Birational automorphisms of Fano hypersurfaces

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To Professor Yurii Ivanovich Manin
on the occasion of his 60th birthday

Introduction

In [IM] V.A.Iskovskikh and Yu.I.Manin proved the famous theorem on birational rigidity of the smooth three-dimensional quartics $V_4 \subset \mathbf{P}^4$. The property of being birationally rigid (for the exact definition, see below) gives an exhaustive characterization of birational maps of a given variety. This result, being one of the most wonderful achievements of the modern birational geometry (probably, the most wonderful one), started a whole new field of research. The techniques, developed by V.A.Iskovskikh and Yu.I.Manin, was later modified, generalized and strengthened in different ways, and made it possible to prove a lot of new theorems, which outlined the general picture of birational classification in dimension three.

Almost immediately after the theorem on the quartic had been proved, several algebraic geometers from the Moscow school (first of all, the very authors of the theory) suggested that a similar result is true for any smooth hypersurface $V_M \subset \mathbf{P}^M$, $M \geq 5$. This conjecture for $M = 5$ was given to the author as the graduate problem. After the four-dimensional quintics have been proved to be birationally rigid [P1], there were no more doubts that the general conjecture was also true. However, the method of maximal singularities in its initial form worked succesfully for the varieties of a small degree only (see [I,P2]). Thus the sphere of its application was very narrow. It seemed that to prove the general conjecture one needed some radically new ideas (see, for instance, [P3], where the author announced a proof of this theorem — the proof which was never completed). Meanwhile, the paper [K] appeared, where J.Kollar, in particular, proved, in the spirit of fashionable p-characteristic tricks, that a general (in the complex-analytic sense) hypersurface of the degree M in $\mathbf{P}^M$ is non-rational (more exactly, that it has no structures of a conic bundle). His paper confirmed indirectly both that the above-spoken conjecture was true, and that to prove it, it was necessary to apply some modern concepts.

The more amazing it is that it turns out to be possible to prove birational rigidity of index 1 hypersurfaces by a direct generalization of the three-dimensional method of V.A.Iskovskikh and Yu.I.Manin! This is done in the present paper. Our construction, developed below, undoubtedly works in a much more general situation (for instance, for the index 1 complete intersections) and makes it possible to obtain the results on birational rigidity for Fano varieties of arbitrary dimension. They will be considered in subsequent papers.

The problem was solved during my stay at Max-Planck-Insitut für Mathematik in Bonn. I think that the creative, stimulating and generally wonderful atmosphere of the Insitute is responsible for the solution being so simple and clear.

1 Regular hypersurfaces $V_M \subset \mathbf{P}^M$

Fix an integer $M \geq 4$. The symbol $\mathbf{P}$ denotes the projective space $\mathbf{P}^M$ over the field $\mathbf{C}$ of complex numbers.

Consider an arbitrary hypersurface $V = V_M$ of the degree M in the projective space $\mathbf{P}$. Let $x \in V$ be a point, $(z_1, \dots, z_M)$ be a system of affine coordinates with the origin at x, and

$$f = q_1 + q_2 + \ldots + q_M$$

be the equation of the hypersurface V with respect to these coordinates, $\deg q_i = i$.

Definition 1. The point $x \in V$ is said to be *k-regular*, where $0 \leq k \leq M$, if the homogeneous polynomials

$$q_1, \dots, q_k$$

make a regular sequence — in other words, the codimension of the algebraic set, defined in $\mathbf{A}^M$ by the system of homogeneous equations

$$q_1 = \ldots = q_k = 0,$$

is equal to k exactly.

Obviously, this property of being k-regular does not depend on the choice of the affine coordinate system on $\mathbf{P}$ with the origin at the point x.

Definition 2. The hypersurface V is said to be *k-regular*, if every point $x \in V$ is k-regular.

The hypersurfaces $q_i = 0$ are obviously cones. Thus for an $(M-1)$-regular point the system of equations $q_1 = \ldots = q_{M-1} = 0$ gives a union of a finite set of lines, passing through the point x. In particular, on an $(M-1)$-regular hypersurface V for any point there is not more than a finite number of lines, passing through it.

It is also obvious that on a smooth hypersurface V the M-regular points form a (non-empty) open subset. On the other hand, there are no M-regular hypersurfaces.

Proposition 1. *$(M-1)$-regular hypersurfaces of the degree M form a dense open set in the space of all hypersurfaces of this degree.*

Proof is obtained by counting constants.

Set $\mathcal{P}_{k,K}$ to be the space of homogeneous polynomials of degree k in K variables, $\dim\mathcal{P}_{k,K}=\binom{k+K-1}{K-1}$. Consider the space of pairs

$$S=\{(x,F)\,\Big|\,x\in\mathbf{P},F\in H^0\left(\mathbf{P},\mathcal{O}_{\mathbf{P}}(M)\right)\}$$

with the natural projections $p\colon S\to\mathbf{P}$ and

$$q\colon S\to H^0\left(\mathbf{P},\mathcal{O}_{\mathbf{P}}(M)\right),$$

and inside it the incidence hypersurface

$$I=\{(x,F)|F(x)=0\}.$$

Let $Y\subset I$ be the set of all the pairs (x,F) such that the point x is not $(M-1)$-regular on the hypersurface $F=0$. To prove Proposition 1, it is sufficient to show that $q(Y)$ is a proper closed subset of the space of homogeneous polynomials $\mathcal{P}_{M,M+1}$. This last statement, in its turn, follows from

Lemma 1. *The codimension of the set $Y(x)=Y\cap p^{-1}(x)$ with respect to the ambient fiber $p^{-1}(x)=H^0\left(\mathbf{P},\mathcal{O}_{\mathbf{P}}(M)\right)$ is equal to $M+1$.*

Proof of the Lemma. Fix an arbitrary point $x\in\mathbf{P}$ and a system of affine coordinates $(z_1,\dots,z_M)$ with the origin at x. Set

$$Y_k(x)=\left\{(q_1,\dots,q_k)\in\prod_{i=1}^{k}\mathcal{P}_{i,M}\,\middle|\,\mathrm{codim}\{q_1=\ldots=q_k=0\}<k\right\}.$$

Obviously, $Y(x)=Y_{M-1}(x)$, so that it is sufficient to show that $\mathrm{codim}\,Y_k(x)=M+1$ for $k=1,\dots,M-1$ (the codimension is taken with respect to the fiber $p^{-1}(x)$). We shall do it by induction on k.

For $k=1$ this fact is obvious (smooth hypersurfaces do exist!).

Assume that $\mathrm{codim}\,Y_k(x)=M+1$ for some $k\in\{1,\dots,M-2\}$. Let us prove that the same is true for $k+1$, too.

Set

$$R_k=\prod_{i=1}^{k}\mathcal{P}_{i,M}\setminus Y_k.$$

Let

$$\pi\colon\prod_{i=1}^{k+1}\mathcal{P}_{i,M}=\prod_{i=1}^{k}\mathcal{P}_{i,M}\times\mathcal{P}_{k+1,M}\to\prod_{i=1}^{k}\mathcal{P}_{i,M}$$

be the natural projection. Since the intersection $I\cap p^{-1}(x)$ has codimension 1 with respect to the fiber $p^{-1}(x)$, it is sufficient to show that

$$\mathrm{codim}\,Y_{k+1}\cap\pi^{-1}(R_k)\geq M,$$

where the codimension is taken with respect to the space of polynomials $\prod\limits_{i=1}^{k}\mathcal{P}_{i,M}$. This inequality, in its turn, follows from another estimate:

$$\mu_{k+1}=\mathrm{codim}_{\pi^{-1}(q_*)}(\pi^{-1}(q_*)\cap Y_{k+1})\geq M,$$

where $q_* = (q_1, \ldots, q_k) \in R_k$ is an arbitrary regular sequence. It is this very inequality that we shall actually prove.

Naturally, $\pi^{-1}(q_*) \cong \mathcal{P}_{k+1,M}$. The condition

$$q_{k+1} \in \pi^{-1}(q_*) \cap Y_{k+1}$$

means that q_{k+1} vanishes along one of the components of the closed algebraic set

$$\{q_1 = \ldots = q_k = 0\} \subset \mathbf{A}^M.$$

Since the sequence q_* is regular, all the components of this set are $(M-k)$-dimensional cones with the vertices at the point $x = (0, \ldots, 0)$. Projectivizing, we get a closed projective set

$$Q = \bigcup Q_i \subset \mathbf{P}^{M-1},$$

all the components of which have codimension k.

Let $S \subset \mathbf{P}^{M-1}$ be a $(k-1)$-plane of general position, so that the restriction of the corresponding linear projection

$$\pi_S \colon \mathbf{P}^{M-1} - - \to \mathbf{P}^{M-1-k}$$

onto each component Q_i of the set Q is a finite morphism. Obviously, restricting the space of homogeneous polynomials

$$\pi_S^* H^0(\mathbf{P}^{M-1-k}, \mathcal{O}_{\mathbf{P}^{M-1-k}}(k+1))$$

onto each component Q_i, we get no non-trivial kernel. It means that we have constructed explicitly a space of homogeneous polynomials $W_{k+1} \subset \mathcal{P}_{k+1,M}$, which intersects Y_{k+1} by zero only. Consequently,

$$\mu_{k+1} \geq \dim W_{k+1} = \binom{M}{k+1} \geq M.$$

Q.E.D. for Lemma 1 and Proposition 1.

Corollary 1. *Let $|\lambda_k|$ be the linear system of divisors on V, which is cut out on V by the polynomials*

$$f_1 p_{k-1} + f_2 p_{k-2} + \ldots + f_k p_0,$$

where $1 \leq k \leq M-1$, $f_i = q_1 + \ldots + q_i$ and $p_i \in \mathcal{P}_{i,M}$ are arbitrary homogeneous polynomials of degree i in the variables z_*. Then the following conditions hold:

(i) $\operatorname{codim}_V \operatorname{Bs} |\lambda_k| \geq k-1$,

(ii) $|\lambda_k| \subset |-kK_V|$, *i.e. $|\lambda_k|$ is cut out on V by hypersurfaces of degree k in $\mathbf{P}$,*

(iii) $\operatorname{mult}_x |\lambda_k| \geq k+1$.

2 Birationally rigid varieties

Following [P3,P4], we introduce some terminology, which is convenient for exact description of the birational type of the variety V.

Definition 1. A pair (W,Y), where W is a projective variety such that $\dim W = \dim V$, $\operatorname{codim} \operatorname{Sing} W \geq 2$ and Y is a divisor on W is said to be a *test pair*, if the following conditions hold:
a) $|Y|$ is free from fixed components,
b) there exists a number $\alpha \in \mathbf{R}_+$ such that for any $\beta > \alpha, \beta \in \mathbf{Q}$ the linear system

$$|M(Y + \beta K_W)|$$

is empty for $M \in \mathbf{Z}_+, M\beta \in \mathbf{Z}$ (the adjunction break condition).

The minimal $\alpha \in \mathbf{R}_+$ satisfying the condition b) is said to be the *index* (or *threshold*) of the pair (W,Y). We shall denote it by $\alpha(W,Y)$.

Our aim is to describe all the possible birational maps

$$\chi : V - - \to W.$$

Here are the principal examples of test pairs with an explanation what do we need them for:

($\mathbf{P}^n$, hyperplane) — to decide whether V is rational or not; note that the index of this pair is equal to $1/(n+1)$;

($\varphi : W \to S$ — a Fano fibration, $Y = \varphi^{-1}$(very ample divisor on S)) — to decide whether there are structures of Fano fibrations on V; for instance, take a conic bundle or a Del Pezzo fibration; the index here is obviously zero;

$(V, -K_V)$ — to describe the group of birational automorphisms $\operatorname{Bir} V$; replacing here V by any other member of the same family of Fano varieties, we get some means of birational classification within the limits of the family.

Now we can make the first step of our study. Assume that there exists a birational map

$$\chi : V - - \to W.$$

Take the proper inverse image $|\chi| \subset |D|$ of the linear system $|Y|$ on V with respect to $|\chi|$. Denote by $\operatorname{Bs}|\chi|$ its base subscheme.

This system $|\chi|$ and this subscheme $\operatorname{Bs}|\chi|$ are the principal objects of our study.

Denote by $n(\chi)$ the index (threshold) of the pair (V,D).

Definition 2. V is said to be *birationally rigid*, if for any test pair (W,Y) and any birational map $\chi : V - - \to W$ there exists $\chi^* \in \operatorname{Bir} V$ such that

$$n(\chi \circ \chi^*) \leq \alpha(W,Y).$$

If, moreover, $\operatorname{Bir} V = \operatorname{Aut} V$, then V is said to be *birationally superrigid.*

Thus for a birationally superrigid variety V the inequality

$$n(\chi) \leq \alpha(W, Y)$$

is always satisfied. It is easy to see ([P4]) that birationally rigid varieties with $\mathrm{Pic} \cong \mathbf{Z}$ have no structures of a Fano fibration. In particular, they have no structures of a conic bundle or a Del Pezzo fibration.

Now we are ready to formulate our principal result.

Theorem. *Any $(M-1)$-regular hypersurface $V = V_M \subset \mathbf{P}^M$ is a birationally superrigid variety.*

Corollary. *The group of birational (=biregular=projective) automorphisms* $\mathrm{Bir}\, V = \mathrm{Aut}\, V$ *is finite, generically trivial. V has no structures of a Fano fibration. In particular, it is non-rational.*

3 Proof of the theorem

We use the language of discrete valuations in the form, in which it was developed in [P3,P4].

Fix an $(M-1)$-regular hypersurface $V = V_M \subset \mathbf{P}$ and a birational map

$$\chi: V - - \to W.$$

Assume that the statement of the theorem is not true: $n(\chi) > \alpha(W, Y)$. Let us show that this cannot be the case.

Note that for a smooth hypersurface $V \subset \mathbf{P}$ the threshold $n = n(\chi)$ has a simple meaning: the linear system $|\chi|$ is cut out on V by hypersurfaces of degree n.

Proposition 1 (Noether-Fano inequality). *There exists a (geometric [1]) discrete valuation $\nu \in \mathcal{N}(V)$, satisfying the inequality*

$$\nu(|\chi|) > nK(V, \nu).$$

Here $K(V, \nu)$ is the canonical multiplicity (discrepancy) of the valuation ν.

Proof: see, for instance, [P3] or [P4].

Obviously, there is an alternative: either the centre $Z(V, \nu)$ of the valuation ν on V is an irreducible cycle B of positive dimension, or $Z(V, \nu) = x \in V$ is a point.

In the first case we get the inequality

$$\mathrm{mult}_B\, |\chi| > n.$$

However, it was shown in [P3], that the multiplicity of $|\chi|$ along any cycle of positive dimension is not greater than n. Consequently, the first case does not realize.

In the second case by means of the higher-dimensional generalization of Iskovskikh-Manin's techniques, which was developed in [P2] and in its most clear form in [P4], we get the following

[1] Geometric means realizable by a prime Weil divisor on some model of the function field; not all the discrete valuations satisfy this property, as S.Mori pointed out to the author.

Proposition 2. *Let $D_i \in |\chi|$, $i=1,2$, be general divisors, $Z=(D_1 \bullet D_2)$ be the algebraic cycle of their scheme-theoretic intersection. Then*

$$\mathrm{mult}_x Z > 4n^2.$$

Let $Z=\sum a_i Z_i$, $a_i \in \mathbf{Z}_+$. Since $\deg Z = Mn^2$, where the degree is understood in the sense of the ambient projective space $\mathbf{P} \supset \mathrm{Supp}\, Z$, there exists an irreducible component $B = Z_i \subset V$, satisfying the inequality

$$\frac{\mathrm{mult}_x B}{\deg B} > \frac{4}{M}.$$

If $M=4$, we get an immediate contradiction: we have reproduced the original proof of birational rigidity of the three-dimensional quartic, which was given in [IM]. Assume that $M \geq 5$. It is at this step that the condition of the hypersurface V being $(M-1)$-regular starts working.

Let us construct a special sequence of irreducible cycles on V,

$$B_1, \dots, B_{M-3},$$

where $\mathrm{codim}\, B_i = i+1$ (with respect to V), satisfying the following property:

$$\frac{\mathrm{mult}_x B_i}{\deg B_i} > \frac{i+3}{M}.$$

For $i=1$ we take $B_1 = B$. Assume that the cycles $B_1, \dots, B_k$ are already constructed. In accordance with the Corollary 1 from Section 1, the base set of the linear system $|\lambda_{k+3}|$ is of codimension $\geq k+2$. Consequently, for a general divisor $L \in |\lambda_{k+3}|$ we get

$$B_k \not\subset L.$$

Thus the cycle $(L \bullet B_k)$ is effective and of codimension $k+2$. Furthermore,

$$\mathrm{mult}_x (L \bullet B_k) \geq (\mathrm{mult}_x L)(\mathrm{mult}_x B_k),$$

$$\deg(L \bullet B_k) = (k+3) \deg B_k.$$

Using the properties of the cycle B_k and of the linear system $|\lambda_{k+3}|$, we get the inequality

$$\mathrm{mult}_x (L \bullet B_k) > \frac{k+4}{M} \deg(L \bullet B_k).$$

Thus there exists an irreducible component B_{k+1} of the cycle $(L \bullet B_k)$, satisfying the required inequality

$$\frac{\mathrm{mult}_x B_{k+1}}{\deg B_{k+1}} > \frac{k+4}{M}.$$

Set $C = B_{M-3}$ to be the last cycle of the sequence just constructed. It is a curve on V, satisfying the inequality $\mathrm{mult}_x C > \deg C$. This is impossible!

The proof of the Theorem is complete.

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