

**Hamiltonian Flows of  
Multivalued Hamiltonians on  
Closed Orientable Surfaces**

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## Abstract

Consider a Morse closed 1-form  $\omega$  on a closed orientable surface  $M_g^2$ . We study a Hamiltonian flow on  $M_g^2$  determined by a multivalued Hamiltonian  $\omega$ . In generic situation, when  $\omega$  has maximal rank, (i.e., all periods of  $\omega$  are rationally independent) corresponding Hamiltonian system splits into several components filled with periodic trajectories, and several minimal components. Each minimal component is a surface of genus  $g_i$  with several holes; the sum  $g_1 + \dots + g_k$  of genres of minimal components equals original genus  $g$ .

We show, that for a generic Morse 1-form  $\omega$  the flow on each minimal component is ergodic, and is equivalent to a special flow over interval exchange transformation of  $4g_i - 4$  intervals on a circle, where  $g_i$  is a genus of the minimal component. The proof is based on a theorem due to H.Masur and W.Veech on unique ergodicity of a generic interval exchange transformation.

## 1 Structure of Hamiltonian flow, determined by a multivalued Hamiltonian.

Consider a closed 1-form  $\omega$  on a closed orientable surface  $M_g^2$  of genus  $g$ . We assume, that  $\omega$  is a Morse form, i.e,  $\omega$  has a finite collection of nondegenerate critical points (which in our case are minima, maxima, and saddles). We also assume, that different critical points of  $\omega$  do not belong to the same critical values, i.e., any two different saddles are not connected by a level of  $\omega$ . We consider only generic case, when  $\omega$  is of maximal rank,  $\text{rk } \omega = 2g$ , i.e, all periods of  $\omega$  are rationally independent.

**Proposition 1.1** *Hamiltonian system determined by a multivalued Hamiltonian  $\omega$  on a closed orientable surface  $M_g^2$  splits into several components filled with periodic trajectories, and several minimal components  $M_{g_1}^2, \dots, M_{g_k}^2$ ,  $1 \leq k \leq g$ ; each nonsingular trajectory is dense on a minimal component. Each minimal component is a surface of genus  $g_i$  with several holes; the sum  $g_1 + \dots + g_k$  of genres of minimal components equals original genus  $g$ .*

The flow on a minimal component  $M_{g_i}^2$ ,  $i = 1, \dots, k$ , is isomorphic to a special flow over interval exchange transformation of  $4g_i - 4$  intervals on a circle — a closed line in  $M_{g_i}^2$ , transversal to the flow.

For a generic Morse 1-form  $\omega$  the corresponding interval exchange transformation is uniquely ergodic; ergodic measure on the circle is determined by restriction of  $\omega$  to the closed transversal, representing the circle.

A notion of a generic Morse 1-form is precisely specified below. Proposition 1.1 is proved in sections 1.1–1.4.

When genus  $g_i$  of a minimal component equals 1, one should understand an “interval exchange transformation of 0 intervals” just as a rotation of a circle; the case of genus one is considered by V.I.Arnold, in [2]. Paper of Ya.G.Sinai and K.M.Khanin [16] proves, that these kind of special flows over rotation of a circle possesses mixing property.

**Problem 1.1** Consider generic special flow over interval exchange transformation on a circle as in Proposition 1.1. Does it possess mixing property?

\* \* \*

After distributing a preliminary version of the paper the author realized, that a major part of Proposition 1.1 is somehow known to those specializing in flows on surfaces or dynamical properties of leaves of closed 1-forms. In particular one can find a comment on unique ergodicity of a generic closed 1-form in the Thesis of G.Levitt [8]. Still, since part of the story belongs rather to un verbalized mathematical folklore, than to some accessible sources of information, one has to take care of the proofs, once he needs rigorous results. Besides there are many specialists, for whom the results are unknown yet. This exposition is addressed to them.

The author wishes to thank V.I.Arnold — his articles [2] and [3] and stimulating discussions with him inspired me to continue a study in this area. I am much obliged to A.Katok for several very essential conversations. In particular, it was his suggestion to use Sataev’s example. Besides the proof of unique ergodicity of a generic closed 1-form was in fact outlined by A.Katok. I wish to thank G.Levitt, who clarified me, which parts of the preliminary version of the paper have overlaps with the facts already described in the literature. I am grateful to S.P.Novikov, who once introduced me to this area, and whose challenging problems and conjectures stimulated this work. I am grateful to Ya.G.Sinai, who convinced me to print this paper. I would like to express my gratitude to Max-Planck-Institut für Mathematik in Bonn for hospitality while preparation of this and other papers.

## 1.1 Decomposition of the surface into components.

A connected component of a singular level of  $\omega$  passing through a saddle point may have a loop  $\gamma$ , or even two loops. Integral of  $\omega$  over  $\gamma$  is obviously zero. Since  $\omega$  is of maximal rank, it means, that cycle  $\gamma$  is homologous to zero,  $[\gamma] = 0$ ,  $[\gamma] \in H_1(M_g^2, \mathbb{Z})$ . Hence, when we cut  $M_g^2$  over  $\gamma$  we obtain two components<sup>1</sup>  $M_g^2 = W_1 \cup W_2$ . It may happen, that restriction  $\omega|_{W_1}$  or  $\omega|_{W_2}$  is exact on one of the components. Component  $W$ , where the form is exact, may contain other components  $W'_i$ , obtained by the cuts over loops on levels passing through the other saddles inside  $W$ . We will call (after V.I. Arnold, see [2]) a maximal component  $W$  of that type by a *trap*. Since by definition  $\omega$  is exact on any trap, all trajectories are closed inside the trap. Each trap is homeomorphic to a disk  $D^2$ .

Let us cut off all the traps. Since we did not lose any nontrivial cycles, we obtain a surface  $M'_g$  of genus  $g$  with several holes. The boundaries of the holes are closed loops on singular levels of  $\omega$ . When  $g > 1$  we may still have closed trajectories inside  $M'_g$ . Note, that we got rid of minima and maxima — all critical points of  $\omega$  on  $M'_g$  are of saddle type only. Let us count the number of inner saddles, without taking into consideration the saddles, which belong to the cuts.

**Lemma 1.1** *There are  $2g - 2$  inner saddles of  $\omega$  on  $M'_g$ .*

◁ Let us paste temporarily the holes on  $M'_g$  with disks. Consider a smooth extension of  $\omega$  to the disks, placing a single additional critical point on each disk — maximum or minimum. Count the Euler characteristic of a closed surface of genus  $g$  thus obtained as an algebraic sum of numbers of critical points of our extended 1-form:

$$2g - 2 = (\#\text{inner saddles}) \\ + [(\#\text{saddles on the cuts}) - (\#\text{minima} + \#\text{maxima})]$$

Since by construction there is a one-to-one correspondence between saddles on the cuts and cuts, the number inside rectangular brackets equals zero. ▷

Let us proceed with a surgery on  $M'_g$ . Some of  $2g - 2$  inner saddles on  $M'_g$  may have loops of singular levels, passing through these saddles. Let us cut  $M'_g$  over all such loops. We remind, that any such loop  $\gamma$  is homologous to zero in  $M'_g$ . Consider those connected components  $\hat{M}_{g_1}^2, \dots, \hat{M}_{g_k}^2$  thus obtained, for which  $\omega$  restricted to  $\hat{M}_{g_i}^2$  is not exact. Each  $\hat{M}_{g_i}^2$  is a surface of genus  $g_i \geq 1$  with several holes.

**Lemma 1.2** *The following equation is valid:*

$$g_1 + \dots + g_k = g$$

◁ To prove Lemma it is sufficient to show, that we can construct a basis of cycles on  $M_g^2$ , which does not intersect any cuts  $\gamma_j$ . Since all  $\gamma_j$  are homologous to zero, a desired basis can be easily constructed. ▷

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<sup>1</sup>Here and below we always consider *closed* components.

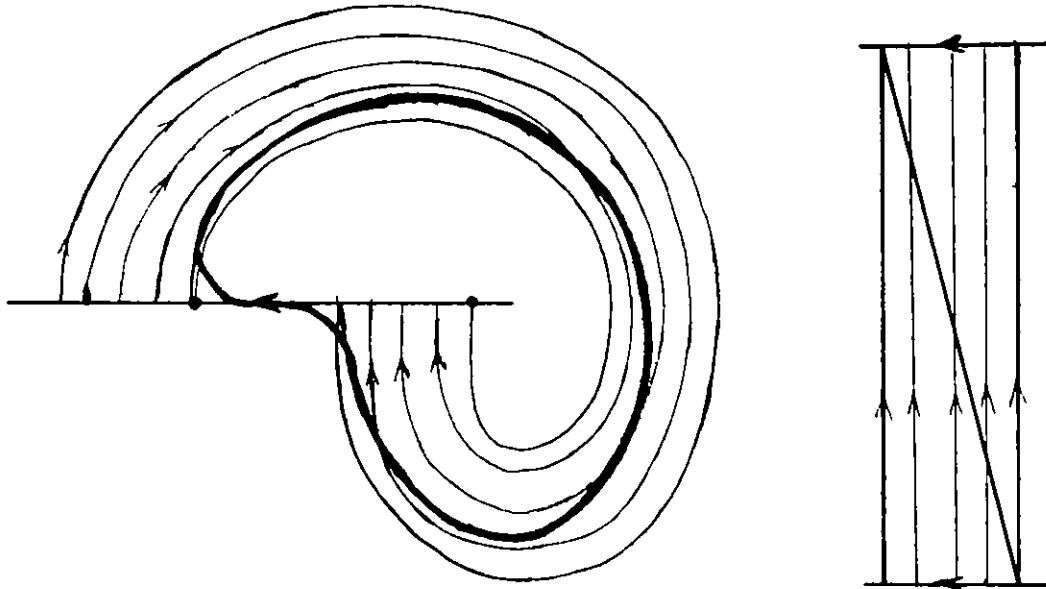


Figure 1: Construction of a closed transversal.

It is easy to see (cf. Lemma 1.1), that the number of inner saddles on the component  $\overset{\circ}{M}_{g_i}^2$ , of genus  $g_i$ ,  $i = 1, \dots, k$ , equals  $2g_i - 2$ . By *necks* we call maximal components  $N_q$  obtained by cutting of  $M'_g$  over loops of critical levels, for which restriction  $\omega|_{N_q}$  is exact. Each neck is diffeomorphic to a cylinder with several holes. Necks are of no interest for us, since all trajectories are closed on the necks.

Consider now some component  $\overset{\circ}{M}_{g_i}^2$ ,  $i = 1, \dots, k$ .

**Lemma 1.3** *The Hamiltonian flow defined by  $\omega$  on  $M_g^2$  preserves traps, necks, and components  $\overset{\circ}{M}_{g_i}^2$ ,  $i = 1, \dots, k$ . Each component  $\overset{\circ}{M}_{g_i}^2$  is a minimal component of the flow.*

*One can construct a closed transversal on each component  $\overset{\circ}{M}_{g_i}^2$  (which obviously does not intersect neither boundary of  $\overset{\circ}{M}_{g_i}^2$ , nor critical points of  $\omega$ ).*

In fact, the first part of the statement is trivial, and the second one is well-known. As for me, I know this method of constructing of a transversal from D.B.Fuchs, and from A.Katok.

◁ By construction each component  $\overset{\circ}{M}_{g_i}^2$  does not contain any closed nonsingular trajectory. By assumption there are no saddle connections by trajectories between saddles inside  $\overset{\circ}{M}_{g_i}^2$ . Hence each nonsingular trajectory is dense in  $\overset{\circ}{M}_{g_i}^2$ , i.e.,  $\overset{\circ}{M}_{g_i}^2$  is a minimal component.

Let us remind the way of constructing a closed transversal on  $\overset{\circ}{M}_{g_i}^2$ . Consider a small open interval  $I$  inside  $\overset{\circ}{M}_{g_i}^2$ , which is transversal to the flow, which neither intersects the boundary of  $\overset{\circ}{M}_{g_i}^2$ , nor contains singular points of  $\omega$ . Consider a nonsingular trajectory  $x = x(t)$ , starting at some point  $x_0 \in I$ ,  $x_0 = x(0)$ . Due to Poincaré recurrence theorem after some time  $t_1$  our trajectory will come back and will intersect  $I$ . Let  $x_1 \in I$ ,  $x_1 = x(t_1)$ ,  $t_1 > 0$  be this first intersection of our trajectory  $x = x(t)$  with  $I$ . Consider a narrow tubular neighborhood of our piece of trajectory  $x(t)$ ,  $0 \leq t \leq t_1$ . We suppose, that the neighborhood is narrow enough, so that it avoids critical points

of  $\omega$ . Hence, we may assume, that it is bounded from “bottom” by subinterval  $I_0 \subset I$ ,  $x_0 \in I_0$ ; it is bounded from “top” by  $I_1 \subset I$ ,  $x_1 \in I_1$ ; it is bounded from sides by pieces of trajectories. We suppose, that  $I_0 \cap I_1 = \emptyset$ . Deform now our piece of trajectory  $x(t)$ ,  $0 \leq t \leq t_1$  inside the tubular neighborhood as indicated on the figure 1. Smoothing conjugations with corresponding piece of  $I$  we complete a construction of a smooth closed line  $\tau$  transversal to our flow.  $\triangleright$

Hamiltonian flow determined by  $\omega$  generates a mapping  $T_{g_i} : \tau \rightarrow \tau$  of our closed transversal  $\tau$  to itself. Recall another well-known fact.

**Lemma 1.4** *Mapping  $T_{g_i}$  of a closed transversal  $\tau$  to itself generated by the flow on the minimal component  $\overset{\circ}{M}_{g_i}^2$  is an interval exchange transformation of  $4g_i - 4$  intervals on a circle  $\tau$ .*

$\triangleleft$  Suppose for a while, that  $\overset{\circ}{M}_{g_i}^2$  does not have any holes at all. Recall, that  $\overset{\circ}{M}_{g_i}^2$  is a minimal component of the flow, and there are no saddle connections by singular trajectories inside  $\overset{\circ}{M}_{g_i}^2$ . Hence all trajectories, starting at the points of  $\tau$ , except  $4g_i - 4$  ones, which hit the saddles (two trajectories for each saddle), would come back to  $\tau$  at a finite time. As for these  $4g_i - 4$  points at  $\tau$ , they are essentially singular — a left hand side limit of  $T_{g_i}$  at such point differs from a right hand side limit. Consider a measure  $\mu$  on  $\tau$ , determined by restriction of  $\omega$  to  $\tau$ . By construction of  $\tau$  the form  $\omega$  is nondegenerate on  $\tau$ . Hence our measure is isomorphic to a canonical measure on a circle. It is easy to see, that the mapping  $T_{g_i}$  preserves this measure. Hence, in the case when  $\overset{\circ}{M}_{g_i}^2$  does not have any holes at all,  $T_{g_i}$  is an interval exchange transformation of  $4g_i - 4$  intervals on the circle  $\tau$ .

Now recall, that we may have some holes on  $\overset{\circ}{M}_{g_i}^2$ . Actually we cheated slightly in formulation of the lemma. Mapping  $T_{g_i}$  may have several additional singularities inside exchanging intervals. These singularities correspond to the saddles on the boundary of  $\overset{\circ}{M}_{g_i}^2$ , i.e., on the singular levels of  $\omega$ , which bound our holes. But the mapping  $T_{g_i}$  can be easily extended to these singularities — for each singular point of that kind a left hand side limit of  $T_{g_i}$  at this point coincides with a right hand side limit.  $\triangleright$

**Lemma 1.5** *The flow determined on a minimal component  $\overset{\circ}{M}_{g_i}^2$  is isomorphic to a special flow over interval exchange transformation  $T_{g_i}$  of  $4g_i - 4$  intervals on a circle.*

*A function, determining a special flow, is a time interval between two consecutive intersections of the transversal by a trajectory. This function is smooth, except a finite collection of points, where it has logarithmical singularities. The set of singularities is a union of two sets — a set of singular points of the interval exchange transformation, and a set of singularities inside exchanged intervals; singularities of the second type correspond to saddles on the boundary of  $\overset{\circ}{M}_{g_i}^2$ .*

$\triangleleft$  The proof is analogous to the proof of Theorem 4 in [2].  $\triangleright$

Lemma 1.5 proves the first part of Proposition 1.1. Note that this result is very close to a more general one obtained in [10].



## 1.2 . Properties of interval exchange transformations $T_{g_i}$ .

To avoid multiple indexes suppose for a while, that we have only one minimal component  $\mathring{M}_g^2$  of the flow determined by  $\omega$ . Consider inner saddles of  $\omega$  inside  $\mathring{M}_g^2$  and chose all but one from them. We will denote these  $2g - 3$  chosen saddles by  $S_1, \dots, S_{2g-3}$ . We will study now in more details an interval exchange transformation  $T$  induced by the flow on a smooth closed curve  $\tau \subset \mathring{M}_g^2$  transversal to the flow.

Let us cut our closed transversal  $\tau$  at some point to get an interval  $X$ . It would be convenient to cut  $\tau$  at a singular point of  $T$  corresponding to a “nonchosen” saddle  $S_{2g-2}$  (we recall, that each inner saddle on a minimal component  $\mathring{M}_g^2$  determines two singular points of the mapping  $T$ ). Mapping  $T$  on  $\tau$  determines an interval exchange transformation of  $4g - 4 + 1 = 4g - 3$  subintervals of the interval  $X$ , which we will denote by  $T$  as well. We remind, that  $T$  preserves a natural measure  $\mu$  on  $X$ , defined by restriction  $\omega|_\tau$ .

We remind briefly the basic properties of interval exchange transformations. Fix the permutation  $\sigma$ , corresponding to the interval exchange transformation  $T$  on  $X$ . Let  $\lambda_1, \dots, \lambda_{4g-3}$  be lengths of exchanging subintervals measured by means of the measure  $\mu$ . Interval exchange transformations having the same permutation  $\sigma$  of subintervals are parametrized by points of a simplex

$$\Delta^{4g-4} = \{(l_1, \dots, l_{4g-3}) \mid l_i \geq 0; l_0 + \dots + l_{4g-3} = \text{const}\}$$

where  $l_1, \dots, l_{4g-3}$  represent lengths of exchanged subintervals. It is convenient to chose a constant, determining the simplex to be equal to  $\int_\tau \omega$ . In this notation, our interval exchange transformation  $T$  is identified with a point  $\lambda = (\lambda_1, \dots, \lambda_{4g-3}) \in \Delta^{4g-4}$ .

An interval exchange transformation  $T \in \Delta^{4g-4}$  translates each of  $4g-3$  subintervals by some shift  $\beta_i$ ,  $i = 1, \dots, 4g - 3$  as follows.

$$T(x) = \begin{cases} x + \beta_1 & \text{for } x \in X_1 \\ \dots & \dots \\ x + \beta_{4g-3} & \text{for } x \in X_{4g-3} \end{cases}$$

Vector  $\beta = (\beta_1, \dots, \beta_{4g-3})$  is represented by

$$\beta = A(\sigma)\lambda$$

where linear transformation  $A(\sigma)$  depends only on permutation  $\sigma$ . The explicit form of  $A(\sigma)$  is as follows:

$$A(\sigma)_{ij} = \begin{cases} 1 & \text{if } i < j \text{ and } \sigma^{-1}(i) > \sigma^{-1}(j) \\ -1 & \text{if } i > j \text{ and } \sigma^{-1}(i) < \sigma^{-1}(j) \\ 0 & \text{otherwise} \end{cases}$$

Matrix  $A(\sigma)$  is a skew-symmetric  $(4g - 3) \times (4g - 3)$  matrix.

**Lemma 1.6** *Let permutation  $\sigma$  be obtained by construction above. Then*

$$rk A(\sigma) = 2g$$

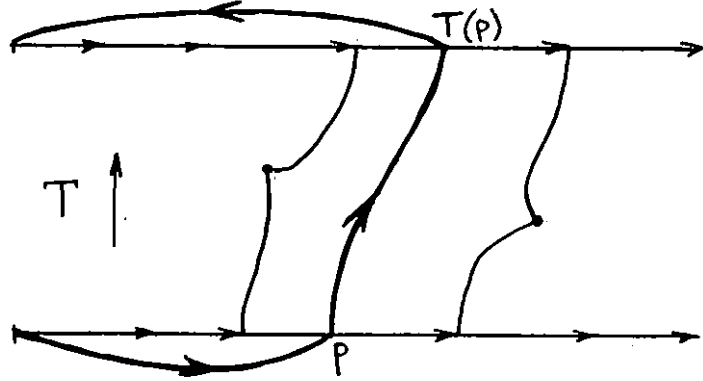


Figure 2: Construction of a special basis of cycles

◁ Lemma 1.6 follows immediately from lemmas 1.8, 1.9 and lemma 1.7 below. ▷

**Lemma 1.7** All numbers  $\beta_1, \dots, \beta_{4g-3}$  represent integrals of  $\omega$  over cycles in  $\mathring{M}_g^2$ . Dimension of a rational span of the numbers  $\beta_1, \dots, \beta_{4g-3}$  equals  $2g$ :

$$\dim\langle\beta_1, \dots, \beta_{4g-3}\rangle_{\mathbb{Q}} = 2g$$

◁ Consider a  $2g$ -dimensional space  $Q^{2g} \cong \mathbb{Q}^{2g} \hookrightarrow \mathbb{R}$  over  $\mathbb{Q}$  generated by rational span of integrals  $\alpha_i = \int_{c_i} \omega$  of  $\omega$  over some basis of cycles  $c_i$ ,  $i = 1, \dots, 2g$ ,  $c_i \in H_1(\mathring{M}_g^2, \mathbb{Z})$ , where  $Q^{2g} = \langle\alpha_1, \dots, \alpha_{2g}\rangle_{\mathbb{Q}}$ . We are going to prove, that

$$\langle\beta_1, \dots, \beta_{4g-3}\rangle_{\mathbb{Q}} = Q^{2g} = \langle\alpha_1, \dots, \alpha_{2g}\rangle_{\mathbb{Q}} \quad (1.1)$$

We suppose for simplicity that  $\mathring{M}_g^2$  does not have any holes at all, i.e.,  $\mathring{M}_g^2 = M_g^2$ . The proof is easily generalized to the case  $\mathring{M}_g^2$ . First let us prove the inclusion

$$\langle\beta_1, \dots, \beta_{4g-3}\rangle_{\mathbb{Q}} \hookrightarrow Q^{2g} \quad (1.2)$$

To prove this we will construct cycles  $b_1, \dots, b_{4g-3}$  on  $M_g^2$ , such that  $\int_{b_i} \omega = -\beta_i$ . The construction of the cycle  $b_i$  is illustrated at figure 2. Our cycle  $b_i$  is a composition of three paths. The first one — from the left endpoint of our interval  $X$ , i.e., from the point, where we cut our closed transversal  $\tau$ , to some inner point  $p \in X_i$  of subinterval  $X_i$ . The next one — a piece of trajectory, starting at this inner point of  $p \in X_i$  and coming back to  $X$  at the point  $T(p)$ . The last one connects  $T(p)$  with the left endpoint of  $X$  along  $X$ .

At figure 2 intervals  $X$  and  $T(X)$  are distinct, but actually we have to identify them, so we really obtain a cycle in  $M_g^2$ . A piece of trajectory gives no impact to the integral over cycle  $b_i$ . Hence  $\int_{b_i} \omega$  equals the difference of the lengths of the first and the third pieces of our path, which equals  $-\beta_i$ . We proved inclusion 1.2.

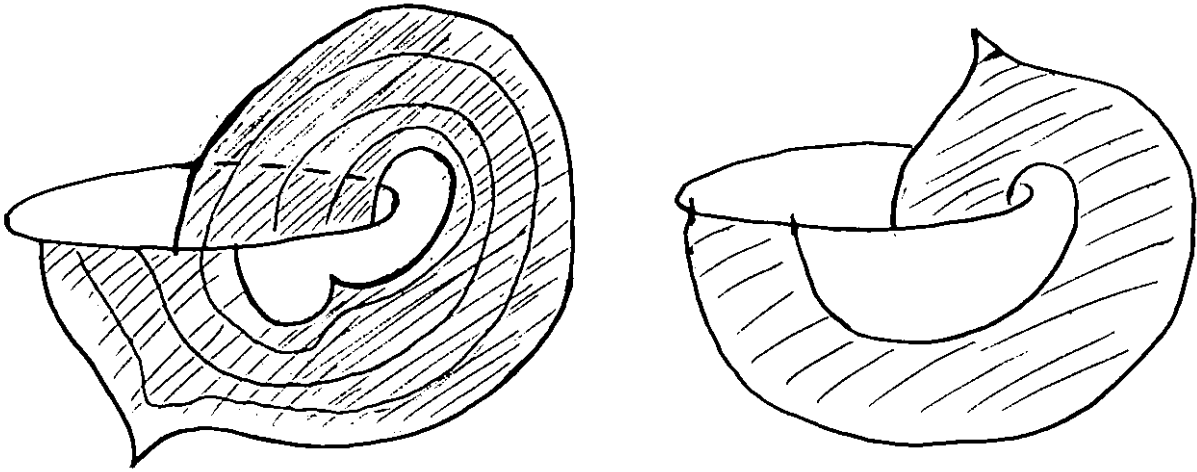


Figure 3: Construction of a cell complex.

To prove 1.1 we will construct a specific cell decomposition of  $M_g^2$ . Consider subinterval  $X_i$ . Consider a closure of a union of pieces of nonsingular trajectories, which start at the points  $p \in X_i$  and end at the points  $T(p) \in T(X_i)$  (see fig. 3). We get a closure of a 2-cell of our complex. It may be of two types, as indicated at figure 3.

Consider 2-cells determined by all subintervals  $X_i$ ,  $i = 1, \dots, 4g - 3$ . They provide us with a 2-dimensional skeleton of our cell complex. Cells of dimension 1 and 0 are obviously determined by construction. Any 1-cycle on  $M_g^2$  can be represented by a 1-cycle of our cell complex. It is not difficult to represent now any cycle on  $M_g^2$  as a combination of cycles  $b_i$ . As an illustration we show how to get a cycle represented by our closed transversal  $\tau$  as a combination of  $b_i$ . Recall, that when we cut  $\tau$  we had to complete our initial set of  $4g - 4$  singular points of interval exchange transformation on  $\tau$  with additional point  $T^{-1}(P_1)$ , where  $P_1$  is a point of the cut. (By convention we cut  $\tau$  over a singular point of  $T$ .) Let  $T^{-1}(P_1) = P_{k+1}$ . By construction  $\beta_k = \lambda_{k+1} + \lambda_{k+2} + \dots + \lambda_{4g-3}$  and  $\beta_{k+1} = -\lambda_1 - \lambda_2 - \dots - \lambda_k$  (see fig. 4). It is easy to see, that  $[\tau] = [b_{k+1}] - [b_k]$ , where cycles  $b_k$  and  $b_{k+1}$  are as constructed above.  $\triangleright$

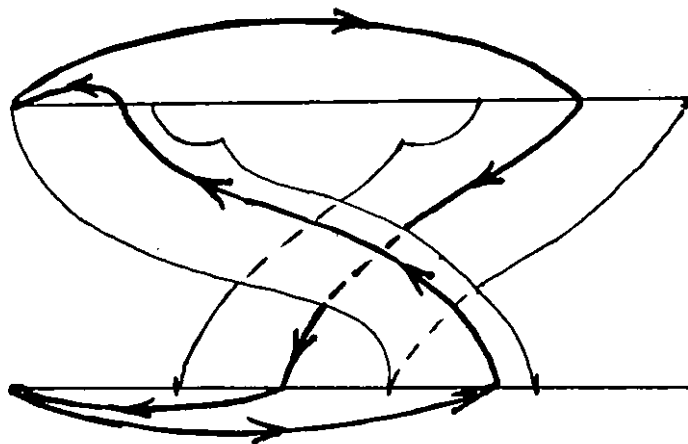


Figure 4: Cycle  $[b_{k+1}] - [b_k]$  represents the closed transversal  $\tau$ .

### 1.3 Deformations of the embedding.

As it was shown in [14] a flow on  $M_{g_i}^2$  like ours may have  $k$  nontrivial invariant ergodic measures for any  $k$ ,  $1 \leq k \leq g_i$ . On the other hand, due to [9] and [17] a generic interval exchange transformation is uniquely ergodic. It is rather natural to conjecture, that a generic closed 1-form  $\omega$  of maximal rank on  $M_g^2$  determines generic, i.e., uniquely ergodic interval exchange transformations on transversals  $\tau_i$  related to components  $M_{g_i}^2$ . The last statement is exactly the formulation of Proposition 1.1. To prove the proposition we have to specify a notion of a “generic closed 1-form  $\omega$  of maximal rank”.

In order to do this we shall construct some natural finite-dimensional space of smooth deformations of  $\omega$ . A small deformation of  $\omega$  does not change the topology of decomposition of  $M_g^2$  into “traps”, “necks”, and minimal components — a slight change of their boundaries, caused by deformation of  $\omega$  does not affect the structure of this decomposition. In this sense the deformation would not change the flow on the components filled with periodic orbits. Still the behavior of the flow on minimal components may change drastically. We are going to prove, that those forms, which determine a uniquely ergodic interval exchange transformation are represented by a set of a complete measure in the space of deformations of the initial form.

Note, that due to V.I. Arnold (see [2]) the flow is always ergodic on the minimal components of genus one, provided that pairs of periods of the form on these components are rationally independent.

\* \* \*

Consider a vector space  $\mathbb{R}^n$  provided with a cubic lattice. Consider a natural projection  $p : \mathbb{R}^n \rightarrow Tor^n = \mathbb{R}^n/\mathbb{Z}^n$ . A differential  $dH$  of any linear function  $H \in \mathbb{R}^{n*}$  can be induced by  $p^*$  from a uniquely determined closed 1-form on  $Tor^n$ , which we will denote by  $dH$  as well. Fix coordinates on  $Tor^n$  corresponding to natural coordinates in  $\mathbb{R}^n$ . We will call closed differential 1-forms as described above by *linear* 1-forms on  $Tor^n$ .

Having arbitrary closed 1-form on a surface  $M_g^2$  of genus  $g$  one can easily construct an embedding  $i : M_g^2 \rightarrow Tor^n$  and a linear form  $dH$  on the torus, such that  $\omega$  coincides with the restriction  $i^*(dH)$  to  $M_g^2$  (see remark 2 in [2]). A dimension of the torus in this construction is rather large:  $n = 2g + 3$ , but it is easy to see, that it can not be seriously reduced. Since for a generic  $\omega \in \Omega^1(M_g^2, \mathbb{R})$ ,  $\text{rk}(\omega) = 2g$ , i.e., integrals of  $\omega$  over  $2g$  basic cycles on  $M_g^2$  are rationally independent, in order to induce  $\omega$  from a 1-form on a torus, one has to consider only those embeddings, which induce monomorphism of the first homology groups:  $i_* : H_1(M_g^2, \mathbb{Z}) \rightarrow H_1(Tor^n, \mathbb{Z}) = \mathbb{Z}^n$ . Hence  $n \geq 2g$ .

Let 1-form  $\omega$  on  $M_g^2$  be defined by a pair  $(i_0, dH_0)$ , where  $i_0 : M_g^2 \hookrightarrow Tor^n$  is an embedding, which induces monomorphism of first homology groups,  $dH_0$  is a linear 1-form on the torus, and  $\omega = i_0^*(dH_0)$ . We assume, that  $\text{rk}(\omega) = 2g$ . We assume also, that all critical points of  $\omega$  are nondegenerate, and distinct critical points of  $\omega$  belong to distinct critical levels.

We will construct a space of small deformations of  $\omega$ , which will be locally described

as a direct product of two spaces:

$$\mathcal{D}(i_0) \times \mathcal{D}(\omega) = (\text{deformations of } i_0) \times (\text{linear deformations of linear function } H_0)$$

We start with deforming the embedding  $i_0$ .

Consider a local deformation of the embedding  $i_0$  which deforms embedding only in some small neighborhood of  $j$ -th inner saddle point  $S_j$ ,  $1 \leq j \leq 2g-3$ . (Note, that we skipped one of our saddles.) We suppose, that each deformation does not add any new critical points, it just shifts slightly one particular saddle. By  $d_j(\epsilon)$  we will denote a 1-parameter family of deformations as above, where parameter  $\epsilon$  indicates, that the  $j$ -th critical point  $S_j$  moved from original level  $h_0 = H(S_j)$  to level  $h_0 + \epsilon = H(S'_j)$ , where  $\omega = dH$ , and  $S'_j$  is the corresponding saddle point of  $dH$  on a deformed submanifold  $M_g^2$ . We suppose, that the neighborhoods of the critical points  $S_j$ , where our deformation  $d_j(\epsilon)$  act, do not intersect for different  $j$ , and that values of parameters of deformation  $\epsilon$  are small enough. Hence we may suppose, that our closed transversal  $\tau$  does not pass through any of this neighborhoods. Moreover, we assume, that deformed interval exchange transformation on the transversal is described by the same permutation  $\sigma$  as the original one.

Consider an action of deformation  $d_j(\epsilon)$  on the interval exchange transformation of  $4g-4$  intervals on our closed transversal, or, what is almost the same, the action on the interval exchange transformation of  $4g-3$  intervals on the interval  $X$ , obtained by cutting  $\tau$ . (We remind, that by construction, we cut  $\tau$  over one of those two singular points of interval exchange transformation, which correspond to the saddle  $S_{2g-2}$ , which we excluded from the set of saddles under deformation.) Deformation  $d_j(\epsilon)$  determines some translation of the initial point  $\lambda_0 = (\lambda_1, \dots, \lambda_{4g-3}) \in \Delta^{4g-4}$ ,  $\lambda_1 + \dots + \lambda_{4g-3} = \text{const} = \int_\tau \omega$  inside the simplex  $\Delta^{4g-4}$  parametrizing the space of interval exchange transformations with fixed permutation  $\sigma$ . We will denote this action by the same symbol  $d_j(\epsilon) : \lambda_0 \mapsto \lambda' \in \Delta^{4g-3}$ .

Consider a saddle  $S_j$ ,  $1 \leq j \leq 2g-3$ . There are exactly two trajectories, starting at  $\tau$ , which hit  $S_j$ , i.e., there are only two singular points  $P_k$  and  $P_l$  of interval exchange transformation  $T$  corresponding to the saddle  $S_j$ . It is easy to see, that  $P_k$  and  $P_l$  are separated from both sides on  $\tau$  by other singular points of  $T$ . Hence there are four distinct subintervals  $X_{k-1}$ ,  $X_k$ ,  $X_{l-1}$ ,  $X_l$  having one of the points  $P_k$ ,  $P_l$  as an endpoint. We assume, that numeration of the subintervals is compatible with the natural orientation on  $\tau$ . The action of  $d_j(\epsilon)$  on the point  $\lambda_0 = (\lambda_1, \dots, \lambda_{4g-3})$  is just a shift by  $\epsilon$

$$\begin{aligned} d_j(\epsilon) : (\lambda_1, \dots, \lambda_{4g-3}) \mapsto \\ (\lambda_1, \dots, \lambda_{k-1} + \epsilon, \lambda_k - \epsilon, \lambda_{k+1}, \dots, \lambda_{l-1} + \epsilon, \lambda_l - \epsilon, \lambda_{l+1}, \dots, \lambda_{4g-3}) \end{aligned}$$

along vector

$$V_j = (0, \dots, 0, 1, -1, 0, \dots, 0, 1, -1, 0, \dots, 0) \quad (1.3)$$

The first pair of nontrivial components of our vector has indices  $k-1$  and  $k$ , the other one — indices  $l-1$  and  $l$ . (Note, that by construction we will never get vector like  $(-1, 0, \dots, 0, 1, -1, 0, \dots, 0, 1)$ , since we never deform the saddle  $S_{2g-2}$ , corresponding to a point of cut of transversal  $\tau$ .)

**Lemma 1.8** *Vectors  $V_1, \dots, V_{2g-3}$  are linearly independent.*

◁ Consider a linear combination  $q_{i_1}V_{i_1} + \dots + q_{i_n}V_{i_n}$ ,  $q_{i_s} \neq 0$ . Consider a subset in  $\{1, 2, \dots, 4g-3\}$  obtained as a union of numbers, enumerating nonzero components of  $V_{i_1}, \dots, V_{i_n}$ . Consider a maximal number in this subset, denote it by  $m$ . Then the  $m$ -th component of our linear combination in usual coordinates  $(\lambda_1, \dots, \lambda_{4g-3})$  is nonzero. Indeed, there are at most two vectors  $V_{s_1}$  and  $V_{s_2}$  having nonzero  $m$ -th component. By choice of  $m$  only one of them can occur in our linear combination. ▷

**Lemma 1.9** *Vectors  $V_1, \dots, V_{2g-3}$  belong to  $\text{Ker}A(\sigma)$ , i.e.,*

$$A(\sigma)V_i = 0, \quad i = 1, \dots, 2g-3$$

◁ Since our deformation affects only some small neighborhoods of a finite number of points (i.e., our saddles), we may chose a basis of cycles in  $H_1(\overset{\circ}{M}_g^2, \mathbb{Z})$  represented by curves, which do not intersect with deformed domains. Hence our deformations preserve the periods of  $\omega$ , i.e., they preserve the cohomology class of  $\omega$ .

If some  $V_i$  does not belong to the kernel of  $A(\sigma)$ , then  $\beta(\epsilon) = A(\sigma)(\lambda_0 + \epsilon V_i)$  is nonconstant with respect to  $\epsilon$ . But as we showed in Lemma 1.7 all components of  $\beta$  represent integrals of  $\omega$  over some cycles in  $\overset{\circ}{M}_g^2$ , which are supposed to be preserved. The contradiction obtained proves the lemma. ▷

Combining results of lemmas 1.9 and 1.8 with result of lemma 1.6 we obtain

**Corollary 1.10** *Vectors  $V_1, \dots, V_{2g-3}$  generate a basis of  $\text{Ker}A(\sigma)$ .*

**Corollary 1.11** *A family of deformations  $\mathcal{D}^{2g-3}(i_0)$ , generated by deformations  $d_j(\epsilon)$ , embeds naturally into affine subspace of the space  $\Delta^{4g-4}$  parallel to the  $(2g-3)$ -dimensional kernel of  $A(\sigma)$ .*

## 1.4 Deformations of the cohomology class.

Consider now a space  $\mathcal{D}^{2g-1}(\omega)$  of deformations of the closed 1-form  $\omega$  as follows. By convention we define our 1-form  $\omega$  as a restriction to  $M_g^2$  of a linear 1-form  $dH_0$  in  $Tor^n$ ,  $\omega = i_0^*(dH_0)$ , where  $i_0 : M_g^2 \rightarrow Tor^n$  is our initial embedding. Consider an affine hyperplane  $\mathcal{H}^{2g-1}$  in the vector space  $H^1(M_g^2, \mathbb{R})$  generated by cohomology classes  $[\rho] \in \mathcal{H}^{2g-1}$ , which have the same coupling with the cycle  $[\tau]$  represented by our closed transversal:  $\int_\tau \rho = \text{const} = \int_\tau \omega$ . Since homomorphism of cohomology groups  $i_0^* : H^1(Tor^n, \mathbb{R}) \rightarrow H^1(M_g^2, \mathbb{R})$  is an epimorphism in our case, we can chose a  $(2g-1)$ -dimensional affine subspace in  $H^1(Tor^n, \mathbb{R})$ , which maps onto  $\mathcal{H}^{2g-1}$ . We chose the affine subspace so that  $dH_0$  belongs to it. We will denote this affine subspace by the same symbol  $\mathcal{H}^{2g-1} \subset H^1(Tor^n, \mathbb{R})$ . We define now  $\mathcal{D}^{2g-1}(\omega)$  as a small neighborhood of  $dH_0$  in  $\mathcal{H}^{2g-1}$ . By "small" we mean the following. We identify the linear spaces  $H_1(Tor^n, \mathbb{R})$  and the space of linear 1-forms on  $Tor^n$ . We assume, that for any linear 1-form  $dH \in \mathcal{D}^{2g-1}(\omega)$  the form  $i_0^*(dH)$  has the same number of critical points of the

same type as  $i_0^*(dH_0)$ , that the quantitative structure of decomposition of  $M_g^2$  into minimal and periodic components does not change, that our closed transversal  $\tau$  is transverse to  $i_0^*(dH)$ , and that the permutation, determining interval exchange transformation on  $\tau$  induced by the flow corresponding to  $i_0^*(dH)$  coincides with permutation  $\sigma$  corresponding to  $dH_0$ . Roughly speaking we assume that the whole quantitative picture remains the same for all forms  $i_0^*(dH)$ , where  $dH \in \mathcal{H}^{2g-1}$ .

Let us study now how deformations from  $\mathcal{D}^{2g-1}$  affect corresponding interval exchange transformations, i.e., consider a mapping  $\mathcal{D}^{2g-1}(\omega) \rightarrow \Delta^{4g-4}$ , where  $\Delta^{4g-4}$  parametrizes interval exchange transformations on the interval  $X$ , obtained by cutting our closed transversal  $\tau$  over a point (see above). Recall that in the proof of Lemma 1.7 we represented shifts  $\beta_i = (A(\sigma)\lambda)_i$  of the exchanged intervals as integrals over the cycles  $b_j$  in  $M_g^2$  (up to multiplication by  $-1$ ). A small deformation of the form does not change the homology classes of cycles  $b_j$ . Since cycles  $b_j$  generate a basis in  $H_1(M_g^2, \mathbb{R})$ , a set of integrals  $-\beta_j$  of the form  $\omega = i_0^*(dH)$  over these cycles determines the cohomology class  $[\omega] \in H^1(M_g^2, \mathbb{R})$ . Thus we get a linear isomorphism between  $\text{Im}(A(\sigma))$  and  $H^1(M_g^2, \mathbb{R})$ . By construction a composition  $\mathcal{D}^{2g-1}(\omega) \rightarrow \Delta^{4g-4} \xrightarrow{A} \text{Im}(A(\sigma)) \rightarrow H^1(M_g^2, \mathbb{R}) \supset \mathcal{D}^{2g-1}(\omega)$  maps  $\mathcal{D}^{2g-1}(\omega)$  onto itself, and this mapping is just the identity mapping.

Hence we proved

**Lemma 1.12** *The mapping  $\mathcal{D}^{2g-1}(\omega) \rightarrow \Delta^{4g-4}$  is an embedding. It is transversal to the fibration of  $\Delta^{4g-4}$  by affine  $(2g-3)$ -dimensional subspaces parallel to the space  $\text{Ker} A(\sigma)$ .*

Combining results of lemma 1.12 and corollary 1.11 we get the following picture. Our simplex  $\Delta^{4g-4} \subset \mathbb{R}^{4g-3}$  is naturally fibered by intersections with the family of affine planes in  $\mathbb{R}^{4g-3}$  parallel to  $\text{Ker} A(\sigma)$ . Dimension of a generic fiber is  $2g-3$ . Deformations  $\mathcal{D}^{2g-3}(i_0)$  of the embedding  $i_0$  result translations along a fiber, while deformations  $\mathcal{D}^{2g-1}(\omega)$  of the cohomology class produce some transversal section of the fibration.

Let us construct a local section  $\mathcal{D}^{2g-1}(\omega)$  for each point of a neighborhood  $\mathcal{D}^{2g-3}(i_0)$  of the fiber passing through  $\lambda$ . It is easy to see, that local sections, corresponding to different points of  $\mathcal{D}^{2g-3}(i_0)$  do not intersect. Hence we obtain a diffeomorphism between  $\mathcal{D}^{2g-3}(i_0) \times \mathcal{D}^{2g-1}(\omega)$  and some neighborhood of the initial point  $\lambda \in \Delta^{4g-4}$ . Since due to theorem of H.Masur [9] and W.Veech [17] the set of uniquely ergodic interval exchange transformation has a complete measure in  $\Delta^{4g-3}$ , we conclude, that almost all deformations from  $\mathcal{D}^{2g-3}(i_0) \times \mathcal{D}^{2g-1}(\omega)$  determine a uniquely ergodic interval exchange transformation on the transversal  $\tau$ . This proves Proposition 1.1 for the case, when we have the only one minimal component  $\hat{M}_g^2$ . In general case, when we have several minimal components  $\hat{M}_{g_1}^2, \dots, \hat{M}_{g_k}^2$  we just repeat all constructions for each minimal component separately, and consider corresponding direct products of spaces and mappings arising here. Note, that we just skip minimal components of genus 1,  $\hat{M}_{g_i}^2$ ,  $g_i = 1$ , since the flow is uniquely ergodic on these components, whenever pairs of periods on these components are rationally independent (see [2]).

Proposition 1.1 is proved.

**Remark** Note, that considerable part of our construction is reduced to the following. A choice of transversal provides us with a collection of (co)cycles in the first relative (co)homology of the surface with respect to subset of saddle points. In fact we proved, that this collection forms a basis in  $H^1(M_g^2, \{\text{saddles}\}; \mathbb{R})$ . Note, that values  $\lambda_i$  represent integrals over the basic relative 1-cycles. Note also, that values  $\beta_i$  represent integrals of the 1-form over cycles  $b_i$  (see Lemma 1.7). Consider the following terms of exact sequence of a pair (set of saddle points)  $\subset$  (Riemann surface  $M_g^2$ ):

$$\dots \rightarrow H^0(\text{saddles}; \mathbb{R}) \rightarrow H^1(M_g^2, \{\text{saddles}\}; \mathbb{R}) \rightarrow H^1(M_g^2; \mathbb{R}) \rightarrow H^1(\text{saddles}; \mathbb{R}) = 0$$

Under identification with cohomology suggested above operator  $A(\sigma)$  can be considered as a mapping from relative to absolute cohomology from the exact sequence of the pair, while the set of vectors  $V_i$  defined by equation 1.3 represents

$$\begin{aligned} \text{Im} \left( H^0(\{\text{saddles}\}; \mathbb{R}) \rightarrow H^1(M_g^2, \{\text{saddles}\}; \mathbb{R}) \right) = \\ \text{Ker} \left( H^1(M_g^2, \{\text{saddles}\}; \mathbb{R}) \rightarrow H^1(M_g^2; \mathbb{R}) \right) \end{aligned}$$

## 1.5 Identification with homology.

Consider a vector space  $\mathbb{R}^{4g-3}$  and linear operator  $A(\sigma)$  acting on  $\mathbb{R}^{4g-3}$ . Consider subspace  $\text{Im} A(\sigma) \subset \mathbb{R}^{4g-3}$ . According to Lemma 1.6 dimension of the subspace  $\text{Im} A(\sigma)$  equals  $2g$ . Fix natural coordinates in  $\mathbb{R}^{4g-3}$ . Subspace  $\text{Im} A(\sigma)$  is spanned by rows of matrix  $A(\sigma)$ . Consider a linear function  $\Lambda$  on our subspace  $\text{Im} A(\sigma)$  obtained as a restriction of a linear function in  $\mathbb{R}^{4g-3}$  which has components  $(\lambda_1, \dots, \lambda_{4g-3})$ . (We stress, that  $\Lambda$  is a function on  $2g$ -dimensional space.) Since  $A(\sigma)$  is skew-symmetric, we have

$$-\beta^T = -(A(\sigma)\lambda)^T = -\lambda^T A^T(\sigma) = \lambda^T A(\sigma)$$

Hence

$$\Lambda(A_i) = \lambda_1 A_{i,1} + \dots + \lambda_{4g-3} A_{i,4g-3} = -\beta_i$$

and according to lemma 1.7  $\text{rk} \langle \beta_1, \dots, \beta_{4g-3} \rangle_{\mathbf{Q}} = 2g$ .

Consider now one more  $2g$ -dimensional vector space  $H_1(M_g^2, \mathbb{R})$ . Consider a set of cycles  $b_1, \dots, b_{4g-3}$  in  $H_1(M_g^2, \mathbb{R})$  as constructed in the proof of lemma 1.7. By construction the values of the cocycle  $[\omega]$  (i.e., of a linear function on the space  $H_1(M_g^2, \mathbb{R})$ ) on cycles  $b_i$  are as follows:  $\omega(b_i) = -\beta_i, i = 1, \dots, 4g-3$ . We have two  $2g$ -dimensional vector spaces, a collection of  $4g-3$  vectors, in each of them, and a linear functions, having the same values  $\beta_i$  on corresponding vectors. The condition  $\text{rk} \langle \beta_1, \dots, \beta_{4g-3} \rangle_{\mathbf{Q}} = 2g$  enables us to construct an isomorphism  $h$  of the two spaces,

$$h : \text{Im} A(\sigma) \cong H_1(M_g^2, \mathbb{R})$$

which identifies corresponding vectors from our collections, and identifies linear functions. (Note, that instead of vector spaces, we could deal with corresponding lattices; in this case we would get isomorphism with homology group  $H_1(M_g^2, \mathbb{Z})$  with integer coefficients.)



**Lemma 1.13** *Consider a nonzero vector  $q = (q_1, \dots, q_{4g-3})$  with nonnegative components  $q_i \geq 0$ ,  $i = 1, \dots, 4g - 3$ . The following inequality is valid:*

$$A(\sigma)q \neq 0$$

◁ The structure of  $\text{Ker } A(\sigma)$  is described by lemma 1.9 and corollary 1.10. It follows immediately from equation 1.3, that for any vector from  $\text{Ker } A(\sigma)$  the sum of its components equals zero. Since by assumption  $q_1 + \dots + q_{4g-3} > 0$ , vector  $q$  does not belong to  $\text{Ker } A(\sigma)$ . ▷

**Corollary 1.14** *A linear combination of cycles  $b_i$ ,  $i = 1, \dots, 4g - 3$ , with nonnegative coefficients, which do not vanish simultaneously determines a nontrivial cycle in  $H_1(M_g^2, \mathbb{R})$ .*

◁ The statement of corollary is just reformulation of lemma 1.13 after identification of cycles  $b_i$  with rows  $A(\sigma)_i$  of matrix  $A(\sigma)$  constructed above. ▷

## 1.6 Behavior of trajectory in the covering space.

Consider a space  $\mathbb{R}^n$  provided with a cubic lattice. Consider a simply-connected surface  $\hat{M}_g^2$  in  $\mathbb{R}^n$  invariant under translations over vectors of the lattice. We do not assume that  $\hat{M}_g^2$  is necessarily connected. Consider lines of intersection of  $\hat{M}_g^2$  with a family of parallel affine hyperplanes in  $\mathbb{R}^n$ , determined by a linear hyperplane of generic direction.

**Proposition 1.2** *Almost all nonclosed intersection lines of a generic simply-connected periodic surface in  $\mathbb{R}^n$  with a generic family of parallel hyperplanes have some fixed direction as they go to  $\pm\infty$ . The set of possible directions is finite for a given surface and family of hyperplanes; nonclosed lines of intersection passing nearby have the same direction.*

The proposition follows from existence of asymptotic cycle proved in [15]. Another way to prove the proposition is to use proposition 1.1 and Ergodic theorem. Indeed, if we consider a quotient of  $\mathbb{R}^n$  by the action of  $\mathbb{Z}^n$ , we get a torus, and a compact surface embedded in it. Hyperplanes would be projected to levels of corresponding closed 1-form on the torus with constant coefficients. Hence hyperplanes crosssections of initial surface would project to leaves of the restriction of the 1-form to the surface, and we can assume, that we are under assumptions of proposition 1.1.

Consider a piecewise constant function  $\vec{F}$  on the interval  $X$ , where  $X$  is obtained by cutting our closed transversal  $\tau$  at some point, as described above. Function  $\vec{F}$  is a “vector-valued” function with values in  $H_1(M_g^2, \mathbb{R})$ . The value of  $\vec{F}$  on the  $j$ -th exchanging subinterval  $X_j$  equals the cycle  $b_j \in H_1(M_g^2, \mathbb{R})$ . Note, that we can identify  $H_1(M_g^2, \mathbb{R})$  with a linear  $2g$ -dimensional subspace of the space  $\mathbb{R}^n$  of universal cover  $\mathbb{R}^n \rightarrow \text{Tor}^n$ . Consider a piece of trajectory of the flow on  $M_g^2$ , which starts at some

point  $p \in X$  and comes back to  $X$  after  $k$  iterations of the mapping  $T$ . Consider a lift of this piece to the manifold  $\hat{M}_g^2 \subset \mathbb{R}^n$ , covering  $M_g^2$ . Consider a vector  $\vec{v}(p, k)$  joining the endpoints of corresponding nonclosed curve. Let us represent  $\vec{v}(p, k)$  as a sum

$$\vec{v}(p, k) = \vec{F}(p) + \vec{F}(T(p)) + \dots + \vec{F}(T^k(p)) + \vec{\epsilon} \quad (1.4)$$

Let us estimate an error  $\vec{\epsilon}$ . Consider a parametrization  $\gamma : [0, 1] \rightarrow \tau \subset M_g^2$ ,  $\gamma(0) = \gamma(1) = P_1$ , of our closed transversal  $\tau$ , where  $P_1$  is a point, where we cut our closed transversal to get an interval  $X$ . Consider a corresponding lift  $\hat{\gamma} : [0, 1] \rightarrow \hat{M}_g^2 \subset \mathbb{R}^n$ . Let  $d = \max_{0 \leq t \leq 1} \|\hat{\gamma}(t) - \hat{\gamma}(0)\|$ , where norm is understood in the sense of usual norm in  $\mathbb{R}^n$ . Then  $\|\vec{\epsilon}\| \leq d$ . Note, that we got an estimate of an error in 1.4, which does not depend neither on the point  $p \in X$ , nor on the number of iterations  $k$ .

Consider limits

$$\lim_{k \rightarrow +\infty} \frac{1}{k} \vec{v}(p, k) \quad \text{and} \quad \lim_{k \rightarrow -\infty} \frac{1}{k} \vec{v}(p, k)$$

It is natural to call these limits by positive and negative directions of trajectory passing through the point  $p$ . In the case, when corresponding interval exchange transformation  $T$  is ergodic under natural measure on  $X$ , we use the ergodic theorem to get

$$\begin{aligned} \lim_{k \rightarrow +\infty} \frac{1}{k} \vec{v}(p, k) = \lim_{k \rightarrow -\infty} \frac{1}{k} \vec{v}(p, k) = \lim_{k \rightarrow -\infty} \frac{1}{k} \left( \sum_{i=0}^k \vec{F}(T^i(p)) + \vec{\epsilon} \right) = \int_X \vec{F}(p) dp = \\ \lambda_1 b_1 + \dots + \lambda_{4g-3} b_{4g-3} \end{aligned} \quad (1.5)$$

Due to corollary 1.14 the latter sum represents a nonzero vector in  $\mathbb{R}^n$ . Hence in the case, when the flow is uniquely ergodic on a minimal component  $\hat{M}_{g_i}^2$ , almost all lifts of trajectories from  $\hat{M}_{g_i}^2$  to the covering space have one and the same direction, determined by formula 1.5. Note that the homology class in (1.5) represent Poincare dual to the cocycle  $[\omega]$  (cf. [15]).

Consider now our initial  $\mathbb{Z}^n$ -invariant surface  $\hat{M}_g^2 \subset \mathbb{R}^n$  covering compact surface  $M_g^2 \subset \text{Tor}^n$ . Consider a connected component of a submanifold in  $\hat{M}_g^2$ , covering minimal component  $\hat{M}_{g_j}^2$ . It is easy to see, that this connected component belongs to a tubular neighborhood of a  $2g_j$ -dimensional affine plane in  $\mathbb{R}^n$ , where the ‘‘thickness’’ of the tubular neighborhood is uniformly bounded by some constant. The direction of the  $2j$ -dimensional plane is determined by the image of  $H_1(\hat{M}_{g_j}^2, \mathbb{R})$  under embedding into  $\mathbb{R}^n$  (see above). Hence, each intersection of the closed component of the cover of  $\hat{M}_{g_j}^2$  with a hyperplane  $H_0 = \text{const}$  in  $\mathbb{R}^n$ ,  $\omega = i_0^*(dH_0)$ , belongs to a tubular neighborhood of an affine  $(2g_j - 1)$ -dimensional plane. Note, that these intersection lines, are exactly lifts of trajectories. Direction of the  $(2g_j - 1)$ -dimensional plane is defined by intersection of  $2g_j$ -dimensional plane in  $\mathbb{R}^n$  determined by the image of  $H_1(\hat{M}_{g_j}^2, \mathbb{R})$ , with hyperplane  $H_0 = \text{const}$ .  $\triangleright$

## 2 Example of a nontypical flow.

In this section we remind a peculiar Hamiltonian flow on a surface of genus two determined by a multivalued Hamiltonian  $\omega$ . This example is due to E.Sataev [14]. We use

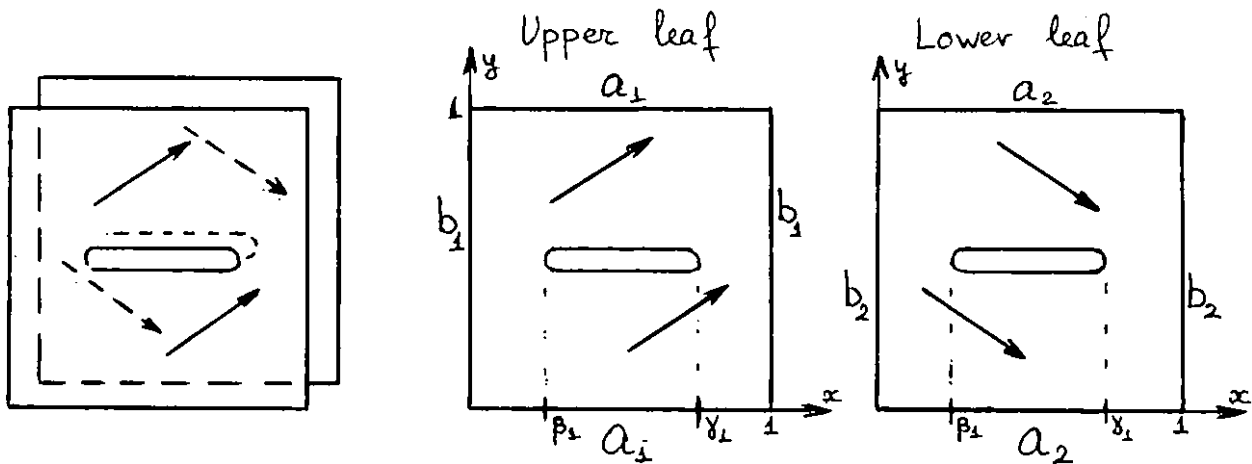


Figure 5: Example of a nontypical flow on a surface of genus 2 due to E.Sataev.

this example to illustrate a nontypical behavior of hyperplane sections of a  $\mathbb{Z}^4$ -invariant surface in  $\mathbb{R}^4$ . The Morse form  $\omega$  under consideration has only two critical points — two saddles; the flow does not have any “traps” or “necks” at all; the whole  $M_2^2$  represents a unique minimal component of the flow. The interval exchange transformation arising in the construction is not uniquely ergodic in this example, this explains a peculiar behavior of the flow.

First we remind briefly Sataev’s example [14] of a topologically transitive flow of class  $C^\infty$  on a surface of genus 2, having two nontrivial invariant normalized ergodic measures and two fixed points, which are nondegenerate saddle points.

We consider a “Riemann surface” of genus 2 having two tori as leaves, which are pasted over cuts as shown at figure 5.

The flow on the upper leaf is generated by a constant vector field  $(\dot{x}, \dot{y}) = (\alpha, 1)$ , where  $\alpha$  is some special irrational number; the flow on the lower leaf is generated by a vector field  $(\dot{x}, \dot{y}) = (-\alpha, 1)$ . (One can see [14], for a more detailed smooth realization of this flow.) In fact, the flow has only two critical points — endpoints of the cut. Each of these two critical points joins four trajectories, so the critical points are of a saddle type.

Lengths of the sides of squares, representing the tori at figure 5 are equal to 1. The cuts are chosen to belong to a circle  $y = \text{const}$ ;  $x$ -coordinates of their endpoints are  $\beta_1$  and  $\gamma_1$ . It is shown in [14], that under special choice of numbers  $\alpha$ ,  $\beta_1$ , and  $\gamma_1$ , the flow has exactly two nontrivial invariant ergodic normalized measures. Roughly speaking, under this special choice of parameters, some trajectories would live mostly on the upper leaf, and the other — mostly on the lower one.

More specifically the flow is described in [14] as a special flow over an automorphism  $T_{h,\alpha}^1$  of two disjoint circles. The automorphism is defined as follows. Consider two disjoint circles  $S^1 \times \{0, 1\}$  of length 1. Let  $h(x)$  be the following function on  $S^1$  with values in the group  $\{1, \sigma_1\}$  of permutations of two elements:

$$h(x) = \begin{cases} \sigma_1 & x \in [\beta_1, \gamma_1] \\ 1 & x \in S^1 \setminus [\beta_1, \gamma_1] \end{cases}$$

Then

$$T_{h,\alpha}^1(x, i) = (x + \alpha, h(x) \cdot i), \quad \text{where } (x, i) \in S^1 \times \{0, 1\}$$

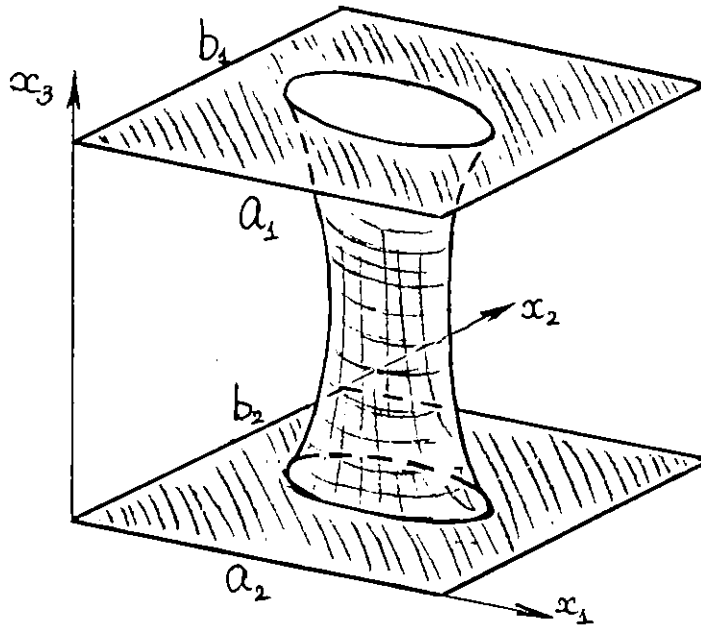


Figure 6: Basis of cycles on the surface of genus two, embedded into  $Tor^3$ .

It is shown in [14], that under appropriate choice of  $\alpha, \beta_1, \gamma_1$  the automorphism  $T_{h,\alpha}^1$  is topologically transitive and has two ergodic components.

Let us construct a realization of the described flow by a Hamiltonian flow, generated by a closed 1-form  $\omega$  on our surface  $M_2^2$  of genus two. Our form  $\omega$  would have the following integrals over the cycles  $a_1, b_1, a_2, b_2$  (see figures 5 and 6)

$$\int_{a_1} \omega = -1 \quad \int_{b_1} \omega = \frac{1}{\alpha} \quad \int_{a_2} \omega = 1 \quad \int_{b_2} \omega = \frac{1}{\alpha}$$

Consider now an auxiliary closed 1-form  $\omega_1$  of rank 1 on  $M_2^2$ , having the following integrals over basic cycles:

$$\int_{a_1} \omega_1 = \int_{a_2} \omega_1 = 0 \quad \int_{b_1} \omega_1 = \int_{b_2} \omega_1 = 1$$

We can obtain the form  $\omega_1$  as a restriction  $\omega_1 = i^*(dx_2)$  of the form  $dx_2$  under embedding  $i : M_2^2 \hookrightarrow Tor^3$  illustrated at figure 6, where by  $x_1, x_2, x_3$  we denote natural coordinates on  $Tor^3$ . The immersion  $i$  maps cycles  $a_1$  and  $a_2$  to the cycle  $e_1$  in  $Tor^3$ ; cycles  $b_1$  and  $b_2$  to the cycle  $e_2$  in  $Tor^3$ .

Let us approximate now the irrational number  $\frac{1}{\alpha}$  by a rational one  $\frac{1}{\alpha} = \frac{p}{q} + \epsilon$ . Then a 1-form  $\omega_2 = \omega - \epsilon\omega_1$  would have rational integrals over cycles on  $M_2^2$ . Consider a mapping  $f_{\omega_2} : M_2^2 \rightarrow S^1$  as follows. Fix a point  $x_0 \in M_2^2$ , and for any  $x_1 \in M_2^2$  let  $x_1 \mapsto \int_{x_0}^{x_1} \omega_2 \pmod{\frac{1}{q}}$ . Consider now an embedding  $M_2^2 \hookrightarrow Tor^4$  obtained as a product of mappings  $i$  and  $f_{\omega_2}$ , i.e.,  $i \times f_{\omega_2} : M_2^2 \hookrightarrow Tor^3 \times Tor^1 = Tor^4$ . Consider a connected component of a surface  $\hat{M}_2^2 \subset \mathbb{R}^4$  induced in the universal covering  $\mathbb{R}^4 \rightarrow Tor^4$ . We obtained a  $\mathbb{Z}^3$ -periodic surface in  $\mathbb{R}^4$  invariant under translations over vectors  $\vec{e}_1, \vec{e}_2$ , and  $\vec{e}_4$ . By construction our initial form  $\omega$  on  $M_2^2 \hookrightarrow Tor^4$  is a restriction of a linear 1-form  $dx_4 + \epsilon dx_2$  (see analogous construction in [2]). Hence level curves of  $\omega$  lifted to  $\hat{M}_2^2$  are represented by intersections of  $\hat{M}_2^2$  with hyperplanes  $x_4 + \epsilon x_2 = \text{const}$ .

Let us describe the images of the cycles  $a_1, b_1, a_2, b_2$  in  $Tor^4$  under the embedding

$$i \times f_{\omega_2} : M_2^2 \hookrightarrow Tor^4.$$

$$\begin{array}{ll} i : a_1 \mapsto e_1 & f_{\omega_2} : a_1 \mapsto -qe_4 \\ i : b_1 \mapsto e_2 & f_{\omega_2} : b_1 \mapsto pe_4 \\ i : a_2 \mapsto e_1 & f_{\omega_2} : a_2 \mapsto qe_4 \\ i : b_2 \mapsto e_2 & f_{\omega_2} : b_2 \mapsto pe_4 \end{array}$$

Intersection of a 2-plane, spanned by  $\text{Im } a_1 = \vec{e}_1 - q\vec{e}_4$  and  $\text{Im } b_1 = \vec{e}_2 + q\vec{e}_4$  with a hyperplane  $x_4 + \epsilon x_2 = \text{const}$  is a line having direction

$$\vec{v}_1 = \frac{p + \epsilon}{q} \vec{e}_1 + \vec{e}_2 - \epsilon \vec{e}_4$$

Intersection of a 2-plane, spanned by  $\text{Im } a_2 = \vec{e}_1 + q\vec{e}_4$  and  $\text{Im } b_2 = \vec{e}_2 + q\vec{e}_4$  with a hyperplane  $x_4 + \epsilon x_2 = \text{const}$  is a line having direction

$$\vec{v}_2 = \frac{-p + \epsilon}{q} \vec{e}_1 + \vec{e}_2 - \epsilon \vec{e}_4$$

Roughly speaking, trajectories of one type on  $\hat{M}_2^2$  would go in  $\mathbb{R}^4$  mostly in direction close to  $\vec{v}_1$ , and trajectories of the other type would go mostly in direction close to  $\vec{v}_2$ ; the two directions are obviously different. Note, that now our trajectories can not be restricted inside any cylinder  $D^3 \times \mathbb{R}$  — one can find pieces of trajectories of arbitrary “length”, where trajectory behaves in a “wrong” way, i.e., goes along the alien direction.

**Remark** We would like to mention, that example by H.B.Keynes and D.Newton [7] of a minimal, non-uniquely ergodic interval exchange transformation is closely related to Sataev’s construction.

One can consider a transformation  $T$  on either of two components of the cut, induced by a transformation  $T_{h,\alpha}^1$ . It is easy to see, that the transformation  $T$  is a square of an interval exchange transformation of three intervals, with permutation  $(1, 3)$ . Varying length of the cut and angle  $\alpha$ , we may get specific values  $\lambda_1, \lambda_2, \lambda_3$  of exchanging intervals as in [7], and hence get an interval exchange transformation with two nontrivial ergodic measures constructed there.

**Remark** We have to admit, that the form  $\omega$  considered in the example does not satisfy our usual assumptions — it is not of a maximal rank.

## References

- [1] D.V.Anosov, *How curves on the universal covering plane that cover nonselfintersecting curves on a closed surface can go to infinity*, Proc. of the Steklov Inst. of Math., **191:2** (1989), 35–46.
- [2] V.I.Arnold, *Topological and ergodic properties of closed 1-forms with rationally independent periods*, English transl. in Functional Anal. Appl. **25** (1991).

- [3] V.I.Arnold, *Problems on singularities and dynamical systems*, Progress in Soviet Math., Chapman and Hall, 1992.
- [4] I.P.Cornfeld, S.V.Fomin, Ya.G.Sinai, *Ergodic Theory*, Springer-Verlag, 1982.
- [5] A.B.Katok, *Invariant measures of flows on oriented surfaces*, Soviet Math. Dokl. **14** (1973), 1104–1108.
- [6] M.Keane *Interval exchange transformations*, Math. Z. **141** (1975), 21–31.
- [7] H.B.Keynes, D.Newton, *A “minimal”, non-uniquely ergodic interval exchange transformation*, Math. Z., **148** (1976), 101–105.
- [8] G. Levitt, *Feuilletages des surfaces*. These de Doctorat D’Etat, Universite Paris VII, 1983.
- [9] H. Masur, *Interval exchange transformations and measured foliations*, Ann. of Math., **115:1** (1982), 169–200.
- [10] A.G.Maier, *Trajectories on the closed orientable surfaces*, Mat.Sb. **12 (54)** (1943), 71–84 (in Russian).
- [11] S.P.Novikov, *The Hamiltonian formalism and a many-valued analogue of Morse theory*, Russian Math. Surveys, **37:5**, (1982), 1–56.
- [12] S.P.Novikov, *Critical points and level surfaces of multivalued functions*, Proc. of the Steklov Inst. of Math., **166** (1984).
- [13] S.P.Novikov, *Quasiperiodic structures in topology*, Proc. of the Conference on Topological Methods in Math. , dedicated to 60-th anniv. of J.Milnor, Stony Brook, 1991.
- [14] E.A.Sataev, *On the number of invariant measures for flows on orientable surfaces*, Math. USSR Izvestija, **9:4** (1975), 813–830.
- [15] S.Schwartzman, *Asymptotic cycles*, Annals of Mathematics **66** (1957), 270–284.
- [16] Ya.G.Sinai, K.M.Khanin, *Mixing of some classes of special flows over rotation of a circle*, Functional Anal. Appl., **26:3** (1992).
- [17] W.A.Veech, *Gauss measures for transformations on the space of interval exchange maps*, Ann. of Math., **115:1** (1982), 201–242.
- [18] A.Zorich, *The S.P.Novikov Problem on Semiclassical motion of electron in homogeneous Magnetic Field*, Russian Math. Surveys, **39:5** (1984), 287–288.

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