Versal families and the existence of stable sheaves on a Del Pezzo surface

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MPI 96-84

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July 5, 1996

Abstract

Let $r(F), c_1(F), c_2(F)$ be rank and Chern classes of an algebraic coherent sheaf F on a Del Pezzo surface X. We will call a tuple $\bar{c}(F) = (r(F), c_1(F), c_2(F))$ the Chern datum for the sheaf F. In the paper we write down several necessary conditions on the Chern datum of a non-exceptional stable sheaf on a Del Pezzo surface which generalize the conditions found by Drezet and Le Potier for sheaves on $\boldsymbol{P}^2$ and we use them to define a set $\mathtt{D}_X$. After that we prove that if $\bar{c}(F) \in \mathtt{D}_X$ and F can be included in a smooth restricted versal family of sheaves on X then there are stable sheaves in the family so F can be deformed into a stable sheaf with the same Chern datum. We provide a way to construct such families and as an application we prove that for any $c \in \mathtt{D}_X$ it exists a stable sheaf F such that $\bar{c}(F) = c$ provided X is $P^2_{(1)}$ - a Del Pezzo surface which arises by blowing up a point in $\boldsymbol{P}^2$.

*The research described in the paper was partially supported by ISF grant MKU000 and by INTAS grant.

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1 Introduction

Originally the notion of stability belongs to the geometric invariant theory and it was used in algebraic geometry since 60-s, when [Mu] appeared, in order to construct moduli varieties. Later it was discovered that stability of a vector bundle is not only related with the construction of moduli varieties but also with the existence of Kähler-Einstein metrics in the bundle ([LT]) and with other properties as well. It is good to know whether one could find a stable algebraic structure in a given topological vector bundle and this is what we work on in the paper.

More precisely let X be a Del Pezzo surface over an algebraically closed field $\boldsymbol{k}$ and K_X canonical class of X. It is known that the divisor $-K_X$ defines an embedding of X into a projective space and we choose this embedding to define stabity so we work in the following with semistability and stability in respect to the anticanonical polarization.

Let $\bar{c}(F) = (r(F), c_1(F), c_2(F))$ be a tuple which consists of rank and Chern classes of an algebraic coherent sheaf F. We consider the tuple as an element from $\mathtt{M}_X = \boldsymbol{Z} \times \mathrm{Pic}X \times \boldsymbol{Z}$ and we call this element a Chern datum for the sheaf F . To fix Chern datum of a vector bundle (or a coherent sheaf) on a Del Pezzo surface is the same as to fix topological type of the bundle or to fix its image in the Grothendieck group of algebraic coherent sheaves category.

It is know that exceptional sheaves on a Del Pezzo surface are stable ([Go],[KO]) and let $\mathtt{Ex}_X$ be a set of Chern data for non torsion exceptional sheaves on X.

We define below a subset $\mathtt{D}_X$ in $\mathtt{M}_X^+$ and show that it contains Chern data of nonexceptional stable sheaves on X with a discriminant greater than $1/2$ and we prove that if a Chern datum for a sheaf F belongs to $\mathtt{D}_X$ and F can be included into a smooth restricted versal family of sheaves on X, then there are stable sheaves with the same Chern datum in the family. We know that the discriminant of nonexceptional stable sheaf on X is always greater or equal to $1/2$ and for the latter case we prove the existence of semistable sheaves in the family.

Futher in this article we present a way to construct such families and as an application we prove that it exists a stable sheaf F with a Chern datum c for any $c \in \mathtt{D}_X$ when X is $P^2_{(1)}$ - a Del Pezzo surface which arises by blowing up a point in $\boldsymbol{P}^2$. Similar results when X is a projective plane or quadric surface were proven before ([DL], [R2]). The same methods are used in [R5] to study vector bundles on other Del Pezzo surfaces.

Exact statements of the results one could find in the body of the article.

I would like to express my gratitute to S.A.Kuleshov, A.L.Gorodentsev and S.Yu.Zuzina with whom I have several helpful discussions, to Mahtematical department of University of Tokyo where part of this article was made and to prof.E.Horikawa for encouragement and hospitality, and to Max-Planck-Institute where the last version of the text was prepared.

2 Preliminaries

We will work with algebraic coherent sheaves on a smooth projective Del Pezzo surface X over an algebraically closed field $\boldsymbol{k}$. Let us denote the canonical class (or canonical sheaf) of X as K_X. Del Pezzo surfaces are those surfaces, where anticanonical class $-K_X$ is ample. It is well known that over an algebraically closed field they are either projective plane $\boldsymbol{P}^2$ or quadric surface Q or surfaces which are made by blowing up $t < 9$ points in general position on $\boldsymbol{P}^2$, ([Ma]).

We use the name "vector bundle" both for a geometrically defined vector bundle and for the sheaf of sections of a geometric vector bundle which is the same as any locally free coherent sheaf.

As usual Pic$X$ denotes the Picard group of X and $r(F)$ (or r_F), $c_1(F)$, $c_2(F)$ are rank and Chern classes of a sheaf F.

We use notations :

$$\mathtt{M}_X = \boldsymbol{Z} \times \mathrm{Pic}X \times \boldsymbol{Z},$$

$$\mathtt{M}_X^+ = \boldsymbol{N} \times \mathrm{Pic}X \times \boldsymbol{Z}.$$

An element $c = (r, c_1, c_2) \in \mathtt{M}_X$ is said to be the Chern datum for a sheaf F,

$$c = (r, c_1, c_2) = Chd(F), \text{ when } \quad r = rk(F), \quad c_i = c_i(F).$$

It is convenient to consider the Chern data set $\mathtt{M}_X$ as an abelian group in a way that

$$Chd(F_1 \oplus F_2) = Chd(F_1) + Chd(F_2).$$

This means according to standard Chern class properties that

$$(r\,,a\,,b)+(r',a',b')=(r+r',\;a+a',\;b+b'+a\cdot a')$$

where $\cdot$ denotes the intersection pairing on Pic$X$.

So the function Chd is an additive function on sheaves. And this resulted in an isomorphism

$$\mathtt{M}_X = K_0(Csh\;X)$$

of the Chern data group and the Grothendieck group for algebraic coherent sheaves on a Del Pezzo surface.

We will keep the following notations from [R1], [R2]:

$$\chi(A,B)=\sum(-1)^i \dim \mathrm{Ext}^i(A,B),$$

$$\nu_F=\nu(F)=\frac{c_1(F)}{r(F)}\in \mathrm{Pic}X\otimes \boldsymbol{Q},$$

$$m(F)=c_1(F)\cdot(-K_X)\in \boldsymbol{Z},$$

$$\mu_F=\mu(F)=\nu(F)\cdot(-K_X)=\frac{m(F)}{r(F)},$$

$$p(F)=\left(\frac{1}{2}c_1^{\cdot 2}-c_2\right)(F)\in\frac{1}{2}\boldsymbol{Z},$$

$$\pi(F)=\frac{p(F)}{r(F)},\in \boldsymbol{Q},$$

$$\Delta_F=\left(\frac{1}{r}(c_2-\frac{r-1}{2r}c_1^{\cdot 2})\right)(F).$$

The Riemann-Roch theorem states that for sheaves on a Del Pezzo surface:

$$\chi(A,B)=r_Ar_B+\frac{1}{2}(r_Am(B)-r_Bm(A))+r_Ap(B)+r_Bp(A)-c_1(A)\cdot c_1(B),$$

or if $r_A\neq 0,\ r_B\neq 0$ then:

$$\chi(A,B)=r_Ar_B\left(\frac{\nu_{A,B}\cdot(\nu_{A,B}-K_X)}{2}+1-\Delta_A-\Delta_B\right),$$

where $\nu_{A,B} = \nu_B - \nu_A$.

It is important to mention that the functions ν, μ, p, π, Δ and χ depend only on Chern data of sheaves which are their arguments so we will consider those functions as functions on $\mathtt{M}_X$.

The Serry duality theorem for sheaves on a smooth surface can be stated in a form ([DL]):

$$\mathrm{Ext}^i(A,B)^* = \mathrm{Ext}^{2-i}(B, A \otimes K_X).$$

We will often use it throughout the text.

We are to use exceptional sheaves and let us recall the definition.

Definition 2.1 *A sheaf E is called exceptional if $Hom(E,E) = \boldsymbol{k}$ and $Ext^i(E,E) = 0$ for $i > 0$.*

We say that $\mathcal{F}$ is a family of sheaves on X with a base (or parameters) Z when $\mathcal{F} \longrightarrow \mathcal{X}$ is a flat sheaf over $\mathcal{X} = X \times Z$. Sheaves $F_{(z)} = \mathcal{F}|_{X \times z}$ are members of the family. We suppose if it is not mentioned the opposite that Z is connected.

An element $c \in \mathtt{M}_X^+$ is said to be a Chern datum for the family $\mathcal{F}$ if for any $z \in Z$ there is $Chd(F_{(z)}) = c$.

Let us recall some properties of cohomologies $\mathrm{Ext}^i_{F,+}(\,,\,)$ and $\mathrm{Ext}^i_{F,-}(\,,\,)$ for sheaves or complexes of sheaves with decreasing filtrations ([DL]). (It is important to mention that we use decreasing filtrations while they use increasing filtrations in [DL]):

Proposition 2.2 *Let K be a sheaf or a complex (bounded on the left) of sheaves with a finite decreasing filtration*

$$K = F^0K \supset F^1K \supset \ldots \supset F^NK \supset F^{N+1}K = 0$$

and let G_iK be factors of the filtration in K.

1. *There is an exact sequence*

$$\to Ext^i_{F,-}(K,K) \to Ext^i(K,K) \to Ext^i_{F,+}(K,K) \to Ext^{i+1}_{F,-}(K,K) \to$$

2. *There is a spectral sequence abutting to* $Ext^{\cdot}_{F,-}(K,K)$ *such that*

$$E_1^{p,q} = \begin{cases} \prod\limits_i Ext^{p+q}(G_iK, G_{i+p}K) & \text{for } p \geq 0, \\ 0 & \text{for } p < 0. \end{cases}$$

3. *There is a spectral sequence abutting to* $Ext^{\cdot}_{F,+}(K,K)$ *such that*

$$E_1^{p,q} = \begin{cases} 0 & \text{for } p \geq 0, \\ \prod\limits_i Ext^{p+q}(G_iK, G_{i+p}K) & \text{for } p < 0. \end{cases}$$

4. *Let* F^NK *be the last member of the filtration* F *in* K, $\tilde{K} = K/F^NK$, *and the filtration in* $\tilde{K}$ *be induced from* K. *Then there exist an exact sequence*

$$\to Ext^i_{F,+}(\tilde{K},\tilde{K}) \to Ext^i_{F,+}(K,K) \to Ext^i(F^NK,\tilde{K}) \to Ext^{i+1}_{F,+}(\tilde{K},\tilde{K}) \to$$

3 Around stability

We use two kinds of stability in the sequel so it is practical for us to describe in the beginning the stability for algebraic coherent sheaves on a variety X in general terms.

As it is usual, the stability matters are discussed in the following only for *sheaves without torsion.*

Definition 3.1 *Let it be defined for nontorsion sheaves on* X *a map* $A \longrightarrow \gamma(A)$ *into a totally ordered set* Γ *such that:*

(a) γ *depends only on the Chern datum of an argument which means if images of* A *and* B *in* $K_0(Csh\ X)$ *coincide then* $\gamma(A) = \gamma(B)$;

(b) if B *is a subsheaf in* A *such that factor* A/B *is a nonzero torsion sheaf then* $\gamma(B) < \gamma(A)$;

(c) if A, B, C *are nonzero sheaves without torsion and there is an exact sequence*

$$0 \longrightarrow B \longrightarrow A \longrightarrow C \longrightarrow 0$$

then $\gamma(B) < \gamma(A)$ is equivalent to $\gamma(A) < \gamma(C)$ and $\gamma(B) = \gamma(A)$ is equivalent to $\gamma(A) = \gamma(C)$.

A sheaf A (without torsion) is called stable (relative to the order defined by the map γ) if for a subsheaf B in A with a nontorsion factor-sheaf we have $\gamma(B) < \gamma(A)$.

A sheaf A (without torsion) is called semistable if for a subsheaf B we have $\gamma(B) \leq \gamma(A)$.

In the following we write $A <_s B$ instead of $\gamma(A) < \gamma(B)$ and call an order on sheaves which we get this way *the stability order.*

A general property of stability is the following.

Proposition 3.2 *Let A, B be semi-stable sheaves and $B <_s A$. Then $Hom(A, B) = 0$.*

Let it be a morphism $\psi : A \longrightarrow B$ and let us consider its splitting into short exact sequences:

$$0 \to K \to A \to I \to 0, \quad 0 \to I \to B \to C \to 0.$$

If $K = 0$ then $I = A$ and this contradicts semi-stability of B. Therefore if $\psi \neq 0$, then $K \neq 0$, $I \neq 0$, and hence $K <_s A$, $A <_s I$. But $I <_s B$ so $A <_s I <_s B$, which is impossible by assumptions.□

The following is a kind of the Schur lemma in our context.

Lemma 3.3 *Let A, B be stable sheaves such that $B \leq_s A$ then either $Hom(A, B) = 0$ or $A = B$ and $Hom(A, B) = \boldsymbol{k}$.*

By the proposition above only the case when $B =_s A$ are to be proven. The arguments which were used to prove the proposition also show that a nonzero morphism from $\mathrm{Hom}(A, B)$ should be a monomorphism onto a subsheaf in B with a torsion factorsheaf. But such a subsheaf coincides with B because of the property (b) of Definition 3.1.

Now $\mathrm{Hom}(A, B) = \mathrm{Hom}(B, B)$ is a finite-dimensional division algebra over $\boldsymbol{k}$ and it coincides with $\boldsymbol{k}$ as soon as the field is algebraically closed which is we need.□

S.Kuleshov pointed out to us the importance of property (b) and the fact that conditions (a-c) imply the existence of a "Harder-Narasimhan filtration" for a sheaf without torsion.

Theorem 3.1 *For a sheaf A without torsion there exists a filtration:*

$$A = F^0 A \supset F^1 A \supset \ldots \supset F^{N+1} A = 0$$

such that its factors $G_i(A)$ are semistable and

$$G_0(A) <_s G_1(A) <_s \ldots <_s G_N(A)$$

and

$$F^0 A <_s F^1 A <_s \ldots <_s F^N A.$$

This filtration is unique (having the stability chosen) and it is uniquely defined by the above properties of its factors; we call it *Harder-Narasimhan filtration of A (with respect to the stability).*

The proof is quite standard. It relies on some lemmas.

Lemma 3.4 *There is no infinitely increasing chain of subsheaves in A.*

It is well know "noetherian" property of an algebraic coherent sheaf category.

Lemma 3.5 *There is no infinitely decreasing chain*

$$B_0 \supset B_1 \supset \ldots$$

of subsheaves of nonzero rank in A such that $B_0 <_s B_1 <_s \ldots$

Because of (b) the sequence of ranks $\{r(B_i)\}$ is strictly decreasing, hence the chain should be finite.

Lemma 3.6 *There is a unique subsheaf A_{max} in A such that $B \leq_s A_{max}$ for any subsheaf B in A of nonzero rank (for $B = A$ also) and that $B \subset A$ whenever $B =_s A_{max}$.*

Proof of the lemma. . Let us mention first that any subsheaf in A could be included into a subsheaf with nontorsion factor such that it has the same rank and whence is greater with respect to the order the order because of (b).

So in searching for A_{max} we shall look only at nonzero subsheaves B with nontorsion factors, let us call them nontrivial subsheaves. If A_{max} exists it is necessary nontrivial or equal to A itself.

Let us suppose that F is a counterexample to the lemma with a minimal rank. Then for any nontrivial subsheaf C in F there exist C_{max} but there is also a nontrivial subsheaf B in F such that $C_{max} <_s B$ (thus B is not a subsheaf in C).

But $B \cap C$ is either a sheaf of nonzero rank or zero. In the latter case $C <_s (B+C) = C'$ because of an exact sequence

$$0 \longrightarrow C \longrightarrow B + C \longrightarrow B \longrightarrow 0,$$

In the former one

$$B \cap C <_s C_{max} <_s B$$

and an exact sequence

$$0 \longrightarrow B \cap C \longrightarrow B \longrightarrow B/B \cap C \longrightarrow 0$$

show us that $B <_s B/B \cap C$ so $C <_s B/B \cap C$. Now taking into account an exact sequence

$$0 \longrightarrow C \longrightarrow B + C \longrightarrow B/B \cap C \longrightarrow 0$$

we conclude that

$$C <_s (B + C) = C^{'}.$$

where $C^{'}$ strictly include C because B is not a subsheaf of C.

Thus we could make an infinite increasing chain

$$C \subset C^{'} \subset C^{''} \dots$$

which provide us with a contradiction.□

Now turning to the proof of the theorem, A_{max} is surely semistable and if $A \neq A_{max}$ then

$$A/A_{max} <_s A <_s A_{max}$$

which permits us to prove the theorem by induction.□

There are also filtrations in semistable sheaves, which are not in general uniquelly defined by their properties.

Proposition 3.7 *For a (s)-semistable sheaf A (without torsion) there exists a filtration:*

$$A = F^0A \supset F^1A \supset \ldots \supset F^{N+1}A = 0$$

such that its factors $G_i(A)$ are (s)-stable and

$$G_0(A) =_s G_1(A) =_s \ldots =_s G_N(A) =_s A.$$

Any filtration in A of this type is called *Jordan-Hölder filtration in A.*

For the proof of the proposition one should take a subsheaf $A^{\#}$ that is a minimal one among subsheaves

$$\{ B \mid B =_s A \},$$

as the last member F^NA of the filtration; then proceed by induction.

There are two stability orders which we need for our work with sheaves on Del Pezzo surfaces. The first one corresponds to Giesecker stability. It could be described as follows.

Definition 3.8 *Let us consider a map which assigns to a nontorsion sheaf A a tuple*

$$(\mu(A), \pi(A))$$

and define an order in a way that

$$A <_g B \Longleftrightarrow (\mu(A), \pi(A)) <_{lex} (\mu(B), \pi(B)),$$

and "$=_g \Leftrightarrow =_{lex}$"*, where* $<_{lex}$ *means "lexicographically less" .*

The second stability we will name the *extended stability.* It is defined by an order which is constructed as follows.

Definition 3.9 *Let us fix a system of ample divisor classes*

$$D = -K_X, D_1, \ldots, D_t \in \ PicX$$

such that they form a base for $PicX \otimes \mathbf{Q}$ *and denote:*

$$\mu_i(F) = \frac{c_1(F) \cdot D_i}{r(F)} = \nu(F) \cdot D_i.$$

To make the extended stability map we assign to a nontorsion sheaf A *a tuple*

$$(\mu(A), \pi(A), \mu_1(A), \ldots, \mu_t(A))$$

and we define the order in a way that $A <_e B$ *means*

$$(\mu(A), \pi(A), \mu_1(A), \ldots, \mu_t(A)) <_{lex} (\mu(B), \pi(B), \mu_1(B), \ldots, \mu_t(B))$$

and $" =_e \ \Leftrightarrow \ =_{lex} "$.

To prove condition (b) for our Gieseker and extended stabilities one should remember that we are on a surface and the slope μ is defined with respect to an ample divisor so if support of A/B is one-dimensional then $\mu_B < \mu_A$ and if the support is zero-dimensional then $\mu_B = \mu_A$ but $p(B) < p(A)$. This results in $B < A$ for both stability orders.

We leave it to the reader to check that properties (a), (c) are also valid.

For the extended stability there is a following property, which is stronger than the condition (a):

Lemma 3.10 $A =_e B$ *if and only if* A *and* B *have proportional Chern data.*

This is a direct consequence of the e-stability definition.

It is important to have in mind that if A is g-stable then it is e-stable and if A is e-semi-stable then it is g-semi-stable.

Lemma 3.11 *Suppose* A, B *are* g*-semi-stable (or e-semi-stable) and*

$$\mu(A) < \mu(B) + K_X^{\cdot 2},$$

or A, B *are* g*-stable,* $A \otimes K_X \neq B$ *and*

$$\mu(A) \leq \mu(B) + K_X^{\cdot 2},$$

then $Ext^2(A, B) = 0$.

One can notice that even Mumford-Takemoto stability is enough for the first statement here but we choose to rectrict our considerations only to g-stability and e-stability through the paper.

Proof of the lemma. . By the Serre duality $\mathrm{Ext}^2(A, B) = \mathrm{Hom}(B, A \otimes K_X)^*$ and

$$\mu(B) > \mu(A) - K_X^2 = \mu(A \otimes K_X).$$

Thus $B >_g A \otimes K_X$ and it is clear that sheaves B and $A \otimes K_X$ are semi-stable so there is no morphisms by Proposition 3.2. Similarly the second statement follows via Serre duality from Lemma 3.3.□

Lemma 3.10 is crucial to prove the following proposition, which is the main reason why we need the extended stability.

Proposition 3.12 *If $\chi(B, B) > 0$ for a e-semistable sheaf B then B is isomorphic to $E \oplus E \oplus \cdots \oplus E$ where E is exceptional.*

Proof of the proposition. . Looking at an e-Jordan-Hölder filtration for B we see that the factors G_i are e-stable and their Chern data are proportional by Lemma 3.10.

Let us denote E one of them. Then $Chd(B) = a\,Chd(E)$ and it follows that

$$\chi(B, B) = a^2 \chi(E, E), \qquad \text{thus } \chi(E, E) > 0.$$

But $\dim \mathrm{Hom}(E, E) = 1$ by Lemma 3.3 and $\mathrm{Ext}^2(E, E) = 0$ by Lemma 3.11. Hence the condition $\chi(E, E) > 0$ implies that $\mathrm{Ext}^1(E, E) = 0$, so we conclude that E is exceptional and $\chi(E, E) = 1$.

Now a is uniquelly defined by the equation

$$\chi(B, B) = a^2,$$

so all the Chern data are equal and the factors are exceptional hence they are isomorphic as exceptional sheaves are uniquelly defined by their Chern data ([Go]).

Then

$$\mathrm{Ext}^1(G_i, G_j) = \mathrm{Ext}^1(E, E) = 0,$$

so the filtration splits and $B = E \oplus E \oplus \cdots \oplus E$.□

4 Families with stable sheaves

We state our main results in this section. The stability in the following means Gieseker stability relative to anticanonical polarization.

Definition 4.1 *We call a coherent sheaf F on a Del Pezzo surface X a restricted sheaf if it has no torsion and*

$$\mu(G_{max}F) - \mu(G_{min}F) \leq K_X^{\cdot 2},$$

where $G_{min}F$ and $G_{max}F$ are the first and the last factors of a Harder-Narasimham filtration in F.

We will say that a family of sheaves is restricted if all its members are restricted sheaves.

Definition 4.2 *Let $\mathcal{F}$ be a family of sheaves on X with base Z. We call it a smooth versal family if Z is smooth and the following condition is satisfied:*

KS *For each $z \in Z$ the Kodaira-Spencer morphism*

$$\omega : T_z Z \longrightarrow Ext^1(F_{(z)}, F_{(z)})$$

is surjective,

(we denote by $T_z Z$ the tangent space of Z at a point z).

Let us remember that $\mathbf{Ex}_X$ denotes the set of Chern data for nontorsion exceptional sheaves on X.

Definition 4.3 *We will say that an element $c \in M_X^+$ satisfy **DL**-condition (Drezet-Le Potier condition) if:*

DL1. c *is not in* $\mathbf{Ex}_X$,

DL2. $\chi(c, e) \leq 0$ *for any* $e \in \mathbf{Ex}_X$ *such that* $r(e) < r(c)$ *and*

$$\mu(c) \geq \mu(e) \geq \mu(c) - K_X^{\cdot 2},$$

DL3. $\chi(e, c) \leq 0$ *for any* $e \in \mathbf{Ex}_X$ *such that* $r(e) < r(c)$ *and*

$$\mu(c) \leq \mu(e) \leq \mu(c) + K_X^{\cdot 2},$$

DL4. $\Delta(c) > \frac{1}{2}$.

Let us denote by D_X *the set of elements in* M_X^+ *satisfying the* ***DL****-condition.*

The conditions above are not independent. The Riemann-Roch formula shows that Ex_X belongs to the subset $\{c \in M_X^+ \mid \Delta(c) < \frac{1}{2}\}$ so DL4 implies DL1. Also one could notice that DL2 and DL3 are equivalent. Really it follows from Serre duality that $\chi(e,c) = \chi(c, e \otimes K_X)$, but as $e \otimes K_X \in \mathrm{Ex}_X$ so we get what needed.

We would like to keep DL1 and DL3 in the definition anyway as it looks more symmetric and it would be more convenient for future references.

Drezet and Le Potier have proved ([DL]) that for $X = \boldsymbol{P^2}$ the condition DL4 follows from DL2 but an example in [R2] shows that this is not so for a general Del Pezzo surface.

It is possible to prove that DL2, DL3 imply $\Delta(c) \geq \frac{1}{2}$ for any Del Pezzo surface. On the other hand it was shown in [R5] that $\Delta(c) \geq 1$ implies DL2, DL3 so as a result it implies the **DL**-condition.

Proposition 4.4 *If* $c = Chd(F)$ *for a nonexceptional stable sheaf* F *then DL1, DL2 and DL3 are valid.*

As F is not exceptional DL1 is valid. To prove DL2 let us remember that nontorsion exceptional sheaves are torsion free and stable ([Go], [KO]). Suppose $e = Chd(E)$ and $F <_g E <_g F \otimes K_X$, then Lemmas 3.3, 3.11 give us that what is needed. Of course

$$\mu(c) < \mu(e) < \mu(c) - K_X^2 \,.$$

implies that $F <_g E <_g F \otimes K_X$ so need to check DL2 only for e such that $\mu(e) = \mu(c)$ or $\mu(e) = \mu(c) - K_X^2$. As the reasonings are similar we will consider only the first possibility.

The Riemann-Roch formula shows that $\chi(e,c) = \chi(c,e)$ in this case. On the other hand either $E >_e F$ and then $\chi(e,c) = 0$, or $E <_e F$ and then $\chi(c,e) = 0$, or $E =_e F$. But the latter case implies that $c = \alpha\, e$ by Lemma 3.10 so

$$\chi(F,F) = \chi(c,c) = \alpha^2\, \chi(e,e) > 0$$

and Proposition 3.12 shows that $F = E$ as it is e-stable which means a contradiction.□

Main results of the paper are the following theorems which show that under certain conditions the converse of the above is also true:

Theorem 4.1 *Let X be a Del Pezzo surface. Let $\mathcal{F}$ be a restricted smooth versal family of sheaves on X with a parameter space Z. Suppose that its Chern datum $c = Chd(\mathcal{F})$ satisfies **DL**-condition.*
Then it exists a nonempty open set $U \subset Z$ such that sheaves $F_{(u)}$ are stable for $u \in U$.

Theorem 4.2 *Let X be a Del Pezzo surface. Let $\mathcal{F}$ be a restricted smooth versal family of sheaves on X with a parameter space Z. Suppose that its Chern datum $c = Chd(\mathcal{F})$ satisfies DL1, DL2, DL3.*
Then sheaves $F_{(u)}$ are e-semi-stable for u in a nonempty open set $U \subset Z$.

Let us mention that sheaves of rank 1 without torsion are always stable and they are exceptional as soon as they are localy free. The discriminant in this case is nonnegative integer and it is greater than $\frac{1}{2}$ if and only if the sheaf is nonexceptional.

Thus the question of the existence of stable sheaves in a family when $r(c) = 1$ becomes trivial so

while proving Theorems 4.2, 4.1 *we suppose that* $r(c) \geq 2$.

We prove Theorem 4.2 in the following section. Now we derive Theorem 4.1 from it.

Proof of Theorem 4.1. We could substitute the base Z of the family in question by its open subset which exists by Theorem 4.2, so let us suppose that sheaves $F_{(z)}$ are e-semi-stable for any $z \in Z$.

As a first step we shall prove a similar result about e-stability namely the following.

Lemma 4.5 *There exist a nonempty open set $U \subset Z$ such that sheaves $F_{(u)}$ are e-stable for $u \in U$.*

If a sheaf is e-semi-stable but not e-stable, then there exists (at least one nontrivial) e-Jordan-Hölder filtration in it whose factors are e-stable and equivalent in respect to e-stability order.

There is a possibility to control filtrations having N factors with fixed Hilbert polynomials $H_1, \ldots, H_N$ for the sheaves of a family with the help of a generalized flag variety which is a projective variety over the base Z:

$$\delta_{H_1,\ldots,H_N} : \mathrm{Drap}^{H_1,\ldots,H_N}(\mathcal{F}) \longrightarrow Z$$

and which represents the functor "set of the filtrations" (see [DL, p.202] or [Gr]).

Thus a point $z \in Z$ belongs to $\mathrm{Im}(\delta_{H_1,\ldots,H_N})$ if and only if there exists a filtration in $F_{(z)}$ having factors with these Hilbert polynomials.

Propositions (1.5), (1.7) from [DL] give us a way to evaluate $\mathrm{Im}(\delta_{H_1,\ldots,H_N})$. We restate them as the following lemma.

Lemma 4.6 *Let $\mathcal{F}$ be a family of sheaves on X with parameters Z, $z \in Z$, and*

$$f \in Drap^{H_1,\ldots,H_N}(\mathcal{F}).$$

Then f induces a filtration in $F_{(z)}$ and there is an exact sequence

$$0 \to Ext^0_{F,+}(F_{(z)}, F_{(z)}) \to T_f Drap^{H_1,\ldots,H_N}(\mathcal{F}) \to T_z Z \xrightarrow{\omega_+} Ext^1_{F,+}(F_{(z)}, F_{(z)}),$$

where the last morphism ω_+ is a composition of a Kodaira-Spencer morphism and a morphism from the exact sequence of Proposition 2.2

$$T_z Z \to Ext^1(F_{(z)}, F_{(z)}) \to Ext^1_{F,+}(F_{(z)}, F_{(z)}).$$

Provided that the family is smooth versal and

$$Ext^2_{F,-}(F_{(z)}, F_{(z)}) = 0,$$

the morphism ω_+ is epimorphism, the variety $Drap^{H_1,\ldots,H_N}(\mathcal{F})$ is smooth at the point f, and the codimension of its image in Z is equal to

$$\dim Ext^1_{F,+}(F_{(z)}, F_{(z)}).$$

In order to use this in our situation let us prove the following.

Lemma 4.7 *Let A be a e-semi-stable sheaf such that $\Delta = \Delta(A) > \frac{1}{2}$ and let $A = F^0A \supset F^1A \supset \ldots \supset F^{N+1}A = 0$ be an e-Jordan-Hölder filtration in A with factors G_i. Then for this filtration*

$$Ext^2_{F,-}(A,A) = 0 \qquad \text{and} \qquad Ext^1_{F,+}(A,A) \neq 0.$$

To prove the first statement we can use the spectral sequence from Proposition 2.2 in order to evaluate $\mathrm{Ext}^2_{F,-}(A,A)$. As sheaves G_i are e-stable and $\mu(G_i) = \mu(G_j)$, so we have

$$\mathrm{Ext}^2(G_i, G_j) = 0$$

and we get what needed.

Proving the second statement we need to establish first that in our special situation we have $\Delta(G_i) = \Delta(G_j) = \Delta$. By the definition of the discriminant we have for a sheaf G

$$\Delta(G) = \frac{1}{2}(\nu(G))^2 - \pi(G).$$

We know that the factors of the filtration in question are equivalent in respect to the e-stability order. Hence we have

$$\pi(G_i) = \pi(G_j) = \pi(A), \qquad \mu(G_i) = \mu(G_j) = \mu(A)$$

and

$$\mu_s(G_i) = \mu_s(G_j) = \mu_s(A) \quad \text{for} \quad s = 1, \ldots, t.$$

As a result $\nu(G_i)$ are uniquelly defined in $\mathrm{Pic}X \otimes \boldsymbol{Q}$ and have to be equal to $\nu(A)$. Thus $\Delta(G_i) = \Delta(A) = \Delta$ as we stated.

Now let us look at Euler characteristic in the spectral sequence for $\mathrm{Ext}^{\cdot}_{F,+}(A,A)$:

$$\sum_i (-1)^i \dim \mathrm{Ext}^i_{F,+}(A,A) = \sum_{i>j} \chi(G_i, G_j) = \sum_{i>j} r(G_i)\, r(G_j)\, (1 - 2\Delta) < 0$$

(we have used the Riemann-Roch theorem here).

This implies that $\mathrm{Ext}^1_{F,+}(A,A) \neq 0$ as it was needed.□

To finish the proof of Lemma 4.5 let us have in mind that a system of Hilbert polynomials for factors of a e-Jordan-Hölder filtration in $F_{(z)}$ is defined by ranks $r_1, \ldots, r_N$ of the factors (because their slopes and discriminants are uniquelly defined by the Chern datum of the family in question). And we have

$$r = r_1 + \ldots + r_N$$

so there are only finite number of possibilities for these systems of Hilbert polynomials. Hence all the points z for which sheaves $F_{(z)}$ could have a nontrivial e-Jordan-Hölder filtration belong to a finite union of subvarieties of nonzero codimension. Therefore the subset U which is a complement to this union gives us what was needed for the lemma.□

Moving forward in our proof of the theorem we can without loss of generality suppose that sheaves $F_{(z)}$ of the family are e-stable.

And even more so: we can suppose that they are e-stable for several different e-stabilities resulted in the different choices of divisors $D_1, \ldots, D_t$ in the definition of an e-stability order.

But from relatively simple geometrical considerations for Pic$X \otimes \boldsymbol{Q}$ it follows that there exists such a system of e-stabilities that if a sheaf F is e-stable in respect to all of them then it is g-stable. This finishes the proof of the theorem.□

5 Families and filtrations

In this section we shall prove Theorem 4.2 after some preliminary considerations.

In the definition of stable sheaf there is a condition on slopes of its subsheaves. One could generalize this to a condition on systems of subsheaves or filtrations. It was done in the paper [DL] in respect to Gieseker stability. There it was defined the weight for a filtration in a sheaf and the properties of the weight are extablished. We deneralize this here to the extended stability along the guidelines from [R2].

Let us recall that for a sheaf F of nonzero rank we have

$$m(F) = r_F\, \mu(F), \quad m_i(F) = r_F\, \mu_i(F).$$

Definition 5.1 *In order to define g-weight of a filtration*

$$F^{\cdot}A: \; A = F^0A \supset F^1A \supset \ldots \supset F^{N+1}A = 0,$$

that has no factors of zero rank let us consider points

$$(\, r(F^iA),\, m(F^iA),\, p(F^iA)\,)$$

as vertices for the graph of a piecewise linear mapping

$$\bar{g}_{F^{\cdot}A}: \; [0, r] \longrightarrow \boldsymbol{R}^3,$$

where $r = r_F$. This mapping is called weight of the filtration with respect to Gieseker stability or g-weight of the filtration.

Definition 5.2 *In order to define e-weight of a filtration*

$$F^{\cdot}A: \; A = F^0A \supset F^1A \supset \ldots \supset F^{N+1}A = 0$$

that has no factors of zero rank let us consider points

$$(\, r(F^iA),\, m(F^iA),\, p(F^iA),\, m_1(F^iA), \ldots, m_t(F^iA)\,)$$

as vertices for the graph of a piecewise linear mapping

$$\bar{e}_{F^{\cdot}A}: \; [0, r] \to \boldsymbol{R}^{3+t},$$

where $r = r_F$.

This mapping is called weight of the filtration with respect to the extended stability or e-weight of the filtration.

We will omit the reference to a stability if it is clear from the context which stability is considered.

A mapping $\bar{n}: [0,r] \to \boldsymbol{R}^k$ is called *convex* if for $a, b \in [0,r]$

$$\bar{n}((a+b)/2) \geq_{lex} (\bar{n}(a) + \bar{n}(b))/2$$

(here "lex" stands for lexicographic order).

For mappings $\bar{n_1}, \bar{n_2}$ we say $\bar{n_1} \leq \bar{n_2}$ if $\bar{n_1}(a) \leq_{lex} \bar{n_2}(a)$ for any $a \in [0,r]$.

Here and below "(s)-" refers either to Gieseker or to the extended stability.

Definition 5.3 *A filtration is called (s)-convex if corresponding (s)-weight mapping is convex.*

Proposition 5.4 *Let F be a sheaf on X.*

1. *An (s)-Harder-Narasimhan filtration in F is (s)-convex.*
2. *The weight of a (s)-Harder-Narasimhan filtration dominates the (s)-weight of any other filtration in F.*
3. *There is a finite number of weights for (s)-convex filtrations in F.*
4. *Let $\mathcal{F}$ be a flat family of sheaves on X with a base Z. Consider for any $z \in Z$ the (s)-weight for an·(s)-Harder-Narasimhan filtration in a sheaf $F_{(z)}$. Then while z varies in Z these weights belong to a finite set.*

The first two statements follow immediately from the basic properties of Harder-Narasimhan filtrations.

To prove the third statement it is sufficient to show that there is a finite number of possibilities for the image of the stability map for any member of the filtration. For this it is sufficient to notice that slopes μ, μ_i of subsheaves in a sheaf F are upper bounded and that p is upper bounded on subsheaves in F having μ is bounded below. (As a consequence p, μ_i are bounded on a set of subsheaves where μ is bounded below). This could be checked easily for a sheaf of rank one and general case follows by induction on rank F.

For the last statement it is also sufficient to establish that slopes μ, μ_i of subsheaves in sheaves $F_{(t)}$ are upper bounded and that p is upper bounded provided that μ is bounded below on the subsheaves. One could prove this by induction on rank of $\mathcal{F}$.

Let it be fixed for the following that $\mathcal{F}$ is a family from Theorem 4.2.

We will denote by $w_{HN(z)}$ the weight of e-Harder-Narasimhan filtration in $F_{(z)}$ and let

$$S^e Z(\bar{n}) = \{z | w_{HN(z)} = \bar{n}\},$$
$$\Omega^e Z(\bar{n}) = \{z | w_{HN(z)} \leq \bar{n}\}.$$

Proposition 5.5 *1. Subsets $S^e Z(\bar{n})$ constitute a stratification of Z.*

2. Subsets $\Omega^e Z(\bar{n})$ are open.

3. $S^e Z(\bar{n})$ is a smooth close subvariety in $\Omega^e Z(\bar{n})$ having its normal space at a point z isomorphic to

$$Ext^1_{F,+}(F_{(z)}, F_{(z)}),$$

where the filtration in $F_{(z)}$ is chosen to be its e-Harder-Narasimhan filtration.

The similar result for Gieseker stability was proved in [DL]. As the proof for the extended stability uses the same arguments we only briefly present it here stressing those moments where some specific properties of e-stability are needed.

Lemma 5.6 *Let A be a restricted sheaf provided with its e-Harder-Narasimhan filtration. Then for this filtration*

$$Ext^0_{F,+}(A, A) = 0 \quad \text{and} \quad Ext^2_{F,-}(A, A) = 0.$$

Let G_i denote factors of the filtration in A. By definition of a Harder-Narasimhan filtration they are e-semistable and $G_i <_e G_j$ for $i < j$ so

$$\mathrm{Ext}^0(G_j, G_i) = 0.$$

Looking at the spectral sequence from Proposition 2.2 for $\mathrm{Ext}^{\cdot}_{F,+}(A, A)$ we conclude that the entries $E_1^{p,q}$ for $p + q = 0$ are equal to zero. Thus

$$\mathrm{Ext}^0_{F,+}(A, A) = 0.$$

As A is restricted whence

$$\mathrm{Ext}^2(G_j, G_i) = 0$$

for any pair i, j by Lemma 3.11 .

Applying this to the computation of $E_1^{p,q}$ in the spectral sequence of Proposition 2.2 for $\mathrm{Ext}^{\cdot}_{F,-}(A, A)$ we conclude that

$$\mathrm{Ext}^2_{F,-}(A, A) = 0$$

and this finishes the proof.□

Proof of Proposition 5.5. The first statement is clear so let us prove the second one. Let $z \in \Omega^e Z(\bar{n})$ and f be a point in a generalized flag variety $\mathrm{Drap}^{H_1,\dots,H_N}(\mathcal{F})$ which corresponds to the Harder-Narasimhan filtration in $F_{(z)}$. From Lemmas 5.6 and 4.6 we conclude that $\mathrm{Drap}^{H_1,\dots,H_N}(\mathcal{F})$ is smooth at f and that there is an exact sequence:

$$0 \to T_f \mathrm{Drap}^{H_1,\dots,H_N}(\mathcal{F}) \to T_z Z \to \mathrm{Ext}^1_{F,+}(F_{(z)}, F_{(z)}) \to 0.$$

Let us denote by D an irreducible component of $\mathrm{Drap}^{H_1,\dots,H_N}(\mathcal{F})$ containing f. It is clear that if z' belongs to the image of D by the canonical proper morphism

$$\mathrm{Drap}^{H_1,\dots,H_N}(\mathcal{F}) \longrightarrow Z$$

then the sheaf $F_{(z')}$ has a filtration with the same e-weight as f. Hence the weight of e-Harder-Narasimhan filtration in $F_{(z')}$ is no less than the weight of f. Thus the set

$$\bigcap_{\bar{m} \geq \bar{n}} S^s Z(\bar{m})$$

is closed because it is the union of a finite number of such images.

So $\Omega^e Z(\bar{n})$ is open.

Now we are to prove the third statement in the proposition.
Let $\Omega = \Omega^e Z(\bar{n})$ and $S = S^e Z(\bar{n})$. Clearly $S \subset \Omega$ and if $z \in S$ then the e-weight of the Harder-Narsimhan filtration in $F_{(z)}$ is equal to $\bar{n}$. Moreover if $F_{(z)}$ has a filtration of weight equal to $\bar{n}$ then it is a Harder-Narasimhan filtration and $z \in S$.

Suppose $z \in S$ and $H_1, \ldots, H_N$ are Hilbert polynomials of the factors of an e-Harder-Narasimhan filtration in $F_{(z)}$. It is clear that S belongs to the image of a canonical morphism

$$\mathrm{Drap}^{H_1,\ldots,H_N}(\mathcal{F}|_\Omega) \longrightarrow \Omega$$

and coincides with a component of it.

Hence from Lemma 4.6 (or Propositions (1.5),(1.7) in [DL]) and from Lemma 5.6 it follows that the normal space to S at z can be computed by an exact sequence

$$0 \to T_f \mathrm{Drap}^{H_1,\ldots,H_N}(\mathcal{F}) \to T_{z'}Z \to \mathrm{Ext}^1_{F,+}(F_{(z')}, F_{(z')}) \to 0$$

and thus it is isomorphic to $\mathrm{Ext}^1_{F,+}(\mathcal{F}_{(z)}, \mathcal{F}_{(z)})$.□

Proof of Theorem 4.2. We work with e-stability through the proof.

The first step of the proof is to consider the stratification of Proposition 5.5 for Z. As the semi-stability is equivalent to triviality of a Harder-Narasimhan filtration or to linearity of the corresponding weight, hence by the proposition the set of parameters for semistable sheaves is open. The task is now to prove that it is not empty.

In order to prove that the stratum corresponding to semistable sheaves exists it is sufficient to prove that all the strata related to nontrivial Harder-Narasimhan filtrations (with nonlinear weight functions) have nonzero codimensions. This would be done if we show that their normal spaces are nonzero.

So by the same proposition 5.5 it is enough to prove the "key lemma":

Lemma 5.7 *Suppose for a restricted sheaf A:*

1. *$Chd(A) = c$ satisfies **DL**-condition,*

2. *an e-Harder-Narasimhan filtration in A is nontrivial.*

Then for this filtration $Ext^1_{F,+}(A, A) \neq 0$.

Proof of the lemma. . Let us suppose the contrary that $\mathrm{Ext}^1_{F,+}(A,A)=0$ for A in respect to the filtration. Let G_i denote factors of the filtration in A and $i=1,\dots,N$. The factors are e-semistable and $G_i <_e G_j$ for $i<j$ so

$$\mathrm{Ext}^0(G_j,G_i)=0.$$

As A is restricted hence

$$\mathrm{Ext}^2(G_j,G_i)=0$$

for any pair i,j by Lemma 3.11 .

Looking at the spectral sequence for $\mathrm{Ext}^{\cdot}_{F,+}(A,A)$ we conclude that it degenerates at E_1 and that the terms $E_1^{p,q}$ for $p+q=1$, $p>0$ are equal to zero.

That means

$$\chi(G_j,G_i)=0 \text{ for } i<j.$$

The additivity of the Euler characteristic permits us to derive from this the equalities:

$$\begin{aligned} \chi(G_N,\ A\ ) &= \chi(G_N,G_N), && (1)\\ \chi(\ A\ ,\ G_1) &= \chi(G_1,G_1), && (2)\\ \chi(G_N,G_1) &= 0. && (3) \end{aligned}$$

Lemma 5.8 *Provided (1),(2),(3) either* $\chi(G_1,G_1)>0$ *or* $\chi(G_N,G_N)>0$.

Suppose that Lemma 5.8 has been proved. Then from Proposition 3.12 we conclude that either G_N or G_1 is isomorphic to a direct sum $E\oplus E\oplus\cdots\oplus E$ where E is an exceptional sheaf.

Then from (1) and (2) it follows that

$$\text{either} \quad \chi(E,A)>0 \quad \text{or} \quad \chi(A,E)>0$$

But because A is restricted we have

$$0\geq \mu(G_N)-\mu(A)\geq K_X^{\cdot 2},$$

and

$$0\geq \mu(A)-\mu(G_0)\geq K_X^{\cdot 2}$$

so there is a contradiction to **DL**-condition.

Thus to finish the proof we need only to establish Lemma 5.8. This can be made through some calculations as follows.

By the second form of the Riemann-Roch theorem we have

$$\chi(G_N, G_1) = r(G_N)r(G_1)\left(\frac{1}{2}(\nu - K_X)\cdot\nu + 1 - \Delta' - \Delta''\right) \qquad (4)$$

where $\nu = \nu_{G_1} - \nu_{G_N}$, $\Delta' = \Delta_{G_1}$, $\Delta'' = \Delta_{G_N}$.

Let us denote

$$\rho = -\frac{1}{2}K_X,\ \nu = a\rho + \varepsilon$$

where $\varepsilon \cdot \rho = 0$. Then as A is restricted so

$$-K_X^{\cdot 2} < \nu \cdot (-K_X) \leq 0.$$

Hence

$$-2 < a \leq 0$$

and it is important to mention that if $a = 0$ then $\varepsilon \neq 0$. Therefore we can conclude that

$$(\nu - K_X)\cdot\nu = (\nu + \rho)^{\cdot 2} - \rho^{\cdot 2} = ((a+1)^2 - 1)\rho^{\cdot 2} + \varepsilon^{\cdot 2} \leq 0$$

By Hodge index theorem $\varepsilon^{\cdot 2} \leq 0$ and if $\varepsilon \neq 0$ then $\varepsilon^{\cdot 2} < 0$; so

$$(\nu - K_X)\cdot\nu < 0.$$

Then it follows from (3) and (4) that

$$1 - \Delta' - \Delta'' > 0.$$

We can rewrite this as

$$1 - \Delta' - \Delta'' = \frac{1}{2}(1 - 2\Delta') + \frac{1}{2}(1 - 2\Delta'') > 0.$$

Hence we conlude that

$$\text{either} \quad 1 - 2\Delta' > 0 \quad \text{or} \quad 1 - 2\Delta'' > 0$$

and as a result either $\chi(G_1, G_1) > 0$ or $\chi(G_N, G_N) > 0$ by the Riemann-Roch theorem and this proves the lemma.

So we have finished the proof of the theorem 4.2.□

6 Restricted families

In this section we will discuss the more constructive appoach to the task to determine weither a family of sheaves on a Del Pezzo surface is restricted or not.

Definition 6.1 *Let us say that there is given a splitting of the anticanonical divisor into lines if there is a set of lines* $\{P_s\}$ *such that*

$$-K_X = \sum P_s.$$

It is easy to construct a splitting when $K_X^2 > 1$ (or $t < 8$). But if $K_X^2 = 1$ when we cannot find a splitting (at least if the lines in question are supposed to be smooth). This is because we have $P_s \cdot (-K_X) \geq 1$ as $-K_X$ is ample, so

$$(-K_X)^2 = (\sum P_s) \cdot (-K_X) \geq \text{ number of lines } \geq 2.$$

Proposition 6.2 *Given a splitting of* $-K_X$ *into lines* $\{P_s\}$ *suppose that for a sheaf* F *restrictions* $F|_{P_s}$ *are rigid. Then the sheaf* F *is restricted.*

We need two lemmas.

Lemma 6.3 *Let*

$$0 \longrightarrow A \longrightarrow F \longrightarrow B \longrightarrow 0$$

be exact and B *have no torsion. Then for a line* P *the sequence*

$$0 \to A|_P \to F|_P \to B|_P \to 0$$

is exact also.

Proof of the lemma. . Clearly there are exact sequences:

$$\mathrm{Tor}_1(\mathcal{O}|_P, B) \longrightarrow A|_P \longrightarrow F|_P \longrightarrow B|_P \longrightarrow 0$$

and

$$0 \to \mathrm{Tor}_1(\mathcal{O}|_P, B) \to \mathcal{O}(-P) \otimes B \xrightarrow{\alpha} \mathcal{O} \otimes B \to \mathcal{O}|_P \otimes B \to 0,$$

where α is a multiplication on a section of $\mathcal{O}(P)$ hence it has no kernel when B has no torsion. So $\mathrm{Tor}_1(\mathcal{O}|_P, B)$ is zero and this proves the lemma.□

Lemma 6.4 *Let P be a line and F be a sheaf on X such that its restriction $F|_P$ on P has no torsion. Suppose the restriction $F|_P$ is rigid. Then there exists a number k such that for any subsheaf A of F and for any factorsheaf without torsion B of F we have*

$$\nu(A) \cdot P \leq k+1 \quad \text{and} \quad \nu(B) \cdot P \geq k.$$

It is important to mention that the sheaf $F|_P$ on a projective line is rigid if and only if for some $k \in \boldsymbol{Z}$

$$F\ |_P \simeq r_1\mathcal{O}(k) \oplus r_2\mathcal{O}(k+1).$$

This means that a Harder-Narasimhan filtration in $F|_P$ (with respect to Giesecker stability on P) has two factors $r_1\mathcal{O}(k)$ and $r_2\mathcal{O}(k+1)$ with slopes k and $k+1$ respectively. Therefore the slope of a subsheaf in $F|_P$ is less or equal to $k+1$ and the slope of a factorsheaf is greater or equal to k.

Now in order to prove the lemma let us suppose that we have an exact sequence

$$0 \longrightarrow A \longrightarrow F \longrightarrow B \longrightarrow 0$$

and B has no torsion. This implies that

$$0 \to A|_P \to F|_P \to B|_P \to 0$$

is exact.

Then the slope of a subsheaf $A|_P$ (which is equal to $\nu(A) \cdot P$) has to be $\leq k+1$ and the slope of a factorsheaf $B|_P$ (which is equal to $\nu(B) \cdot P$) is $\geq k$.□

Let us again denote by $G_{min}F$ and $G_{max}F$ the first and the last factors of the Harder-Narasimhan filtration in F.

Proof of the proposition. . We can apply the previous lemma to $A = G_{max}F$ and $B = G_{min}F$ and $P = P_s$. Then for $\nu = \nu(A) - \nu(B)$ we get

$$\nu \cdot P_s \leq 1.$$

But as $-K_X$ is ample so

$$\nu \cdot P_s \leq 1 \leq P_s \cdot (-K_X).$$

Thus we can make the following calculation

$$\mu(G_{max}F) - \mu(G_{min}F) = \nu \cdot (-K_X) = \sum \nu \cdot P_i \leq \sum P_s \cdot (-K_X) = K_X^{\cdot 2}.$$

This provide us with what is needed to conclude that F is restricted.□

Proposition 6.5 *Let us suppose that we have a smooth versal family $\mathcal{F}$ of sheaves without torsion on X with parameters Z and there is a splitting of $-K_X$ into lines $\{P_s\}$ such that for any $z \in Z$ and $P = P_s$*

$$Ext^2(F_{(z)}, F_{(z)}(-P)) = 0.$$

Then there exists a nonempty open subset $U \subset Z$ such that the family $\mathcal{F}|_U$ is restricted.

Proof of the proposition. . We follow here the reasoning of Drezet and Le Potier [DL, p.231].

Consider for $z \in Z$ a moduli space M of deformations for $F_{(z)}|_P$. M is a smooth variety with a tangent space isomorphic to $\mathrm{Ext}^1(F_{(z)}|_P, F_{(z)}|_P)$.

Therefore we have a morphism for some neighborhood Z' of z

$$\rho : Z' \longrightarrow M,$$

with a corresponding morphism of tangent spaces

$$T_z Z' \longrightarrow \mathrm{Ext}^1(F_{(z)}|_P, F_{(z)}|_P).$$

But $T_z Z' = T_z Z$ and the above morphism fits into a commutative diagram

$$\begin{array}{ccc} T_z Z & \longrightarrow & \mathrm{Ext}^1(F_{(z)}, F_{(z)}) \\ \| & & \downarrow \\ T_z Z' & \longrightarrow & \mathrm{Ext}^1(F_{(z)}|_P, F_{(z)}|_P) \end{array}$$

where the upper horizontal arrow is a Kodaira-Spencer morphism and the right vertical arrow is a morphism from the following exact sequence related to the restriction onto P:

$$\mathrm{Ext}^1(F_{(z)}, F_{(z)}) \to \mathrm{Ext}^1(F_{(z)}|_P, F_{(z)}|_P) \to \mathrm{Ext}^2(F_{(z)}, F_{(z)}(-P))$$

and it is surjective by our presuppositions. This implies that ρ is a submersion in some neighborhood of z, hence there is a nonempty open set in Z', which is mapped by ρ onto nonempty open subset in M of points corresponding to rigid sheaves on P.

As we have a finite number of lines in question, so there is a nonempty open set U such that for $z \in U$ all the restrictions $F_{(z)}|_{P_i}$ are rigid. Then from the proposition above we conclude that $F_{(z)}$ is restricted.□

7 Versal families

There is a general way to construct families of sheaves which could be considered as a variant of the monad technique related to exceptional systems. We will use it here to produce smooth restricted versal families.

Let us recall the definition of exceptional systems ([Go],[KO],[R4]).

Definition 7.1 *Sheaves $E_0, \ldots, E_m$ are called exceptional system if they are exceptional and for $i < j$*

$$Ext^k(E_j, E_i) = 0$$

for all k.

An exceptional system is called complete if it generates the derived category; then its image in the Grothendieck group of sheaves provides a base for the group. It follows from results of Orlov ([Or],[KO]) that for a Del Pezzo surface X over an algebraically closed field an exceptional system $E_0, \ldots, E_m$ is complete if and only if $m = t + 2$.

Proposition 7.2 *Suppose that there are given:*
a complete exceptional system $E_0, \ldots, E_m$ of vector bundles on X,
sets I^+, I^-, where $I^+ \cup I^- = \{0, \ldots, m\}$ and $I^+ \cap I^- = \emptyset$,
and nonnegative integer numbers n_i, $i = 0, \ldots, m$
such that they satisfy the following conditions:

Hm *$Ext^q(E_i, E_j) = 0$ for $i < j$ and $q \neq 0$;*

Gls *sheaves* $\mathcal{H}om(E_i, E_j)$ *for* $i \in I^-$ *and* $j \in I^+$ *are generated by global sections;*

Rk $\sum_{i\in I^+} n_i r(E_i) - \sum_{i \in I^-} n_i r(E_i) \ > 1.$

Then there exists a nonempty open set U,

$$U \subset Hom(\bigoplus_{i\in I^-} n_i E_i, \bigoplus_{i\in I^+} n_i E_i),$$

which consists of monomorphisms and such that for inverse images $\mathcal{E}_i$ *of* E_i *onto* $X \times U$ *there is an exact sequence*

$$0 \longrightarrow \bigoplus_{i\in I^-} n_i\mathcal{E}_i \xrightarrow{\Phi} \bigoplus_{i\in I^+} n_i\mathcal{E}_i \longrightarrow \mathcal{F} \longrightarrow 0 \tag{5}$$

where:
(a) the morphism Φ *is defined so that for* $u \in U$ *the restriction* Φ *to* $X \times u$ *coincides with a morphism:*

$$u : \bigoplus_{i\in I^-} n_i E_i \longrightarrow \bigoplus_{i\in I^+} n_i E_i\,;$$

(b) $\mathcal{F}$ *is a smooth versal family of sheaves on* X *with the base* U.

Suppose in addition that it is given a splitting of $-K_X$ *into lines*

$$-K_X = \sum P_s$$

and that for any line $P = P_s$:

R1 $Ext^1(E_i, E_j(-P)) = 0$ *for* $i \in I^-$ *and* $j \in I^+$,

R2 $Ext^2(E_i, E_j(-P)) = 0$ *for either* $i, j \in I^-$ *or* $i, j \in I^+$.

Then it is possible to find the set U *above such that the family* $\mathcal{F}$ *happens to be a restricted smooth versal family.*

Combining Theorem 4.2 and the proposition we conclude that there is a way to find stable sheaves on X with given Chern data. Let us write down this conclusion as a theorem.

Theorem 7.1 *Let X be a Del Pezzo surface and $-K_X = \sum P_s$ be a splitting of $-K_X$ into lines . Suppose that there are given: a complete exceptional system $E_0, \dots, E_m$ on X and an element $c \in M_X^+$ that satisfies **DL**-condition. Suppose that sets I^+, I^- and numbers n_i, $i = 0, \dots, m$ are defined in a way that*

$$c = \sum_i a_i\, Chd(E_i) \ \ \textit{and}\ a_i = n_i > 0 \ \textit{for}\ i \in I^+, \quad a_i = -n_i < 0 \ \textit{for}\ i \in I^-.$$

*If for the system $E_0, \dots, E_m$ and for I^+, I^-, $\{n_i\}$ constructed above the conditions: **Hm** , **Gls**, **Rk**, **R1**, **R2** are valid, then there exist a stable sheaf F on X with $Chd(F) = c$.*

All that we need is to prove the proposition.

Proof of Proposition 7.2. Let it be

$$S = \mathrm{Hom}(\bigoplus_{i \in I^-} n_i E_i, \bigoplus_{i \in I^+} n_i E_i)$$

and let us consider a set Y of points $(x, s) \in X \times S$ such that the restriction $s(x)$ of s is on the fiber at a point x of the vector bundles is not a monomorphism. Clearly Y is an algebraic subset in $X \times S$ and because of **Gls**. its codimension could be calculated as

$$\sum_{i \in I^+} n_i r(E_i) - \sum_{i \in I^-} n_i r(E_i) \ + \ 1.$$

So it is bigger than 2 by assumption **Rk**.

Hence the projection of Y on S, which coincides with the set of non-monomorphisms, has positive codimension. Thus there is nonempty open set U of monomorphisms in S.

The construction of Φ is fairly standard and the exact sequence (5) provides us with a flat family $\mathcal{F}$. To check that $\mathcal{F}$ is smooth versal we should first prove that

$$\mathrm{Ext}^k(F_{(u)}, F_{(u)}) = 0 \text{ for } k = 2 \text{ and } u \in U.$$

But $F_{(u)}$ is quasi-isomorphic to a complex $R_{(u)}$,

$$R_{(u)} = \left[\ \dots \longrightarrow 0 \longrightarrow \bigoplus_{i \in I^-} n_i E_i \xrightarrow{\ u\ } \bigoplus_{i \in I^+} n_i E_i \longrightarrow 0 \longrightarrow \dots \right]$$

and it is well known that the cohomologies $\mathrm{Ext}^k(F_{(u)}, F_{(u)})$ coincides the hyperhomology of $R_{(u)}$ and the latter can be evaluated via the following spectral sequence.

Lemma 7.3 *Let $A^{\cdot}$, $B^{\cdot}$ be bounded complexes and suppose that A^m are locally free then there is a spectral sequence abutting to hypercohomology*

$$\boldsymbol{H}^k(\mathcal{H}om^{\cdot}(A^{\cdot}, B^{\cdot})),$$

(here $\mathcal{H}om^{\cdot}$ is a complex of sheaves of local homomorphisms) and such that

$$E_1^{p,q} = \bigoplus_i Ext^q(A^i, B^{i+p}).$$

In our situation higher cohomologies between E_i and E_j are trivial hence we conclude that the above spectral sequence degenerates at E_1 and

$$\mathrm{Ext}^k(F_{(u)}, F_{(u)}) = \mathrm{H}^k(\mathrm{Hom}^{\cdot}(R_{(u)}, R_{(u)}))$$

(here $\mathrm{Hom}^{\cdot}$ is a complex of global homomorphisms).

This way we get at once that $\mathrm{Ext}^2(F_{(u)}, F_{(u)}) = 0$ and that the natural morphism

$$\mathrm{Hom}(\bigoplus_{i \in I^-} n_i E_i, \bigoplus_{i \in I^+} n_i E_i) \longrightarrow \mathrm{Ext}^1(F_{(u)}, F_{(u)})$$

which arise in this computation is an epimorphism. Then Lemma (1.6) in [DL] states that this morphism coincides with a Kodaira-Spencer morphism so we have got proved the property **KS**.

According to Proposition 6.5 in order to get a restricted family it is sufficient to check the cohomological conditions:

$$\mathrm{Ext}^2(F_{(u)}, F_{(u)}(-P)) = 0,$$

but these cohomologies could also be computed as hypercohomologies and we can apply the above lemma.

Conditions **R1** and **R2** imply that for the corresponding spectral sequence we have got

$$\mathrm{E}_1^{p,q} = 0 \text{ for } p + q = 2,$$

hence $\mathrm{Ext}^k(F_{(u)}, F_{(u)}(-L)) = 0$ for $k = 2$ and this is what needed.□

8 Stable sheaves with given Chern data

Here we prove that it is possible to find a stable sheaf with a given Chern datum for some Del Pezzo surfaces.

Theorem 8.1 *Let $X = P^2_{(1)}$ (a Del Pezzo surface arising by blowing up a point in $\boldsymbol{P^2}$).*

*If $c \in M^+_X$ satisfies the **DL**-condition, then there exist a stable sheaf F on X with $Chd(F) = c$.*

If $c \in M^+_X$ satisfies the conditions DL1, DL2, DL3, then there exist a semi-stable sheaf F on X with $Chd(F) = c$.

Corollary 8.1 *If $X = P^2_{(1)}$, $c \in M_X$, $r(c) \geq 1$ and $\Delta(c) \geq 1$ then there exist a stable sheaf F on X with $Chd(F) = c$.*

The same result was proved for $X = \boldsymbol{P^2}$ in [DL] and for $X = Q$ in [R2].[1]

Proof of the theorem. . As it was mentioned after Theorem 4.2 the existence of stable sheaves for $r(c) = 1$ is trivial so we will suppose that $r(c) > 1$ for the rest of the proof.

Because of Theorem 7.1 all we have to do is to find an appropriate exceptional system. For this we need to make some calculations so let us first fix notations.

Let L be the blown up line in $X = P^2_{(1)}$ and H be a preimage in X of a general line in $\boldsymbol{P^2}$. Then H, L is a base for Pic$X$ with the following intersection numbers:

$$H \cdot H = 1, \quad H \cdot L = 0, \quad L \cdot L = -1,$$

The canonical divisor K_X is equal to $K_X = -3H + L$, and $K_X^2 = 8$.

In order to fix a splitting for $-K_X$ we put $P_1 = H - L$, $P_2 = P_3 = H$. We can chose the exceptional system $E_1, \ldots, E_4$ as

$$\mathcal{O}(-2H + L + D),\ \mathcal{O}(-H + D),\ \mathcal{O}(-H + L + D),\ \mathcal{O}(D)$$

where the choice of a divisor D depends on c and it is specified in the following lemma.

[1]The had announced in [R3] that it is true for any Del Pezzo surface but the proof happened to be incomplete.

Lemma 8.2 *For any c from the theorem there exist $D \in PicX$ and nonnegative numbers $\{n_i | i = 1, \ldots, 4\}$ such that*

$$\sum_{i>1} n_i\, Chd(E_i) - n_1\, Chd(E_1) = c$$

It follows from here that $I^- = \{1\}$, $I^+ = \{2, 3, 4\}$.

We postpone the proof of the lemma for a little while and continue with the theorem. Let us first check the conditions on an exceptional system according to Theorem 7.1 The homomorphism and Ext spaces between E_i do not depend on the shift by D, whence it is sufficient to prove the conditions providing $D = 0$.

Condition **Hm**:
As it was proven in [Go] for an exceptional pair A, B on X the property

$$\mathrm{Ext}^k(A, B) = 0 \text{ for } k > 0$$

is equivalent to $\mu(A) \le \mu(B)$.

Therefore we need to calculate values of μ for the elements of the system. The result is

$$\begin{array}{ll} \mu(\mathcal{O}(-2H + L) = -5, & \mu(\mathcal{O}(-H) = -3, \\ \mu(\mathcal{O}(-H + L) = -2, & \mu(\mathcal{O}) = 0 \end{array}$$

From this it follows that **Hm** is valid.

Condition **Gls**:
It is clear that sheaves

$$\begin{array}{l} \mathcal{H}om(E_1, E_2) = \mathcal{O}(H - L), \\ \mathcal{H}om(E_1, E_3) = \mathcal{O}(H), \\ \mathcal{H}om(E_1, E_4) = \mathcal{O}(2H - L) \end{array}$$

are generated by global sections.

Condition **Rk**:
It follows from Lemma 8.2 that

$$\sum_{i \in I^+} n_i r(E_i) - \sum_{i \in I^-} n_i r(E_i) \;=\; r(c)$$

but as $r(c) > 1$, we conclude that **Rk** is valid.

Condition **R2**:
Is is proved in [G] that exceptional sheaves on X are stable, whence by Proposition 3.11 in order to prove that

$$\mathrm{Ext}^2(E_i, E_j(-P)) = 0 \quad \text{for } P = P_k$$

it is sufficient to check that $\mu(E_j) - 3 > \mu(E_i) - 8$ or that

$$\mu(E_i) - \mu(E_j) < 5.$$

But the above calculations show that if $i, j \in I^+$ or $i, j \in I^-$ then

$$\mu(E_i) - \mu(E_j) \leq 3 < 5$$

thus we have got what is needed here.

Condition **R1**:
We are to check that $\mathrm{Ext}^1(E_1, E_j(-P)) = 0$ for $j = 2, 3, 4$. This amounts to show that 1-cohomology for the following sheaves are equal to zero:

$$\begin{array}{lll} \mathcal{O}, & \mathcal{O}(L), & \mathcal{O}(H), \\ \mathcal{O}(-L), & \mathcal{O}, & \mathcal{O}(H-L) \end{array}$$

This is just an elementary computation.

So we get the needed conditions checked and in order to finish the proof all we are to do is to prove Lemma 8.2.

Proof of Lemma 8.2. There is a way to compute $\{n_i\}$ by means of the right dual exceptional system ([Go])

$$E_4^*,\ E_3^*,\ E_2^*,\ E_1^*$$

which is an exceptional system having the property:

$$\chi(E_i^*,\ E_j) = \varepsilon_i \delta_j^i \quad \text{where } \varepsilon_i = +1 \text{ or } -1.$$

For our case one can easily check that the dual system is:

$$\mathcal{O}(D),\ \mathcal{O}(H - L + D),\ \mathcal{O}(L + D),\ \mathcal{O}(H + D),$$

and

$$\varepsilon_1 = +1,\ \varepsilon_2 = -1,\ \varepsilon_3 = -1,\ \varepsilon_4 = +1.$$

So we get the following:

$$\begin{aligned} n_1 &= -\chi(\mathcal{O}(H+D), c), \\ n_2 &= -\chi(\mathcal{O}(L+D), c), \\ n_3 &= -\chi(\mathcal{O}(H-L+D), c), \\ n_4 &= +\chi(\mathcal{O}(D), c), \end{aligned}$$

and now our task is to show that it is possible to find D such that n_i are nonnegative. It is the same as to find a solution for the system of inequalities:

$$\left\{ \begin{array}{rl} \chi(\mathcal{O}(H), & \mathcal{O}(c-D)) \leq 0 \\ \chi(\mathcal{O}(L), & \mathcal{O}(c-D)) \leq 0 \\ \chi(\mathcal{O}(H-L), & \mathcal{O}(c-D)) \leq 0 \\ \chi(\mathcal{O}, & \mathcal{O}(c-D)) \geq 0 \end{array} \right. \tag{6}$$

By means of the Riemann-Roch formula it is possible to rewrite the system in more explicite form as follows. Let us denote

$$\nu = \nu(c),\ \Delta = \Delta(c),\ \rho = \frac{1}{2}K_X,$$

and $Z = \rho + \nu - H - D = xH + yL$.
Then

$$\chi(\mathcal{O}(N), c - D) = r(c)\left(\frac{1}{2}(Z-N)^{\cdot 2} - \frac{1}{2}\rho^{\cdot 2} + 1 - \Delta\right)$$

As we have $\frac{1}{2}\rho^{\cdot 2} = 1$ so system (6) is equivalent to

$$\left\{ \begin{array}{ll} (Z-H)^{\cdot 2} & \leq \varepsilon \\ (Z-L)^{\cdot 2} & \leq \varepsilon \\ (Z-H+L)^{\cdot 2} & \leq \varepsilon \\ (Z)^{\cdot 2} & \geq \varepsilon \end{array} \right. \tag{7}$$

where $\varepsilon = 2\Delta$.
In a coordinate form this means

$$\left\{ \begin{array}{cccl} (x-1)^2 & - & y^2 & \leq \varepsilon \\ x^2 & - & (y-1)^2 & \leq \varepsilon \\ (x-1)^2 & - & (y+1)^2 & \leq \varepsilon \\ x^2 & - & y^2 & \geq \varepsilon \end{array} \right. \tag{8}$$

It is sufficient to prove that for any $z' \in \boldsymbol{R}^2$ one could find $n \in \boldsymbol{Z}^2$ such that $z = z' + n$ is a solution for the system (8).

Let S be the set of solutions for (8) in $\boldsymbol{R}^2$. It is enough to prove that there are $M \subset \boldsymbol{R}^2$ and $\{m_p\} \subset \boldsymbol{Z}^2$ with a property:

for any $z' \in \boldsymbol{R}^2$ there exists $n \in \boldsymbol{Z}^2$ such that

$$z = z' + n \in \bigcup_p (S \cap M - m_p) \subset \bigcup_p (S - m_p).$$

For the following we take

$$M = \{(x, y) \mid 0 < x + y \leq 1\}$$

and we leave to the reader to check that $S \cap M$ coincides with the set of solutions for the following system:

$$\left\{ \begin{array}{ccccc} (x-1)^2 & - & (y+1)^2 & \leq \varepsilon \\ x^2 & - & y^2 & \geq \varepsilon \\ & 0 < x + y \leq 1 & & \end{array} \right. \tag{9}$$

Let us put $m_p = (p, -p, 0, 0)$ for $p \in \boldsymbol{Z}$. It is clear from the inequalities (9) that

$$\bigcup_p (S \cap M - m_p) = M.$$

But for any $z' \in \boldsymbol{R}^2$ there exists $n \in \boldsymbol{Z}^2$ such that $z = z' + n \in M$, and we have got what was needed.□

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ALGEBRAIC STABILITY: SCHUR LEMMA AND CANONICAL FILTRATIONS.

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June 15, 1996

ABSTRACT.
The main goal of the article is to give the general definition of algebraic stability that would permit to consider stalility not only for algebraic vector bundles or torsion-free coherent sheaves but for the whole category of coherent sheaves in an unified way.

We present an axiomatic description of the algebraic stability on an abelian category and prove some general results. Then the stability for coherent sheaves on a projective variety is constructed which generalizes Gieseker stability. Stabilities for graded modules and for quiver representations are also discussed. The constructions could be used for other abelian categories as well.

The idea to generalize stability has appealed to the author because it is quite inconvenient when stability considerations were restricted to the torsion-free sheaf subcategory that is not abelian (see for example [OSS], ch.2). Here in the section 2 we present the definition of stability for coherent sheaves in general.[1]

The section 1 is devoted to the definition and basic properties of a general algebraic stability. Then we discuss possible ways to construct stabilities.

The author would like to thank E.Schrödinger International Institute where the first version of the text was written.[2]

[1]When a preliminary version of this text had been written the author found the article [M] where stability for "coherent sheaves of pure dimension d" (thus for torsion sheaves as well) is considered. Although definitions of the stability proposed in [M] and in this paper are different there is some commonality between them and the sets of stable sheaves appear to be the same in both approaches. Hence the results of [M] about the moduli spaces for stable coherent sheaves are valid for stable sheaves in our sense as well.

[2]The research was partly supported by INTAS grant.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

1. General algebraic stability.

Let $\mathcal{A}$ be an abelian category.

Remark. We will discuss later the cases when $\mathcal{A}$ is the category of algebraic coherent sheaves on a projective variety over a field $\mathbf{k}$, the category finitely generated graded R-modules over a polynomial $\mathbf{k}$-algebra R, and the category of representations of a quiver.

The main ingredient needed to define stability in $\mathcal{A}$ is a stability order on the objects of $\mathcal{A}$.

Definition 1.1. An order on nonzero objects on $\mathcal{A}$ is called a stability order if:
Given an exact sequence of nonzero objects

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

we have
(SS): (seesaw property)

$$\begin{aligned} A \prec B &\Leftrightarrow A \prec C \Leftrightarrow B \prec C, \\ A \succ B &\Leftrightarrow A \succ C \Leftrightarrow B \succ C, \\ A \asymp B &\Leftrightarrow A \asymp C \Leftrightarrow B \asymp C, \end{aligned}$$

Remark. We imply that for $A, B \in Obj\, \mathcal{A}$ either $A \prec B$, or $A \succ B$, or $A \asymp B$ is valid and that it is possible to have $A \asymp B$ even when $A \neq B$.

One can also deduce from the definition the following property.

Lemma 1.2. Given an exact sequence of nonzero objects

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

and an object D we have
(CM): (center of mass property)

$$\begin{aligned} A \prec D \text{ and } C \prec D &\Rightarrow B \prec D, \\ A \succ D \text{ and } C \succ D &\Rightarrow B \succ D, \\ A \asymp D \text{ and } C \asymp D &\Rightarrow B \asymp D. \end{aligned}$$

We leave it to the reader to prove the lemma.

Definition 1.3. Let us call B stable when B is nonzero and for a nontrivial subobject $A \subset B$ we have $A \prec B$.

Definition 1.4. Let us B call semi-stable when B is nonzero and for a nontrivial subobject $A \subset B$ we have $A \preccurlyeq B$.

Because of the seesaw property of the order one can use factorobjects in the above definitions as well:
B is stable if and only if $B \prec C$ for a nontrivial factorobject C,
B is semi-stable means $B \preccurlyeq C$ for a nontrivial factorobject C.

In a sense stable objects are similar to irreducible ones and we have a general Schur lemma type result.

Theorem 1. *Let A, B be semi-stable objects from $\mathcal{A}$ such that $A \succcurlyeq B$ and suppose there is a nonzero morphism $\varphi : A \to B$. Then:*

(a) $A \asymp B$,
(b) *if B is stable then φ is an epimorphism,*
(c) *if A is stable then φ is a monomorphism,*
(d) *if both A, B are stable then φ is an isomorphism.*

Corollary (Schur lemma). *Suppose that $\mathrm{Hom}(A, B)$ are finite dimensional vector spaces over a field $\mathbf{k}$ and that $\mathbf{k}$ is algebraically closed. Let A, B be stable objects such that $A \succcurlyeq B$. Then*

$$\text{if } \mathrm{Hom}(A, B) \neq 0 \text{ then } A \simeq B \text{ and } \mathrm{Hom}(A, B) = \mathrm{Hom}(A, A) = \mathbf{k}.$$

Remark. For our examples of coherent sheaves and graded R-modules Hom-s are finite dimensional vector spaces so the Schur lemma is valid.

To derive Corollary from the theorem we need only to mention the classical fact that a finite dimensional associative algebra, where a nonzero element is invertible, over an algebraically closed field is necessary the field itself.

Proof of Theorem 1. Let us consider the usual ker-im and im-coker exact sequences for φ

$$0 \longrightarrow K \longrightarrow A \longrightarrow I \longrightarrow 0, \qquad 0 \longrightarrow I \longrightarrow B \longrightarrow C \longrightarrow 0.$$

As $\varphi \neq 0$ so $I \neq 0$. By the definition of semi-stability

$$I \preccurlyeq B, \quad \text{and} \quad A \preccurlyeq I \quad \text{so} \quad A \preccurlyeq B.$$

But $A \succcurlyeq B$, so $A \asymp I \asymp B$, thus (a) is proved.

For (b) we need to mention that $I \neq B$ implies $I \prec B$ (because B is stable) in contradiction with $I \asymp B$ that we have got above. We proceed similarly with (c) and (d). □

We can also generalize the Harder-Narasimhan theorem for algebraic vector bundles in the following way.

Let us use in the following the convenient shorthand notations like $A \subset; \preccurlyeq B$ instead of writing $A \subset B$ and $A \preccurlyeq B$ (with obvious variations).

As usual we call B noetherian if an ascending chain in B stabilizes and say $\mathcal{A}$ is noetherian when any object of $\mathcal{A}$ is noetherian.

Definition 1.5. Let us call B quasi-noetherian (or q-noetherian) if a chain

$$A_1 \subset; \preccurlyeq A_2 \subset; \preccurlyeq \ldots$$

in B has to stabilize.

Of course the condition of being q-noetherian is weaker than being noetherian.

Definition 1.6. Let us call B weakly artinian (or w-artinian) if

(wa1): a chain

$$A_1 \supset; \prec A_2 \supset; \prec \ldots$$

in B has to be finite;

(wa2): a chain

$$A_1 \supset; \asymp A_2 \supset; \asymp \ldots$$

in B has to stabilize.

We call $\mathcal{A}$ w-artinian if any object A in $\mathcal{A}$ is w-artinian.

Theorem 2. *Suppose $\mathcal{A}$ is w-artinian and noetherian and B is an object of $\mathcal{A}$. Then B has a filtration*

$$B = F^0 \supset F^1 \supset \cdots \supset F^m \supset F^{m+1} = 0$$

such that:

(i) *factors $G^i = F^i/F^{i+1}$ are semistable,*
(ii) $G^0 \prec G^1 \prec \cdots \prec G^m$,

and the filtration is uniquelly defined by the properties (i),(ii).

We need to prove some propositions to get the theorem.

Proposition 1.7. *Let B be q-noetherian and w-artinian then it exist a subobject $B^{\#}$ in B such that:*

(a) *if $A \subset B$ is a subobject in B then $A \preccurlyeq B^{\#}$,*
(b) *if $A \subset B$ and $A \asymp B^{\#}$ then $A \subset B$,*

and it is defined uniquelly by these properties.

Clearly $B^{\#}$ would be semi-stable and B is semi-stable iff $B = B^{\#}$.

Let B be under conditions of Proposition 1.7 further on.

Lemma 1.8. *Let $A \subset B$. Then either A is semi-stable or there is a semi-stable $A' \subset B$ such that $A' \succ A$.*

Proof of the lemma. Let $A_1 = A$. If A_1 is not semi-stable then there is A_2 such that

$$A_1 \supset; \prec A_2$$

The same is valid for A_2 and so on. We have to come to a semi-stable subobject after a finite number of steps because the infinite chain

$$A_1 \supset; \prec A_2 \supset; \prec \ldots$$

does not exist in the w-artinian B.

Lemma 1.9. *Let C be a subobject in B. If there is $A \subset B$ satisfying $A \succ C$ then it exists $C' \subset B$ such that $C' \supset; \succ C$.*

Proof of the lemma. By Lemma 1.8 we can suppose that A is semi-stable. Now we have two standard exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A \cap C & \longrightarrow & A & \longrightarrow & U & \longrightarrow & 0, \\ 0 & \longrightarrow & C & \longrightarrow & A + C & \longrightarrow & U & \longrightarrow & 0. \end{array}$$

Because A is semistable, $A \cap C \preccurlyeq A$. Thus $A \preccurlyeq U$ by the seesaw property applied to the first sequence. But $C \prec A$ so $C \prec U$. Hence the second sequence implies that $C \prec (A+C)$ because of the seesaw property.

We see that $C' = A + C$ satisfies the lemma.

Proof of Proposition 1.7. The uniqueness of $B^{\#}$ is clear.

To prove the existence suppose to the contrary that for any subobject $B^{\#}$ in B either (a) or (b) is wrong.

Let B_0 be a subobject in B. If (a) is wrong for B_0 then by Lemma 1.9 it exists $B_1 \supset; \succ B_0$ and B_1 is strictly larger then B_0.

If (a) is valid for B_0 but (b) is wrong then it exists A, $A \asymp B_0$, A is not a subobject in B_0 and we can suppose that A is semi-stable by Lemma 1.8. Let $B_1 = B_0 + A$. Again it is easy to show that $B_1 \succcurlyeq B_0$ and B_1 is also strictly large than B_0.

So we have got $B_0 \subset; \preccurlyeq B_1$ anyway with B_1 is strictly larger then B_0. Repeating these arguments we find $B_2, B_3, \ldots$, such that

$$B_0 \subset; \preccurlyeq B_1 \subset; \preccurlyeq B_2 \ldots$$

with strict inclusion on every step. This is impossible because B is q-noetherian. $\square$

Suppose that $\mathcal{A}$ satisfies the conditions of Theorem 2.

Proposition 1.10. *Let B have a filtration with the properties (i),(ii) from Theorem 2. Then $B^{\#} = F^m$.*

Proof of the proposition. We can proceed by induction on m. For $m = 0$ the statement is trivial. So let us consider the general case.

Let A be a subobject in B. By induction $F^{m-1}/F^m = (A/F^m)^{\#}$, thus

$$A/(F^m \cap A) \preccurlyeq F^{m-1}/F^m = G^{m-1}.$$

But $G^{m-1} \prec G^m$ so $A/(F^m \cap A) \prec F^m$.

Notice that $(F^m \cap A) \preccurlyeq F^m$ because F^m is semi-stable. Then by the property (CM) we have

$$A \preccurlyeq F^m,$$

so F^m satisfies the condition (a) from Proposition 1.7.

To prove that F^m satisfies (b) consider $A \asymp F^m$. Now we have $(F^m \cap A) \preccurlyeq F^m \asymp A$. By (SS)-property this implies

$$A/(F^m \cap A) \succcurlyeq A,$$

provided that $A/(F^m \cap A) \neq 0$. But $A \asymp F^m = G^m \succ G^{m-1}$, hence

$$A/(F^m \cap A) \succ G^{m-1},$$

which is impossible by induction. Whence $A/(F^m \cap A) = 0$ and $F^m \cap A = A$. Thus we conclude that $A \subset F^m$ so F^m satisfies (b), and the uniqueness statement from Proposition 1.7 gives us exactly what is needed. □

Proof of Theorem 2. To construct the filtration let us define

$$F^0 = 0, \quad F^{-1} = B^{\#} \quad \text{and} \quad F^{-(i+1)} = \text{preimage } (B/F^{-i})^{\#}.$$

Clearly a factor $G^{-(i+1)} = (B/F^{-i})^{\#}$ is semi-stable and $G^{-(i+2)} \prec G^{-i+1}$ by (SS)-property applied to the sequence

$$0 \longrightarrow G^{-i+2} \longrightarrow F^{-i+2}/F^{-i} \longrightarrow G^{-i+1} \longrightarrow 0.$$

Since B is noetherian so $F^{-(m+1)} = B$ for some m and we have only to shift the indices to get the filtration as it is needed for the theorem.

To prove the uniqueness let us notice first that the last term of a filtration is uniquely defined by Proposition 1.10. From this it is easy to get the result by induction. □

One can also constract a Jordan-Hölder filtration in a semi-stable object.

Theorem 3. *Suppose $\mathcal{A}$ is w-artinian and noetherian and B is a semi-stable object of $\mathcal{A}$. Then B has a filtration*

$$B = F^0 \supset F^1 \supset \cdots \supset F^m \supset F^{m+1} = 0$$

such that:

(i) *factors $G^i = F^i/F^{i+1}$ are stable,*

(ii) $G^0 \asymp G^1 \asymp \cdots \asymp G^m$,

and the set $\{G_i\}$ of factors is uniquelly defined by the properties (i),(ii).

Proof of the theorem. Clearly the subobjects X in B such that $X \asymp B$ satisfy the ascending and descending chain conditions. So the result becomes the standard fact of basic algebra. □

2. Polynomial stability.

It is well known that the category of algebraic coherent sheaves on a projective variety is noetherian. The same is the category of finitely generated graded R-modules where the algebra R is commutative and finitely generated over a field $\Bbbk$. We would like to construct a natural stability order for these categories.

In both cases an object of a category has "a characteristic function". For a sheaf A on a variety X it is:

$$P_{[A]}(n) = \dim_{\Bbbk} \mathrm{H}^0(X, A(n)).$$

For a graded module $A = \oplus_{q \in \mathbf{Z}} A_q$ let it be the Hilbert-Samuel function:

$$P_{[A]}(n) = \dim_{\Bbbk} \oplus_{q > -\infty}^{q \leq n} A_q.$$

This justifies the following definition.

Definition 2.1. We say that a category $\mathcal{A}$ has a characteristic function if for any object A a function $P_{[A]} : \mathbb{Z} \to \mathbb{Z}$ is defined with the properties:

a) given an exact sequence $0 \to A \to B \to C \to 0$ we have

$$P_{[B]}(n) = P_{[A]}(n) + P_{[C]}(n) \quad \text{for} \quad n >> 0;$$

b) $P_{[A]} = 0$ iff $A = 0$;

c) for $n >> 0$ the function $P_{[A]}$ becomes a polynomial which has a positive highest coefficient when $A \neq 0$.

Remark. The functions discussed above for coherent sheaves and R-modules have these properties.

It follows from the definition that if $A \subset B$ then

$$P_{[A]}(n) \leq P_{[B]}(n) \quad \text{for} \quad n >> 0.$$

Without loss of generality we can suppose from now on that $P_{[A]}$ denotes the polynomial obtained via condition c) of the definition.

Definition 2.2. Let A, B be nonzero objects of $\mathcal{A}$ and

$$P_{[A]}(n) = \sum_{i=0}^{m} a_i n^i, \qquad P_{[B]}(n) = \sum_{i=0}^{m} b_i n^i$$

be the corresponding polynomials (m being unspecified large number). Denote

$$\lambda_{i,j} = \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}$$

and let

$$\Lambda_{(A,B)} = (\lambda_{m,m-1}, \lambda_{m,m-2}, \ldots, \lambda_{m,0}, \lambda_{m-1,m-2}, \ldots, \lambda_{2,1})$$

be the line of 2x2-minors of the matrix $\begin{bmatrix} a_m, & a_{m-1}, & \ldots, & a_0 \\ b_m, & b_{m-1}, & \ldots, & b_0 \end{bmatrix}$.

The polynomial order is define by conditions:

$A \asymp B \quad \Leftrightarrow \quad \Lambda_{(A,B)} = 0$

$A \prec B \quad \Leftrightarrow \quad$ the first nonzero term in $\Lambda_{(A,B)}$ is positive.

We have to check transitivity and the (SS) property.

Lemma 2.3. *If* $\deg P_{[A]} > \deg P_{[B]}$ *then* $A \prec B$.

Clearly the first nonzero minor in $\Lambda_{(A,B)}$ will be equal to the product of the highest cofficients of $P_{[A]}$ and $P_{[B]}$ which are positive.

Lemma 2.4. *If* $\deg P_{[A]} = \deg P_{[B]} = d$ *then* $A \prec B$ *if and only if*

$$\left(\frac{a_{d-1}}{a_d}, \frac{a_{d-2}}{a_d}, \dots, \frac{a_0}{a_d}\right) <_{lex} \left(\frac{b_{d-1}}{b_d}, \frac{b_{d-2}}{b_d}, \dots, \frac{b_0}{b_d}\right)$$

(where "$<_{lex}$" is used for "lexicographically less").

This amounts to the straight checking according to the definition.

It follows from Lemmas 2.3, 2.4 that the order is transitive.

Lemma 2.5. *The polynomial order is a stability order.*

Proof of the proposition. Let $0 \to A \to B \to C \to 0$ be an exact sequence. Then

$$P_{[B]}(n) = P_{[A]}(n) + P_{[C]}(n).$$

Hence

$$\begin{vmatrix} a_j & a_i \\ b_j & b_i \end{vmatrix} = \begin{vmatrix} a_j & a_i \\ a_j + c_j & a_i + c_i \end{vmatrix} = \begin{vmatrix} a_j & a_i \\ c_j & c_i \end{vmatrix} = \begin{vmatrix} a_j + c_j & a_i + c_i \\ c_j & c_i \end{vmatrix} = \begin{vmatrix} b_j & b_i \\ c_j & c_i \end{vmatrix}$$

and this implies the seesaw property. □

Proposition 2.6. *If the characteristic function with the properties a)-c) is defined for* $\mathcal{A}$, *then* $\mathcal{A}$ *is w-artinian.*

Proof of the proposition. By the contrary let us have an infinite chain

$$A_1 \supset; \preccurlyeq A_2 \supset; \preccurlyeq \dots,$$

with strict inclusions and let

$$P_r = \sum a_i^{[r]} x^i$$

be the corresponding polynomials. As $A_r \supset A_{r+1}$ strictly so

$$P_r(n) > P_{r+1}(n) \quad \text{for} \quad n >> 0.$$

Hence $\deg P_r \geq \deg P_{r+1}$ and therefore $\deg P_r = \deg P_{r+1} = \dots = d$ for large enough r.

Since the polynomials have positive integer values for $n >> 0$ so their highest coefficients $a_d^{[r]}$ belong to $\frac{1}{d!}\mathbb{N}$ and $a_d^{[r]} \geq a_d^{[r+1]}$ by the same reason so $a_d^{[r]} = a_d^{[r+1]} = \dots = q$ for some large r.

Then the property $P_r(n) > P_{r+1}(n)$ for $n >> 0$ is equivalent to

$$(q, a_{d-1}^{[r]}, a_{d-2}^{[r]}, \dots, a_0^{[r]}) >_{lex} (q, a_{d-1}^{[r+1]}, a_{d-2}^{[r+1]}, \dots, a_0^{[r+1]})$$

and this is the same as

$$\left(\frac{a_{d-1}^{[r]}}{q}, \frac{a_{d-2}^{[r]}}{q}, \dots, \frac{a_0^{[r]}}{q}\right) >_{lex} \left(\frac{a_{d-1}^{[r+1]}}{q}, \frac{a_{d-2}^{[r+1]}}{q}, \dots, \frac{a_0^{[r+1]}}{q}\right).$$

Because of Lemma 2.4 this means $A_r \succ A_{r+1}$ which contradicts to the presupposition that $A_r \preccurlyeq A_{r+1}$. □

3. Ratio of additive functions stability.

Another, perhaps more usual way to define a stability order ([F],[K],[LT],[OSS]) is via a ratio of two additive functions in a way that we are going to discuss in this section.

Definition 3.1. Let c and r be two additive functions on $\mathcal{A}$ and let $r(A) > 0$ for any nonzero object A of $\mathcal{A}$. We call the ratio

$$\mu(A) = \frac{c(A)}{r(A)}$$

the $(c{:}r)$-slope of A and define the slope order by conditions:

$$A \prec B \Leftrightarrow \mu(A) < \mu(B),$$
$$A \asymp B \Leftrightarrow \mu(A) = \mu(B).$$

This way stability for algebraic vector bundles is usually defined ([OSS],[M],[LT]).

Lemma 3.2. *The $(c{:}r)$-slope order is a stability order.*

Proof of the lemma. Let us notice that

$$\frac{c(A)}{r(A)} - \frac{c(B)}{r(B)} = \frac{1}{r(A)r(B)} \begin{vmatrix} r(B) & c(B) \\ r(A) & c(A) \end{vmatrix}.$$

So the ordering between A and B is determined by the positivity, negativity or nullity of the determinant

$$\begin{vmatrix} r(B) & c(B) \\ r(A) & c(A) \end{vmatrix}.$$

Now it is easy to see that the same transformations of determinants that were used in the proof of Lemma 2.5 also work here. We leave details to the reader. ☐

Remark. The function c is not obliged to take values in $\mathbb{Z}$. For example, $\mathbb{Q}$, $\mathbb{C}$ or an ordered $\mathbb{Z}$-module could be the target set as well. The latter one was the case for the stability used in ([R]).

A.D.King, [K] has used the notion of stability to construst moduli spaces of the representations of a quiver. In his case stability is discussed only for representations with a fixed K_0-image α and it depends on a choice of an additive function θ such that $\theta(\alpha) = 0$. This approach makes it possible to construct a moduli space but at the same moment it does not allow to compare stable representations with different α as their stabilities often have to be defined with respect to different functions θ.

In order to relate the King's definition with ours let us first remind the definition from the King's paper.

Definition 3.3. ([K],p.516) Let $\mathcal{A}$ be an abelian category and $\theta : K_0(\mathcal{A}) \to \mathbb{R}$ an additive function on the Grothendieck group. An object $M \in \mathcal{A}$ is called θ-semistable if $\theta(M) = 0$ and every subobject $M' \subset M$ satisfies $\theta(M') \geq 0$. Such an M is called θ-stable if the only subobjects M' with $\theta(M') = 0$ are M and 0.

Proposition 3.4. *Given a stability for an abelian category $\mathcal{A}$ that is defined via the $(c:r)$-slope order and $M \in \mathcal{A}$ let us consider an additive function θ such that*

$$\theta = -c + \frac{c(M)}{r(M)} r.$$

Then $\theta(M) = 0$ and M is stable by the $(c:r)$-stability if and only if it is θ-stable in the sense Definition 3.3.

Proof. Let us notice that

$$\theta(M') \geq 0 \quad \Leftrightarrow \quad - c(M') + \frac{c(M)}{r(M)} r(M') \geq 0 \quad \Leftrightarrow \quad \frac{c(M')}{r(M')} \leq \frac{c(M)}{r(M)}. \quad \square$$

So the King's results about moduli spaces θ-stable objects are relevant to our stability. The existence theorems from ([K]) for moduli spaces of θ-stable representations of a finite dimensional algebra imply the existence theorems for moduli spaces of $(c:r)$-stable representations.

Remark. The filtration of Theorem 2 depends on the stability. This is easy to check with the following example.

Let $(1) \longrightarrow (2) \longrightarrow (3)$ be a quiver of type A_3 and

$$V = \{V_1 \longrightarrow V_2 \longrightarrow V_3\}$$

be the representation of the quiver (for the definitions consult for example [K]).

We take $r(V) = \sum \dim V_i$, $c(V) = \sum a_i \dim V_i$. Let V' be the representation where $\dim V_i' = 1$ and the maps are isomorphisms.

The subobjects of V' are the following two:

$$V^{[1]} = \{V_1^{[1]} = 0,\ V_2^{[1]} = 0,\ V_3^{[1]} = V_3'\};$$
$$V^{[2]} = \{V_1^{[2]} = 0,\ V_2^{[2]} = V_2',\ V_3^{[2]} = V_3'\}.$$

As a result we conclude that if $a_1 = 3$, $a_2 = 2$, $a_3 = 1$ then V' is stable. But if $a_i = i$ then V' is not stable and

$$V' \supset V^{[2]} \supset V^{[1]} \supset 0$$

is the Harder-Narasimhan filtration in V'.

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