 0$, then (4.5) can be written as

$$\varphi(u,y) = \sum_{\mu_j > 0} a_j\, e^{-\mu_j u}\, \phi_j(y). \tag{4.6}$$

Let $\mu_{j_0} > 0$ be the smallest positive eigenvalue of A such that $\mu_{j_0} > |\lambda|$. Then (4.5) implies

$$\| \varphi(u,y) \| \leq C\, e^{-\sqrt{\mu_{j_0}^2 - \lambda^2}\, u/2}, \quad u \geq 0,$$

for some constant $C > 0$. Thus we have proved

Proposition 4.7. *Let $\varphi \in L^2(Z,S)$ be an eigensection of $\mathcal{D}$. There exist $C, c > 0$ such that, on $\mathbf{R}^+ \times Y$, we have*

$$\| \varphi(u,y) \| \leq C\, e^{-cu}.$$

We turn now to the study of the continuous spectrum of $\mathcal{D}$. First we note that the operator $\mathcal{D}_0$ defined in section 3 has no point spectrum. Indeed, suppose that $\varphi \in C^\infty(\mathbf{R}^+ \times Y, S)$ satisfies $D_0\varphi = 0$ and $\Pi_+^\sigma(\varphi(0,\cdot)) = 0$. Then φ has an expansion of the form

$$\varphi(u,y) = \sum_{\mu_j \geq 0} c_j\, e^{\mu_j u}\, \gamma\phi_j(y)$$

so that φ can not be square integrable unless $\varphi = 0$. Thus $\mathcal{D}_0$ has pure absolutely continuous spectrum.

Let J be the canonical inclusion of $L^2(\mathbf{R}^+ \times Y, S)$ into $L^2(Z,S)$. Consider the wave operators

$$W_\pm(\mathcal{D}, \mathcal{D}_0) = s - \lim_{t \to \pm\infty} e^{it\mathcal{D}}\, J\, e^{-it\mathcal{D}_0}. \tag{4.8}$$

Theorem 3.7 together with the Kato–Rosenblum theory [K1] and the Birman–Kato invariance principle of the wave operators [K2] imply

Proposition 4.9. *The wave operators $W_{\pm}(\mathcal{D}, \mathcal{D}_0)$ exist and are complete.*

Thus $W_{\pm}(\mathcal{D}, \mathcal{D}_0)$ establishes a unitary equivalence of $\mathcal{D}_0$ and the absolutely continuous part $\mathcal{D}_{ac}$ of $\mathcal{D}$.

Another method to establish the existence and completeness of the wave operators is based on the method of Enß (cf. [Gu]). As a byproduct one obtains that the singularly continuous spectrum of $\mathcal{D}$ is empty. Thus we have

Theorem 4.10. (a) *$\mathcal{D}$ has no singularly continuous spectrum.*
(b) *The absolutely continuous part $\mathcal{D}_{ac}$ of $\mathcal{D}$ is unitarily equivalent to $\mathcal{D}_0$.*

The wave operators can be described more explicitly in terms of generalized eigensections (cf. [Gu]). Let ω be the set of all non-negative eigenvalues of A. Let $\mu \in \omega$. If $\mu > 0$, let $\mathcal{E}(\mu)$ denote the μ–eigenspace. If $\mu = 0$, put $\mathcal{E}(\mu) = \operatorname{Ker}(\sigma - 1)$. Let Σ^s be the Riemann surface associated to the functions $\sqrt{\lambda \pm \mu}$, $\mu \in \omega$, such that $\sqrt{\lambda \pm \mu}$ has positive imaginary part for μ sufficiently large. Thus Σ^s is a ramified double covering $\pi^s : \Sigma^s \to \mathbf{C}$ with ramification locus $\{\pm\mu \mid \mu \in \omega\}$. To each $\mu \in \omega$ and $\phi \in \mathcal{E}(\mu)$ there is associated a smooth section $E(\phi, \Lambda)$ of S which is a meromorphic function of $\Lambda \in \Sigma^s$ and satisfies

$$D\, E(\phi, \Lambda) = \pi^s(\Lambda) E(\phi, \Lambda), \quad \Lambda \in \Sigma^s.$$

(cf. [Gu] for details). The half–plane $\operatorname{Im}(\lambda) > 0$ can be identified with an open subset FP^s of Σ^s, the physical sheet. Each section $E(\phi, \Lambda)$ is regular on $\partial FP^s \cong \mathbf{R}$. In particular, $E(\phi, \lambda)$ is regular for $\lambda \in (-\infty, -\mu] \cup [\mu, \infty)$. This is the generalized eigensection attached to ϕ. If ϕ_j, $j \in \mathbf{N}$, is the basis of $\operatorname{Ran}(\Pi_+^\sigma)$ chosen above, then the $E(\phi_j, \lambda)$ form a complete system of generalized eigensections of $\mathcal{D}$. More precisely, this statement means the following. Let $\varphi \in C_0^\infty(Z, S)$. Put

$$\hat{\varphi}_j(\lambda) = \int_Z E(\phi_j, \lambda, z)\, \overline{\varphi(z)}\, dz, \quad j \in \mathbf{N}.$$

For $\mu \in \omega$ define the measure $d\tau_\mu$ by

$$d\tau_\mu(\lambda) = \frac{\sqrt{\lambda^2 - \mu^2}}{2\pi\lambda}\, d\lambda.$$

Then, for any $m \in \mathbf{N}$, the function $\lambda \mapsto (1 + \lambda^2)^m \hat{\varphi}_j(\lambda)$ belongs to $L^2([\mu_j, \infty); d\tau_{\mu_j})$ as well as to $L^2((-\infty, -\mu_j]; d\tau_{\mu_j})$ and the orthogonal projection φ_{ac} of φ onto the absolutely continuous subspace $L^2_{ac}(Z, S)$ of $\mathcal{D}$ has the following expansion

$$(4.11)\quad \varphi_{ac}(z) = \sum_{j=1}^{\infty} \left\{ \int_{\mu_j}^{\infty} E(\phi_j, \lambda, z)\, \hat{\varphi}_j(\lambda)\, d\tau_{\mu_j}(\lambda) + \int_{\mu_j}^{\infty} E(\phi_j, -\lambda, z)\, \hat{\varphi}_j(-\lambda)\, d\tau_{\mu_j}(\lambda) \right\}.$$

We shall now consider more closely the generalized eigensections $E(\phi, \lambda)$ attached to $\phi \in \operatorname{Ker}(\sigma - 1)$. Let $\psi \in \operatorname{Ker} A$ and define $h(\psi, \lambda) \in C^\infty(\mathbf{R}^+ \times Y, S)$ by

$$h(\psi, \lambda, (u, y)) = e^{-i\lambda u}\, \psi(y), \quad \lambda \in \mathbf{C}.$$

Let $\chi \in C^\infty(\mathbf{R})$ such that $\chi(u) = 0$ for $u \leq 1$ and $\chi(u) = 1$ for $u \geq 2$. We regard χ as a function on $\mathbf{R}^+ \times Y$ in the obvious way and then extend it by zero to a smooth function on Z. Observe that $(D^2 - \lambda^2)(\chi h(\psi, \lambda))$ is a smooth section with compact support. In particular, it is contained in $L^2(Z, S)$. Put

$$(4.12) \qquad F(\psi, \lambda) = \chi\, e^{-i\lambda u}\psi \; - \; (\mathcal{D}^2 - \lambda^2)^{-1}\big((D^2 - \lambda^2)(\chi h(\psi, \lambda))\big), \quad \operatorname{Im}(\lambda) > 0.$$

Then $F(\psi, \lambda)$ belongs to $C^\infty(Z, S)$ and satisfies

$$D^2 F(\psi, \lambda) = \lambda^2 F(\psi, \lambda), \quad \operatorname{Im}(\lambda) > 0.$$

The function $\lambda \mapsto F(\psi, \lambda)$ admits also a meromorphic continuation to Σ^s [Gu]. Let $\mu_1 > 0$ be the smallest positive eigenvalue of A and put

$$(4.13) \qquad \Sigma_1 = \mathbf{C} - \big\{(-\infty, -\mu_1] \cup [\mu_1, \infty)\big\}.$$

Then, in particular, $F(\psi, \lambda)$ is a meromorphic function of $\lambda \in \Sigma_1$. We explain this in more detail. Let $H^1(Z, S)$ denote the 1^{st} Sobolev space. Let $\psi_1, ..., \psi_{2r}$ be an orthonormal basis for $\operatorname{Ker} A$. For any $b \geq 0$ we introduce a closed subspace of $H^1(Z, S)$ by

$$(4.14) \quad H^1_b(Z, S) = \big\{\varphi \in H^1(Z, S) \mid \langle \varphi(u, \cdot), \psi_j \rangle = 0 \quad \text{for} \quad u \geq b \quad \text{and} \quad j = 1, ..., 2r\big\}.$$

Consider the quadratic form

$$(4.15) \qquad q(\varphi) = \| D\varphi \|^2, \quad \varphi \in H^1_b(Z, S).$$

Let $\mathcal{H}_b$ be the closure of $H^1_b(Z, S)$ in $L^2(Z, S)$. Then the quadratic form (4.15) is represented by a positive self–adjoint operator H_b in $\mathcal{H}_b$. This operator is analogous to the pseudo–Laplacian used by Colin de Verdiere [Co]. Similarly to Theorem 1 of [Co], the domain of H_b can be described as follows. For j, $1 \leq j \leq 2r$, we define the distribution T^j_b by

$$T^j_b(\varphi) = \langle \tilde{\psi}(b, \cdot), \psi_j \rangle, \quad \psi \in C^\infty_0(Z, S)$$

where $\tilde{\psi}$ denotes the restriction of ψ to $\mathbf{R}^+ \times Y$. Then $\phi \in H^1_b(Z, S)$ belongs to the domain of H_b iff there exist $C_1, ..., C_{2r} \in \mathbf{C}$ such that $D^2\varphi - \sum_j C_j T^j_b$ belongs to $L^2(Z, S)$. Here $D^2\varphi$ is taken in the sense of distributions. If φ is in the domain of H_b, then $H_b\varphi = D^2\varphi - \sum_j C_j T^j_b$.

Lemma 4.16. *The essential spectrum of H_b equals $[\mu_1^2, \infty)$ where $\mu_1 > 0$ is the smallest positive eigenvalue of A.*

Proof. We introduce Dirichlet boundary conditions on $\{b\} \times Y$. This gives rise to a self–adjoint operator $H_{b,0}$. Since Y is compact, it follows that $\exp -tH_b \; - \; \exp -tH_{b,0}$

is of the trace class for $t > 0$. Hence, H_b and $H_{b,0}$ have the same essential spectrum. By definition, we have $H_{b,0} = H_{b,int} \oplus H_{b,\infty}$ where $H_{b,int}$ acts in $L^2(M_b, S)$ and $H_{b,\infty}$ in $L^2(\mathbf{R}^+ \times Y, S)$. The operator $H_{b,int}$ is obtained from D^2, acting in $C^\infty(M_b, S)$, by imposing Dirichlet boundary conditions. Therefore, $H_{b,int}$ has pure point spectrum. The operator $H_{b,\infty}$ can be analyzed by applying separation of variables. This shows that the essential spectrum of $H_{b,\infty}$ equals $[\mu_1^2, \infty)$. Q.E.D.

In particular, H_b has pure point spectrum in $[0, \mu_1^2)$. Therefore, $(H_b - \lambda^2)^{-1}$ is a meromorphic function of $\lambda \in \Sigma_1$. Now we may proceed in the same way as in the proof of Theorem 4 in [Co]. Fix $b \geq 2$ and put

$$\tilde{G}(\psi, \lambda) = \chi\, e^{-i\lambda u}\psi \; - \; (H_b - \lambda^2)^{-1}\big((D^2 - \lambda^2)(\chi\, h(\psi, \lambda))\big), \quad \mathrm{Im}(\lambda) > 0.$$

This is a meromorphic function of $\lambda \in \Sigma_1$. On $\mathbf{R}^+ \times Y$, it has the form $\tilde{G}_0 + \tilde{G}_1$ where $\tilde{G}_1$ is smooth and square integrable and

$$\tilde{G}_0(\psi, \lambda) = \begin{cases} e^{-i\lambda u}\psi, & u \geq b; \\ e^{-i\lambda u}\, C_1(\lambda)\psi \; + \; e^{i\lambda u}\, C_2(\lambda)\psi, & u \leq b. \end{cases}$$

Here $C_1(\lambda), C_2(\lambda) : \mathrm{Ker}\, A \to \mathrm{Ker}\, A$ are linear operators which depend meromorphically on $\lambda \in \Sigma_1$. Let f_b denote the characteristic function of $[b, \infty) \times Y$. Put

$$G(\psi, \lambda) = \tilde{G}(\psi, \lambda) \; + \; f_b(e^{-i\lambda u}\, C_1(\lambda)\psi \; + \; e^{i\lambda u}\, C_2(\lambda)\psi \; - \; e^{-i\lambda u}\psi).$$

Then G is in $C^\infty(Z, S)$ and satisfies $D^2 G = \lambda^2 G$. Moreover, it is easy to see that $C_1(\lambda)$ is invertible and

$$F(\psi, \lambda) = G(C_1(\lambda)^{-1}\psi, \lambda). \tag{4.17}$$

The right hand side provides the meromorphic continuation of $F(\psi, \lambda)$ to Σ_1. Put

$$C(\lambda) = C_2(\lambda) \circ C_1(\lambda)^{-1}, \quad \lambda \in \Sigma_1. \tag{4.18}$$

This is a linear operator in $\mathrm{Ker}\, A$ which is a meromorphic function of $\lambda \in \Sigma_1$. For $\mu \in \omega$, $\mu > 0$, there exist also linear operators

$$T_\mu(\lambda) : \mathrm{Ker}\, A \to \mathcal{E}(\mu) \oplus \mathcal{E}(-\mu) \tag{4.19}$$

which depend meromorphically on $\lambda \in \Sigma_1$ such that, on $\mathbf{R}^+ \times Y$, we have

$$F(\psi, \lambda) = e^{-i\lambda u}\, \psi \; + \; e^{i\lambda u}\, C(\lambda)\psi \; + \; \sum_{\mu > 0} e^{-\sqrt{\mu_2 - \lambda^2}\, u}\, T_\mu(\lambda)\psi, \quad \lambda \in \Sigma_1. \tag{4.20}$$

For $\lambda \in \mathbf{R}$, the operator $C(\lambda)$ is regular and unitary. It equals the "scattering matrix" for $|\lambda| < \mu_1$. Furthermore, the following functional equations hold

$$C(\lambda)\, C(-\lambda) = \mathrm{Id}, \quad \lambda \in \Sigma_1 \tag{4.21}$$

(4.22) $$F(C(\lambda)\psi, -\lambda) = F(\psi, \lambda), \quad \psi \in \operatorname{Ker} A.$$

There are also functional equations for the T_μ (cf. [Gu]).

Let $\phi \in \operatorname{Ker}(\sigma - 1)$. Put

(4.23) $$E(\phi, \lambda) = F(\phi, \lambda) + \frac{1}{\lambda} DF(\phi, \lambda) = F(\phi - i\gamma\phi, \lambda), \quad \lambda \in \Sigma_1.$$

Then $E(\phi, \lambda)$ satisfies

$$DE(\phi, \lambda) = \lambda E(\phi, \lambda).$$

This is the generalized eigensection of $\mathcal{D}$ attached to ϕ. If we apply (4.20) to $F(\phi - i\gamma\phi, \lambda)$, it follows that, on $R^+ \times Y$, we have

(4.24) $$E(\phi, \lambda) = e^{-i\lambda u}(\phi - i\gamma\phi) + e^{i\lambda u} C(\lambda)(\phi - i\gamma\phi) + \theta(\phi, \lambda), \quad \lambda \in \Sigma_1,$$

where θ is square integrable and $\theta(\phi, \lambda, (u, \cdot))$ orthogonal to $\operatorname{Ker} A$. If we compare (4.24) with the expansion of $F(\phi, \lambda) + \lambda^{-1} DF(\phi, \lambda)$, we obtain

(4.25) $$C(\lambda)\gamma = -\gamma C(\lambda), \quad \lambda \in \Sigma_1.$$

Together with the functional equation (4.21) we get

Proposition 4.26. *The operator $C(0) : \operatorname{Ker} A \to \operatorname{Ker} A$ is unitary and satisfies*

$$C(0)^2 = \mathrm{Id} \quad \text{and} \quad C(0)\gamma = -\gamma C(0).$$

Thus there exists always a distinguished unitary involution σ of $\operatorname{Ker} A$ – the on-shell scattering matrix $C(0)$ – which anticommutes with γ. This involution is determined by the operator D.

We remark that the on–shell scattering matrix $C(0)$ is closely related to the so–called limiting values of extended L^2–sections φ of S satisfying $D\varphi = 0$ (cf. [APS1,p.58]). Let $L_\pm$ denote the ± 1–eigenspaces of $C(0)$. It follows from Proposition 4.26 that γ switches L_+ and L_-. Thus $L_\pm \oplus \gamma L_\pm = \operatorname{Ker} A$ is an orthogonal splitting of $\operatorname{Ker} A$. By the prescription $\Phi(\psi_1, \psi_2) = \langle \gamma\psi_1, \psi_2 \rangle$, $\psi_1, \psi_2 \in \operatorname{Ker} A$, we get a canonical symplectic structure on $\operatorname{Ker} A$. Then an equivalent statement is that L_+ and L_- are Lagrangian subspaces of $\operatorname{Ker} A$. Let $\phi \in L_+$. It follows from (4.24) that, on $\mathbf{R}^+ \times Y$, we have $E(\phi, 0) = 2\phi + \theta$ where θ is square integrable. Put $\varphi = \frac{1}{2} E(\phi, 0)$. Then $\phi \neq 0$ and it satisfies $D\varphi = 0$. If we use the notation of [APS1,p.58], this means that ϕ is the limiting value of the extended solution φ of $D\varphi = 0$. Using Lemma 8.5, it follows that every limiting value arises in this way, that is, L_+ is precisely the space of all limiting values of L^2–extended sections φ of S satisfying $D\varphi = 0$.

Finally, we recall a special case of the Maaß–Selberg relations. We define the constant term $E_0(\phi, \lambda) \in C^\infty(\mathbf{R}^+ \times Y, S)$ by

(4.27) $$E_0(\phi, \lambda) = e^{-i\lambda u}(\phi - i\gamma\phi) + e^{i\lambda u} C(\lambda)(\phi - i\gamma\phi).$$

For $a \geq 0$ let χ_a denote the characteristic function of $[a, \infty) \times Y \subset Z$. Set

$$\tilde{E}_a(\phi, \lambda) = E(\phi, \lambda) - \chi_a E_0(\phi, \lambda), \quad \lambda \in \Sigma_1. \tag{4.28}$$

By (4.24), $\tilde{E}_a(\phi, \lambda)$ is square integrable . Its norm can be computed as follows. Pick $\lambda' \in \Sigma_1$ such that $\overline{\lambda'} \neq \lambda$. Then

$$\langle \tilde{E}_a(\phi, \lambda), \tilde{E}_a(\phi, \lambda') \rangle = \frac{1}{\lambda - \overline{\lambda'}} \left\{ \langle D\tilde{E}_a(\phi, \lambda), \tilde{E}_a(\phi, \lambda') \rangle - \langle \tilde{E}_a(\phi, \lambda), D\tilde{E}_a(\phi, \lambda') \rangle \right\}.$$

Now apply Green's formula together with (4.20) and take the limit $\lambda' \to \lambda$. this gives

$$\| \tilde{E}_a(\phi, \lambda) \|^2 = 4a \| \phi \|^2 - i\langle C(-\lambda) C'(\lambda)(\phi - i\gamma\phi), \phi - i\gamma\phi \rangle, \quad \lambda \in (-\mu_1, \mu_1), \tag{4.29}$$

where $C'(z) = (d/dz)C(z)$. This is a special case of the Maaß–Selberg relations.

5. The Large Time Asymptotic

In this section we shall study the behaviour of $\int_Z \operatorname{tr} E(x,x,t)\,dx$ as $t \to \infty$. The main difficulty arises from the continuous spectrum of $\mathcal{D}$; in particular, if the continuous spectrum has no gap at zero. By Theorem 4.10, this case occurs iff $\operatorname{Ker} A \neq \{0\}$.

We start with some auxiliary result. Let

$$\mathcal{G} = \Big\{ f : \mathbf{R} \to \mathbf{R} \mid f \in L^1 \quad \text{and} \quad \int_{-\infty}^{\infty} |\hat{f}(\lambda)|\,(1+|\lambda|)\,d\lambda < \infty \Big\}.$$

We denote the trace norm of a trace class operator T in some Hilbert space by $\| T \|_1$.

Lemma 5.1. *Let T_1, T_2 be self-adjoint operators in a Hilbert space. Suppose that $T_1 - T_2$ is trace class. Then, for every $f \in \mathcal{G}$, $f(T_1) - f(T_2)$ is trace class and*

$$\| f(T_1) - f(T_2) \|_1 \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |\lambda \hat{f}(\lambda)|\,d\lambda \ \| T_1 - T_2 \|_1 .$$

For the proof see [Th, p.161]. Note that $C_0^\infty(\mathbf{R}) \subset \mathcal{G}$.

Proposition 5.2. *Let $\phi \in C_0^\infty(\mathbf{R})$. Then $\phi(\mathcal{D}) - \phi(\mathcal{D}_0)$ is of the trace class.*

Proof. Let $\alpha \in C_0^\infty(\mathbf{R})$. Then, by Theorem 3.7 and Lemma 5.1,

$$\alpha\big(\mathcal{D}e^{-t\mathcal{D}^2}\big) \ - \ \alpha\big(\mathcal{D}_0 e^{-t\mathcal{D}_0^2}\big) \ \text{ is of the trace class for } \ t > 0. \tag{5.3}$$

Given $\phi \in C_0^\infty(\mathbf{R})$, choose $t > 0$ such that $\operatorname{supp}\phi$ is contained in $(-1/\sqrt{2t}, 1/\sqrt{2t})$. The map $f(\lambda) = \lambda \exp -t\lambda^2$ is a diffeomorphism of the interval $(-1/\sqrt{2t}, 1/\sqrt{2t})$ onto the interval $(-e^{-1/2}/\sqrt{2t}, e^{-1/2}/\sqrt{2t})$. Let $\alpha(u) = \phi(f^{-1}(u))$. Then $\alpha \in C_0^\infty(\mathbf{R})$ with support contained in $(-e^{-1/2}/\sqrt{2t}, e^{-1/2}/\sqrt{2t})$. Moreover $\alpha\big(\mathcal{D}e^{-t\mathcal{D}^2}\big) = \phi(\mathcal{D})$ and $\alpha\big(\mathcal{D}_0 e^{-t\mathcal{D}_0^2}\big) = \phi(\mathcal{D}_0)$. By (5.3) our result follows. Q.E.D.

Corollary 5.4. *Let $\alpha \in C^\infty(\mathbf{R})$ and suppose that $\alpha(\lambda) = 1$ for $|\lambda| \geq C$. Then*

$$\alpha(\mathcal{D})e^{-t\mathcal{D}^2} \ - \ \alpha(\mathcal{D}_0)e^{-t\mathcal{D}_0^2} \quad \textit{and} \quad \alpha(\mathcal{D})\mathcal{D}e^{-t\mathcal{D}^2} \ - \ \alpha(\mathcal{D}_0)\mathcal{D}_0 e^{-t\mathcal{D}_0^2}$$

are of the trace class for $t > 0$.

Proof. Let $\phi = 1 - \alpha$, $\phi_1(\lambda) = \phi(\lambda)e^{-t\lambda^2}$ and $\phi_2(\lambda) = \phi(\lambda)\lambda e^{-t\lambda^2}$, $t > 0$. Then $\phi_1, \phi_2 \in C_0^\infty(\mathbf{R})$ and, by Proposition 5.2, $\phi_i(\mathcal{D}) - \phi_i(\mathcal{D}_0)$, $i = 1, 2$, are trace class operators. Moreover

$$\alpha(\mathcal{D})\,e^{-t\mathcal{D}^2} \ - \ \alpha(\mathcal{D}_0)\,e^{-t\mathcal{D}_0^2} = e^{-t\mathcal{D}^2} \ - \ e^{-t\mathcal{D}_0^2} - \big(\phi_1(\mathcal{D}) - \phi_1(\mathcal{D}_0)\big)$$

which is trace class by Theorem 3.7. The second case is similar. Q.E.D.

Proposition 5.5. *Let $\alpha \in C^\infty(\mathbf{R})$. Suppose that there exist $0 < a < b$ such that $\alpha(\lambda) = 0$ for $|\lambda| \leq a$ and $\alpha(\lambda) = 1$ for $|\lambda| \geq b$. Then there exist $C, c > 0$ such that*

$$\| \alpha(\mathcal{D})e^{-t\mathcal{D}^2} - \alpha(\mathcal{D}_0)e^{-t\mathcal{D}_0^2} \|_1 \leq C\, e^{-ct} \tag{5.6}$$

and

$$\| \alpha(\mathcal{D})\mathcal{D}e^{-t\mathcal{D}^2} - \alpha(\mathcal{D}_0)\mathcal{D}_0 e^{-t\mathcal{D}_0^2} \|_1 \leq C\, e^{-ct} \tag{5.7}$$

for $t \geq 1$.

Proof. The function α can be written as $\alpha = \alpha_+ + \alpha_-$ where $\alpha_+(\lambda) = 0$ for $\lambda < a$ and $\alpha_-(\lambda) = 0$ for $\lambda > -a$, $a > 0$. Suppose that $\alpha = \alpha_+$. For $t > 0$ put

$$\phi_t(u) = \begin{cases} \alpha(\sqrt{-\log u}\,)\, u^t, & 0 \leq u \leq 1; \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Then we have

$$\phi_t(e^{-\mathcal{D}^2}) = \alpha(\mathcal{D})e^{-t\mathcal{D}^2} \quad \text{and} \quad \phi_t(e^{-\mathcal{D}_0^2}) = \alpha(\mathcal{D}_0)e^{-t\mathcal{D}_0^2}. \tag{5.8}$$

Moreover, ϕ_t is smooth on $\mathbf{R} - \{0\}$ with support contained in $(0,1)$. For $t > 3$, ϕ_t belongs to $C_0^3(\mathbf{R})$. Therefore

$$\int_{-\infty}^{\infty} |\hat{\phi}_t(\lambda)|\,(1 + |\lambda|)\, d\lambda < \infty \quad \text{for} \quad t > 3.$$

By Theorem 3.7, Lemma 5.1 and (5.8) we get

$$\begin{aligned} \| \alpha(\mathcal{D})e^{-t\mathcal{D}^2} - \alpha(\mathcal{D}_0)e^{-t\mathcal{D}_0^2} \|_1 &\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |\lambda\hat{\phi}_t(\lambda)|\, d\lambda \; \| e^{-\mathcal{D}^2} - e^{-\mathcal{D}_0^2} \|_1 \\ &\leq C \int_{-\infty}^{\infty} |\lambda\hat{\phi}_t(\lambda)|\, d\lambda \end{aligned}$$

for $t > 3$. To estimate the integral we split it as follows

$$\int_{-1}^{1} + \int_{-\infty}^{-1} + \int_{1}^{\infty}.$$

For the first integral we obtain

$$\int_{-1}^{1} |\lambda\hat{\phi}_t(\lambda)|\, d\lambda \leq \int_{-1}^{1} |\hat{\phi}_t(\lambda)|\, d\lambda \leq 2 \int_{-\infty}^{\infty} |\phi_t(u)|\, du.$$

If $\lambda \neq 0$ and $t \geq 3$, integration by parts gives

$$\hat{\phi}_t(\lambda) = -\frac{1}{(i\lambda)^3} \int_{-\infty}^{\infty} \frac{d^3}{du^3} \phi_t(u)\, e^{i\lambda u}\, du$$

which can be used to estimate the second and the third integral. Putting our estimates together, we get

$$\int_{-\infty}^{\infty} |\lambda \hat{\phi}_t(\lambda)|\, d\lambda \leq \int_{-\infty}^{\infty} |\phi_t(u)|\, du \;+\; 2 \int_{-\infty}^{\infty} \left| \frac{d^3}{du^3} \phi_t(u) \right| du.$$

By definition of α, we have $\operatorname{supp} \phi_t \subset (-\varepsilon, \varepsilon)$ for some $\varepsilon < 1$. Hence

$$\int_{-\infty}^{\infty} |\lambda \hat{\phi}_t(\lambda)|\, d\lambda \leq C\, e^{-t|\log \varepsilon|}\,.$$

If $\alpha = \alpha_-$, we set $\beta(u) = \alpha_-(-u)$ and then proceed as above. This establishes (5.6). The proof of (5.7) is analogous. For $\alpha = \alpha_+$ we put

$$\phi_t(u) = \begin{cases} \alpha(\sqrt{-\log u}\,)\, \sqrt{-\log u}\, u^t, & 0 \leq u \leq 1; \\ 0 & , \quad \text{otherwise.} \end{cases}$$

If $t \geq 4$, this function is three times continuously differentiable with support contained in $(0, 1)$, and (5.7) follows in the same way as above. Q.E.D.

If $\operatorname{Ker} A \neq \{0\}$, the continuous spectrum of $\mathcal{D}$ fills the whole real line. Our next goal is to isolate the contribution to $\int_Z \operatorname{tr} E(x, x, t)\, dx$ given by the continuous spectrum near zero.

Proposition 5.9. *Let $\mu_1 > 0$ be the smallest positive eigenvalue of A. Let $\alpha \in C_0^\infty(\mathbf{R})$ be even and suppose that $\operatorname{supp} \alpha \subset (-\mu_1, \mu_1)$. Furthermore, let $\mathcal{D}_{ac}$ denote the absolutely continuous part of $\mathcal{D}$. Then we have*

$$\operatorname{Tr}\big(\alpha(\mathcal{D}_{ac})\mathcal{D}_{ac} e^{-t\mathcal{D}_{ac}^2} \;-\; \alpha(\mathcal{D}_0)\mathcal{D}_0 e^{-t\mathcal{D}_0^2}\big) = -\frac{1}{2\pi} \int_0^{\mu_1} \alpha(\lambda)\lambda e^{-t\lambda^2} \operatorname{Tr}(\gamma\, C(-\lambda)\, C'(\lambda))\, d\lambda$$

where $C(\lambda)$ is the scattering operator (4.18), $C'(z) = (d/dz)C(z)$ and γ is defined by (1.1).

Proof. Let $E_\alpha^{ac}(x, y, t)$ be the kernel of $\alpha(\mathcal{D}_{ac})\mathcal{D}_{ac} e^{-t\mathcal{D}_{ac}^2}$ and $E_\alpha^0(x, y, t)$ the kernel of $\alpha(\mathcal{D}_0)\mathcal{D}_0 e^{-t\mathcal{D}_0^2}$. Let $\phi_1, \ldots, \phi_r$ be an orthonormal basis for $\operatorname{Ker}(\sigma - 1)$ and $E(\phi_j, \lambda)$, $j = 1, \ldots, r$, the corresponding generalized eigensections. It follows from (4.11) that the kernel E_α^{ac} has the following expansion in terms of generalized eigensections.

(5.10)

$$E_\alpha^{ac}(x, y, t) = \frac{1}{4\pi} \sum_{j=1}^{r} \left\{ \int_0^{\mu_1} \alpha(\lambda)\lambda e^{-t\lambda^2}\, E(\phi_j, \lambda, x) \otimes \overline{E(\phi_j, \lambda, y)}\, d\lambda \right.$$
$$\left. - \int_0^{\mu_1} \alpha(\lambda)\lambda e^{-t\lambda^2}\, E(\phi_j, -\lambda, x) \otimes \overline{E(\phi_j, -\lambda, y)}\, d\lambda \right\}.$$

A similar formula holds for the kernel $E^0_\alpha(x,y,t)$. Let

$$e(\phi_j,\lambda,(u,y)) = \sin(\lambda u)\,\phi_j(y) \;+\; \cos(\lambda u)\,\gamma\phi_j(y), \quad (u,y)\in \mathbf{R}^+\times Y.$$

Then

$$\begin{aligned} E^0_\alpha(x,y,t) = \frac{1}{4\pi}\sum_{j=1}^{r}\Big\{\int_0^{\mu_1} \alpha(\lambda)\lambda e^{-t\lambda^2}\, e(\phi_j,\lambda,x)\otimes\overline{e(\phi_j,\lambda,y)}\,d\lambda \\ - \int_0^{\mu_1}\alpha(\lambda)\lambda e^{-t\lambda^2}\, e(\phi_j,-\lambda,x)\otimes\overline{e(\phi_j,-\lambda,y)}\,d\lambda\Big\}. \end{aligned} \tag{5.11}$$

Let $\mathcal{H}_0 = L^2(\mathbf{R}^+)\otimes \operatorname{Ker} A \subset L^2(\mathbf{R}^+\times Y, S)$ and $\mathcal{H}_1$ the orthogonal complement of $\mathcal{H}_0$ in $L^2(Z,S)$. Let $\tilde{E}(\phi_j,\lambda) = \tilde{E}_0(\phi_j,\lambda)$ be the generalized eigensection $E(\phi_j,\lambda)$ truncated at level 0 (cf. (4.24)). Furthermore, let $\varphi\in C_0^\infty(Z,S)$ and suppose that $\varphi\perp\mathcal{H}_0$. then we have

$$\langle E(\phi_j,\lambda),\varphi\rangle = \langle \tilde{E}(\phi_j,\lambda),\varphi\rangle \quad\text{and}\quad \langle e(\phi_j,\lambda),\varphi\rangle = 0. \tag{5.12}$$

Put

$$T = \alpha(\mathcal{D}_{ac})\mathcal{D}_{ac}e^{-t\mathcal{D}_{ac}^2} \;-\; \alpha(\mathcal{D}_0)\mathcal{D}_0 e^{-t\mathcal{D}_0^2}.$$

Using (5.10) – (5.12), we obtain

$$\langle T\varphi,\varphi\rangle = \frac{1}{4\pi}\sum_{j=1}^{r}\int_0^{\mu_1}\alpha(\lambda)\lambda e^{-t\lambda^2}\Big\{|\langle\tilde{E}(\phi_j,\lambda),\varphi\rangle|^2 \;-\; |\langle\tilde{E}(\phi_j,-\lambda),\varphi\rangle|^2\Big\}\,d\lambda. \tag{5.13}$$

Observe that $\tilde{E}(\phi_j,\lambda)\in\mathcal{H}_1$. Hence, by continuity, (5.13) holds for all $\varphi\in\mathcal{H}_1$. Let φ_j, $j\in\mathbf{N}$, be an orthonormal basis for $\mathcal{H}_1$. Then (5.13) implies

$$\sum_{j=1}^{\infty}\langle T\varphi_j,\varphi_j\rangle = \frac{1}{4\pi}\sum_{j=1}^{r}\int_0^{\mu_1}\alpha(\lambda)\lambda e^{-t\lambda^2}\Big\{\|\,\tilde{E}(\phi_j,\lambda)\,\|^2 \;-\; \|\,\tilde{E}(\phi_j,-\lambda)\,\|^2\Big\}\,d\lambda. \tag{5.14}$$

Now let $\varphi\in C_0^\infty(\mathbf{R}^+)\otimes\operatorname{Ker} A$. Then we get

$$\begin{aligned}\langle T\varphi,\varphi\rangle = \frac{1}{4\pi}\sum_{j=1}^{r}\int_0^{\mu_1}\alpha(\lambda)\lambda e^{-t\lambda^2}\Big\{|\langle E_0(\phi_j,\lambda),\varphi\rangle|^2 \;-\; |\langle E_0(\phi_j,-\lambda),\varphi\rangle|^2 \\ -\;\big(|\langle e(\phi_j,\lambda),\varphi\rangle|^2 \;-\; |\langle e(\phi_j,-\lambda),\varphi\rangle|^2\big)\Big\}\,d\lambda\end{aligned}$$

where $E_0(\phi_j,\lambda)$ is the constant term of $E(\phi_j,\lambda)$ defined by (4.27). Using the unitarity of $C(\lambda)$ for λ real together with (4.25), a direct computation shows that $\langle T\varphi,\varphi\rangle = 0$. By (5.14) and (4.29) we finally get

$$\begin{aligned}\operatorname{Tr}(T) = -\frac{i}{4\pi}\int_0^{\mu_1}\alpha(\lambda)\lambda e^{-t\lambda^2}\sum_{j=1}^{r}\Big\{\langle C(-\lambda)\,C'(\lambda)(\phi_j - i\gamma\phi_j),\phi_j - i\gamma\phi_j\rangle \\ -\;\langle C(\lambda)\,C'(-\lambda)(\phi_j - i\gamma\phi_j),\phi_j - i\gamma\phi_j\rangle\Big\}\,d\lambda.\end{aligned}$$

Since $\phi_1, ..., \phi_r, \gamma\phi_1, ..., \gamma\phi_r$ is an orthonormal basis for $\operatorname{Ker} A$, the sum equals

$$\operatorname{Tr}(C(-\lambda)\, C'(\lambda)) \; - \; \operatorname{Tr}(C(\lambda)\, C'(-\lambda)) \; + \; i\operatorname{Tr}(\gamma\, C(\lambda)\, C'(-\lambda)) \; - \; i\operatorname{Tr}(\gamma\, C(-\lambda)\, C'(\lambda)).$$

By the functional equation (4.21) we have

$$C'(\lambda)\, C(-\lambda) \; - \; C(\lambda)\, C'(-\lambda) = 0. \tag{5.15}$$

Therefore the first two traces cancel. If we employ (4.25) and (5.15) to rewrite the remaining terms, we get the equality claimed by the Proposition. Q.E.D.

Corollary 5.16. *Let $\mu_1 > 0$ be the smallest positive eigenvalue of A. There exists $c > 0$ such that*

$$\int_Z \operatorname{tr} E(x,x,t)\, dx = -\frac{1}{2\pi} \int_0^{\mu_1} \lambda e^{-t\lambda^2} \operatorname{Tr}(\gamma\, C(-\lambda)\, C'(\lambda))\, d\lambda \; + \; O(e^{-ct})$$

for $t \geq 1$.

Proof. Let $\alpha \in C_0^\infty(\mathbf{R})$ be an even real valued function such that $\operatorname{supp}\alpha \subset (-\mu_1, \mu_1)$ and $\alpha(u) = 1$ for $|u| < \delta$. Put $\beta = 1 - \alpha$. Then, by Proposition 3.11, we get

$$\begin{aligned}\int_Z \operatorname{tr} E(x,x,t)\, dx = &\sum_j \alpha(\lambda_j)\lambda_j e^{-t\lambda_j^2} \; + \; \operatorname{Tr}\big(\alpha(\mathcal{D}_{ac})\mathcal{D}_{ac} e^{-t\mathcal{D}_{ac}^2} \; - \; \alpha(\mathcal{D}_0)\mathcal{D}_0 e^{-t\mathcal{D}_0^2}\big) \\ &+ \operatorname{Tr}\big(\beta(\mathcal{D})\mathcal{D} e^{-t\mathcal{D}^2} \; - \; \beta(\mathcal{D}_0)\mathcal{D}_0 e^{-t\mathcal{D}_0^2}\big).\end{aligned}$$

Note that the sum over the eigenvalues is finite. By Proposition 5.5, the second trace on the right hand side decays exponentialy as $t \to \infty$. Then we apply Proposition 5.9 to the first trace. For the asymptotic expansion we may replace α by 1. Q.E.D.

Corollary 5.17. *Suppose that* $\operatorname{Ker} A = \{0\}$. *Then there exist constants $C, c > 0$ such that*

$$\left| \int_Z \operatorname{tr} E(x,x,t)\, dx \right| \leq C\, e^{-ct}, \quad t \geq 1.$$

Observe that, by (4.25), γ commutes with $C(-\lambda)\, C'(\lambda)$. Therefore, the integral on the right hand side of the equality of Corollary 5.16 can be rewritten as

$$\frac{1}{2\pi i} \int_0^{\mu_1} \lambda e^{-t\lambda^2} \Big\{ \operatorname{Tr}\big(C(-\lambda)\, C'(\lambda) | \operatorname{Ker}(\gamma - i)\big) \; - \; \operatorname{Tr}\big(C(-\lambda)\, C'(\lambda) | \operatorname{Ker}(\gamma + i)\big) \Big\}\, d\lambda$$

Furthermore, recall that $C(\lambda)$ is real analytic for $\lambda \in (-\mu_1, \mu_1)$. Moreover, using (4.25) and (5.15), it follows that

$$\operatorname{Tr}(\gamma\, C(-\lambda)\, C'(\lambda)) = -\operatorname{Tr}(\gamma\, C(\lambda)\, C'(-\lambda)).$$

In particular, this function vanishes at $\lambda = 0$. Using this observation, we get an asymptotic expansion, as $t \to \infty$, of the form

$$\int_0^{\mu_1} \lambda e^{-t\lambda^2} \operatorname{Tr}(\gamma C(-\lambda) C'(\lambda)) \, d\lambda \sim \sum_{k=1}^{\infty} c_k \, t^{-(k+2)/2}$$

where

$$c_k = \frac{1}{2} \frac{\Gamma(k/2+1)}{k!} \frac{d^k}{d\lambda^k} \operatorname{Tr}(\gamma C(-\lambda) C'(\lambda)) \big|_{\lambda=0}.$$

Therefore, Corollary 5.16 leads to

Corollary 5.19. *As $t \to \infty$, there exists an asymptotic expansion of the form*

$$\int_Z \operatorname{tr} E(x,x,t) \, dx \sim -\frac{1}{2\pi} \sum_{k=1}^{\infty} c_k \, t^{-(k+2)/2}$$

and the coefficients c_k are given by (5.18).

Remark. In contrast to the asymptotic expansion at $t = 0$, the coefficients c_k are nonlocal. They are determined by global properties of the continuous spectrum at $\lambda = 0$.

6. Eta Ivariants for Manifolds with Cylindrical Ends

We are now ready to define the eta function of $\mathcal{D}$. Let $a > 0$. For $\mathrm{Re}(s) > n$ put

$$(6.1) \qquad \eta^a(s,\mathcal{D}) = \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^a t^{(s-1)/2} \int_Z \operatorname{tr} E(x,x,t)\, dx\, dt.$$

By Proposition 3.12, the integral is absolutely converging in the half-plane $\mathrm{Re}(s) > n$ and admits a meromorphic continuation to the whole complex plane. Similarly, for $\mathrm{Re}(s) < 2$, we put

$$(6.2) \qquad \eta_a(s,\mathcal{D}) = \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_a^\infty t^{(s-1)/2} \int_Z \operatorname{tr} E(x,x,t)\, dx\, dt.$$

By Corollary 5.19, the t –integral is absolutely converging for $\mathrm{Re}(s) < 2$ and admits also a meromorphic continuation to $\mathbf{C}$. Now observe that the meromorphic function $\eta^a(s,\mathcal{D}) + \eta_a(s,\mathcal{D})$ is independent of $a > 0$ and, therefore, we may define the eta function of $\mathcal{D}$ by

$$(6.3) \qquad \eta(s,\mathcal{D}) = \eta^a(s,\mathcal{D}) \; + \; \eta_a(s,\mathcal{D}).$$

Then $\eta(s,\mathcal{D})$ is a meromorphic function with simple poles at $s = j,\ j \in \mathbf{Z}$. The poles at $s = j,\ j \geq 2$, may not be given as the integral of a local density.

Remark. In view of Theorem 3.11 we may regard $\eta(s,\mathcal{D})$ also as a relative eta function $\eta(s;\mathcal{D},\mathcal{D}_0)$ attached to $\mathcal{D},\mathcal{D}_0$.

If $\eta(s,\mathcal{D})$ is regular at $s = 0$, we define the eta invariant of $\mathcal{D}$ to be $\eta(0,\mathcal{D})$. There are two special cases

(a) $\operatorname{Ker} A = \{0\}$. Then $\int_Z \operatorname{tr} E(x,x,t)\, dx$ decays exponentially as $t \to \infty$ and $\eta(s,\mathcal{D})$ can be defined in the half-plane $\mathrm{Re}(s) > n$ by

$$(6.4) \qquad \eta(s,\mathcal{D}) = \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^\infty t^{(s-1)/2} \int_Z \operatorname{tr} E(x,x,t)\, dx\, dt.$$

(b) Suppose that D is a compatible Dirac type operator and $\dim Z$ is odd. By Proposition 3.12, we have $\int_Z \operatorname{tr} E(x,x,t)\, dx = O(t^{1/2})$ as $t \to 0$ and $\eta(s,\mathcal{D})$ can be defined by formula (6.4) in the strip $2 > \mathrm{Re}(s) > -2$. In particular, $\eta(s,\mathcal{D})$ is regular at $s = 0$ and the eta invariant of $\mathcal{D}$ is given by

$$(6.5) \qquad \eta(0,\mathcal{D}) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} \int_Z \operatorname{tr} E(x,x,t)\, dx\, dt.$$

The case where (a) and (b) are both satisfied has been studied also by Klimek and Wojciechowski [KW]. In this paper we shall not attempt to answer the question of the regularity of $\eta(s,\mathcal{D})$ at $s=0$ in general. Next we derive a variational formula for compactly supported perturbations. Let D_v be a smooth one–parameter family of first order elliptic differential operators on Z which satisfies the same assumptions as in section 2. In particular, $D_v = \gamma(\partial/\partial u + A)$ on $\mathbf{R}^+ \times Y$. Let $\dot{D}_v = dD_v/dv$.

Lemma 6.6. *For $t>0$, the operator $\dot{D}_v e^{-t\mathcal{D}_v^2}$ is of the trace class.*

Proof. Let U_X be the operator defined in the proof of Theorem 3.7. Then we may write

$$\dot{D}_v e^{-t\mathcal{D}_v^2} = \dot{D}_v e^{-t/2\mathcal{D}_v^2} \circ U_X^{-1} \circ U_X \circ e^{-t/2\mathcal{D}_v^2}.$$

In the course of the proof of Theorem 3.7 it was shown that $U_X \circ \exp -t/2\mathcal{D}_v^2$ is a Hilbert–Schmidt operator. By assumption, $\dot{D}_v = 0$ on $\mathbf{R}^+ \times Y$. If we use (3.5) it is easy to see that $\dot{D}_v \exp -t/2\mathcal{D}_v^2 \circ U_X^{-1}$ is Hilbert–Schmidt too. Q.E.D.

Let $E_v(x,y,t)$ be the kernel of $\mathcal{D}_v \exp -t\mathcal{D}_v^2$. Using (3.10), it is easy to see that $\int_Z \operatorname{tr} E_v(x,x,t)\,dx$ is a smooth function of v. If we employ Proposition 3.11 and then proceed as in the proof of Proposition 2.1, we obtain

Lemma 6.7. *For $t>0$, we have*

$$\frac{\partial}{\partial v}\int_Z \operatorname{tr} E_v(x,x,t)\,dx = \Big(1 + 2t\frac{\partial}{\partial t}\Big)\operatorname{Tr}\big(\dot{D}_v e^{-t\mathcal{D}_v^2}\big).$$

To continue we have to determine the asymptotic behaviour of $\operatorname{Tr}\big(\dot{D}_v e^{-t\mathcal{D}_v^2}\big)$ as $t\to 0$ and $t\to\infty$. Since $\dot{D}_v = 0$ on $\mathbf{R}^+ \times Y$, the small time asymptotic is reduced to the compact case. Using (3.5) and Lemma 1.7.7 of [Gi1], we get an asymptotic expansion of the form

$$(6.8) \qquad \operatorname{Tr}\big(\dot{D}_v e^{-t\mathcal{D}_v^2}\big) \sim \sum_{j=0}^{\infty} c_j(D_v)\, t^{(j-n-1)/2}$$

as $t\to 0$.

Now we come to the large time behaviour. Let P_v be the orthogonal projection of $L^2(Z,S)$ onto $\operatorname{Ker}\mathcal{D}_v$. Since 0 may not be an isolated point of the spectrum of $\mathcal{D}_v$, the following Lemma is non–trivial.

Lemma 6.9. *Suppose that* $\dim(\operatorname{Ker}\mathcal{D}_v)$ *is constant. Then P_v depends smoothly on v.*

Proof. For $b\ge 0$, let $H_b(v)$ be the operator which represents the quadratic form (4.15) defined by D_v. By Lemma 4.16, $H_b(v)$ has pure point spectrum in $[0,\mu_1^2)$ where $\mu_1>0$ is the smallest positive eigenvalue of A. By Proposition 8.7, we have $\operatorname{Ker} H_b(v) = \operatorname{Ker}\mathcal{D}_v^2$. Moreover, it is clear that $\operatorname{Ker}\mathcal{D}_v^2 = \operatorname{Ker}\mathcal{D}_v$. Using the definition of $H_b(v)$, it is easy to see that $H_b(v)$ depends smoothly on v. Since $\dim(\operatorname{Ker} H_b(v))$ is constant and 0 is an isolated point in the spectrum of $H_b(v)$, the orthogonal projection of $\mathcal{H}_b$ onto $\operatorname{Ker} H_b(v)$ depends smoothly on v. Now observe that $\operatorname{Ker}\mathcal{D}_v$ is contained in $\mathcal{H}_b$ and the orthogonal complement of $\mathcal{H}_b$ in $L^2(Z,S)$ is independent of v. This proves our claim. Q.E.D.

Assume that $\dim(\operatorname{Ker}\mathcal{D}_v)$ is constant. Then P_v is smooth in v. Since $D_vP_v = 0$, it follows that

$$\dot{D}_vP_v = -D_v\dot{P}_v. \tag{6.10}$$

To begin with we consider the contribution of the eigenvalues first. Let $\mathcal{D}_{v,d}$ be the restriction of $\mathcal{D}_v$ to the subspace of $L^2(Z,S)$ spanned by the eigensections of $\mathcal{D}_v$. Since $\dot{P}_v$ has finite rank and $\| D_v \exp -t\mathcal{D}_{v,d}^2 \| \leq C\, e^{-ct}$, it follows from (6.10) that

$$\left|\operatorname{Tr}\left(\dot{D}_v \exp -t\mathcal{D}_{v,d}^2\right)\right| \leq C\, e^{-ct}$$

for some constants $C, c > 0$.

To estimate the contribution of the continuous spectrum, we pick $\alpha \in C_0^\infty(\mathbf{R})$ as in Proposition 5.9. Put $\beta = 1-\alpha$. Since $\beta(u) = 0$ for $|u| < \delta$, the spectral theorem implies that $\| \beta(\mathcal{D}) \exp -t\mathcal{D}^2 \| \leq e^{-t\delta}$, $t \geq 0$. Hence, for $t > 1$, we get

$$\left|\operatorname{Tr}\left(\dot{D}_v\beta(\mathcal{D}_v)e^{-t\mathcal{D}_v^2}\right)\right| \leq \| \dot{D}_v e^{-\mathcal{D}_v^2} \|_1 \cdot \| \beta(\mathcal{D}_v)e^{-(t-1)\mathcal{D}_v^2} \| \leq C\, e^{-t\delta}. \tag{6.11}$$

Let $\mathcal{D}_{ac}(v)$ denote the absolutely continuous part of $\mathcal{D}_v$. We use (4.11) to construct the kernel of $\dot{D}_v\alpha(\mathcal{D}_{ac}(v))\exp -t\mathcal{D}_{ac}(v)^2$. It is given by an expression similar to (5.10). Using this kernel, we get

$$\begin{aligned}\operatorname{Tr}\left(\dot{D}_v\alpha(\mathcal{D}_{ac}(v))e^{-t\mathcal{D}_{ac}(v)^2}\right) = \frac{1}{4\pi}\sum_{j=1}^{\infty}\int_0^{\mu_1} \alpha(\lambda)e^{-t\lambda^2}\Big\{&\langle \dot{D}_vE_v(\phi_j,\lambda), E_v(\phi_j,\lambda)\rangle \\ &+ \langle \dot{D}_vE_v(\phi_j,-\lambda), E_v(\phi_j,-\lambda)\rangle\Big\}\, d\lambda,\end{aligned}$$

where $E_v(\phi,\lambda)$ denotes the generalized eigensection of $\mathcal{D}_v$ attached to $\phi \in \operatorname{Ker}(\sigma-1)$. Since $\dim(\operatorname{Ker} H_b(v))$ is constant, it follows that $(H_b(v)-\lambda^2)^{-1}$ is smooth for $|\lambda|$ sufficiently small. Using the construction of the analytic continuation of $E_v(\phi,\lambda)$, $\lambda \in \Sigma_1$, it follows that $E_v(\phi,\lambda)$ depends smoothly on v for $|\lambda|$ sufficiently small. More precisely, for each u_0 there exists $\delta > 0$ such that, for $|\lambda| < \delta$, $E_v(\phi,\lambda)$ is a smooth function of v for $|v-v_0| < \delta$. Differentiating the equation $D_vE_v(\phi,\lambda) = \lambda E_v(\phi,\lambda)$ with respect to v, we get

$$\dot{D}_vE_v(\phi,\lambda) = -(D_v-\lambda)\frac{\partial}{\partial v}E_v(\phi,\lambda), \quad |\lambda| < \delta.$$

If we use Green's formula together with (4.20) and (4.24), we get

$$\langle \dot{D}_vE_v(\phi_j,\lambda), E_v(\phi_j,\lambda)\rangle_{M_a} = \langle \gamma\frac{\partial}{\partial v}C_v(\lambda)(\phi - i\gamma\phi), C_v(\lambda)(\phi-i\gamma\phi)\rangle + O(e^{-ca})$$

for some $c > 0$. Choose α such that $\operatorname{supp}\alpha \subset (-\delta, \delta)$. Then

$$\begin{aligned}
&\operatorname{Tr}\big(\dot{D}_v \alpha(\mathcal{D}_{ac}(v)) e^{-t\mathcal{D}_{ac}(v)^2}\big) \\
&= -\frac{1}{4\pi}\int_0^{\mu_1} \alpha(\lambda) e^{-t\lambda^2}\Big\{\operatorname{Tr}\big(\gamma\, C_v(-\lambda)\frac{\partial}{\partial v} C_v(\lambda)\big) \;+\; \operatorname{Tr}\big(\gamma\, C_v(\lambda)\frac{\partial}{\partial v} C_v(-\lambda)\big) \\
&\qquad\qquad - \; i\operatorname{Tr}\big(C_v(-\lambda)\frac{\partial}{\partial v} C_v(\lambda)\big) \;-\; i\operatorname{Tr}\big(C_v(\lambda)\frac{\partial}{\partial v} C_v(-\lambda)\big)\Big\}\, d\lambda.
\end{aligned}$$

The functional equation (4.21) implies

$$\left(\frac{\partial}{\partial v} C_v(\lambda)\right) C_v(-\lambda) \;+\; C_v(\lambda)\left(\frac{\partial}{\partial v} C_v(-\lambda)\right) = 0. \tag{6.12}$$

Therefore, the right hand side equals

$$-\frac{1}{2\pi}\int_0^{\mu_1} \alpha(\lambda) e^{-t\lambda^2} \operatorname{Tr}\big(\gamma\, C_v(-\lambda)\frac{\partial}{\partial v} C_v(\lambda)\big)\, d\lambda.$$

Since $\operatorname{Tr}\big(\gamma\, C_v(-\lambda)\frac{\partial}{\partial v} C_v(\lambda)\big)$ is an analytic function near $\lambda = 0$, we get an asymptotic expansion

$$\operatorname{Tr}\big(\dot{D}_v e^{-t\mathcal{D}_v^2}\big) \sim \sum_{j=1}^{\infty} b_j\, t^{-j/2}$$

as $t \to \infty$. The first coefficient is given by

$$b_1 = -\frac{2}{\pi}\operatorname{Tr}\big(\gamma\, C_v(0)\frac{\partial}{\partial v} C_v(0)\big).$$

Put

$$\xi_1(s, \mathcal{D}_v) = \frac{1}{\Gamma\left(\frac{s+1}{2}\right)}\int_0^1 t^{(s-1)/2} \operatorname{Tr}\big(\dot{D}_v e^{-t\mathcal{D}_v^2}\big)\, dt, \quad \operatorname{Re}(s) > n$$

and

$$\xi_2(s, \mathcal{D}_v) = \frac{1}{\Gamma\left(\frac{s+1}{2}\right)}\int_1^{\infty} t^{(s-1)/2} \operatorname{Tr}\big(\dot{D}_v e^{-t\mathcal{D}_v^2}\big)\, dt, \quad \operatorname{Re}(s) < 0.$$

Then $\xi_1(s, \mathcal{D}_v)$ and $\xi_2(s, \mathcal{D}_v)$ admit meromorphic continuations to the whole complex plane. Summarizing our results, we have proved

Proposition 6.13. *Let D_v be a smooth one-parameter family of first order differential operators on Z satisfying the assumptions above. Suppose that $\dim(\operatorname{Ker}\mathcal{D}_v)$ is constant. Then $\eta(s, \mathcal{D}_v)$ is differentiable with respect to v and*

$$\frac{\partial}{\partial v}\eta(s, \mathcal{D}_v) = -s\big(\xi_1(s, \mathcal{D}_v) \;+\; \xi_2(s, \mathcal{D}_v)\big).$$

Corollary 6.14. *Suppose that* $\dim(\operatorname{Ker}\mathcal{D}_v)$ *is constant. Then the residue of* $\eta(s,\mathcal{D}_v)$ *at* $s=0$ *is independent of* v.

Since D_v has continuous spectrum, we can not procced as in the proof of Corollary 2.12 to eliminate the condition on $\operatorname{Ker}\mathcal{D}_v$. Eigenvalues embedded into the continuous spectrum are usually unstable under perturbations. We have to understand how this is compensated by the continuous spectrum. We claim without proof that Corollary 6.14 remains true without any assumption on $\operatorname{Ker}\mathcal{D}_v$.

Corollary 6.15. *Assume that* $\dim(\operatorname{Ker}\mathcal{D}_v)$ *is constant and* $\eta(s,\mathcal{D}_v)$ *is regular at* $s=0$. *Then*

$$\frac{\partial}{\partial v}\eta(0,\mathcal{D}_v) = -\frac{2}{\sqrt{\pi}}\,c_n(D_v) \;+\; \frac{1}{2\pi}\operatorname{Tr}\big(\gamma\, C_v(0)\,\frac{\partial}{\partial v}C_v(0)\big)$$

where $c_n(D_v)$ *is the* n*–th coefficient in the asymptotic expansion (6.8).*

Using (4.25) and (6.12), we get

$$\operatorname{Tr}\big(\gamma\, C_v(0)\,\frac{\partial}{\partial v}C_v(0)\big) = 2i\operatorname{Tr}\big(C_v(0)\,\frac{\partial}{\partial v}C_v(0)|\operatorname{Ker}(\gamma-i)\big).$$

If we compare the variational formulas given by Corollary 2.9, Theorem 2.21 and Corollary 6.15, we get

Proposition 6.16. *Let* D_v *be a smooth one–parameter family of compatible Dirac type operators as above. Suppose that* $\dim(\operatorname{Ker}\mathcal{D}_v)$ *is constant. Let* $\tau_v = C_v(0)$. *Then* $\eta(0,(D_v)_{\tau_v})$ *and* $\eta(0,\mathcal{D}_v)$ *are smooth functions of* v *and*

$$\frac{\partial}{\partial v}\eta(0,(D_v)_{\tau_v}) = \frac{\partial}{\partial v}\eta(0,\mathcal{D}_v).$$

If the kernel of $\mathcal{D}_v$ is not constant, $\eta(0,\mathcal{D}_v)$ will have discontinuities which we are going to study next. Let $T>0$ be given. It follows from (3.10) and Proposition 3.12 that $\int_0^T t^{-1/2}\int_Z \operatorname{tr} E(x,x,t)\,dx\,dt$ is a smooth function of v. Now consider the integral from T to ∞. Since we vary $\mathcal{D}_v$ on a compact set, it follows that the constants occurring on the right hand side of (3.5) can be chosen to be uniform for $v\in(-\varepsilon,\varepsilon)$. This implies

$$\| e^{-\mathcal{D}_v^2} - e^{-\mathcal{D}_0^2} \|_1 \le C_1 \tag{6.17}$$

for some constant $C_1>0$ and $|v|<\varepsilon$. Let β be as in (6.11). Then (6.17) implies that

$$\int_T^\infty t^{-1/2}\operatorname{Tr}\big(\mathcal{D}_v\beta(\mathcal{D}_v)e^{-t\mathcal{D}_v^2} - \mathcal{D}_0\beta(\mathcal{D}_0)e^{-t\mathcal{D}_0^2}\big)\,dt$$

depends smoothly on v.

Next we have to consider the contribution of the continuous spectrum near zero. It follows from Proposition 5.9 that this contribution is given by

$$-\frac{1}{2\pi}\int_0^{\mu_1} \alpha(\lambda)\,\mathrm{sign}\lambda\,\mathrm{Tr}\big(\gamma\, C_v(-\lambda)\, C_v'(\lambda)\big)\, d\lambda \tag{6.18}$$

where $\mathrm{supp}\,\alpha$ is contained in $(-\mu_1, \mu_1)$.

Lemma 6.19. *There exists $\varepsilon > 0$ such that $\mathrm{Tr}\big(\gamma\, C_v(-\lambda)\, C_v'(\lambda)\big)$ is a smooth function of v for $|v| < \varepsilon$, $|\lambda| < \varepsilon$.*

Proof. Using the functional equation (4.21) it follows that the singularities of the meromorphic matrix valued function $C_v(-z)C_v'(z)$ are simple poles with residues of the form $-m\mathrm{Id}$, $m \in \mathbf{N}$. Since $\mathrm{Tr}(\gamma) = 0$, it follows that $\mathrm{Tr}\big(\gamma\, C_v(-\lambda)\, C_v'(\lambda)\big)$ is an entire function of z. Let $\Gamma \subset \mathbf{C}$ be a circle with center at the origin such that all poles$\neq 0$ of $C_0(-z)C_0'(z)$ are contained in the domain exterior to Γ. It follows from the construction of the analytic continuation of the generalized eigenfunctions that $C_v(-z)C_v'(z)$ will be a smooth function of $z \in \Gamma$ and v, $|v| < \varepsilon$, for $\varepsilon > 0$ sufficiently small. Our claim follows now from Cauchy's theorem. Q.E.D.

If we choose α with support sufficiently small, it follows from Lemma 6.19 that (6.18) is a smooth function of v for $|v| < \varepsilon$. Combining our results, we see that the only possible discontinuities of $\eta(0, \mathcal{D}_v)$ may arise from the small eigenvalues. There are two possibilities. Either eigenvalues disappear and become resonances, i.e., poles of the scattering matrix, or they remain eigenvalues but cross zero. In the former case eigenvalues must disappear in pairs of positive and negative eigenvalues. Indeed, the definition of the generalized eigenfunctions immediately implies that the scattering matrix satisfies the following relation:

$$\overline{C(\lambda)} = C(-\overline{\lambda}), \quad \lambda \in \Sigma_1.$$

Thus, poles of $C(\lambda)$ appear in pairs $\{z, -\overline{z}\}$. Hence, disappearing eigenvalues do not cause discontinuities. Next observe that (4.25) implies that $C_v(0)$ has exactly $\frac{1}{2}\dim(\mathrm{Ker}\, A)$ eigenvalues equal to 1. Hence, by Proposition 8.10, we have $\dim\,\mathrm{Ker}\big((D_v)_r\big) = \dim\,\mathrm{Ker}(\mathcal{D}_v) + \frac{1}{2}\dim\,\mathrm{Ker}(A)$. This implies

Proposition 6.20. *Let D_v be a smooth one-parameter family of compatible Dirac type operators satisfying the properties above. Then $\eta(0, (D_v)_r) - \eta(0, \mathcal{D}_v)$ is a continuous function of v.*

7. Convergence Results for Eta Invariants

Throughout this section we shall assume that D is a compatible Dirac type operator on Z satisfying the assumptions above. Then the various eta invariants are well–defined. Let σ be a unitary involution of $\operatorname{Ker} A$ as in (1.5). Our main purpose is to relate the eta invariant $\eta(0, D_\sigma)$ to the eta invariant $\eta(0, \mathcal{D})$. If $\operatorname{Ker} A = \{0\}$, this problem was studied in [DW].

For $a \geq 0$, consider the restriction of $D(a)$ of D to the compact manifold $M_a = M \cup ([0, a] \times Y)$. By Proposition 2.16, we have $\eta(0, D_\sigma) = \eta(0, D(a)_\sigma)$, $a \geq 0$. We shall now study the behaviour of $\eta(0, D(a)_\sigma)$ as $a \to \infty$. Since D is a compatible Dirac type operator, $\eta(0, D(a)_\sigma)$ is given by (1.30). Then we may write

$$\begin{aligned} \eta(0, D(a)_\sigma) =& \frac{1}{\sqrt{\pi}} \int_0^{\sqrt{a}} t^{-1/2} \operatorname{Tr}\big(D(a)_\sigma e^{-tD(a)_\sigma^2}\big)\, dt \\ &+ \frac{1}{\sqrt{\pi}} \int_{\sqrt{a}}^{\infty} t^{-1/2} \operatorname{Tr}\big(D(a)_\sigma e^{-tD(a)_\sigma^2}\big)\, dt. \end{aligned} \tag{7.1}$$

The first integral can be treated in essentially the same way as in §7 of [DW]. For our purpose we shall use a slightly different approach. Let $e_{1,\sigma}$ be the kernel (1.13) and e_2^a the restriction of the heat kernel K of $\partial/\partial t + \mathcal{D}^2$ to M_a. We change coordinates so that $M_a = M \cup ([-a, 0] \times Y)$ where the boundary of M is identified with $\{-a\} \times Y$. Let $\phi_1, \phi_2, \psi_1, \psi_2$ be the functions defined by (1.14) and put

$$\phi_i^a(u) = \phi_i(u/a) \quad \text{and} \quad \psi_i^a(u) = \psi_i(u/a), \quad i = 1, 2.$$

Again we regard these functions as functions on the cylinder $[-a, 0] \times Y$ and then extend them to M_a in the obvious way. Put

$$e_\sigma^a = \phi_1^a\, e_{1,\sigma}\, \psi_1^a \;+\; \phi_2^a\, e_2^a\, \psi_2^a. \tag{7.2}$$

This is the parametrix for the kernel K_σ^a of $\exp -tD(a)_\sigma^2$ and K_σ^a is obtained from e_σ^a by a convergent series of the form

$$K_\sigma^a = e_\sigma^a \;+\; \sum_{m=1}^{\infty} (-1)^m\, c_m^a * e_\sigma^a$$

where the notation is similar to (1.16). Using (1.13) and (3.5), it is easy to see that, for $m \in \mathbf{Z}$, there exist $C_1, C_2, C_3 > 0$ such that

$$\| \, D_x^k\big(K_\sigma^a(x, y, t) \;-\; e_\sigma^a(x, y, t)\big)\big|_{x=y} \, \| \leq C_1 \exp\big(C_2 t - C_3 \frac{a^2}{t}\big) \tag{7.3}$$

for $k \leq m$, $x \in M_a$, $t \in \mathbf{R}^+$. If we use (3.10) and follow the proof of Proposition 3.12, it is easy to see that

$$\Big| \int_{M_a} \operatorname{tr} E(x, x, t)\, dx \Big| \leq C\, t^{1/2}$$

for $0 \leq t \leq 1$ and some constant $C > 0$ independent of a. Together with (7.3) this implies that the first integral on the right hand side of (7.1) equals

$$\frac{1}{\sqrt{\pi}} \int_0^{\sqrt{a}} t^{-1/2} \int_{M_a} \operatorname{tr} E(x,x,t)\, dx\, dt \;+\; O(\exp(-C_4 a^{3/2})) \tag{7.5}$$

for some $C_4 > 0$ and $a \to \infty$.

Proposition 7.6. *We have*

$$\lim_{a\to\infty} \frac{1}{\sqrt{\pi}} \int_0^{\sqrt{a}} t^{-1/2} \int_{M_a} \operatorname{tr} E(x,x,t)\, dx\, dt = \eta(0,\mathcal{D}).$$

Proof. It follows from Corollary 5.19 that

$$\lim_{a\to\infty} \frac{1}{\sqrt{\pi}} \int_0^{\sqrt{a}} t^{-1/2} \int_{Z} \operatorname{tr} E(x,x,t)\, dx\, dt = \eta(0,\mathcal{D}).$$

Therefore, it is sufficient to prove that

$$\lim_{a\to\infty} \frac{1}{\sqrt{\pi}} \int_0^{\sqrt{a}} t^{-1/2} \left| \int_{[a,\infty)\times Y} \operatorname{tr} E\big((u,y),(u,y),t\big)\, dy\, du \right| dt = 0.$$

Let $b > 0$. Note that the support of th right hand side of (3.10) is contained in $M = M_0$. Hence, by (3.5), it follows that

$$\lim_{a\to\infty} \int_0^{b} t^{-1/2} \int_{[a,\infty)\times Y} \operatorname{tr} E(x,x,t)\, dx\, dt = 0. \tag{7.7}$$

Pick $\alpha \in C_0^\infty(\mathbf{R})$ such that $\alpha(u) = \alpha(-u)$, $0 \leq \alpha \leq 1$, $\alpha(u) = 1$ for $|u| < \mu_1/4$ and $\alpha(u) = 0$ for $|u| \geq \mu_1/2$. Set $\beta = 1 - \alpha$. Let E_α (resp. E_β) denote the kernel of $\alpha(\mathcal{D})\mathcal{D}\exp -t\mathcal{D}^2$ (resp. $\beta(\mathcal{D})\mathcal{D}\exp -t\mathcal{D}^2$). Then

$$E = E_\alpha \;+\; E_\beta.$$

Let χ_a denote the characteristic function of $[a,\infty)\times Y$ in Z. By following the proof of Proposition 3.11, one can show thaat

$$\begin{aligned}\int_{[a,\infty)\times Y} \operatorname{tr} E_\beta(x,x,t)\, dx &= \int_Z \chi_a \operatorname{tr} E_\beta(x,x,t)\, dx \\ &= \operatorname{Tr}\big(\chi_a(\beta(\mathcal{D})\mathcal{D}e^{-t\mathcal{D}^2} \;-\; \beta(\mathcal{D}_0)\mathcal{D}_0 e^{-t\mathcal{D}_0^2})\big).\end{aligned}$$

Let $1 \leq b \leq \sqrt{a}$. Then Proposition 5.5 implies

$$\int_b^{\sqrt{a}} t^{-1/2} \left| \int_{[a,\infty)\times Y} \operatorname{tr} E_\beta(x,x,t)\, dx \right| dt \leq C \int_b^{\infty} t^{-1/2}\, e^{-ct}\, dt \leq C\, \frac{e^{-cb}}{b^{3/2}}. \tag{7.9}$$

Now we turn to the kernel E_α. First, observe that

$$E_\alpha(x,y,t) = \sum_{|\lambda_j|<\mu_1/2} \alpha(\lambda_j)\lambda_j e^{-t\lambda_j^2}\,\varphi_j(x)\otimes\overline{\varphi_j(y)} \;+\; E_\alpha^{ac}(x,y,t)$$

where E_α^{ac} is the absolutely continuous part of E_α, λ_j runs over the eigenvalues of $\mathcal{D}$ and φ_j are the corresponding orthonormalized eigensections. By Proposition 4.7, the contribution of the discrete part to the integral in question can be estimated by

$$\sum_{|\lambda_j|<\mu_1/2} |\lambda_j| \int_0^\infty t^{-1/2}\,e^{-t\lambda^2}\,dt \int_{[a,\infty)\times Y} \left|\varphi_j(u,y)\right|^2 dy\,du \le C\,e^{-ac_1}.$$

The kernel E_α^{ac} is given by (5.10). If we use this formula, we obtain
(7.10)

$$\begin{aligned}&\int_Y \operatorname{tr} E_\alpha^{ac}\big((u,y),(u,y),t\big)\,dy\\ &= \frac{1}{4\pi}\sum_{j=1}^{r}\int_0^{\mu_1} \alpha(\lambda)\lambda e^{-t\lambda^2}\int_Y\Big\{\|\ E(\phi_j,\lambda,(u,y))\ \|^2 \;-\; \|\ E(\phi_j,-\lambda,(u,y))\ \|^2\Big\}\,dy\,d\lambda.\end{aligned}$$

Now we use (4.24) to compute the integral over Y. Note that $\phi_j - i\gamma\phi_j$ belongs to the $+i$–eigenspace of γ and, in view of (4.25), $C(\lambda)(\phi_j - i\gamma\phi_j)$ belongs to the $-i$–eigenspace of γ. Hence $\phi_j - i\gamma\phi_j$ is orthogonal to $C(\lambda)(\phi_j - i\gamma\phi_j)$. Moreover, recall that $C(\lambda)$ is unitary for λ real. Therefore, we get

$$\int_Y \|\ E(\phi,\lambda,(u,\lambda))\ \|^2\ dy = 4\ \|\ \phi\ \|^2 \;+\; \int_Y \|\ \theta(\phi,\lambda,(u,y))\ \|^2\ dy.$$

By (4.20), it follows that

$$\int_Y \|\ \theta(\phi,\lambda,(u,y))\ \|^2\ dy \le C\,\exp\!\big(-2\sqrt{\mu_1^2-\lambda^2}\,u\big).$$

If we apply this to (7.10), we get

$$\int_b^{\sqrt{a}} t^{-1/2}\left|\int_{[a,\infty)\times Y} \operatorname{tr} E_\alpha^{ac}(x,x,t)\,dx\right|dt \le C\,e^{-\mu_1 a}.$$

Putting our estimates together, it follows that there exist $C,c>0$ such that

$$\int_b^{\sqrt{a}} t^{-1/2}\left|\int_{[a,\infty)\times Y} \operatorname{tr} E(x,x,t)\,dx\right|dt \le C(e^{-ca}+e^{-cb})$$

for $0<b<\sqrt{a}$. Combined with (7.7) this proves our claim. Q.E.D.

It remains to study the second integral in (7.1). First note that, for $\mu > 0$, one has

$$(7.11) \qquad \int_{\sqrt{a}}^{\infty} t^{-1/2} \mu\, e^{-t\mu^2} = 2 \int_{\mu a^{1/4}}^{\infty} e^{-x^2}\, dx \leq 2\, e^{-\mu^2 \sqrt{a}}.$$

Let $\lambda_j = \lambda_j(a)$ run over the eigenvalues of $D(a)_\sigma$. Let $0 < \kappa < 1/4$. Then we may split the trace as follows

$$(7.12) \qquad \operatorname{Tr}\big(D(a)_\sigma e^{-tD(a)_\sigma^2}\big) = \sum_{|\lambda_j| \geq a^{-\kappa}} \lambda_j\, e^{-t\lambda_j^2} + \sum_{|\lambda_j| < a^{-\kappa}} \lambda_j\, e^{-t\lambda_j^2}.$$

Using (7.11), it follows that

$$(7.13) \qquad \int_{\sqrt{a}}^{\infty} t^{-1/2} \sum_{|\lambda_j| \geq a^{-\kappa}} \lambda_j\, e^{-t\lambda_j^2}\, dt \leq C\, e^{-a^{1/2-2\kappa}} \operatorname{Tr}\big(e^{-D(a)_\sigma^2}\big)$$

(cf. (7.2) in [DW]). Using Theorem 4.1 of [DW] (which holds without any restriction on A), we see that $\operatorname{Tr}(\exp -D(a)_\sigma^2)$ can be estimated by $C \operatorname{Vol}(M_a) \leq C_1\, a$ where $C_1 > 0$ is independent of a. Hence (7.13) can be estimated by $C_2\, a \exp -a^{1/2-2\kappa}$ which tends to zero as $a \to \infty$.

It remains to study the contribution made by the eigenvalues λ_j which satisfy $|\lambda_j| < a^{-\kappa}$. If $\operatorname{Ker} A = \{0\}$ it was proved in [DW], Theorem 6.1, that the non-zero spectrum of $D(a)_{\Pi_-}$ has a positive lower bound as $a \to \infty$. In this case it follows from our estimates that $\eta(0, D(a)_{\Pi_-})$ converges to $\eta(0, \mathcal{D})$ as $a \to \infty$. Combined with Proposition 2.16 we obtain

$$(7.14) \qquad \eta(0, D(a)_{\Pi_-}) = \eta(0, \mathcal{D}).$$

8. The Small Eigenvalues

Suppose that $\operatorname{Ker} A \neq 0$. The scattering matrix $C(\lambda)$ acts in this vector space and, for $\lambda = 0$, we get a unitary involution $\tau = C(0)$ of $\operatorname{Ker} A$ which anticommutes with γ (cf. Proposition 4.26). In this section we shall use τ to define the boundary conditions. Thus

$$L_{\pm} = \operatorname{Ker}(C(0) \mp \mathrm{Id}). \tag{8.1}$$

We shall employ the following notation. Let $P_{\pm}$ denote the orthogonal projection of $\operatorname{Ker} A$ onto $L_{\pm}$. Let ϕ_j, $j \in \mathbf{N}$ be an orthonormal basis for $\operatorname{Ran}(\tilde{\Pi}_+)$ consisting of eigensections of A with eigenvalues $\mu_j > 0$.

Our main purpose is to investigate the small eigenvalues of $D(a)_\tau$. More precisely, we pick $0 < \kappa < 1$ and study the eigenvalues λ of $D(a)_\tau$ which satisfy $|\lambda| < a^{-\kappa}$. We shall employ the self-adjoint operator H_b defined by the quadratic form (4.15). Recall that H_b has pure point spectrum in $[0, \mu_1^2)$. The description of the spectrum of H_b in $[0, \mu_1^2)$ is analogous to Theorem 5 in [Co]. Here we shall discuss only the kernel of H_b. For this purpose we need some preparation. If we put $\lambda = 0$ in (5.15), it follows that

$$C'(0)\, C(0) = C(0)\, C'(0) \tag{8.2}$$

and, therefore, $\operatorname{Ker} A$ admits a decomposition into common eigenspaces of $C(0)$, $C'(0)$. Given $b \in \mathbf{R}$, put

$$V_b = \{\phi \in \operatorname{Ker} A \mid C(0)\phi = -\phi,\ C'(0)\phi = 2ib\phi\}. \tag{8.3}$$

Lemma 8.4. *If $V_b \neq \{0\}$, then $b < 0$.*

Proof. Suppose that $V_b \neq \{0\}$ and $b \geq 0$. Let $\phi \in V_b$, $\phi \neq 0$. Consider the generalized eigensection $E(\gamma\phi, \lambda)$ of $\mathcal{D}$ attached to $\gamma\phi \in L_+$. Let $\tilde{E}_b(\gamma\phi, \lambda)$ be the truncated section (4.28). Employing (4.29), we get

$$\| \tilde{E}_b(\gamma\phi, 0) \|^2 = 4b \| \phi \|^2 - i\langle C(0)\, C'(0)(\gamma\phi + i\phi), \gamma\phi + i\phi\rangle = 4(b-b) \| \phi \|^2 = 0.$$

But $\tilde{E}_b(\gamma\phi, 0) \neq 0$. Q.E.D.

Lemma 8.5. *Let $\varphi \in C^\infty(Z, S)$ be a solution of $D^2\varphi = 0$ and suppose that, on $\mathbf{R}^+ \times Y$, φ takes the form $\varphi = \phi + \varphi_1$ where $\varphi_1 \in L^2$ and $\phi \in \operatorname{Ker} A$. Then ϕ satisfies $C(0)\phi = \phi$.*

Proof. Since φ_1 is square integrable and satisfies $D^2\varphi_1 = 0$, we have

$$\varphi_1 = \sum_{\mu_j > 0} c_j\, e^{-\mu_j u}\, \phi_j. \tag{8.6}$$

This implies $D\varphi = 0$ on $\mathbf{R}^+ \times Y$. If we apply Green's formula to M_a, it follows that $D\varphi = 0$ on Z. Thus $\phi \in \operatorname{Ker} A$ is the limiting value of φ in the sense of [APS1]. We may write ϕ as $\phi = \phi_+ + \phi_-$ where $C(0)\phi_\pm = \pm\phi_\pm$. Now consider the generalized eigensection $E(\gamma\phi_-, \lambda)$ of $\mathcal{D}$ attached to $\gamma\phi_- \in L_+$. Put $\psi = \frac{1}{2}E(\gamma\phi_-, 0)$. Then ψ is a smooth section of S and satisfies $D\psi = 0$. Using (4.24), it follows that, on $\mathbf{R}^+ \times Y$, $\psi = \gamma\phi_- + \theta$, $\theta \in L^2$. Moreover, θ is smooth and satisfies $\| \theta(u,y) \| \leq Ce^{-cu}$. Using Green's formula and (8.6), we get

$$0 = \langle D\varphi, \psi\rangle_{M_a} = \int_Y \langle \gamma\varphi(a,y), \psi(a,y)\rangle \, dy \; + \; \langle \varphi, D\psi\rangle_{M_a} = \| \gamma\phi_- \|^2 \; + O(e^{-ca}).$$

Hence $\phi_- = 0$. Q.E.D.

Proposition 8.7. *For $b \geq 0$, we have* $\operatorname{Ker} H_b = \operatorname{Ker} \mathcal{D}^2$.

Proof. If $\varphi \in L^2(Z, S)$ satisfies $D^2\varphi = 0$, then, on $\mathbf{R}^+ \times Y$, φ has an expansion of the form (8.6). This expansion shows that φ belongs to the domain of H_b and satisfies $H_b\varphi = 0$. To establish equality, consider $\varphi \in \operatorname{Ker} H_b$. From the description of the domain of H_b given in §4, it follows that φ is smooth in the complement of $\{b\} \times Y$ and there it satisfies $D^2\varphi = 0$. Hence, on $\mathbf{R}^+ \times Y$, φ can be written as follows

$$\varphi = \varphi_0 \; + \sum_{\mu_j > 0} e^{-\mu_j u} \, \phi_j$$

where

$$\varphi_0(u,y) = \begin{cases} 2i(u-b)\phi, & u \leq b; \\ 0 \quad\quad , & u > b; \end{cases}$$

for some $\phi \in \operatorname{Ker} A$. Let χ_b be the characteristic function of $[b, \infty) \times Y$ and set

$$\tilde{\varphi} = \varphi \; + \; \chi_b \, 2i(u-b) \, \phi.$$

Then $\tilde{\varphi} \in C^\infty(Z, S)$, $D^2\tilde{\varphi} = 0$ and, on $\mathbf{R}^+ \times Y$, we have

$$\tilde{\varphi} = 2i(u-b)\,\phi + \varphi_1 \tag{8.8}$$

where $\varphi_1 \in L^2$. We may write ϕ as $\phi = \phi_+ + \phi_-$ where $C(0)\phi_\pm = \pm\phi_\pm$. Let $F(\phi_\pm, \lambda)$ be the corresponding eigensection and put

$$\psi = \tilde{\varphi} + ibF(\phi_+, 0) + \frac{\partial}{\partial\lambda} F(\phi_-, \lambda)\big|_{\lambda=0}.$$

Then $\psi \in C^\infty(Z, S)$, $D^2\psi = 0$ and, on $\mathbf{R}^+ \times Y$, we have

$$\psi = 2iu\phi_+ \; + \; C'(0)\phi_- \; - \; 2ib\phi_- \; + \; \psi_1, \tag{8.9}$$

$\psi_1 \in L^2$. Now consider $D\psi$. By (8.9), we have $D\psi = 2i\gamma\phi_+ + D\psi_1$, $D\psi_1 \in L^2$, on $\mathbf{R}^+ \times Y$ and Lemma 8.5 implies $\gamma\phi_+ = 0$. Since $C'(0)\phi_- - 2ib\phi_-$ belongs to the -1–eigenspace of $C(0)$, Lemma 8.5 implies also that $C'(0)\phi_- = 2ib\phi_-$. Thus $\phi = \phi_-$ is contained in V_b. By Lemma 8.4, $\phi = 0$ and, therefore, $\varphi = \tilde{\varphi}$ is square integrable and satisfies $D^2\varphi = 0$. Q.E.D.

Now we can start the investigation of the small eigenvalues. First, consider the eigenvalue $\lambda = 0$. Let $\varphi \in \operatorname{Ker} D(a)_r$. On $[0,a] \times Y$, φ satifies $\gamma(\partial/\partial u + A)\varphi = 0$ and, therefore, it can be written in the form

$$\varphi = \phi + \sum_{\mu_j > 0} c_j\, e^{-\mu_j u}\, \phi_j$$

where $\phi \in L_+$. We may use this expansion to extend φ to a smooth section $\tilde{\varphi}$ on Z satisfying $D\tilde{\varphi} = 0$. Let $E(\phi, \lambda)$ be the generalized eigensection attached to ϕ. In view of (4.24), $\tilde{\varphi} - \frac{1}{2}E(\phi, 0)$ is square integrable and $D(\tilde{\varphi} - \frac{1}{2}E(\phi, 0)) = 0$, i.e., $\tilde{\varphi} - \frac{1}{2}E(\phi, 0) \in \operatorname{Ker}\mathcal{D}$. This proves

Proposition 8.10. *There is a natural isomorphism*

$$\operatorname{Ker} D(a)_r \cong \operatorname{Ker}\mathcal{D} \oplus \operatorname{Ker}(C(0) - \mathrm{Id}).$$

Now suppose that λ, $|\lambda| < \mu_1$, is an eigenvalue of $D(a)_r$ with eigensection φ. On $[0,a] \times Y$, φ has an expansion of the following form

(8.11)

$$\begin{aligned}\varphi =& e^{-i\lambda u}\psi_1 + e^{i\lambda u}\psi_2 \\ &+ \sum_{j=1}^{\infty} a_j(\lambda)\Big\{\Big(\mathrm{ch}\big(\sqrt{\mu_j^2 - \lambda^2}\,(u-a)\big) - \frac{\mu_j}{\sqrt{\mu_j^2 - \lambda^2}}\mathrm{sh}\big(\sqrt{\mu_j^2 - \lambda^2}\,(u-a)\big)\Big)\phi_j \\ &- \frac{\lambda}{\sqrt{\mu_j^2 - \lambda^2}}\mathrm{sh}\big(\sqrt{\mu_j^2 - \lambda^2}\,(u-a)\big)\gamma\phi_j\Big\}\end{aligned}$$

where $\psi_1 \in \operatorname{Ker}(\gamma - i)$, $\psi_2 \in \operatorname{Ker}(\gamma + i)$ and

$$P_-\psi_2 = -e^{-2i\lambda a}P_-\psi_1. \tag{8.12}$$

Set

$$\varphi_0 = e^{-i\lambda u}\psi_1 + e^{i\lambda u}\psi_2. \tag{8.13}$$

We call ϕ_0 the *constant term* of φ.

Proposition 8.14. *There exist $\delta > 0$, $a_0 > 0$, such that, for $a \geq a_0$, any eigensection $\varphi \neq 0$ of $D(a)_r$ with eigenvalue λ satisfying $0 < |\lambda| < \delta$ has non-vanishing constant term φ_0.*

Proof. Let φ be an eigensection of $D(a)_r$ with eigenvalue λ, $0 < |\lambda| < \mu_1/2$. Suppose that the constant term φ_0 of φ vanishes, i.e., $\psi_1 = \psi_2 = 0$ in (8.11). We assume that $\| \varphi \| = 1$. There is a constant $C > 0$, independent of a, such that

$$\sum_j \left|a_j(\lambda)\right|^2 e^{\mu_j a} \leq C$$

where $a_j(\lambda)$ are the coefficients occurring in (8.11). We extend φ to a section $\tilde{\varphi}$ of S over Z by

$$\tilde{\varphi}(x) = \begin{cases} \varphi(x) & , \quad x \in M_a; \\ \sum_j a_j e^{-\mu_j (u-a)} \phi_j, & x = (u, y) \in [a, \infty) \times Y. \end{cases}$$

Then $\tilde{\varphi}$ is continuous on Z and smooth on $Z - (\{a\} \times Y)$. Moreover, it is easy to see that $\tilde{\varphi}$ belongs to $H_b^1(Z, S)$ for every $b \geq 0$ and satisfies $\left| \| \tilde{\varphi} \| -1 \right| \leq C\, e^{-ca}$. By Proposition 8.7, any $\psi \in \operatorname{Ker} H_b$ is smooth, satisfies $D\psi = 0$ and, on $\mathbf{R}^+ \times Y$, it takes the form (8.6). In particular, ψ satisfies $\Pi_-^\sigma(\psi(u, \cdot)) = 0$ for $u \geq 0$. Using Green's formula, we get

$$\langle \varphi, \psi \rangle_{M_a} = \lambda^{-1} \langle D\varphi, \psi \rangle_{M_a} = \lambda^{-1} \langle \varphi, D\psi \rangle_{M_a} = 0.$$

Furthermore, by definition of $\tilde{\varphi}$, we get

$$\int_{[a,\infty)\times Y} \langle \tilde{\varphi}(x), \psi(x) \rangle \, dx = \sum_{j=1}^{\infty} a_j \bar{b}_j \, \frac{e^{-\mu_j a}}{2\mu_j} \leq C\, e^{-ca}$$

for some constants $C, c > 0$. Hence, $\tilde{\varphi}$ satisfies

(8.15) $$\left| \langle \tilde{\varphi}, \psi \rangle \right| \leq C \, \| \psi \| \, e^{-ca} \quad \text{for} \quad \psi \in \operatorname{Ker} H_b.$$

Now we shall apply the mini–max principle. Recall that by the second representation theorem for quadratic forms (Theorem 2.23 of [K,VI,§2.6]), the domain of $H_b^{1/2}$ equals $H_b^1(Z, S)$. Let

$$\varpi = \min_{\substack{\psi \in H_b^1(Z,S) \\ \psi \perp \operatorname{Ker} H_b}} \frac{\| H_b^{1/2} \psi \|^2}{\| \psi \|^2}.$$

It follows from Lemma 4.16 that $0 < \varpi \leq \mu_1^2$. Using again Theorem 2.23 of [K,VI,§2.6], we get

(8.16) $$\| H_b^{1/2} \tilde{\varphi} \|^2 = \| D\tilde{\varphi} \|^2 = \| D\varphi \|^2_{M_a} = \lambda^2.$$

Let π_b denote the orthogonal projection of $\mathcal{H}_b$ onto $\operatorname{Ker} H_b$. Put $\hat{\varphi} = \tilde{\varphi} - \pi_b \tilde{\varphi}$. Employing (8.15) and (8.16), we get

$$\big| \| \hat{\varphi} \|^2 - 1 \big| \leq C\, e^{-ca} \quad \text{and} \quad \big| \| H_b^{1/2} \hat{\varphi} \|^2 - \lambda^2 \big| \leq C\, e^{-ca}.$$

This impies $\varpi \leq \| H_b^{1/2} \hat{\varphi} \|^2 \,/\, \| \hat{\varphi} \|^2 \leq (1 + C\, e^{-ca})\lambda^2$ and, therefore, we can find $a_0 \geq 0$ such that $\lambda^2 \geq \varpi/2$ for $a \geq a_0$. Put $\delta = (\varpi/2)^{1/2}$. Q.E.D.

The Proposition shows that, for $a \geq a_0$, the eigensections of $D(a)_\tau$ with sufficiently small nonzero eigenvalues are determined by their constant terms. We shall now investigate the constant terms more closely. Pick $\delta > 0$ and $a_0 \geq 0$ as in Proposition 8.14. Suppose that λ with $0 < |\lambda| < \delta$ is an eigenvalue of $D(a)_\tau$, $a \geq a_0$, and φ an eigensection for λ normalized by $\| \varphi \| = 1$. Then the constant term (8.13) of φ does not vanish. We may write ψ_1 as $\psi_1 = \phi_1 - i\gamma\phi_1$ for a uniquely determined $\phi_1 \in L_+$. Put

$$G = \varphi - E(\phi_1, \lambda).$$

Then G is smooth and satisfies $DG = \lambda G$. On $[0, a] \times Y$, it has an expansion of the following form

$$\begin{aligned} G = & e^{i\lambda u}(\psi_2 - C(\lambda)\psi_1) + \sum_{\mu_j > 0} \left\{ c_j(\lambda)\, e^{\sqrt{\mu_j^2 - \lambda^2}\, u} + d_j(\lambda)\, e^{-\sqrt{\mu_j^2 - \lambda^2}\, u} \right\} \phi_j \\ & + \sum_{\mu_j > 0} \left\{ c_j(\lambda) \frac{\mu_j + \sqrt{\mu_j^2 - \lambda^2}}{\lambda} e^{\sqrt{\mu_j^2 - \lambda^2}\, u} + d_j(\lambda) \frac{\mu_j - \sqrt{\mu_j^2 - \lambda^2}}{\lambda} e^{-\sqrt{\mu_j^2 - \lambda^2}\, u} \right\} \gamma\phi_j. \end{aligned}$$

The coefficients $c_j(\lambda)$ and $d_j(\lambda)$ are determined by the expansions (8.11) and (4.20). Using (8.11), (4.20) and (4.29), it follows that these coefficients satisfy

$$\sum_j |a_j(\lambda)|^2\, e^{\mu_j a} \leq C \quad \text{and} \quad |b_j(\lambda)| \leq C$$

for some constants $C > 0$ independent of a and j. By Green's formula, we obtain

$$\begin{aligned} 0 = \langle DG, G \rangle_{M_a} - \langle G, DG \rangle_{M_a} &= \int_Y \langle \gamma G(a, y), G(a, y) \rangle \, dy \\ &= -i \, \| C(\lambda)\psi_1 - \psi_2 \|^2 + O(e^{-ca}). \end{aligned}$$

Hence, we have

$$\| C(\lambda)\psi_1 - \psi_2 \|^2 \leq e^{-ca}. \tag{8.17}$$

Let $I : L_- \to \operatorname{Ker}(\gamma - i)$ be defined by $I(\phi) = \phi - i\gamma\phi$. Put

$$S(\lambda) = P_- \circ C(\lambda) \circ I, \quad \lambda \in \Sigma_1.$$

Observe that there exists a unique $\phi \in L_-$ such that $\psi_1 = \phi - i\gamma\phi$. Then, together with (8.12), inequality (8.17) can be rewritten as

$$\| e^{2i\lambda a} S(\lambda)\phi + \phi \|^2 \leq e^{-ca}. \tag{8.18}$$

Lemma 8.19. *The operator $S(\lambda) : L_- \to L_-$ is unitary for $\lambda \in (-\mu_1, \mu_1)$.*

This is an easy consequence of the unitarity of $C(\lambda)$ for $\lambda \in (-\mu_1, \mu_1)$.

Since $S(\lambda)$ is unitary, the eigenvalues of the linear operator $e^{2i\lambda a}S(\lambda) + \mathrm{Id}$ are of the form $e^{i\theta}+1$, $\theta \in \mathbf{R}$. Let $0 \leq \zeta$ be the smallest eigenvalue of $(e^{2i\lambda a}S(\lambda)+\mathrm{Id})(e^{2i\lambda a}S(\lambda)+\mathrm{Id})^*$. Then

$$\zeta = \min_{\psi \in L_-} \frac{\| (e^{2i\lambda a}S(\lambda) + \mathrm{Id})\psi \|^2}{\| \psi \|^2}.$$

Combined with (8.18), it follows that $\zeta \leq e^{-ca}$. Hence, $e^{2i\lambda a}S(\lambda) + \mathrm{Id}$ has an eigenvalue $e^{i\theta}$ satisfying

$$|1 + \cos\theta| \leq e^{-ca}.$$

Therefore, there exists $k \in \mathbf{Z}$ such that

$$|\pi k - \theta| \leq e^{-ca}.$$

Let $m(\lambda)$ be the multiplicity of the eigenvalue λ. By Proposition 8.14, we get $m(\lambda)$ linearly independent vectors $\phi_1, ..., \phi_{m(\lambda)} \in L_-$ which satisfy (8.18). Summarizing, we get

Proposition 8.20. *Let δ, a_0 be chosen according to Proposition 8.14. Let $a \geq a_0$ and suppose that λ, $0 < |\lambda| < \delta$, is an eigenvalue of $D(a)_\tau$ of multiplicity m. Then there exist m eigenvalues $e^{i\theta_1}, ..., e^{i\theta_m}$ of $e^{2i\lambda a}S(\lambda)$ such that*

$$|e^{i\theta_j} + 1| \leq e^{-ca}, \quad j = 1, ..., m.$$

Next we shall study the zeros of $\det\bigl(e^{2i\lambda a}S(\lambda) \ + \ \mathrm{Id}\bigr)$ near $\lambda = 0$. By (8.2), $C'(0)$ preserves the eigenspace decomposition (8.1). Let $C'_-(0)$ denote the restriction of $C'(0)$ to L_-. Then $S'(0) = C'_-(0)$ and we have

$$S(\lambda) = -\mathrm{Id} \ + \ S'(0)\lambda \ + \ O(\lambda^2). \tag{8.21}$$

In view of Lemma 8.19, we can apply Rellich's Theorem [Ba, p. 142] to study $S(\lambda)$. By choosing $\delta > 0$ sufficiently small, the punctured disc $0 < |z| < \delta$ consists of simple points of $S(z)$ only. Then there exist $r = \dim L_-$ mutually distinct eigenvalues of $S(z)$:

$$\nu_j(z) = -1 + \alpha_{j1} z + \alpha_{j2} z^2 + \cdots, \quad |z| < \delta.$$

The eigenprojectors $P_j(z)$ associated to $\nu_j(z)$ are also holomorphic at $z = 0$ and $S(z)$ takes the form

$$S(z) = \sum_{j=1}^{r} \nu_j(z)\, P_j(z), \quad 0 < |z| < \delta.$$

Let $\psi_j(z)$ be the eigenvectors corresponding to $\lambda_j(z)$. We may assume that $\psi_j(z)$ is holomorphic at $z = 0$. Differentiating the equation $S(z)\psi_j(z) = \nu_j(z)\psi_j(z)$, we obtain

$$S'(0)\psi_j(0) \; + \; S(0)\psi_j'(0) = \nu_j'(0)\psi_j(0) \; + \; \nu_j(0)\psi_j'(0).$$

Since $S(0) = -\mathrm{Id}$ and $\nu_j(0) = -1$, we get

$$S'(0)\psi_j(0) = \nu_j'(0)\psi_j(0). \tag{8.22}$$

Recall that $S(\lambda)$ is unitary for $\lambda \in (-\mu_1, \mu_1)$. Therefore, it follows that there exist real analytic real valued functions $\beta_j(\lambda)$ of $\lambda \in (-\delta, \delta)$ such that

$$\nu_j(\lambda) = -e^{i\beta_j(\lambda)}, \quad \lambda \in (-\delta, \delta) \quad \text{and} \quad \beta_j(0) = 0.$$

Moreover, each $\beta_j(\lambda)$ has an expansion of the form

$$\beta_j(\lambda) = a_{j1}\lambda + a_{j2}\lambda^2 + \cdots, \quad |\lambda| < \delta. \tag{8.23}$$

By (8.22), it follows that the eigenvalues of $S'(0)$ are equal to

$$\nu_j'(0) = i a_{j1}, \quad j = 1, ..., r.$$

Fix δ_1, $0 < \delta_1 < \delta$, and let

$$m_j = \max_{\lambda \in (-\delta_1, \delta_1)} |\beta_j'(\lambda)|. \tag{8.24}$$

Then the function $f(\lambda) = 2a\lambda + \beta_j(\lambda)$ is strictly increasing for $|\lambda| < \delta_1$, $a \geq m_j$. Choose $a_0 \geq \max(m_j, \delta_1^{1/\kappa})$. For $a \geq a_0$ and $k \in \mathbf{Z}$, there exists at most one solution $\rho_k^{(j)}$ of

$$2a\lambda + \beta_j(\lambda) = 2\pi k, \quad |\lambda| < a^{-\kappa}. \tag{8.25}$$

Let $k_{j,max} = k_{j,max}(a)$ be the maximal k for which (8.25) has a solution. Then

$$|k_{j,max}| \leq \frac{1}{\pi} a^{1-\kappa} + C \leq a^{1-\kappa} \quad \text{for} \quad a \geq a_0. \tag{8.26}$$

Furthermore, if $\rho_k^{(j)}$ is a solution of (8.25) for some $k \in \mathbf{Z}$, then

$$\rho_k^{(j)} = \frac{\pi k}{a + a_{j1}/2} \; + \; O(a^{-(1+2\kappa)}). \tag{8.27}$$

Together with (8.26), we get

$$\rho_k^{(j)} = \frac{\pi k}{a} + O(a^{-(1+\kappa)}). \tag{8.28}$$

Lemma 8.29. *Let $a \geq a_0$ and $|k| < k_{j,max}(a)$. Then the solutions $\rho_k^{(j)}$ and $\rho_{-k}^{(j)}$ of (8.25) exist and satisfy*

$$|\rho_k^{(j)} + \rho_{-k}^{(j)}| \leq \frac{C}{a^{1+2\kappa}}$$

for some $C > 0$ independent of a.

This can be easily derived from (8.23) and (8.27).

Given $a \geq 0$, we introduce

$$\Omega(a) = \Big\{\rho \in \mathbf{R} - \{0\} \mid \det\big(e^{2ia\rho}S(\rho) + \mathrm{Id}\big) = 0 \quad \text{and} \quad |\rho| \leq a^{-\kappa}\Big\}. \tag{8.30}$$

For $\rho \in \Omega(a)$, let $m(\rho)$ denote the order of the zero ρ.

Theorem 8.31. *Let $0 < \kappa < 1$. There exists $a_0 \geq 0$ such that, for $a \geq a_0$, we have*

(i) *The zeros $\rho \in \Omega(a)$ are of the form $\rho = \rho_k^{(j)}$ for some j, $1 \leq j \leq r$, and $|k| \leq k_{j,max}(a)$.*

(ii) *There exist $n \in \mathbf{N}$ and $C > 0$ such that, for any two zeros $\rho_1, \rho_2 \in \Omega(a)$ satisfying $\rho_1 \not\equiv \pm\rho_2$, we have $|\rho_1 \pm \rho_2| \geq C/a^n$.*

(iii) *There exists a subset $\Omega'(a) \subset \Omega(a)$ of cardinality $\leq 2r$ with the following property: For any $\rho \in \Omega(a) - \Omega'(a)$, $\rho > 0$ (resp. $\rho < 0$), there exists a unique $\rho' \in \Omega(a)$, $\rho' < 0$ (resp. $\rho' > 0$), such that*

$$|\rho + \rho'| \leq \frac{C}{a^{1+2\kappa}}$$

and $m(\rho) = m(\rho')$. Here $C > 0$ is independent of a.

Proof. Let $\rho \in \Omega(a)$. Then there exist j, $1 \leq j \leq r$, and $k \in \mathbf{Z}$, $|k| \leq k_{j,max}(a)$, such that $\rho = \rho_k^{(j)}$. Hence, $\rho_k^{(j)}$, regarded as solution of (8.25), has multiplicity 1 and satisfies (8.27). This proves (i).

To prove (ii), consider two zeros $\rho, \rho' \in \Omega(a)$ and suppose that $\rho = \rho_k^{(j)}$, $\rho' = \rho_k^{(j')}$. If $k \neq \pm k'$, it follows from (8.28) that

$$|\rho \pm \rho'| \geq \frac{|k \pm k'|}{a} \geq \frac{1}{a} \quad \text{for} \quad a \geq a_0.$$

Assume that $k = k'$, $\beta_{j'} \not\equiv \beta_j$. If $k = k' = 0$, it follows from (8.23) that $\rho = \rho' = 0$. Hence, we may assume that $k = k' \neq 0$. Then we have $2a\rho + \beta_j(\rho) = 2a\rho' + \beta_{j'}(\rho')$. Suppose that the corresponding Taylor coefficients in (8.23) satisfy $a_{j,l} = a_{j',l}$ for $l \le m-1$ and $a_{j',m} \ne a_{j,m}$. Then we get

$$2a(\rho - \rho') + \sum_{l=1}^{\infty} a_{j,l}(\rho^l - \rho'^l) = \sum_{l=m}^{\infty} (a_{j',l} - a_{j,l})\,\rho'^l.$$

Put $c = a_{j',l} - a_{j,l}$. By assumption, $c \ne 0$. Moreover, $|\rho|, |\rho'| < a^{-\kappa}$. This implies

$$|\rho - \rho'|(2a + O(1)) = |\rho'|^m\, |c + O(a^{-\kappa})|.$$

Since $k' \ne 0$, it follows from (8.28) that $|\rho'| \ge a^{-1}$ for $a \ge a_0$. Hence, we obtain

$$|\rho - \rho'| \ge \frac{c}{4}\, a^{-(m+1)}, \quad a \ge a_1.$$

Furthermore, by (8.28), we have $|\rho + \rho'| \ge a^{-1}$. The case $k = -k'$, $\beta_{j'} \not\equiv \beta_j$, can be treated in much the same way. It remains to consider the case $k' = -k$ and $\beta j' = \beta_j$, i.e., $\rho = \rho_k^{(j)}$ and $\rho' = \rho_{-k}^{(j)}$, $k \ne 0$. Then $|\rho - \rho'| \ge a^{-1}$. If $\rho \ne -\rho'$, there exists $n \in \mathbf{N}$ such that $a_{j,2n} \ne 0$. Otherwise the function $\beta_j(\lambda)$ is odd which implies $\rho_k^{(j)} = -\rho_{-k}^{(j)}$. Let $m \in \mathbf{N}$ such that $m\kappa > 2n$. By assumption, we have

$$2a(\rho + \rho') + \beta_j(\rho) + \beta_j(\rho') = 0.$$

We rewrite this as follows

$$2a(\rho + \rho') + \sum_{l=1}^{m} a_{j,2l+1}(\rho^{2l+1} + \rho'^{2l+1}) = -\sum_{p=1}^{\infty} a_{j,2p}(\rho^{2p} + \rho'^{2p}) - \sum_{l=m+1}^{\infty} a_{j,2l+1}(\rho^{2l+1} + \rho'^{2l+1}).$$

This implies

$$|\rho + \rho'|\big(2a + O(a^{-2\kappa})\big) \ge |a_{j,2n}|(\rho^{2n} + \rho'^{2n}) + O(a^{-4n}).$$

Since $k \ne 0$, we have $|\rho|, |\rho'| \ge a^{-1}$ by (8.28). Hence

$$|\rho + \rho'| \ge |a_{j,2n}|\, \frac{1}{a^{2n+1}} + O(a^{-4n}) \ge \frac{C}{a^{2n+1}}.$$

This proves (ii). Finally, the first part of (iii) follows from (i) and Lemma 8.29. The multiplicity $m(\rho)$ of any $\rho \in \Omega(a)$ equals the number of $j's$, $1 \le j \le r$, such that ρ is a solution of (8.25). This shows immediately that $m(\rho) = m(\rho')$. Q.E.D.

We are now ready to prove our main result concerning the small eigenvalues.

Theorem 8.32. *Let $0 < \kappa < 1$ and $a > 0$. Let $\lambda_1(a) \le \lambda_2(a) \le \cdots \le \lambda_{p_*}(a)$ be the nonzero eigenvalues, counted to multiplicity, of $D(a)_\tau$ which satisfy $|\lambda_j(a)| \le a^{-\kappa}$*

and let $\rho_1(a) \leq \rho_2(a) \leq \cdots \leq \rho_{m_a}(a)$ run over the zeros$\neq 0$, counted to multiplicity, of $\det(e^{2i\lambda a}S(\lambda)+\mathrm{Id})$ satisfying $|\rho_j(a)| \leq a^{-\kappa}$. There exists $a_1 \geq 0$ and $c > 0$, independent of a, such that, for $a \geq a_1$, $p_a = m_a$ and

$$|\lambda_j(a) - \rho_j(a)| \leq e^{-ca}, \quad j = 1, \ldots, m_a.$$

Proof. Let $a \geq a_0$ and let λ, $0 < |\lambda| < a^{-\kappa}$, be an eigenvalue of $D(a)_\tau$ of multiplicity $m(\lambda)$. It follows from Proposition 8.20 that there exist $k \in \mathbf{Z}$, $1 \leq j \leq r$, such that

$$|2\lambda a + \beta_j(\lambda) - 2\pi k| \leq e^{-ca}. \tag{8.33}$$

Let ρ_j^k be the unique solution of (8.25). Then (8.33) implies

$$|\lambda - \rho_j^k| \leq e^{-c_1 a}. \tag{8.34}$$

If $m(\lambda) > 1$, there exist pairwise distinct branches $\beta_{j_1}, \ldots, \beta_{j_{m(\lambda)}}$ such that (8.33) holds with the same k. Let $a_0 > 0$ be chosen according to Theorem 8.31. Together with Theorem 8.31 we obtain

Lemma 8.35. *Let $a \geq a_0$ and let λ, $0 < |\lambda| < a^{-\kappa}$, be an eigenvalue of $D(a)_\tau$ of multiplicity $m(\lambda)$. Then there exists a unique $\rho \in \Omega(a)$ such that*

$$|\lambda - \rho| \leq e^{-ca} \quad \text{and} \quad m(\rho) \geq m(\lambda)$$

where $m(\rho)$ denotes the multiplicity of the zero ρ.

By Lemma 8.35, it remains to show that

$$\sum_{\rho \in \Omega(a)} m(\rho) = \sum_{\substack{\lambda \\ 0<|\lambda|<a^{-\kappa}}} m(\lambda)$$

where λ runs over the eigenvalues of $D(a)_\tau$.

Let $a \geq a_0$ and $\rho \in \Omega(a)$. Let $\phi \in L_-$, $\| \phi \| = 1$, such that

$$e^{2ia\rho}S(\lambda)\phi = -\phi. \tag{8.36}$$

Consider the generalized eigensection $E(\phi, \lambda)$ attached to ϕ. Using (4.20), (8.36) and the definition of $S(\lambda)$, it follows that the constant term $E_0(\phi, \rho)$ of $E(\phi, \rho)$ satisfies

$$P_-\big(E_0(\phi, \rho, (a, \cdot))\big) = 0 \quad \text{and} \quad P_+\Big(\frac{\partial}{\partial u} E_0(\phi, \rho, (u, \cdot))\big|_{u=a}\Big) = 0. \tag{8.37}$$

Let $\rho' \in \Omega(a)$, $\rho \neq \rho'$. Choose $\phi' \in L_-$, $\| \phi' \| = 1$, such that $e^{2ia\rho'} S(\rho')\phi' = -\phi'$. By Green's formula, we get

$$\begin{aligned}(8.38)\qquad &\int_{M_a} \langle E(\phi,\rho,x), E(\phi',\rho',x)\rangle\, dx \\ &= \frac{1}{\rho-\rho'} \int_{M_a} \{\langle DE(\phi,\rho,x), E(\phi',\rho',x)\rangle \;-\; \langle E(\phi,\rho,x), DE(\phi',\rho',x)\rangle\}\, dx \\ &= \frac{1}{\rho-\rho'} \int_Y \langle \gamma E(\phi,\rho,(a,\cdot)), E(\phi',\rho',(a,\cdot))\rangle\, dy.\end{aligned}$$

To compute the right hand side, we need the complete expansion of $E(\phi,\lambda)$ on $\mathbf{R}^+ \times Y$. Note that the section $\theta(\phi,\lambda)$ occurring in (4.24) is square integrable and satisfies $D\theta(\phi,\lambda) = \lambda\theta(\phi,\lambda)$. Therefore, it can be expanded in terms of the eigensections (4.3). Let $\lambda \in \Sigma_1$. Together with (4.24), we get

$$\begin{aligned}(8.39)\qquad E(\phi,\lambda) &= e^{-i\lambda u}(\phi - i\gamma\phi) \;+\; e^{i\lambda u} C(\lambda)(\phi - i\gamma\phi) \\ &\quad + \sum_{\mu_j>0} a_j(\lambda) \left\{ e^{-\sqrt{\mu_j^2-\lambda^2}\,u}\,\psi_j^+ \;+\; \frac{\mu_j-\lambda}{\sqrt{\mu_j^2-\lambda^2}}\, e^{-\sqrt{\mu_j^2-\lambda^2}\,u}\,\psi_j^- \right\}.\end{aligned}$$

Using (4.29), it is easy to see that the coefficients $a_j(\lambda)$ satisfy $\sum_j |a_j(\lambda)|^2 \leq C$ for $\lambda \in (-\mu_1/2, \mu_1/2)$ and some $C > 0$. We apply this formula to compute the right hand side of (8.38). Because of (8.37), the constant term makes no contribution and, by Theorem 8.31, (ii), we get

$$(8.40)\qquad \left|\langle E(\phi,\rho), E(\phi',\rho')\rangle_{M_a}\right| \leq C\, e^{-\mu_1 a/2}, \quad a \gg 0.$$

Using the description of $\operatorname{Ker} D(a)_\tau$ given by Proposition 8.10, one can show in the same way that

$$(8.41)\qquad \left|\langle E(\phi,\rho), \psi\rangle_{M_a}\right| \leq C\, e^{-\mu_1 a/2}, \quad a \gg 0,\; \psi \in \operatorname{Ker} D(a)_\tau.$$

Now let $\phi' \in L_-$, $\| \phi' \| = 1$, be a second solution of (8.36). Let $h > 0$ and apply the method above to compute $\langle E(\phi,\rho), E(\phi',\rho+ih)\rangle_{M_a}$. If we pass to the limit $h \to 0$, we get

$$\begin{aligned}(8.42)\qquad \langle E(\phi,\rho), E(\phi',\rho)\rangle_{M_a} &= 4a\,\langle \phi,\phi'\rangle \;-\; i\langle C(-\rho)\,C'(\rho)(\phi - i\gamma\phi), \phi' - i\gamma\phi'\rangle \\ &\quad + O(e^{-\mu_1 a/2}), \quad a \gg 0.\end{aligned}$$

The constant in the remainder term is independent of a, ρ. If $\phi = \phi'$, we get a formula for $\| E(\phi,\rho) \|_{M_a}^2$.

Lemma 8.43. *Let $\rho \in \Omega(a)$ be given and suppose that $\phi_0, \phi_1 \in L_-$ are two solutions of (8.36). If $\langle \phi_0, \phi_1\rangle = 0$, then*

$$\langle C(-\rho)\,C'(\rho)(\phi_0 - i\gamma\phi_0), \phi_1 - i\gamma\phi_1\rangle = 0.$$

Proof. First, observe that $C(\rho)(\phi_j - i\gamma\phi_j)$ belongs to the $(-i)$–eigenspace of γ. Therefore, (8.36) can be rewritten as

$$C(\rho)(\phi_j - i\gamma\phi_j) = -e^{-2i\rho a}(\phi_j + i\gamma\phi_j), \quad j = 0, 1; \tag{8.44}$$

and, we have to show that

$$\langle C'(\rho)(\phi_0 - i\gamma\phi_0), \phi_1 + i\gamma\phi_1\rangle = 0. \tag{8.45}$$

Let $\phi_u \in L_-$, $|u| < \varepsilon$, be a smooth one–parameter family of eigenvectors of $S(\rho + u)$ with eigenvalues $\mu(u)$ such that $\mu(0) = -e^{-2i\rho a}$. As above, this is equivalent to

$$C(\rho + u)(\phi_u - i\gamma\phi_u) = \mu(u)(\phi_u + i\gamma\phi_u).$$

Differentiating this equality, we get

$$C'(\rho)(\phi_0 - i\gamma\phi_0) = \mu'(0)(\phi_0 + i\gamma\phi_0) \; + \; \mu(0)(\dot\phi_0 + i\gamma\dot\phi_0) \; - \; C(\rho)(\dot\phi_0 - i\gamma\dot\phi_0).$$

Hence

$$\begin{aligned}\langle C'(\rho)(\phi_0 - i\gamma\phi_0), \phi_1 + i\gamma\phi_1\rangle &= -e^{-2i\rho a}\langle\dot\phi_0 + i\gamma\dot\phi_0, \phi_1 + i\gamma\phi_1\rangle \\ &\quad - \langle\dot\phi_0 - i\gamma\dot\phi_0, C(-\rho)(\phi_1 + i\gamma\phi_1)\rangle.\end{aligned}$$

Using the functional equation (4.21) and (8.44), we get

$$C(-\rho)(\phi_1 + i\gamma\phi_1) = -e^{2i\rho a}(\phi_1 - i\gamma\phi_1).$$

Finally, since $\dot\phi_0, \phi_1 \in L_-$, we have

$$\langle\dot\phi_0 - i\gamma\dot\phi_0, \phi_1 - i\gamma\phi_1\rangle = \langle\dot\phi_0 + i\gamma\dot\phi_0, \phi_1 + i\gamma\phi_1\rangle.$$

Combining our results, we obtain (8.45). Q.E.D.

Now return to (8.42). Suppose that $\langle\phi, \phi'\rangle = 0$. Then, using Lemma 8.43, we get

$$\langle E(\phi, \rho), E(\phi', \rho)\rangle_{M_a} = O(e^{-\mu_1 a/2}). \tag{8.46}$$

Let $f \in C^\infty(\mathbf{R})$ satisfying $0 \le f \le 1$, $f(u) = 1$ for $u \le 1/2$ and $f(u) = 0$ for $u \ge 1$. Put $f_a(u) = f(u/a)$. We regard f_a as a function on M_a in the obvious way. Furthermore, let χ_a denote the characteristic function of $[0, a] \times Y \subset M_a$. Let $\rho_1 \le \rho_2 \le \cdots \le \rho_{m_a}$ be the zeros in $\Omega(a)$ where each zero is repeated according to its multiplicity. For each j, $1 \le j \le m_a$, we pick $\phi_j \in L_-$ satisfying the following properties

(1) $e^{2i\rho_j a}S(\rho_j)\phi_j = -\phi_j$.

(2) Whenever $\rho_j = \rho_{j+1} = \cdots = \rho_{j+k}$, then $\phi_j, \phi_{j+1}, ..., \phi_{j+k}$ form an orthonormal system of vectors in L_-.

Put

$$\tilde\psi_j = f_a\left(E(\phi_j, \rho_j) - \chi_a E_0(\phi_j, \rho_j)\right) \; + \; \chi_a E_0(\phi_j, \rho_j)$$

and

$$\psi_j = \frac{\tilde{\psi}_j}{\| \tilde{\psi}_j \|}, \quad j = 1, ..., m_a.$$

From the definition follows that each ψ_j is a smooth section of S over M_a and satisfies $\Pi^{\sigma}_{-}(\psi_j | \partial M_a) = 0$. Thus ψ_j belongs to the domain of $D(a)_r$. Furthermore, employing (8.40) – (8.42) and (8.46), it follows that there exist $a_2, C, c > 0$ such that, for $a \geq a_2$,

(8.47) $$\left| \langle \psi_i, \psi_j \rangle \right| \leq C\, e^{-ca}, \quad i \neq j,\ i, j = 1, ..., m_a$$

and

(8.48) $$\left| \langle \psi_i, \psi \rangle \right| \leq C\, e^{-ca}, \quad \psi \in \operatorname{Ker} D(a)_r,\ i = 1, ..., m_a.$$

Let π_a denote the orthogonal projection of $L^2(M_a, S)$ onto $\operatorname{Ker} D(a)_r$. Put

$$\hat{\psi}_j = \psi_j \ - \ \pi_a \psi_j, \quad j = 1, ..., m_a.$$

Since $\dim(\operatorname{Ker} D(a)_r)$ is independent of a, it follows from (8.47) and (8.48) that

(8.49) $$\left| \langle \hat{\psi}_i, \hat{\psi}_j \rangle \ - \ \delta_{ij} \right| \leq C\, e^{-ca}, \quad i \neq j,\ i, j = 1, ..., m_a,\ a \gg 0.$$

By (8.26), we have $m_a \leq r\, a^{1-\kappa}$ for $a \gg 0$. Together with (8.49), it follows that

(8.50) $$\hat{\psi}_1, ..., \hat{\psi}_{m_a} \quad \text{are linearly independent for} \quad a \gg 0.$$

Now let $0 < \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \cdots \leq \tilde{\lambda}_{p_a}$ denote the nonzero eigenvalues, counted with multiplicity, of $D(a)_r^2$ which are less than $a^{-2\kappa}$. Let $m = m_a$ and let $k_1, ..., k_m$ be a permutation of $\{1, ..., m\}$ such that $0 < \rho^2_{k_1} \leq \rho^2_{k_2} \leq \cdots \leq \rho^2_{k_m}$. By the mini–max principle, we have

$$\tilde{\lambda}_j = \min_{W} \max_{\varphi \in W} \frac{\| D(a)_r \varphi \|^2}{\| \varphi \|^2}$$

where W runs over all j–dimensional subspaces of $\operatorname{dom}(D(a)_r)$ which are orthogonal to $\operatorname{Ker} D(a)_r$ (cf. [R-S, p.82]). Let W_j be the subspace of $\operatorname{dom}(D(a)_r)$ spanned by $\hat{\psi}_{k_1}, ..., \hat{\psi}_{k_j}$. By (8.50), we have $\dim W_j = j$ for $a \gg 0$. Moreover, by construction, W_j is orthogonal to $\operatorname{Ker} D(a)_r$. Hence, using (8.47), (8.48) and the definition of $\hat{\psi}_j$, we get

(8.51) $$\tilde{\lambda}_j \leq \max_{\varphi \in W_j} \frac{\| D(a)_r \varphi \|^2}{\| \varphi \|^2} \leq \rho^2_{k_j} \left(1 + C_1 e^{-c_1 a}\right)$$

for some constants $C_1, c_1 > 0$. In particular, this shows that $m_a \leq p_a$. Using Lemma 8.35, we get $m_a = p_a$. Combined with Lemma 8.35 this completes the proof. Q.E.D.

Let $0 < \kappa < 1$. We can now investigate the behaviour of

(8.52) $$\int_{\sqrt{a}}^{\infty} t^{-1/2} \sum_{|\lambda_j| \leq a^{-\kappa}} \lambda_j\, e^{-t\lambda_j^2}\, dt$$

as $t \to \infty$. By Theorem 8.32, we may as well sum over $\rho \in \Omega(a)$. Let $\Omega'(a)$ be defined as in Theorem 8.31, (iii). Let $\rho \in \Omega(a) - \Omega'(a)$, $\rho > 0$. By Theorem 8.31, (iii), there exists a unique $\rho' \in \Omega(a)$, $\rho' < 0$, such that $|\rho + \rho'| \le C\, a^{-(1+2\kappa)}$. Suppose that $\rho > -\rho'$. Then

$$\rho \int_{\sqrt{a}}^{\infty} t^{-1/2}\, e^{-t\rho^2}\, dt \; + \; \rho' \int_{\sqrt{a}}^{\infty} t^{-1/2}\, e^{-t\rho'^2}\, dt = \int_{-\rho' a^{1/4}}^{\rho a^{1/4}} e^{-x^2}\, dx$$
$$\le C|\rho + \rho'|\, a^{1/4} \le C_1\, a^{-3/4-2\kappa}.$$

Thus, (8.52) can be estimated by

$$C_1\, \#\Omega(a)\, r\, a^{-3/4-2\kappa}.$$

By Theorem 8.31, (i), and (8.26), we have $\#\Omega(a) \le r\, a^{1-\kappa}$; and (8.49) can be estimated by $C_2\, a^{1/4-3\kappa}$. Pick κ such that $1/12 < \kappa < 1/4$. Then (8.49) tends to zero as $a \to \infty$. Together with (7.5), Proposition 7.6 and the final estimate for (7.13), we have proved that

$$\lim_{a \to \infty} \eta(0, D(a)_r) = \eta(0, \mathcal{D}).$$

Combined with Proposition 2.16, we get our main result, Theorem 0.1.

We conclude this section by discussing an example – the Dirac operator in dimension one. Consider the following differential operator

$$D(a) = \begin{pmatrix} 0 & \partial/\partial u \\ -\partial/\partial u & 0 \end{pmatrix}$$

acting in $C^\infty([0,a]; \mathbf{C}^2)$, $a > 0$. Then $\gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\mathbf{C}^2$ is equipped with the standard symplectic structure $\Phi(z,w) = z_2\overline{w}_1 - z_1\overline{w}_2$ where $z = (z_1, z_2)$, $w = (w_1, w_2)$. Let $\alpha \in \mathbf{R}$ and consider the complex line $L_\alpha \subset \mathbf{C}^2$ spanned by $(1, -e^{i\alpha})$. Then L_α, $\alpha \in \mathbf{R}$, are Lagrangian subspaces of $\mathbf{C}^2$. Let P_α be the orthogonal projection of $\mathbf{C}^2$ onto L_α. Denote by $D(a)_\alpha$ the operator $D(a)$ with domain

$$\operatorname{dom} D(a)_\alpha = \{\varphi \in C^\infty([0,a]; \mathbf{C}^2) \mid P_0(\varphi(0)) = 0,\ P_\alpha(\varphi(a)) = 0\}.$$

Then $D(a)_\alpha$ is symmetric with self-adjoint closure. A direct computation shows that the eigenvalues of $D(a)_\alpha$ are given by

$$\lambda_k = \frac{1}{a}\left(\pi k - \frac{\alpha}{2}\right) = \frac{\pi}{a}\left(k - \frac{\alpha}{2\pi}\right), \quad k \in \mathbf{Z}.$$

Put $b = \frac{\alpha}{2\pi}$ and suppose that $0 < b < 1$. Then the eta function of $D(a)_\alpha$ equals

$$\eta(s, D(a)_\alpha) = \left(\frac{a}{\pi}\right)^s \left\{ \sum_{k=1}^{\infty} \frac{1}{|k-b|^s} - \sum_{k=0}^{\infty} \frac{1}{|k+b|^s} \right\}.$$

It follows from [APS2,p.411] that

$$\eta(0, D(a)_\alpha) = 2b - 1 = \frac{\alpha}{2\pi} - 1, \quad 0 < \alpha < 2\pi, \quad \text{and} \quad \eta(0, D(a)_0) = 0. \tag{8.53}$$

In particular, the eta invariant is independent of a as claimed by Proposition 2.16. Now consider $D = D(\infty)$ acting in $L^2([0,\infty); \mathbf{C}^2)$ with domain

$$\operatorname{dom} D = \{\varphi \in C^\infty([0,\infty); \mathbf{C}^2) \mid P_0(\varphi(0)) = 0 \quad \text{and} \quad \varphi(u) = 0 \text{ for } u \gg 0\}.$$

If $\varphi = (f, g)$, $f, g \in C^\infty([0,\infty))$ then the boundary conditions mean that $f(0) = g(0)$. Let $\mathcal{D}$ be the closure of D in L^2. Then $\mathcal{D}$ is self-adjoint. It is easy to see that the kernel of $\exp -t\mathcal{D}^2$ is given by

$$k(u, u', t) = \frac{1}{\sqrt{4\pi t}} \begin{pmatrix} e^{-(u-u')^2/4t} & e^{-(u+u')^2/4t} \\ e^{-(u+u')^2/4t} & e^{-(u-u')^2/4t} \end{pmatrix}.$$

This implies that $\operatorname{tr}(D_u k(u, u', t)|_{u=u'}) = 0$. Hence $\eta(0, \mathcal{D}) = 0$. From (8.53) we get

$$\eta(0, D(a)_0) = \eta(0, \mathcal{D}) \quad \text{and} \quad \eta(0, D(a)_\pi) = \eta(0, \mathcal{D}).$$

Next we determine the scattering matrix associated to $\mathcal{D}$. Let $\phi_1 = (1,0)$ and $\phi_2 = (0,1)$. Then it is easy to see that the corresponding generalized eigenfunctions of $\mathcal{D}^2$ are the following one

$$F(\phi_1, \lambda, u) = e^{-i\lambda u}\phi_1 + e^{i\lambda u}\phi_2 \quad \text{and} \quad F(\phi_2, \lambda, u) = e^{-i\lambda u}\phi_2 + e^{i\lambda u}\phi_1.$$

This implies that the on-shell scattering matrix $C(\lambda) : \mathbf{C}^2 \to \mathbf{C}^2$ is given by

$$C(\lambda) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In particular, the ± 1-eigenspaces of $C(0)$ are equal to L_0 and L_π, respectively. Thus, the possible boundary conditions for which $\eta(0, D(a)_\alpha)$ equals $\eta(0, \mathcal{D})$ are determined by the eigenspaces of $C(0)$.

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