Analytic torsion and R-torsion for unimodular representations

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Introduction

Let M be a closed C^∞-manifold of dimension n. Both R-torsion and analytic torsion are smooth invariants of acyclic orthogonal (or unitary) representations ρ of the fundamental group $\pi_1(M)$. The Reidemeister-Franz torsion (or R-torsion) $\tau_M(\rho)$ of ρ is defined in terms of the combinatorial structure of M given by its smooth triangulations. The analytic torsion $T_M(\rho)$ was introduced by Ray and Singer [RS] as analytic counterpart of R-torsion. In order to define the analytic torsion one has to choose a Riemannian metric on M. Then $T_M(\rho)$ is a certain weighted alternating product of regularized determinants of the Laplacians on differential q-forms on M with values in the flat bundle E_ρ defined by ρ. It was conjectured by Ray and Singer [RS] that $T_M(\rho) = \tau_M(\rho)$ for all acyclic orthogonal (or unitary) representations ρ. This conjecture was proved independently by Cheeger [C] and the author [Mü].

The restriction to orthogonal (or unitary) representations is certainly a limitation of the applicability of this result if $\pi_1(M)$ is infinite because an infinite discrete group will have, in general, many non-orthogonal finite dimensional representations. It is the purpose of the present paper to remove this limitation.

We call a representation $\rho : \pi_1(M) \to \mathrm{GL}(E)$ on a finite dimensional real or complex vector space E *unimodular* if $|\det \rho(\gamma)| = 1$ for all $\gamma \in \pi_1(M)$. Then we define R-torsion and analytic torsion for unimodular representations and the main result is that for odd dimensional manifolds M, the equality of the two torsions extends to all unimodular representations.

Now we shall explain this in more detail. Let $\rho : \pi_1(M) \to \mathrm{GL}(E)$ be an acyclic unimodular representation. Then the definition of R-torsion also makes sense for ρ. Nothing has to be changed. The problem is how to define the analytic torsion. If ρ is an orthogonal (or unitary) representation, then the flat bundle E_ρ over M defined by ρ can be equipped with a natural metric which is compatible with the flat connection. Associated to the metrics on E_ρ and M is the Laplacian Δ_q acting on the space $\Lambda^q(M; E_\rho)$ of E_ρ-valued differential q-forms on M. The zeta function of Δ_q is defined in the usual way by

$$\zeta_q(s;\rho) = \sum_{\lambda_j > 0} \lambda_j^{-s}, \quad \mathrm{Re}(s) > n/2,$$

where the λ_j run through the eigenvalues of Δ_q. It is well known that this function has a meromorphic continuation to $\mathbf{C}$ which is holomorphic at $s = 0$ [Se]. Then the definition of analytic torsion given by Ray and Singer is

$$T_M(\rho) = \exp\left(\tfrac{1}{2} \sum_{q=0}^{n} (-1)^q q \, \frac{d}{ds} \zeta_q(s;\rho)\big|_{s=0} \right). \tag{0.1}$$

For acyclic representations, $T_M(\rho)$ is independent of the metric on M.

For an arbitrary finite dimensional representation $\rho : \pi_1(M) \to \mathrm{GL}(E)$, there is no metric on E_ρ which is compatible with the flat connection. In order to define the analytic torsion in this case we proceed as follows. We choose a metric h on E_ρ. Using the metrics on M and E_ρ, we define an inner product on $\Lambda^*(M; E_\rho)$ which in turn gives rise to the Laplacian Δ_h. Then we define the analytic torsion $T_M(\rho; h)$ by formula (0.1) where $\zeta_q(s;\rho)$ is now the zeta function of the q-component of the new Laplacian Δ_h.

We note that this approach was used by Schwarz [S] in his treatment of abelian Chern-Simons theory and it is actually the origin for our definition of analytic torsion for non-orthogonal representations. We also note that a similar approach has recently been used by Bar-Natan and Witten [BNW] to deal with the perturbative expansion of non-abelian Chern-Simons gauge theory with non-compact gauge group.

The first important result about analytic torsion in the present context is that for an odd dimensional manifold M and an acyclic representation ρ, $T_M(\rho; h)$ is independent of the choice of h on E_ρ. Of course, it is also independent of the Riemannian metric on M and we denote its common value, for any choice of h and g, by $T_M(\rho)$. In general, the variation of $T_M(\rho; h)$ with repsect to h and g is given by an explicit formula. If $\dim M$ is even, the variational formula contains additional terms which are locally computable, that is, they are obtained by integrating densities which in any coordinate system are given by universal polynomials in the components $g_{ij}(x)$ and $h_{ij}(x)$ of the metrics and a finite number of their partial derivatives. There are examples showing that these terms may not vanish even if the representation is acyclic.

Using the inner product induced on the space of harmonic forms, we can also define the R-torsion $\tau_M(\rho; h)$ for non-acyclic representations ρ.

On the first sight, the choice of an arbitrary metric on E_ρ seems to be very artificial, but it is no more artificial than the arbitrary choice of the Riemannian metric on M. In general, there is no distinguished choice of a Riemannian metric on M. Only for special manifolds, like locally symmetric manifolds, do there exist distinguished metrics on M which make analytic torsion for orthogonal representations accessible to computations. In this case, however, there exist also distinguished metrics on E_ρ for many unimodular representations which makes analytic torsion for these representations also accessible to computations.

The main result of the present paper is

Theorem 1. *Let M be a closed smooth Riemannian manifold of odd dimension and let $\rho : \pi_1(M) \to \mathrm{GL}(E)$ be a unimodular representation on a finite dimensional real (or complex) vector space. Let E_ρ be the associated flat bundle. Then for any choice of a metric on E_ρ, we have*

$$T_M(\rho; h) = \tau_M(\rho; h).$$

We remark that, in general, Theorem 1 does not hold in even dimensions.

To prove Theorem 1 we follow essentially Cheeger's proof [C]. First we show that $\log T_M(\rho; h) - \log \tau_M(\rho; h)$ is independent of the metrics h on E_ρ and g on M. The main idea is then to keep track of

$$e_M(\rho) = \log T_M(\rho; h) - \log \tau_M(\rho; h)$$

if one does surgery on an embedded k-sphere in M and finally to reduce everything to the case where one is able to show the equality explicitly. Cheeger's proof is well suited to work for non-orthogonal representations, because the decisive part of the proof consists of local analysis near a given handle of M and the handles considered are such that E_ρ restricted to any of them is trivial as a flat bundle. Since we are free to choose the metrics h and g according to our purpose, we choose h and g to be the standard product metrics near the given handle. Then all the local analysis done by Cheeger in [C] extends without change to the present situation.

Now we describe briefly the content of the paper. In section 1 we review Reidemeister torsion for unimodular representations and we establish some of its properties. In section 2 we introduce analytic torsion for finite dimensional representations and we prove some of its properties. We also consider analytic torsion for manifolds with boundary and prove some results related to the variation of analytic torsion and R-torsion in this case. This is needed in section 3 where we establish the equality of the two torsions for unimodular representations. We explain the main steps of the proof and refer to Cheeger's paper [C] for all details not discussed in this section. Finally, in section 4 we consider two examples where non-orthogonal representations occur in nature and the analytic torsion as defined in section 2 arises naturally in this context. The first example are compact locally symmetric manifolds. The results of Borel and Wallach [BW] can be used to obtain numerous examples of acyclic unimodular representations. The corresponding flat bundles can be equipped with canonical metrics which are locally homogeneous so that methods of harmonic analysis can be applied. The second example is Chern-Simons gauge theory. Witten [W1] has shown that for a compact gauge group, the analytic torsion of flat connections occurs in the perturbative expansion of the path integral defined by the Chern-Simons action. This has been recently extended to non-compact gauge groups by Bar-Natan and Witten [BNW] and we explain how it is related to Theorem 1.

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1.Reidemeister torsion

In this section we recall the definition of Reidemeister torsion and we also collect some of its basic properties. For details we refer to [Mi1].

Given a real vector space E of dimension n, we set

$$\det E = \Lambda^n(E).$$

If $E = \{0\}$, then $\det E = \mathbf{R}$. Furthermore, if L is a one-dimensional vector space, each nonzero element $l \in L$ determines a unique element $l^{-1} \in L^*$ defined by the equation $l^{-1}(l) = 1$ and we shall use the notation $L^{-1} = L^*$.

A volume on E will be a nonzero element $\omega \in \det E$. Any volume determines an isomorphism $\det E \cong \mathbf{R}$. Note also that a volume ω can be written as $e_1 \wedge \cdots \wedge e_n$ for some basis $e_1, ..., e_n$ of E.

Let

$$C_\bullet : 0 \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0 \to 0 \tag{1.1}$$

be a chain complex of finite dimensional real vector spaces and let

$$H_q(C_\bullet) = \ker \partial_q / \mathrm{Im} \partial_{q+1}$$

be the q-th homology group of $C_\bullet$. The *determinant line* of the complex (1.1) is the one-dimensional vector space

$$\det(C_\bullet) = \bigotimes_{q=0}^{n} \left(\det C_q\right)^{(-1)^q}. \tag{1.2}$$

We also set

$$\det H_*(C_\bullet) = \bigotimes_{q=0}^{n} \left(\det H_q(C_\bullet)\right)^{(-1)^q}.$$

Let $b_q = \dim \partial(C_q)$ and $h_q = \dim H_q(C_\bullet)$. For each $q, 1 \leq q \leq n$, we choose $\theta_q \in \Lambda^{b_q}(C_q)$ such that $\partial \theta_q \neq 0$. Furthermore, let $0 \neq \mu_q \in \det H_q(C_\bullet)$ and $\nu_q \in \Lambda^{h_q}(\ker \partial_q)$ be such that $\pi(\nu_q) = \mu_q$ where $\pi : \ker \partial_q \to H_q(C_\bullet)$ is the canonical projection. Let $i : \ker \partial_q \to C_q$ be the natural embedding. Then $\partial \theta_{q+1} \wedge \theta_q \wedge i(\nu_q)$ is a nonzero element of $\det C_q$. Set $\mu = \otimes_{q=0}^{n} (\mu_q)^{(-1)^{q+1}}$. Then we define the *torsion*

$$T(C_\bullet) \in (\det C_\bullet) \otimes \left(\det H_*(C_\bullet)\right)^{-1}$$

of the complex (1.1) by

(1.3)
$$T(C_\bullet) = \left(\partial\theta_1 \wedge i(\nu_0)\right) \otimes \left(\partial\theta_2 \wedge \theta_1 \wedge i(\nu_1)\right)^{-1} \otimes \cdots \otimes \left(\partial\theta_n \wedge \theta_{n-1} \wedge i(\nu_{n-1})\right)^{(-1)^{n-1}}$$
$$\otimes \left(\theta_n \wedge i(\nu_n)\right)^{(-1)^n} \otimes \mu.$$

It is easy to see that $T(C_\bullet)$ is independent of the particular choices of θ_q, μ_q and ν_q.

Now assume that we have chosen $0 \neq \omega \in \det C_\bullet$. Then ω defines an isomorphism $\det C_\bullet \cong \mathbf{R}$ and therefore, also an isomorphism

$$\det C_\bullet \otimes \left(\det H_*(C_\bullet)\right)^{-1} \cong \left(\det H_*(C_\bullet)\right)^{-1}.$$

The image of $T(C_\bullet)$ with respect to this isomorphism will be denoted by

$$T(C_\bullet, \omega) \in \left(\det H_*(C_\bullet)\right)^{-1}.$$

Note that for a 2-term complex $C_\bullet : 0 \to C_0 \xrightarrow{A} C_1 \to 0$ with $C_0 = C_1 = \mathbf{R}^N$ and $\omega_q \in \det C_q$ being the canonical volumes , $T(C_\bullet, \omega) = \det A$.

Definition 1.4. The element $T(C_\bullet, \omega) \in \left(\det H_*(C_\bullet)\right)^{-1}$ defined above is called the *Reidemeister torsion* (or *R-torsion)* of $C_\bullet$ with respect to the volume ω.

If $C_\bullet$ is acyclic, i.e., $H_*(C_\bullet) = \{0\}$, then $\left(\det H_*(C_\bullet)\right)^{-1} = \mathbf{R}$ and $T(C_\bullet, \omega)$ is a real number. More generally, any choice of a volume $\mu \in \det H_*(C_\bullet)$ induces a natural isomorphism

$$\left(\det H_*(C_\bullet)\right)^{-1} \cong \mathbf{R}$$

by sending $\lambda\mu^{-1}$ to $\lambda \in \mathbf{R}$.Thus $T(C_\bullet, \omega)$ can be identified with a real number $T(C_\bullet, \omega, \mu) \in \mathbf{R}$ and we set

$$\tau(C_\bullet, \omega, \mu) = |T(C_\bullet, \omega, \mu)|. \tag{1.5}$$

Next we discuss some of the properties satisfied by R-torsion. Suppose that the C_q are actually equipped with inner products. Let $\partial_q^* : C_{q-1} \to C_q$ be the adjoint of ∂_q with respect to these inner products and set

$$D_q = \partial_q^* \partial_q + \partial_{q+1} \partial_{q+1}^*.$$

This is a symmetric and positive semi-definite operator on C_q. Set$\mathcal{H}_q(C_\bullet) = \ker D_q$. Then we have the following decomposition

$$C_q = \mathcal{H}_q(C_\bullet) \oplus \mathrm{Im}\partial_{q+1} \oplus \mathrm{Im}\partial_q^*$$

and a canonical isomorphism

$$\mathcal{H}_q(C_\bullet) \cong H_q(C_\bullet). \tag{1.6}$$

Let $\overline{D}_q$ be the restriction of D_q to the orthogonal complement of $\ker D_q$ in C_q and set

$$\det{}' D_q = \det \overline{D}_q.$$

Now observe that the innner product on $C_\bullet$ defines a volume $\omega \in \det C_\bullet$. Furthermore, if we use the induced inner product on $\mathcal{H}_q(C_\bullet)$ combined with (1.6), we obtain a volume $\mu \in \det H_*(C_\bullet)$.

Proposition 1.7. *With the notation above, we have the following equality*

$$\tau(C_\bullet,\omega,\mu)^2 = \prod_{q=0}^{n} \left(\det{}' D_q\right)^{q(-1)^q}.$$

Proof. The inner product on $C_\bullet$ gives rise to a canonical splitting $C_\bullet = C'_\bullet \oplus C''_\bullet$, where $C'_\bullet$ is acyclic, $C''_\bullet$ has zero differential and $C''_\bullet \cong H_q(C_\bullet)$. Let ω' be the volume element in $\det C'_\bullet$ induced by the inner product on $C'_\bullet$. There is a natural isomorphism

$$\det C_\bullet \cong \det C'_\bullet \otimes \det C''_\bullet$$

and with respect to this isomorphism we have $\omega = \omega' \otimes \mu$. It is then clear that $\tau(C_\bullet,\omega,\mu) = \tau(C'_\bullet,\omega')$. The rest follows from Proposition 1.5 in [BGS]. Q.E.D.

Now consider a double complex, i.e., a sequence

$$C_{\bullet\bullet} : 0 \to C_{m\bullet} \xrightarrow{\delta} \cdots \xrightarrow{\delta} C_{1\bullet} \xrightarrow{\delta} C_{0\bullet} \to 0 \tag{1.8}$$

of finite complexes. As usually, define the total complex $T_\bullet$ by

$$T_q = \bigoplus_{i+j=q} C_{i,j}$$

with differential $D = \delta + (-1)^i\partial$ (on $C_{i,j}$). The complex $T_\bullet$ has two natural filtrations and associated spectral sequences converging to $H_*(T_\bullet)$. Let $(E^0_\bullet,\partial^0)$, $(E^1_\bullet,\partial^1)$,... be the spectral sequence associated to the filtration of $T_\bullet$ by the columns of $C_{\bullet\bullet}$. Then $E^{r+1}_\bullet \cong H_*(E^r_\bullet)$. The following result is due to D.Freed [F]

Proposition 1.9. *Suppose that (1.8) is an exact sequence , i.e., each complex $C_{\bullet j}$ is acyclic. Then there is a natural isomorphism*

$$\bigotimes_{j=0}^{n}\left(\det(C_{\bullet j})\right)^{(-1)^j} \cong \bigotimes_{r=0}^{\infty} \det(E^r_\bullet)$$

where the product on the right hand side is finite and, with respect to this isomorphism,

$$\otimes_{j=0}^{n} T(C_{\bullet j})^{(-1)^j} = \otimes_{r=0}^{\infty} T(E^r_\bullet).$$

For the proof see [F].

In particular, let

$$0 \to C_{\bullet 2} \longrightarrow C_{\bullet 1} \longrightarrow C_{\bullet 0} \to 0 \tag{1.10}$$

be a short exact sequence of finite complexes and let

$$\mathcal{H}_\bullet : \cdots \to H_q(C_{\bullet 2}) \longrightarrow H_q(C_{\bullet 1}) \longrightarrow H_q(C_{\bullet 0}) \xrightarrow{\delta} H_{q-1}(C_{\bullet 2}) \to \cdots$$

be the long exact homology sequence associated to (1.10). The spectral sequence degenerates at the E^2-term. Moreover,

$$T(E^0_\bullet) = \otimes_{i=0}^2 T(C_{i\bullet})^{(-1)^i} \quad \text{and} \quad T(E^1_\bullet) = T(\mathcal{H}_\bullet).$$

Corollary 1.11. *There is a natural isomorphism*

$$\bigotimes_{j=0}^{n} \left(\det(C_{\bullet j}\right)^{(-1)^j} \cong \bigotimes_{i=0}^{2} \left(\det(C_{i\bullet})\right)^{(-1)^i} \otimes \det(\mathcal{H}_\bullet)$$

and, with respect to this isomorphism,

$$\otimes_{j=0}^n T(C_{\bullet j})^{(-1)^j} = \otimes_{i=0}^2 T(C_{i\bullet})^{(-1)^i} \otimes T(\mathcal{H}_\bullet).$$

Suppose we have chosen volumes $\omega_{qi} \in \det(C_{qi})$ and $\mu_{qi} \in \det H_q(C_{\bullet i})$, $i = 1,2,3$. Set $\omega_{\bullet i} = \otimes_{q=0}^n (\omega_{qi})^{(-1)^q}$, $\omega_{q\bullet} = \otimes_{i=0}^2 (\omega_{qi})^{(-1)^i}$ and $\mu_i = \otimes_{q=0}^n (\mu_{qi})^{(-1)^q}$. Suppose that $T(C_{q\bullet}, \omega_{q\bullet}) = 1$ for all $q = 0, ..., n$. Then Corollary 1.11 implies

$$\tau(C^2_\bullet, \omega_2, \mu_2) = \tau(C^1_\bullet, \omega_1, \mu_1)\, \tau(C^3_\bullet, \omega_3, \mu_3)\, \tau(\mathcal{H}_\bullet, \mu) \tag{1.12}$$

where we regard $\mathcal{H}_\bullet$ as an acyclic complex with volume μ defined by μ_1, μ_2 and μ_3 (cf. [Mi1] for details).

Let $C_\bullet$ and $C'_\bullet$ be two chain complexes with volumes $\omega \in \det C_\bullet, \omega' \in \det C'_\bullet$, $\mu \in \det H_*(C_\bullet)$ and $\mu' \in \det H^*(C'_\bullet)$. Let $C_\bullet \otimes C'_\bullet$ denote the tensor product complex with its standard differential, i.e.,

$$(C_\bullet \otimes C'_\bullet)_q = \bigoplus_{r+s=q} (C_r \otimes C'_s). \tag{1.13}$$

Let $d_p = \dim C_p$ and $d'_q = \dim C'_s$. Then, by (1.13), we have a natural isomorphism

$$\det(C_\bullet \otimes C'_\bullet)_q \cong \bigotimes_{r+s=q} (\det C_r)^{\otimes d'_s} \otimes (\det C'_s)^{\otimes d_r}. \tag{1.14}$$

Now recall that, for any complex $E_\bullet$, one has $\sum (-1)^q \dim E_q = \chi(E_\bullet)$, where $\chi(E_\bullet)$ denotes the Euler characteristic of the complex. Hence (1.14) induces a natural isomorphism

$$\det(C_\bullet \otimes C'_\bullet) \cong (\det C_\bullet)^{\otimes \chi(C'_\bullet)} \otimes (\det C'_\bullet)^{\otimes \chi(C_\bullet)}.$$

Let $\omega \otimes \omega'$ the element of $\det(C_\bullet \otimes C'_\bullet)$ which corresponds to $\omega^{\otimes \chi(C'_\bullet)} \otimes (\omega')^{\otimes \chi(C_\bullet)}$ under this isomorphism. Furthermore, by the Kunneth formula, the homology of $C_\bullet \otimes C'_\bullet$ is given by

$$(1.15) \qquad H_q(C_\bullet \otimes C'_\bullet) = \bigoplus_{r+s=q} H_r(C_\bullet) \otimes H_s(C'_\bullet)$$

and, in the same way as above, this induces a natural isomorphism

$$\det H_*(C_\bullet \otimes C'_\bullet) \cong \det H_*(C_\bullet)^{\otimes \chi(C'_\bullet)} \otimes \det H_*(C'_\bullet)^{\otimes \chi(C_\bullet)}.$$

We let $\mu \otimes \mu'$ denote the element of $\det H_*(C_\bullet \otimes C'_\bullet)$ which corresponds to $\mu^{\otimes \chi(C'_\bullet)} \otimes (\mu')^{\otimes \chi(C_\bullet)}$ under this isomorphism.

Proposition 1.16. *We have the following equality*

$$\tau(C_\bullet \otimes C'_\bullet, \omega \otimes \omega', \mu \otimes \mu') = \tau(C_\bullet, \omega, \mu)^{\chi(C'_\bullet)} \, \tau(C'_\bullet, \omega', \mu')^{\chi(C_\bullet)}$$

where $\chi(E_\bullet)$ denotes the Euler characteristic of the complex $E_\bullet$.

Proof. We introduce inner products on $C_\bullet, C'_\bullet, H_*(C_\bullet)$ and $H_*(C'_\bullet)$ which induce the corresponding volumes. Then we use (1.13) and (1.15) to define inner products on $C_\bullet \otimes C'_\bullet$ and $H_*(C_\bullet \otimes C'_\bullet)$ in the canonical way. It is easy to see that these inner products induce the volumes $\omega \otimes \omega' \in \det(C_\bullet \otimes C'_\bullet)$ and $\mu \otimes \mu' \in \det H_*(C_\bullet \otimes C'_\bullet)$. Now we apply Proposition 1.7 above and Proposition 1.16 in [C]. Q.E.D.

Let

$$C^*_\bullet : 0 \leftarrow C^*_n \xleftarrow{\partial^*_n} C^*_{n-1} \xleftarrow{\partial^*_{n-1}} \cdots \xleftarrow{\partial^*_1} C^*_0 \leftarrow 0$$

be the dual complex to (1.1). Note that there are natural isomorphisms

$$\det C^*_q \cong (\det C_q)^{-1} \quad \text{and} \quad \det H_q(C^*_\bullet) \cong \left(\det H_q(C_\bullet)\right)^{-1}.$$

Hence, we get a natural isomorphism

$$(1.17) \qquad \left(\det C_\bullet \otimes \left(\det H_*(C_\bullet)\right)^{-1}\right) \otimes \left(\det C^*_\bullet \otimes \left(\det H_*(C^*_\bullet)\right)^{-1}\right)^{(-1)^n} \cong \mathbf{R}.$$

Proposition 1.18. *With respect to the isomorphism (1.17), we have*

$$T(C_\bullet) \otimes T(C^*_\bullet)^{(-1)^n} = \pm 1.$$

The proof is straight forward and follows by an easy generalization of the argument used on p.141 of [Mi2] to prove equality (3).

Let $\omega \in \det C_\bullet$ and $\mu \in \det H_*(C_\bullet)$ be volumes. Denote by $\omega^* \in \det C^*_\bullet \cong (\det C_\bullet)^{-1}$ and $\mu^* \in \det H_*(C^*_\bullet) \cong \left(\det H_*(C_\bullet)\right)^*$ the dual volumes determined by ω and μ, respectively.

Corollary 1.19. *We have the following equality*

$$\tau(C_\bullet,\omega,\mu)\,\tau(C^*_\bullet,\omega^*,\mu^*)^{(-1)^n}=1.$$

The R-torsion arises in the following context. Let K be a finite cell complex and $\tilde{K}$ the universal covering space of K with the fundamental group $\pi_1=\pi_1(K)$ acting as deck transformations on $\tilde{K}$. We think of K as being embedded as a fundamental domain in $\tilde{K}$, so that $\tilde{K}$ is the union of the translates of K under π_1. Let $C_q(\tilde{K})$ be the real chain group generated by the q-cells of $\tilde{K}$. Then $C_q(\tilde{K})$ is a module over the real group algebra $\mathbf{R}(\pi_1)$. The q-cells of K form a preferred base for $C_q(\tilde{K})$ as $\mathbf{R}(\pi_1)$-module. Let $\rho:\pi_1\to \mathrm{GL}(E)$ be a representation of π_1 on a real vector space E of dimension N. It defines a flat bundle E_ρ over K and we define the chain group $C_q(K;E)$ of chains with values in the local system E_ρ by

$$C_q(K;E)=C_q(\tilde{K})\underset{\mathbf{R}(\pi_1)}{\otimes}E\,.$$

The boundary operator $\tilde{\partial}_q:C_q(\tilde{K})\to C_{q-1}(\tilde{K})$ induces $\partial_q:C_q(K;E)\to C_{q-1}(K;E)$ and we get a real chain complex

$$C_\bullet(K;E):0\to C_n(K;E)\xrightarrow{\partial_n}C_{n-1}(K;E)\xrightarrow{\partial_{n-1}}\cdots\xrightarrow{\partial_1}C_0(K;E)\to 0$$

Its homology will be denoted by $H_*(K;E)$. Similarly, we have the cochain complex with coefficients in E_ρ:

$$C^\bullet(K;E):0\to C^0(K;E)\xrightarrow{\delta_1}C^1(K;E)\xrightarrow{\delta_2}\cdots\xrightarrow{\delta_n}C^n(K;E)\to 0$$

where $C^q(K;E)=\mathrm{Hom}_{\mathbf{R}(\pi_1)}(C_q(\tilde{K}),E)$ and $\delta=\partial^*$. Note that

$$C^q(K;E)\cong C^q(\tilde{K})\underset{\mathbf{R}(\pi_1)}{\otimes}E.$$

We shall denote the cohomology of this complex by $H^*(K;E)$.

If K is a triangulation of a closed oriented smooth manifold of dimension n then, for each q, we have the Poincaré duality isomorphism

$$\Pi_q:H_q(M;E)\xrightarrow{\sim}H^{n-q}(M;E). \tag{1.20}$$

To define this isomorphism on the level of chains and cochains consider the dual cell complex K^* of K and let $\tilde{K}^*$ be its universal covering space. Then we have the following result due to Reidemeister

Lemma 1.21. *The* $\mathbf{R}(\pi_1)$*-module* $C_{n-q}(\tilde{K}^*)$ *is canonically isomorphic to the dual of the* $\mathbf{R}(\pi_1)$*-module* $C_q(\tilde{K})$*. Furthermore the boundary operator*

$$\partial:C_{n-q}(\tilde{K}^*)\longrightarrow C_{n-q-1}(\tilde{K}^*)$$

is (up to sign) dual to the boundary operator

$$\partial:C_{q+1}(\tilde{K})\longrightarrow C_q(\tilde{K}).$$

For the proof see Lemma 1 in [Mi2].

Corollary 1.22. *For each q, there is a canonical isomorphism*

$$\tilde{\Pi}_q : C_q(K^*; E) = C_q(\tilde{K}^*) \underset{\mathbf{R}(\pi_1)}{\otimes} E \xrightarrow{\sim} (C_{n-q}(\tilde{K}))^* \underset{\mathbf{R}(\pi_1)}{\otimes} E = C^{n-q}(K; E)$$

which satisfies $\delta_{n-q+1} \circ \tilde{\Pi}_q = \tilde{\Pi}_q \circ \partial_q$.

Hence, $\tilde{\Pi}_q$ induces an isomorphism on the homology of the corresponding complexes and this is the Poincaré duality isomorphism Π_q above.

Now assume that a volume $\theta \in \det E$ is given.Let e^q_j, $j = 1, ..., r_q$, be the oriented q-cells of K considered as a preferred base of the $\mathbf{R}(\pi_1)$-module $C_q(\tilde{K})$ and let $x_1, ..., x_N$ be a base of E such that $\theta = \pm x_1 \wedge \cdots \wedge x_N$. Then $(e^q_j \otimes x_k)$ is a preferred base of $C_q(K; E)$ and it defines a volume $\omega_q \in \det C_q(K; E)$.

At this point, the volumes depend on several choices

a) The choice of the embedding of K in the covering space $\tilde{K}$.

b) The orientation and ordering of the cells of K.

c) The choice of the base $x_1, ..., x_N$ of E.

To deal with a), we make the following

Definition 1.23. A representation $\rho : \pi_1(K) \to \mathrm{GL}(E)$ is called *unimodular* if $|\det \rho(\gamma)| = 1$ for all $\gamma \in \pi_1(K)$.

If we assume that ρ is unimodular, then a different choice of the embedding of K into $\tilde{K}$ corresponds to a change of base in $C_q(K; E)$ by a matrix whose determinant is of absolute value one and therefore, the volume ω_q changes at most by sign. Similarly, b) and c) cause only a change of sign. Set $\omega = \otimes_{q=0}^n (\omega_q)^{(-1)^q}$.

Remark. The volume ω depends, of course, on the choice of the volume θ on E. We indicate this by $\omega(\theta)$. If we replace θ by $\lambda\theta$, $\lambda \in \mathbf{R}$, then

$$\omega(\lambda\theta) = \prod_{q=0}^{n} \lambda^{(-1)^q r_q} \omega(\theta) = \lambda^{\chi(K)} \omega(\theta)$$

where $\chi(K)$ is the Euler characteristic of K. Thus, if K is a triangulation of a closed oriented manifold of odd dimension, then ω is independent of the choice of a volume on E. If $\chi(K) \neq 0$ we fix a volume on E and set $\omega = \omega(\theta)$.

Then we define the R-torsion $\tau_K(\rho)$ by

$$\tau_K(\rho) = \left[\tau(C_\bullet(K; E), \omega)\right] \in \left(\det H_*(K; E)\right)^{-1} / \{\pm 1\}$$

and this definition does not depend on the choices we made to define the volume element ω for $C_\bullet(K; E)$. Let $\mu \in \det H_*(K; E)$ be a volume . Then we have a natural isomorphism

$$\left(\det H_*(K; E)\right)^{-1} / \{\pm 1\} \cong \mathbf{R}^+$$

and $\tau_K(\rho)$ can be identified with a positive real number which we denote by $\tau_K(\rho;\mu)$. An important fact is that $\tau_K(\rho;\mu)$ is invariant under subdivision. The proof is based on (1.12). Namely, let K' be a subdivision of K and set $C_\bullet = C_\bullet(K;E)$ and $C'_\bullet = C_\bullet(K';E)$. Then we have a natural injective chain map $C_\bullet \to C'_\bullet$ and we denote by $\hat{C}_\bullet$ the quotient complex $C'_\bullet/C_\bullet$. In this way we get a short exact sequence of chain complexes

$$0 \to C_\bullet \xrightarrow{i} C'_\bullet \longrightarrow \hat{C}_\bullet \to 0.$$

Furthermore, $H_*(C_\bullet) \cong H_*(C'_\bullet)$ and $H_*(\hat{C}_\bullet) = 0$. We choose the volume $\mu' \in \det H_*(C'_\bullet)$ to be equal to μ. Then the torsion of the long exact homology sequence $\mathcal{H}_\bullet$ is 1. Each chain group has a volume defined as above. The rows are acyclic, and it follows from the definitions that the torsion of each row is 1. Hence we can apply (1.12) which gives

$$\tau(C'_\bullet, \omega', \mu') = \tau(C_\bullet, \omega, \mu)\,\tau(\hat{C}_\bullet, \hat{\omega})$$

and it remains to verify that $\tau(\hat{C}_\bullet, \hat{\omega}) = 1$. But this follows from the explicit description of $\hat{C}_\bullet$ (cf. [Mi1] for more details). Thus $\tau_K(\rho)$ is a combinatorial invariant.

Let M be a compact smooth manifold and let $\rho : \pi_1(M) \to \mathrm{GL}(E)$ be a unimodular representation. The manifold M has a distinguished class of triangulations, the so-called smooth triangulations. Any two of these have a common subdivision (cf. [Mu]). Thus, for any choice of a volume $\mu \in \det H_*(M)$, we get an invariant $\tau_M(\rho;\mu) \in \mathbf{R}^+$.

Definition 1.24. The positive real number $\tau_M(\rho;\mu)$ is called the *Reidemeister torsion* (or *R-torsion*) of the manifold M with respect to ρ and μ.

Recall that a representation $\rho : \pi_1(M) \to \mathrm{GL}(E)$ is called *acyclic* if $H^*(M; E_\rho) = 0$.

If ρ is acyclic , the R-torsion $\tau_M(\rho)$ is a positive real number which is an invariant of the manifold M and the representation ρ.

As above, the R-torsion $\tau_M(\rho;\mu)$ has several important properties . Here we recall two of them which we need in section 3 to prove the equality of analytic torsion and R-torsion.

Let M be a closed orientable manifold of dimension n and let $\rho : \pi_1(M) \to \mathrm{GL}(E)$ be a unimodular representation with associated flat bundle E_ρ. Let ρ^* be the contragredient representation. Then ρ^* is also unimodular and the associated flat bundle is the dual vector bundle E^*_ρ of E_ρ. For each q, the Poincaré duality isomorphism Π_q induces an isomorphism $H_q(M;E^*) \xrightarrow{\sim} (H_{n-q}(M;E))^*$ and therefore, an isomorphism

$$\lambda_q : \det H_q(M;E^*) \xrightarrow{\sim} (\det H_{n-q}(M;E))^{-1}.$$

The following result is a slight extension of Milnor's duality theorem [Mi2].

Proposition 1.25. *Assume that, for each q, we have chosen volumes $\mu_q \in \det H_q(M;E)$ and $\mu_q^* \in \det H_q(M;E^*)$ satisfying $\lambda_q(\mu_q^*) = (\mu_{n-q})^{-1}$. Then we have*

$$\tau_M(\rho;\mu)\,\tau_M(\rho^*;\mu^*)^{(-1)^n} = 1.$$

Proof. We may essentially proceed as in [Mi2]. Let K be a smooth triangulation of M and K^* the dual cell complex. Denote by $\tilde{K}$ and $\tilde{K}^*$ the universal covering spaces of K and K^*, respectively. By Corollary 1.22, we get a canonical isomorphism of chain complexes

$$\begin{array}{ccccccccc}
0 \to & C_n(K^*;E^*) & \xrightarrow{\partial} & C_{n-1}(K^*;E^*) & \xrightarrow{\partial} \cdots \xrightarrow{\partial} & C_0(K^*;E^*) & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & \left(C_0(K;E)\right)^* & \xrightarrow{\partial^*} & \left(C_1(K;E)\right)^* & \xrightarrow{\partial^*} \cdots \xrightarrow{\partial^*} & \left(C_n(K;E)\right)^* & \to 0
\end{array}$$

and, for each q, we have a canonical isomorphism

$$\det C_{n-q}(K^*;E^*) \xrightarrow{\sim} \left(\det C_q(K;E)\right)^{-1}. \tag{1.26}$$

Let $\omega_q \in \det C_q(K;E)$ and $\omega_q^* \in \det C_q(K^*;E^*)$ be the canonical volumes constructed above by choosing embeddings of K, K^* in their universal covering spaces and a base $x_1, ..., x_N$ of E such that $\theta = \pm x_1 \wedge \cdots \wedge x_N$. Then, with respect to the isomorphism (1.26), we have $\omega_{n-q}^* = (\omega_q)^{-1}$. Using Corollary 1.19, we obtain

$$\tau_K(\rho, \omega, \mu)\, \tau_{K^*}(\rho^*, \omega^*, \mu^*)^{(-1)^n} = 1.$$

Since K and K^* have a common subdivision, our result follows from the definition of $\tau_M(\rho;\mu)$ and $\tau_M(\rho^*;\mu^*)$. Q.E.D.

Let M_1, M_2 be two compact smooth manifolds and let $\rho_i : \pi_1(M_i) \to \mathrm{GL}(E_i)$, $i = 1, 2$, be unimodular representations with associated flat bundles E_{ρ_1}, E_{ρ_2}. Note that $\pi_1(M_1 \times M_2) = \pi_1(M_1) \times \pi_1(M_2)$ and the representation $\rho_1 \otimes \rho_2 : \pi_1(M_1 \times M_2) \to \mathrm{GL}(E_1 \otimes E_2)$ is unimodular. Denote by $p_i : M_1 \times M_2 \to M_i$, $i = 1, 2$, the canonical projection. Then the flat bundle over $M_1 \times M_2$ defined by $\rho_1 \otimes \rho_2$ equals $p_1^* E_{\rho_1} \otimes p_2^* E_{\rho_2}$. Let $\mu_i \in \det H_*(M_i; E_i)$ $i = 1, 2$, be volumes. By the Kunneth formula, we have

$$H_q(M_1 \times M_2; E_1 \otimes E_2) = \bigoplus_{r+s=q} H_r(M_1; E_1) \otimes H_s(M_2; E_2)$$

and as above, we obtain an isomorphism

$$\det H_*(M_1 \times M_2; E_1 \otimes E_2) = \det H_*(M_1; E_1)^{\chi(M_2;E_2)} \otimes \det H_*(M_2; E_2)^{\chi(M_1;E_1)}.$$

Let $\mu_1 \otimes \mu_2$ be the element of $\det H_*(M_1 \times M_2; E_1 \otimes E_2)$ which corresponds to $\mu_1^{\chi(M_2;E_2)} \otimes \mu_2^{\chi(M_1;E_1)}$ under this isomorphism. As a consequence of Proposition 1.16 we get

Proposition 1.27. *With the notation above, we have the following equality*

$$\tau_{M_1\times M_2}(\rho_1\otimes\rho_2;\mu_1\otimes\mu_2)=\tau_{M_1}(\rho_1;\mu_1)^{\chi(M_2;E_2)}\cdot\tau_{M_2}(\rho_2;\mu_2)^{\chi(M_1;E_1)}.$$

The volumes μ we are considering arise in the following way. We choose metrics on M and on the flat bundle E_ρ defined by ρ. If $\partial M\neq\emptyset$ we also impose boundary conditions for the Laplacian on forms. Then the space of harmonic forms has an inner product and, via the De Rham isomorphism, we get inner products on $H^*(M;E)$ or $H^*(M,\partial M;E)$, depending on the boundary conditions, and these inner products define volumes (cf. section 2 for details). In particular, for this choice of volumes the condition of Proposition 1.25 is satisfied.

Remark. We may also work with complex coefficients and define the R-torsion in this case. As above, a representation $\rho:\pi_1(M)\to\mathrm{GL}(E)$ on a complex finite dimensional vector space E is called unimodular if $|\det\rho(\gamma)|=1$ for all $\gamma\in\pi_1(M)$. For a unimodular representation ρ the R-torsion $\tau_M(\rho;\mu)$ is defined in the same way as in the real case.

Let $i:\mathrm{GL}(N,\mathbf{C})\to\mathrm{GL}(2N,\mathbf{R})$ be the standard embedding which sends $g=A+\sqrt{-1}B$, $A,B\in\mathrm{Mat}(N,\mathbf{R})$ to $\begin{pmatrix}A & -B\\ B & A\end{pmatrix}\in\mathrm{GL}(2N,\mathbf{R})$. If $|\det(g)|=1$ then $|\det(i(g))|=1$. Hence, if $\rho:\pi_1(M)\to\mathrm{GL}(N,\mathbf{C})$ is unimodular, then $i\circ\rho:\pi_1(M)\to\mathrm{GL}(2N,\mathbf{R})$ is also unimodular and we are back to our previous framework.

2. Analytic torsion for unimodular representations

In this section we define the analytic torsion and we prove some of its properties. Let (M, g) be a compact Riemannian manifold of dimension n. For simplicity we assume that M is orientable. Let $\rho : \pi_1(M) \to \mathrm{GL}(E)$ be a representation of the fundamental group of M on a real vector space of dimension N. The representation ρ defines a flat vector bundle E_ρ over M. We choose a metric h on E_ρ. It induces a $\mathbf{R}$-linear isomorphism

$$\# : E_\rho \xrightarrow{\sim} E_\rho^*$$

where E_ρ^* is the dual vector bundle. Let $\Lambda^p(E)$ be the space of C^∞ p-forms on M with values in E_ρ, i.e., the space of smooth sections of the vector bundle $\Lambda^p(T^*M) \otimes E_\rho$. Then $\#$ extends to an isomorphism

$$\# : \Lambda^p(E) \xrightarrow{\sim} \Lambda^p(E^*)$$

for each p. Furthermore, the Riemannian metric on M defines a linear mapping

$$* : \Lambda^p(E) \longrightarrow \Lambda^{n-p}(E)$$

for each p (cf. [MM], §2) which satisfies $** = (-1)^{p(n-p)}$ on $\Lambda^p(E)$ and it is easy to see that $*$ and $\#$ commute. The usual exterior product of differential forms combined with the evaluation map $\mathrm{tr} : E_\rho \otimes E_\rho^* \to \mathbf{R}$ induces the following exterior product for vector valued forms

$$\wedge : \Lambda^p(E) \otimes \Lambda^q(E^*) \longrightarrow \Lambda^{p+q}(M)$$

where $\Lambda^*(M)$ is the space of smooth differential forms on M (cf. [MM],§2). Then an inner product on $\Lambda^p(E)$ is defined by

$$\langle \omega, \omega' \rangle = \int_M \omega \wedge * \circ \# \omega'.$$

Let $L^2\Lambda^p(E)$ denote the completion of $\Lambda^p(E)$ with respect to the norm defined by this inner product. Since E_ρ is flat we have the De Rham complex

$$\Lambda^0(E) \xrightarrow{d_0} \Lambda^1(E) \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} \Lambda^n(E)\,.$$

The formal adjoint of d_p on $\Lambda^p(E)$ is given by

$$\delta_p = (-1)^{np+n+1} * \circ \#^{-1}\, d_p \,\# \circ *\,.$$

Remark: We may, of course, combine $*$ and $\#$ into a single operator $\tilde{*}$. The advantage of the notation above is that the dependence on the Riemannian metric on M and on the Euclidean metric on E_ρ is separated by $*$ and $\#$.

Now we define the Laplacian on p-forms as usual by

$$\Delta_p = \delta_p d_p + d_{p-1}\delta_{p-1}.$$

Note that Δ_p does not depend on the choice of an orientation on M and therefore we can define Δ_p also if M is not orientable. One has to use the formalism of densities (see section 3 of [C] for details).

Now assume that $\partial M = \emptyset$.Then the Laplacian is a symmetric, positive semi-definite, elliptic operator with pure point spectrum

$$0 \leq \lambda_0 \leq \lambda_1 \leq \cdots \to \infty\,.$$

Let $\mathcal{H}^p(E)$ denote the kernel of Δ_p. This is the space of E_ρ-valued harmonic p-forms on M and the De Rham map induces an isomorphism

$$\mathcal{H}^p(E) \xrightarrow{\sim} H^p(M;E) \tag{2.1}$$

where $H^p(M;E)$ is the cohomology of M with coefficients in the flat bundle E_ρ. Let P_q denote the orthogonal projection of $L^2\Lambda^q(E)$ onto the subspace $\mathcal{H}^q(E)$. The zeta function associated to the Laplacian Δ_q on $\Lambda^q(E)$ is defined by

$$\zeta_q(s;\rho) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\operatorname{Tr}\big(e^{-t\Delta_q} - P_q\big)\,dt = \sum_{\lambda_j>0}\lambda_j^{-s}, \quad \operatorname{Re}(s) > n/2.$$

It is proved in [Se] that $\zeta_q(s;\rho)$ extends to a meromorphic function of $s \in \mathbf{C}$ which is holomorphic at $s = 0$. As in the case of an orthogonal representation we define the *analytic torsion* $T_M(\rho;h)$ by

$$T_M(\rho;h) = \exp\bigg(\tfrac{1}{2}\sum_{q=0}^{n}(-1)^q q\,\frac{d}{ds}\zeta_q(s;\rho)\big|_{s=0}\bigg)\,. \tag{2.2}$$

Note that $T_M(\rho;h)$ also depends on the Riemannian metric g , but we do not indicate this dependence explicitly.

First we want to get a formula for the variation of the analytic torsion with respect to the Riemannian and the Euclidean structure. To obtain such a formula we may proceed as in [RS]. Consider a 1-parameter family h_u of Euclidean metrics on E_ρ. Let $\Delta_q(u)$ be the Laplacian and let $P_q(u)$ be the harmonic projection with respect to h_u. Put

$$F(u,s) = \tfrac{1}{2}\sum_{q=0}^{n}(-1)^q q\int_0^\infty t^{s-1}\operatorname{Tr}\big(e^{-t\Delta_q(u)} - P_q(u)\big)\,dt \tag{2.3}$$

for $\operatorname{Re}(s) > n/2$. By the remark above, this function extends to a meromorphic function of $s \in \mathbf{C}$ with a possible simple pole at $s = 0$. We may use the heat expansion

$$\operatorname{Tr}\big(e^{-t\Delta_q(u)}\big) \sim \sum_{j=0}^{\infty} a_{q,j}(u)\,t^{-n/2+j}$$

to construct the explicit analytic continuation for $F(u,s)$ (cf. section 3 of [C]). From the resulting formula it is clear that $\frac{\partial}{\partial u}F(u,s)$ also admits an analytic continuation to $\mathbf{C}$ and

$$\frac{\partial}{\partial u}\log T_M(\rho;h_u)=\frac{\partial}{\partial s}\Big(\frac{1}{\Gamma(s)}\frac{\partial}{\partial u}F(u,s)\Big)\Big|_{s=0}. \tag{2.4}$$

To compute $\frac{\partial}{\partial u}F(u,s)$ we have to determine the variation of the trace of the heat operator. Let $\alpha=\#^{-1}\dot{\#}$ where $\dot{\#}=d\#/du$. By (2.1), $\mathrm{Tr}(P_q(u))=\dim\ H^q(M;E)$ is indepentent of u. Hence

$$\frac{\partial}{\partial u}\mathrm{Tr}\big(e^{-t\Delta_q(u)}-P_q(u)\big)=-t\,\mathrm{Tr}\big(\dot{\Delta}_q\,e^{-t\Delta_q(u)}\big) \tag{2.5}$$

where $\dot{\Delta}_q=d/du(\Delta_q)$. Since $\dot{\#^{-1}}\#=-\#^{-1}\dot{\#}$ and $*\#=\#*$, it follows that $\dot{\Delta}_q=-\alpha\delta d+\delta\alpha d-d\alpha\delta+d\delta\alpha$. Employing $d\Delta_q=\Delta_{q+1}d$, $\delta\Delta_q=\Delta_{q-1}\delta$ and the fact that $\mathrm{Tr}(AB)=\mathrm{Tr}(BA)$ for a trace class operator A and a bounded operator B, we obtain

$$\begin{aligned}\mathrm{Tr}\big(\dot{\Delta}_q e^{-t\Delta_q(u)}\big)&=Tr\big(\alpha d\delta e^{-t\Delta_q(u)}\big)-\mathrm{Tr}\big(\alpha\delta d e^{-t\Delta_{q-1}(u)}\big)\\&\quad+\mathrm{Tr}\big(\alpha d\delta e^{-t\Delta_{q+1}(u)}\big)-\mathrm{Tr}\big(\alpha\delta d e^{-t\Delta_q(u)}\big).\end{aligned}$$

This implies

$$\sum_{q=0}^{n}(-1)^q q\,\mathrm{Tr}\big(\dot{\Delta}_q\,e^{-t\Delta_q(u)}\big)=-\frac{d}{dt}\sum_{q=0}^{n}(-1)^q\,\mathrm{Tr}\big(\alpha\,e^{-t\Delta_q(u)}\big)\,. \tag{2.6}$$

It is clear that

$$\big|\mathrm{Tr}\big(e^{-t\Delta_q(u)}-P_q(u)\big)\big|\le C\,e^{-\varepsilon t} \tag{2.7}$$

for $t\ge 1$ and some constants $C,\varepsilon>0$. Hence, for $\mathrm{Re}(s)>n/2$, we can differentiate the right hand side of (2.3) under the integral and, by (2.5) and (2.6), we obtain

$$\frac{\partial}{\partial u}F(u,s)=\tfrac{1}{2}\sum_{q=0}^{n}(-1)^q\int_0^\infty t^s\,\frac{d}{dt}\mathrm{Tr}\Big(\alpha\big(e^{-t\Delta_q(u)}-P_q(u)\big)\Big)\,dt\,. \tag{2.8}$$

We may regard α as differential operator of order zero. Then it follows from Lemma 1.7.7 in [G] that for $t\to 0$, there exists an asymptotic expansion of the following form

$$\mathrm{Tr}\big(\alpha\,e^{-t\Delta_q(u)}\big)\sim\sum_{k=0}^{\infty}c_{q,k}(u)\,t^{-n/2+k}\,. \tag{2.9}$$

This shows that $\mathrm{Tr}\big(\alpha\big(e^{-t\Delta_q(u)}-P_q(u)\big)\big)$ is $O(t^{-n/2})$ for $t\to 0$ and, by (2.7), it decreases exponentially for $t\to\infty$. Hence, if $\mathrm{Re}(s)>n/2$, we can integrate by parts on the right hand side of (2.8) and we obtain

$$\frac{\partial}{\partial u}F(u,s)=\tfrac{1}{2}\,s\sum_{q=0}^{n}(-1)^{q+1}\int_0^\infty t^{s-1}\,\mathrm{Tr}\Big(\alpha\big(e^{-t\Delta_q(u)}-P_q(u)\big)\Big)\,dt\,. \tag{2.10}$$

By (2.9), this function extends to a meromorphic function of $s \in \mathbf{C}$ which is holomorphic at $s = 0$. Set

$$c_q(u) = \begin{cases} c_{q,n/2}(u), & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

where $c_{q,n/2}(u)$ is the constant term in (2.9). Then

$$\frac{\partial}{\partial u} F(u,s)\big|_{s=0} = \tfrac{1}{2} \sum_{q=0}^{n} (-1)^q \operatorname{Tr}(\alpha P_q(u)) - \tfrac{1}{2} \sum_{q=0}^{n} (-1)^q c_q(u)\,. \tag{2.11}$$

A similar result holds for the variation of the Riemannian metric on M. If we combine (2.4) and (2.11), we obtain

Theorem 2.12. *Let M be a closed oriented manifold of dimension n and let ρ be a representation of $\pi_1(M)$ on E. Let h_u and g_v be 1-parameter families of metrics on E_ρ and M, respectively. Set $\alpha_u = \#_u^{-1}\dot{\#}_u, \beta_v = *_v^{-1}\dot{*}_v$ where $\dot{*} = d*/dv$ and denote by $T_M(\rho; u, v)$ the analytic torsion and by $P_q(u,v)$ the harmonic projection with respect to (h_u, g_v). Let $c_q(u,v)$ be the coefficients defined by the asymptotic expansion (2.9) as above. Similarly, let $d_q(u,v)$ be the corresponding coefficients defined by the asymptotic expansion of $\operatorname{Tr}(\beta e^{-t\Delta_q})$. Then*

$$\frac{\partial}{\partial u} \log T_M(\rho; u, v) = \tfrac{1}{2} \sum_{q=0}^{n} (-1)^q \operatorname{Tr}(\alpha_u P_q(u,v)) - \tfrac{1}{2} \sum_{q=0}^{n} c_q(u,v)$$

$$\frac{\partial}{\partial v} \log T_M(\rho; u, v) = \tfrac{1}{2} \sum_{q=0}^{n} (-1)^q \operatorname{Tr}(\beta_v P_q(u,v)) - \tfrac{1}{2} \sum_{q=0}^{n} d_q(u,v)\,.$$

If n is odd, the local terms c_q and d_q vanish and, by (2.1), we get

Corollary 2.13. *Assume that $\dim M$ is odd and $\rho : \pi_1(M) \to \mathrm{GL}(E)$ is acyclic. Then $T_M(\rho; h)$ has the same value for any choice of a Riemannian metric on M and a metric h on E_ρ.*

Thus, for an odd dimensional manifold M and an acyclic representation ρ, $T_M(\rho; h)$ is an invariant $T_M(\rho)$ of the manifold M and of the representation ρ. This justifies the following

Definition 2.14. *Let M be an odd dimensional closed manifold and ρ an acyclic representation of $\pi_1(M)$. Then the analytic torsion $T_M(\rho)$ of ρ is the common value of $T_M(\rho; h)$ for any choice of a metric h on E_ρ.*

Thus for closed manifolds of odd dimension our definition of analytic torsion works well. The situation is different if the dimension of M is even. For orthogonal (or unitary) representations the individual local terms do not vanish in general, but their alternating sum does. This can be seen as follows. Assume that E is equipped with an inner product and $\rho : \pi_1(M) \to \mathrm{GL}(E)$ preserves the inner product, that is, ρ is orthogonal. Then $*$ commutes with Δ. Hence

$$e^{-t\Delta_{n-q}} *^{-1} = *^{-1} e^{-t\Delta_q}.$$

Multiplying by $\dot{*}$ and taking the trace gives

$$\mathrm{Tr}\big(*^{-1}\dot{*}e^{-t\Delta_{n-q}}\big) = \mathrm{Tr}\big(e^{-t\Delta_{n-q}} *^{-1} \dot{*}\big) = \mathrm{Tr}\big(*^{-1}e^{-t\Delta_q}\dot{*}\big) = \mathrm{Tr}\big(\dot{*}\, *^{-1}\, e^{-t\Delta_q}\big).$$

Now observe that $\dot{*}*^{-1} = -\, *^{-1}\, \dot{*}$. Then

$$\mathrm{Tr}\big(*^{-1}\dot{*}e^{-t\Delta_{n-q}}\big) = -\mathrm{Tr}\big(*^{-1}\dot{*}e^{-t\Delta_q}\big)$$

which implies $d_{q,n/2} = -d_{n-q,n/2}$. Therefore $\sum_{q=0}^{n}(-1)^q d_{q,n/2} = 0$.

This is false, however, if the metric on E_ρ is not compatible with the flat connection. As an example consider a closed surface Σ of genus $g \geq 2$ and choose on Σ the metric of constant curvature -1. Let E_ρ be any flat orthogonal bundle over Σ. We denote by $\langle\cdot,\cdot\rangle$ the canonical metric on E_ρ which is compatible with the flat connection. Let $f \in C^\infty(\Sigma)$ and $u \in \mathbf{R}$. Then we define a new metric h_u by

$$h_u(v, w) = e^{uf(z)}\langle v, w\rangle_z, \quad \text{for} \quad v, w \in E_{\rho,z},\ z \in \Sigma.$$

If we use the canonical metric to identify E_ρ and E_ρ^*, then $\#_u : E_\rho \to E_\rho$ is given by

$$\#_u v = e^{uf(z)} v, \quad v \in E_{\rho,z},\ z \in \Sigma.$$

Hence $\alpha_u = \#^{-1}\dot{\#} = f \cdot Id$. Let $K_{q,u}(z, z', t)$ be the heat kernel on E_ρ-valued q-forms on Σ where E_ρ is equipped with h_u. Then

$$\mathrm{tr}\big(\alpha_u(z)K_{q,u}(z,z,t)\big) = f(z)\,\mathrm{tr}\big(K_{q,u}(z,z,t)\big).$$

Let m be the rank of E_ρ and $K_q(z, z', t)$ the heat kernel for the Laplacian on q-forms. Then, for $u = 0$, $\mathrm{tr}K_{q,0}(z,z,t) = m\,\mathrm{tr}K_q(z,z,t)$.Furthermore, the first three coefficients of the asymptotic expansion of $\mathrm{tr}K_q(z,z,t)$ are well-known (cf. [G], p.330). Since, by assumption, the curvature of Σ is $\equiv -1$, it follows that

$$c_0(0) - c_1(0) + c_2(0) = -\frac{m}{2\pi}\int_\Sigma f(z)\,dz.$$

This shows that in the even dimensional case the local contribution to the variational formula for $T_M(\rho; h)$ is non-zero in general.

The analytic torsion for orthogonal representations satisfies a number of functorial properties (cf. [RS],§2) which reflect known properties of the R-torsion. They continue to hold, with some modifications, for arbitrary finite dimensional representations.

Let E_ρ be a flat bundle over M defined by the representation $\rho : \pi_1(M) \to \mathrm{GL}(E)$. We denote by ρ^* the contragredient representation. The associated flat bundle is the dual bundle E_ρ^* of E_ρ. Any metric h on E_ρ induces a metric on E_ρ^* which we denote by h^*.

Proposition 2.15. *Suppose that M is a closed orientable manifold of dimension n. Let ρ be a finite dimensional real representation of $\pi_1(M)$ and E_ρ the associated flat vector bundle. For any choice of a metric on E_ρ, we have*

$$T_M(\rho; h)\, T_M(\rho^*; h^*)^{(-1)^n} = 1.$$

Proof. We may essentially proceed as in the proof of Theorem 2.3 in [RS]. Let Δ_ρ (resp. Δ_{ρ^*}) denote the Laplacian on E_ρ-valued (resp. E_ρ^*-valued) differential forms on M. The zeta functions $\zeta_q(s;\rho)$ and $\zeta_q(s;\rho^*)$ of the operators $\Delta_{\rho,q}$ and $\Delta_{\rho^*,q}$, respectively, are defined as above. By (2.2), it suffices to show that

$$\sum_{q=0}^{n}(-1)^q q\big(\zeta_q(s;\rho) + (-1)^n \zeta_q(s;\rho^*)\big) \equiv 0. \tag{2.16}$$

Let $\lambda > 0$ be an eigenvalue of $\Delta_{\rho,q}$, and let $\mathcal{E}_q(\lambda;\rho) \subset \Lambda^q(E)$ be the corresponding eigenspace. We introduce the following subspaces

$$\mathcal{E}_q'(\lambda;\rho) = \{\phi \in \mathcal{E}_q(\lambda;\rho) \,|\, d\phi = 0\} \quad \text{and} \quad \mathcal{E}_q''(\lambda;\rho) = \{\phi \in \mathcal{E}_q(\lambda;\rho) \,|\, \delta\phi = 0\}.$$

As in [RS], pp.154-155, it follows that

$$\mathcal{E}_q(\lambda;\rho) = \mathcal{E}_q'(\lambda;\rho) \oplus \mathcal{E}_q''(\lambda;\rho)$$

and $\lambda^{-1/2}d$ defines an isometry of $\mathcal{E}_q''(\lambda;\rho)$ onto $\mathcal{E}_{q+1}'(\lambda;\rho)$, with inverse $\lambda^{-1/2}\delta$. Let $m_q'(\lambda;\rho)$ and $m_q''(\lambda;\rho)$ be the dimension of the spaces $\mathcal{E}_q'(\lambda;\rho)$ and $\mathcal{E}_q''(\lambda;\rho)$, respectively. Then the multiplicity of λ is $m_q(\lambda;\rho) = m_q'(\lambda;\rho) + m_q''(\lambda;\rho) = m_q'(\lambda;\rho) + m_{q+1}'(\lambda;\rho)$ and we obtain

$$\sum_{q=0}^{n}(-1)^q q\, \zeta_q(s;\rho) = \sum_{q=1}^{n}(-1)^q \sum_{\lambda>0} m_q'(\lambda;\rho)\,\lambda^{-s} = -\sum_{q=0}^{n-1}(-1)^q \sum_{\lambda>0} m_q''(\lambda;\rho)\,\lambda^{-s} \tag{2.17}$$

and similarly for ρ^*. Let d, δ denote the differential and the co-differential with respect to E_ρ and d', δ' the corresponding operators with respect to $E_{\rho^*} = E_\rho^*$. Then $* \circ \# : \Lambda^q(E) \to \Lambda^{n-q}(E^*)$ satisfies $* \circ \# \, d\delta = \delta' d' * \circ \#$. Hence, $* \circ \#$ defines an isometry of $\mathcal{E}_q'(\lambda;\rho)$ onto $\mathcal{E}_{n-q}''(\lambda;\rho^*)$ showing that $m_q'(\lambda;\rho) = m_{n-q}''(\lambda;\rho^*)$. Combined with (2.17), we obtain (2.16). Q.E.D.

Since $\big(H_*(M;E)\big)^* \cong H_*(M;E^*)$, the representation ρ is acyclic iff the contragredient representation $\rho^* : \pi_1(M) \to \mathrm{GL}(E^*)$ is acyclic and we get

Corollary 2.18. *Let M be a closed orientable manifold of even dimension and let $\rho : \pi_1(M) \to \mathrm{GL}(E)$ be an acylic representation. Then*

$$T_M(\rho)\, T_M(\rho^*) = 1.$$

Assume that E is equipped with an inner product so that $E \cong E^*$. Then ρ is orthogonal iff $\rho^* = \rho$. Hence, for orthogonal representations, Corollary 2.18 implies $T_M(\rho) = 1$ which agrees with Theorem 2.3 in [RS].

Next consider two closed oriented Riemannian manifolds $M_i, i = 1, 2$, and let E_{ρ_i} be a flat bundle over M_i defined by a representation $\rho_i : \pi_1(M_i) \to GL(E_i)$. We choose a metric h_i on E_{ρ_i}. Furthermore, let $p_i : M_1 \times M_2 \to M_i, i = 1, 2$, be the canonical projection. The flat bundle $p_1^*(E_{\rho_1}) \otimes p_2^*(E_{\rho_2})$ over $M_1 \times M_2$ is associated to the representation $\rho_1 \otimes \rho_2$ and we denote by $h_1 \times h_2$ the product metric on this bundle. We also assume that $M_1 \times M_2$ is equipped with the product metric.

Proposition 2.19. *With the notation above we have the following equality*

$$\log T_{M_1 \times M_2}(\rho_1 \otimes \rho_2; h_1 \times h_2) = \chi(M_2; E_2) \log T_{M_1}(\rho_1; h_1) + \chi(M_1; E_1) \log T_{M_2}(\rho_2; h_2),$$

where $\chi(M_i; E_i)$ denotes the Euler characteristic.

This is proved by an easy generalization of the proof of Theorem 2.5 of [RS].

Now assume that the representation $\rho : \pi_1(M) \to \mathrm{GL}(E)$ is unimodular . Then the Reidemeister torsion $\tau_M(\rho)$ is defined to be an element of the one-dimensional vector space $\left(\det H_*(M; E)\right)^{-1}$. In order to get a real number we have to choose a volume $\mu \in \det H_*(M; E)$. In the present context we choose μ as follows. Let h be a metric on E_ρ. Then the inner product on $\Lambda^*(E)$ induces an inner product on the space of E_ρ-valued harmonic forms $\mathcal{H}^*(E)$ and we use the De Rham isomorphism combined with Poincaré duality to introduce an inner product on $H_*(M; E)$. The inner product on $H_q(M; E)$ defines a volume μ_q which, up to sign, is uniquely determined by the metric h on E_ρ.

Definition 2.20. Let $\mu_h = \otimes_{q=0}^n (\mu_q)^{(-1)^q} \in \det H_*(M; E)$ be the volume defined by the metric h on E. Then we set

$$\tau_M(\rho; h) = \tau_M(\rho; \mu_h)$$

where the latter is the R-torsion of M with respect to ρ and μ_h (cf. section 1).

Again the R-torsion depends not only on h, but also on g, unless ρ is acyclic. To compute the variation of the R-torsion we follow the proof of Theorem 7.6 of [RS]. This, however, requires some preparation. First we have the following version of Proposition 6.4 of [RS] in our context.

Proposition 2.21. *Let M and E_ρ be as above. Let h_u and g_v be 1-parameter families of metrics on E_ρ and M, respectively. For all values of u and v of the parameters, let $\mathcal{H}_{u,v}$ be the space of E_ρ-valued harmonic forms with respect to the metrics h_u and g_v. Let v_0 be fixed. For each u there is an orthonormal base $\{\varphi_j(u)\}$ of $\mathcal{H}_{u,v_0}$ such that for each j, $\varphi_j(u)$ is a differentiable function of u, $\dot{\varphi}_j = d/du\varphi_j(u)$ is closed , and*

$$(\varphi_j, \dot{\varphi}_j) = -\tfrac{1}{2}(\varphi_j, \alpha\varphi_j) \tag{2.22}$$

where $\alpha = \#^{-1}\dot{\#}$. A similar result holds if we fix u_0 and vary v. Then in (2.22), α has to be replaced by $\beta = *^{-1}\dot{*}$.

The proof is analogous to the proof of Proposition 6.4 of [RS].

We also need some results concerning the De Rham map. Let K be a smooth triangulation of M and denote by K^* the dual cell complex. Given a q-simplex σ and $\varphi \in \Lambda^q(E)$, set $A^q(\varphi)(\sigma) = \int_\sigma \omega$. Then $A^q(\varphi)$ extends to a π_1-equivariant map $C_q(\tilde{K}) \to E$, and hence, we get an element $A^q(\varphi)$ in $C^q(K; E)$. This defines the De Rham map

$$A^q : \Lambda^q(E) \to C^q(K; E)$$

and it satisfies $A^q(d\varphi) = \delta A^q(\varphi)$. In case it is necessary to indicate which simplicial complex we are working with we shall write A^q_K in place of A^q. If we replace K by K^*, we get a similar map $A^q_{K^*}$. Let E^*_ρ be the dual flat vector bundle associated to the contragredient representation $\rho^* : \pi_1(M) \to \mathrm{GL}(E^*)$ and h^* the metric on $E^*_\rho = E_{\rho^*}$ defined by h in the canonical way. Then $\Lambda^*(E^*)$ is also equipped with an inner product and for each q, the map

$$* \circ \# : \Lambda^q(E) \longrightarrow \Lambda^{n-q}(E^*)$$

is an isometry. Let $\tilde{\Pi}_q : C_q(K; E^*) \to C^{n-q}(K^*; E^*)$ be the isomorphism of Corollary 1.22 with the roles of K and K^* switched. We define the map

$$A_q : \Lambda^q(E) \longrightarrow C_q(K; E^*)$$

to be

$$A_q = (-1)^{(n-1)q}\, \tilde{\Pi}_q^{-1} \circ A^{n-q}_{K^*} \circ * \circ \#.$$

This map satisfies $A_q(\delta\varphi) = \partial A_q(\varphi)$. Again, we shall write A^K_q for this map if we wish to indicate the dependence on the particular simplicial complex K. Note that $C^q(K; E) \cong \left(C_q(K; E^*)\right)^*$ and therefore, we have a canonical pairing $\langle x, y\rangle$ of $x \in C^q(K; E)$ and $y \in C_q(K; E^*)$. Let $Z_q(K; E^*) \subset C_q(K; E^*)$ be the subspace of cycles and $Z^q(K; E) \subset C^q(K; E)$ the subspace of cocycles. Then we can state De Rham's theorem in our context as follows.

Proposition 2.23. *Let $\varphi, \psi \in \Lambda^q(E)$ and assume that $d\varphi = 0$ and $\delta\psi = 0$. Then $A^q(\varphi) \in Z^q(K; E)$, $A_q(\psi) \in Z_q(K; E^*)$ and*

$$(\varphi, \psi) = \langle A^q(\varphi), A_q(\psi)\rangle.$$

The proof is the same as for the case of orthogonal representations (cf. Proposition 4.2 of [RS]).

Now we are ready to derive the variational formula for R-torsion.

Theorem 2.24. *Let M be a closed oriented manifold and let $\rho : \pi_1(M) \to \mathrm{GL}(E)$ be a unimodular representation. Let h_u and g_v be 1-parameter families of metrics on E_ρ and M, respectively. Set $\alpha_u = \#_u^{-1}\dot{\#}_u$, $\beta_v = *_v^{-1}\dot{*}_v$ and denote by $\tau_M(\rho; u, v)$ the R-torsion and by $P_q(u, v)$ the harmonic projection with respect to (h_u, g_v). Then*

$$\frac{\partial}{\partial u} \log\, \tau_M(\rho; u, v) = \tfrac{1}{2} \sum_{q=0}^{n} (-1)^q\, \mathrm{Tr}\big(\alpha_u P_q(u, v)\big)$$

$$\frac{\partial}{\partial v} \log\, \tau_M(\rho; u, v) = \tfrac{1}{2} \sum_{q=0}^{n} (-1)^q\, \mathrm{Tr}\big(\beta_v P_q(u, v)\big)\, .$$

Proof. To begin with, we recall the definition of $\tau_M(\rho; h)$. Let K be a smooth triangulation of M and set $C_\bullet = C_\bullet(K; E)$. We choose a lift of K to its universal covering space $\tilde{K}$. Furthermore, we have chosen a volume $\theta \in \det E$. Let $x_1, ..., x_N$ be a base of E such that $\theta = x_1 \wedge \cdots \wedge x_N$. Then the oriented q-simpleces of $K \subset \tilde{K}$ together with the base $\{x_j\}$ determine a preferred base of C_q and a corresponding volume $\omega_q \in \det C_q$. Assume that h is a metric on E_ρ. It determines a metric h^* on $E_\rho^* = E_{\rho^*}$. Let $\varphi_1, ..., \varphi_r$ be an orthonormal base of $\mathcal{H}^q(E^*)$. Then the homology classes of the cycles $A_q(\varphi_1), ..., A_q(\varphi_r)$ form a base of $H_q(M; E)$ and $\mu_q = \big[A_q(\varphi_1)\big] \wedge \cdots \wedge \big[A_q(\varphi_r)\big]$ is the volume defined above. With these choices of volumes, we have $\tau_M(\rho; h) = |T(C_\bullet, \omega, \mu)|$. For each q, pick a base $\{\theta_{q-1}^j\}$ of ∂C_q and for each j, choose $\tilde{\theta}_{q-1}^j \in C_q$ such that $\partial\tilde{\theta}_{q-1}^j = \theta_{q-1}^j$. Then $\{\theta_q^i, \tilde{\theta}_{q-1}^j, A_q(\varphi_k)\}$ is a base for C_q. Let B_q be the matrix of the change from this base to the preferred base constructed above. Using the definition of $T(C_\bullet, \omega, \mu)$, it follows that

$$\log \tau_M(\rho; h) = \sum_{q=0}^{n} (-1)^q \log|\det B_q|.$$

Let $\mathcal{H}_{u,v}^q(E)$ be the space of E_ρ-valued harmonic forms with respect to the metrics h_u and g_v. We fix v and for each u, we choose an orthonormal base $\{\varphi_j(u)\}$ for $\mathcal{H}_{u,v}^q(E)$ according to Proposition 2.21. The coefficients of B_q that depend on u are the coefficients of $A_q(\varphi_j(u))$ which are smooth functions of u. Therefore, $\tau_M(\rho; u, v)$ is a differentiable function of u and

$$\frac{\partial}{\partial u} \log \tau_M(\rho; u, v) = \sum_{q=0}^{n} (-1)^q\, \mathrm{Tr}\big(B_q^{-1} \frac{d}{du} B_q\big).$$

In order to compute the traces we consider the matrix $D_q = ({}^t B_q)^{-1}$ which can be described as follows. Note that $C^q(K; E^*) \cong \big(C_q(K; E)\big)^*$. Let $x_1^*, ..., x_N^* \in E^*$ be the dual base of $x_1, ..., x_N \in E$ and consider the preferred base of the $\mathbf{R}(\pi_1)$-module $C^q(\tilde{K})$ which is defined by the q-simpleces of $K \subset \tilde{K}$. Then, as above, we get a preferred base of $C^q(K; E^*)$ which is dual to the preferred base of $C_q(K; E)$. Then consider the base of $C^q(K; E^*)$ which is dual to $\{\theta_q^i, \tilde{\theta}_{q-1}^j, A_q(\varphi_k)\}$. Using Proposition 2.23, it is easy to see that this base is of the form $\{\eta_i^q, \tilde{\eta}_j^{q+1}, A^q(\varphi_k) + \delta c_k\}$ where $\{\eta_i^q\}$ is a base of $\delta C^{q-1}(K; E^*)$ and the $\tilde{\eta}_j^{q+1} \in C^q(K; E^*)$

are such that $\delta\tilde{\eta}_j^{q+1} = \eta_j^{q+1}$. The matrix of the change from this base to the preferred base is D_q. Since $\partial A_q(\varphi_k) = 0$, we get

$$\begin{aligned}\operatorname{Tr}(B_q^{-1}\frac{d}{du}B_q) = \operatorname{Tr}({}^tD_q\frac{d}{du}B_q) &= \sum_{j=1}^{r}\langle A^q(\varphi_j) + \delta c_j, \frac{d}{du}A_q(\varphi_j)\rangle \\ &= \sum_{j=1}^{r}\langle A^q(\varphi_j), \frac{d}{du}A_q(\varphi_j)\rangle.\end{aligned}$$

Now observe that by Proposition 2.23, $\langle A^q(\varphi_j), A_q(\varphi_j)\rangle = 1$ and therefore,

$$\langle A^q(\varphi_j), \frac{d}{du}A_q(\varphi_j)\rangle = -\langle\frac{d}{du}A^q(\varphi_j), A_q(\varphi_j)\rangle = -\langle A^q(\dot{\varphi}_j), A_q(\varphi_j)\rangle.$$

Using Proposition 2.23 and (2.22), we obtain

$$-\sum_{j=1}^{r}\langle A^q(\dot{\varphi}_j), A_q(\varphi_j)\rangle = -\sum_{j=1}^{r}(\varphi_j, \dot{\varphi}_j) = \tfrac{1}{2}\sum_{j=1}^{r}(\varphi_j, \alpha\varphi_j) = \tfrac{1}{2}\operatorname{Tr}\big(\alpha P_q(u,v)\big)$$

which completes the proof of the first equality.

The variation with respect to v is computed in the same way. Q.E.D.

If we combine Theorem 2.12 and Theorem 2.24, we get

Corollary 2.25. *Let M be a closed oriented manifold and let $\rho : \pi_1(M) \to \mathrm{GL}(E)$ be a unimodular representation. If $\dim M$ is odd, then*

$$\log T_M(\rho; h) - \log \tau_M(\rho; h)$$

has the same value for any choice of a metric h on E_ρ and a Riemannian metric on M. If $\dim M$ is even, then the variation of $\log T_M(\rho; h) - \log \tau_M(\rho; h)$ can be computed as follows. Let h_u and g_v be 1-parameter families of metrics on E_ρ and M, respectively. Let the notation be the same as in Theorem 2.12. Then

$$\frac{\partial}{\partial u}(\log T_M(\rho; u,v) - \log \tau_M(\rho; u,v)) = \tfrac{1}{2}\sum_{q=0}^{n}(-1)^{q+1}c_q(u,v)$$

$$\frac{\partial}{\partial v}(\log T_M(\rho; u,v) - \log \tau_M(\rho; u,v)) = \tfrac{1}{2}\sum_{q=0}^{n}(-1)^{q+1}d_q(u,v).$$

The example discussed above shows that for even dimensional manifolds the variation of $\log T_M(\rho; h) - \log \tau_M(\rho; h)$ may not be zero.

As in the case of orthogonal representations Corollary 2.25 is the key result to prove the equality of $T_M(\rho; h)$ and $\tau_M(\rho; h)$ for closed manifolds of odd dimension. For this purpose it is essential to have the proper generalization of Corollary 2.25 to the case $\partial M \neq \emptyset$. To define the analytic torsion in this case we have to introduce boundary conditions for Δ_q. Let $i : \partial M \to M$ be the inclusion and let $i^* : \Lambda^*(E) \to \Lambda^*(E|\partial M)$ be the induced map on E_ρ-valued differential forms. A differential form $\omega \in \Lambda^*(E)$ is said to satisfy *absolute* boundary conditions if

$$i^*(*\omega) = 0 \quad \text{and} \quad i^*(*d\omega) = 0,$$

and if ω satisfies absolute boundary conditions then $*\omega$ is said to satisfy *relative* boundary conditions.

Let $\mathrm{Dom}(\Delta_a)$ (resp. $\mathrm{Dom}(\Delta_r)$) be the subspace of $\Lambda^*(E)$ consisting of all those forms which satisfy absolute (resp. relative) boundary conditions on ∂M. If ω, θ satisfy either boundary conditions, then

$$\langle d\omega, \theta \rangle = \langle \omega, \delta\theta \rangle$$

and the restriction of the Laplacian to the corresponding domains defines symmetric, positive semi-definite, operators

$$\Delta_a : \mathrm{Dom}(\Delta_a) \longrightarrow \Lambda^*(E) \quad \text{and} \quad \Delta_r : \mathrm{Dom}(\Delta_r) \longrightarrow \Lambda^*(E)$$

The corresponding self-adjoint extensions on $L^2\Lambda^*(E)$ have pure point spectrum $0 \leq \lambda_{a,0} \leq \lambda_{a,1} \leq \cdots$ and $0 \leq \lambda_{r,0} \leq \lambda_{r,1} \leq \cdots$, respectively. Let $\mathcal{H}_a^*(E)$ and $\mathcal{H}_r^*(E)$ denote the spaces of harmonic forms for absolute and relative boundary conditions, respectively. Then the De Rham map induces isomorphisms

$$\mathcal{H}_a^*(E) \xrightarrow{\sim} H^*(M; E) \quad \text{and} \quad \mathcal{H}_r^*(E) \xrightarrow{\sim} H^*(M, \partial M; E)\,. \tag{2.26}$$

The analytic torsion $T_M^a(\rho; h)$ (resp. $T_M^r(\rho; h)$)for absolute (resp. relative) boundary conditions is defined by formula (2.2) with Δ_q replaced by $\Delta_{a,q}$ (resp. $\Delta_{r,q}$).

In order to compute the variation of the analytic torsion we may essentially proceed as in the case of a closed manifold. There are, however, additional complications due to the non-empty boundary.

First we have to study the variation of the trace of the heat operator. We fix absolute boundary conditions. The case of relative boundary conditions follows similarly. Let h_u be a 1-parameter family of metrics on E_ρ and let $\Delta_{a,q}(u)$ denote the Laplacian and $P_{a,q}(u)$ the harmonic projection with respect to h_u and the choice of absolute boundary conditions. Let $K_q(t; u)$ denote the kernel of $e^{-t\Delta_{a,q}(u)}$ and let $K_q = K_q' + K_q'' + K_q'''$ denote the Hodge decomposition of K_q into its exact, coexact and harmonic components. To compute the variation of $\mathrm{Tr}(e^{-t\Delta_{a,q}(u)})$ we simply follow the proof of Theorem 3.10 in [C] and we note that Duhamel's principle (3.9) which is used in the course of the proof remains valid in our case. As above, let $\alpha = \#^{-1}\dot{\#}$. Then

$$\begin{aligned} \frac{\partial}{\partial u}\mathrm{Tr}\big(e^{-t\Delta_{a,q}(u)}\big) = t\,\frac{\partial}{\partial t}\Big\{ &\mathrm{Tr}\big(\alpha K_{q+1}'(t; u)\big) - \mathrm{Tr}\big(\alpha K_q''(t; u)\big) \\ &+ \mathrm{Tr}\big(\alpha K_q'(t; u)\big) - \mathrm{Tr}\big(\alpha K_{q-1}''(t; u)\big)\Big\}. \end{aligned} \tag{2.27}$$

Applying (2.27), it follows that

$$\frac{\partial}{\partial u}\sum_{q=0}^{n}(-1)^q q\,\mathrm{Tr}\big(e^{-t\Delta_{a,q}(u)}-P_{a,q}(u)\big)=t\,\frac{\partial}{\partial t}\sum_{q=0}^{n}(-1)^q\,\mathrm{Tr}\Big(\alpha\big(e^{-t\Delta_{a,q}(u)}-P_{a,q}(u)\big)\Big)\,.$$

The asymptotic expansion of $\mathrm{Tr}\big(\alpha e^{-t\Delta_{a,q}(u)}\big)$ contains now additional terms coming from the boundary

$$\mathrm{Tr}\big(\alpha e^{-t\Delta_{a,q}(u)}\big)\sim\sum_{k=0}^{\infty}c^a_{q,k}(u)\,t^{-n/2+k}+\sum_{k=0}^{\infty}b^a_{q,k}(u)\,t^{-n/2+k/2}. \tag{2.28}$$

Now we can proceed in exactly the same way as in the closed case and we get

$$\frac{\partial}{\partial u}\log T^a_M(\rho;h_u)=\tfrac{1}{2}\sum_{q=0}^{n}(-1)^q\big\{\mathrm{Tr}\big(\alpha P_{a,q}(u)\big)-b^a_{q,n}(u)\big\} \tag{2.29}$$

if $\dim M$ is odd.

A similar relation holds for the variation with respect to the Riemannian metric. Then α has to be replaced by $\beta=*_v^{-1}\dot{*}_v$ and the coefficients $b^a_{q,n}(u)$ by $\tilde{b}^a_{q,n}(v)$ occurring in the corresponding asymptotic expansion of $\mathrm{Tr}\big(\beta e^{-t\Delta_{a,q}(u)}\big)$.

To define the R-torsion $\tau^a_M(\rho;h)$ we employ the De Rham isomorphism (2.26) to introduce a volume element for $H^*(M;E)$. The computation of the variation of $\log\tau^a_M(\rho;h_u)$ remains the same

$$\frac{\partial}{\partial u}\log\tau^a_M(\rho;h_u)=\tfrac{1}{2}\sum_{q=0}^{n}(-1)^q\,\mathrm{Tr}\big(\alpha P_{a,q}(u)\big)\,. \tag{2.30}$$

Putting (2.29) and (2.30) together we obtain

Theorem 2.31. *Let M be a closed manifold of odd dimension and E_ρ a flat vector bundle over M. Impose absolute boundary conditions. Let h_u and g_v be 1-parameter families of metrics on E_ρ and M, respectively. Denote by $T^a_M(\rho;u,v)$ the analytic torsion and by $\tau^a_M(\rho;u,v)$ the R-torsion with respect to (h_u,g_v). Then*

$$\frac{\partial}{\partial u}\big(\log T^a_M(\rho;u,v)-\log\tau^a_M(\rho;u,v)\big)=\tfrac{1}{2}\sum_{q=0}^{n}(-1)^{q+1}\,b^a_{q,n}(u,v)$$

$$\frac{\partial}{\partial v}\big(\log T^a_M(\rho;u,v)-\log\tau^a_M(\rho;u,v)\big)=\tfrac{1}{2}\sum_{q=0}^{n}(-1)^{q+1}\,\tilde{b}^a_{q,n}(u,v)$$

where the coefficients $b^a_{q,n}$ and $\tilde{b}^a_{q,n}$ are determined by the asymptotic expansion (2.28) and its counterpart for the variation with respect to g_v. A similar result holds for relative boundary conditions.

An immediate consequence of Theorem 2.31 is the following

Corollary 2.32. *Let the assumption and notation be the same as in Theorem 2.31. Then for any values* (u_2, v_2) *and* (u_1, v_1) *of the parameters the difference*

$$\left(\log T_M^a(\rho; u_2, v_2) - \log \tau_M^a(\rho; u_2, v_2)\right) - \left(\log T_M^a(\rho; u_1, v_1) - \log \tau_M^a(\rho; u_1, v_1)\right)$$

depends only on the germs of $h_{u_2}, h_{u_1}, g_{v_2}, g_{v_1}$ *restricted to* ∂M *and is completely independent of the geometry and topology of* $\mathrm{int}(M)$. *The same holds for relative boundary conditions.*

Proof. Set

$$f(u, v) = \log T_M^a(\rho; u, v) - \log \tau_M^a(\rho; u, v).$$

Then, by Theorem 2.31,

$$\begin{aligned} f(u_2, v_2) - f(u_1, v_1) &= \int_{u_1}^{u_2} \frac{\partial}{\partial u} f(u, v_2)\, du + \int_{v_1}^{v_2} \frac{\partial}{\partial v} f(u_1, v)\, dv \\ &= \tfrac{1}{2} \sum_{q=0}^{n} (-1)^{q+1} \left\{ \int_{u_1}^{u_2} b_{q,n}^a(u, v_2)\, du + \int_{v_1}^{v_2} \tilde{b}_{q,n}^a(u_1, v)\, dv \right\}. \end{aligned}$$

By the construction of the asymptotic expansion (2.28) and its analogue for $\mathrm{Tr}\big(\beta e^{-t\Delta_{a,q}(v)}\big)$ the coefficients $b_{q,n}^a$ and $\tilde{b}_{q,n}^a$ depend only on the germs of $h_{u_2}, h_{u_1}, g_{v_2}, g_{v_1}$ restricted to ∂M. Q.E.D.

3.The equality of analytic torsion and R-torsion

Let M be a closed oriented Riemannian manifold of odd dimension and let $\rho : \pi_1(M) \to \mathrm{GL}(E)$ be a unimodular representation with associated flat bundle E_ρ. Let h be an Euclidean metric on E_ρ. To establish the equality of the analytic torsion $T_M(\rho; h)$ and the R-torsion $\tau_M(\rho; h)$ we shall follow Cheeger's proof for the case of orthogonal representations [C]. As explained in the introduction, this proof is well-suited for this purpose because the relevant analysis is done locally near a given handle and this can easily be adapted to the case of unimodular representations.

We recall the basic steps in Cheeger's proof and indicate how they have to be modified (if at all). By Corollary 2.25, $\log T_M(\rho; h) - \log \tau_M(\rho; h)$ is independent of the choice of h and of the metric on M. Set

$$e_M(\rho) = \log T_M(\rho; h) - \log \tau_M(\rho; h). \tag{3.1}$$

Furthermore, both $T_M(\rho; h)$ and $\tau_M(\rho; h)$ are independent of the choice of orientation.

Let M_0 and M_1 be two closed smooth manifolds of odd dimension n and assume that M_1 is obtained from M_0 by surgery on some embedded k-sphere $S^k \subset M_0$ (see below for the precise definition of this statement). Let E_0 be a flat bundle over M_0 defined by a unimodular representation ρ_0 of $\pi_1(M_0)$ and assume that E_0 extends over the trace of the surgery to a flat bundle E_1 over M_1 defined by a unimodular representation ρ_1 of $\pi_1(M_1)$. By an argument similar to the one used in the introduction of [C], it suffices to show that $e_{M_0}(\rho_0) = 0$ implies $e_{M_1}(\rho_1) = 0$ for all such pairs (M_0, ρ_0), (M_1, ρ_1). A slight complication arises because this can be proved directly only for $0 < k < n-1$.

Now recall that surgery on an embedded k-sphere $S^k \subset M_0$ means the following:

a) The tubular neighborhood $N(S^k)$ is a product

$$N(S^k) = S^k \times D^{n-k}.$$

b) There is an embedded $(n-k-1)$-sphere $S^{n-k-1} \subset M_1$ whose tubular neighborhood is also a product

$$N(S^{n-k-1}) = S^{n-k-1} \times D^{k+1}.$$

c) There is a manifold $\tilde{M}$ with boundary $\partial\tilde{M} = S^k \times S^{n-k-1}$ such that

$$M_0 = N(S^k) \underset{\partial\tilde{M}}{\cup} \tilde{M} \quad \text{and} \quad M_1 = N(S^{n-k-1}) \underset{\partial\tilde{M}}{\cup} \tilde{M}$$

where the union means that the common boundaries of $\tilde{M}$, $N(S^k)$ and $N(S^{n-k-1})$ are identified with the obvious identification and the manifolds are given the standard differentiable structures.

The fact that E_0 extends over the trace of the surgery to give E_1 is equivalent to the existence of a flat unimodular bundle $\tilde{E}$ over $\tilde{M}$ which extends as a flat unimodular bundle over $M_0 = N(S^k) \cup \tilde{M}$ and $M_1 = N(S^{n-k-1}) \cup \tilde{M}$. Recall that a flat bundle over a contractible space is trivial as a flat bundle. Since $S^k \times \{p\} \subset \partial\tilde{M}$ bounds $D^{k+1} \times \{p\}$, it follows that $E_0|S^k$, $S^k \subset M_0$, and therefore , $E_0|N(S^k)$ is trivial as a flat bundle. Similarly, $E_1|N(S^{n-k-1})$ is trivial as a flat bundle. This is the important fact which allows us to extend Cheeger's analysis.

We shall employ the following notation. If X is a Riemannian manifold and $Y \subset X$ a submanifold then we shall denote by $N_u(Y)$ the tubular neighborhood of Y in X consisting of all normal vectors to Y of length $\leq u$.

Now we introduce metrics on M_0 and M_1 such that the tubular neighborhoods $N_1(S^k)$ and $N_1(S^{n-k-1})$ are isometrically the product of the unit spheres S^k, S^{n-k-1} and the unit balls D^{n-k}, D^{k+1} equipped with the standard flat metrics. Furthermore, since $E_0|N_1(S^k)$ and $E_1|N_1(S^{n-k-1})$ are trivial we may fix trivializations

$$E_0|N_1(S^k) \cong N_1(S^k) \times \mathbf{R}^N \quad \text{and} \quad E_1|N_1(S^{n-k-1}) \cong N_1(S^{n-k-1}) \times \mathbf{R}^N \tag{3.2}$$

and we choose metrics h_0, h_1 on E_0, E_1 in such a way that they coincide on $E_0|N_1(S^k)$ and $E_1|N_1(S^{n-k-1})$ with the product metrics given by the trivializations (3.2). Set

$$M_u = \begin{cases} M_0 - N_u(S^k), & \text{if } 0 < u \leq 1/3; \\ M_1 - N_{1-u}(S^{n-k-1}), & \text{if } 2/3 \leq u < 1. \end{cases}$$

$$E_u = \begin{cases} E_0|M_u, & \text{if } 0 < u \leq 1/3; \\ E_1|M_u, & \text{if } 2/3 \leq u < 1. \end{cases}$$

We may think of M_u (resp. E_u) as being $\tilde{M}$ (resp. $\tilde{E}$) equipped with the metric g_u (resp. h_u). For $1/3 \leq u \leq 2/3$ let g_u (resp. h_u) be any smooth family of metrics connecting these two families, but subject to the condition that, near $\partial\tilde{M} = S^k \times S^{n-k-1}$, g_u (resp. h_u) is fixed independent of M_0 , M_1. As above, we denote by M_u (resp. E_u) , $0 < u < 1$, the manifold $\tilde{M}$ (resp. the flat bundle $\tilde{E}$) equipped with g_u (resp. h_u).

The structure of the proof of the equality of the two torsions is now the same as in [C]. Let $\rho : \pi_1(\tilde{M}) \to \mathrm{GL}(E)$ be the unimodular representation that defines $\tilde{E}$ and set

$$e_u = \begin{cases} \log T_M(\rho; h_u) - \log \tau_M(\rho; h_u), & \text{if } 0 < u < 1; \\ \log T_{M_i}(\rho_i; h_i) - \log \tau_{M_i}(\rho_i; h_i), & \text{if } u = i \in \{0,1\} . \end{cases}$$

Then

$$e_1 - e_0 = (e_1 - e_{1-u}) + (e_{1-u} - e_u) + (e_u - e_0) = A_u + B_u + C_u.$$

Let M_0', M_1', E_0', E_1' be a similar set of data where M_1' is again obtained from M_0' by surgery on a k-sphere S^k (with the same k). Then

$$e_1' - e_0' = A_u' + B_u' + C_u'$$

with the obvious notation. By Corollary 2.25, we have $B_u = B_u'$ and the core of the proof is the following

Theorem 3.3. *Let M_0, M_1, E_0, E_1 and M_0', M_1', E_0', E_1' be quatruples as above. Then, if $0 < k \leq n-1$,*

$$\lim_{u \to 0}(A_u - A_u') = 0$$

and, if $0 \leq k < n-1$,

$$\lim_{u \to 0}(C_u - C_u') = 0.$$

By symmetry, it suffices to consider C_u. To prove the Theorem for C_u, we first consider

$$(3.4) \qquad \left(\log T_{M_u}(\rho; h_u) - \log T_{M_0}(\rho_0; h_0)\right) - \left(\log T_{M_u'}(\rho'; h_u') - \log T_{M_0'}(\rho_0'; h_0')\right)$$

and investigate its behaviour as $u \to 0$. Let $\Delta_q(u)$ be the Laplacian on E_u-valued q-forms on M_u, $1 > u > 0$, satisfying absolute boundary conditions. Let $\zeta_q(s; u)$ be the zeta function associated to $\Delta_q(u)$. Then

$$\log T_{M_u}(\rho; h_u) = \tfrac{1}{2} \sum_{q=0}^{n} (-1)^q q \, \frac{\partial}{\partial s} \zeta_q(s; u)\big|_{s=0}$$

and we have to study the behaviour of $\frac{\partial}{\partial s}\zeta_q(s; u)|_{s=0}$ as $u \to 0$.

Recall that there exists an asymptotic expansion

$$(3.5) \qquad \operatorname{Tr}(e^{-t\Delta_q(u)}) = \sum_{j=0}^{m} a_{q,j}(u)\, t^{-n/2+j} + \sum_{j=0}^{2m} b_{q,j}(u)\, t^{-n/2+j/2} + O(t^{-n/2+m+1/2})$$

as $t \to 0$. The coefficients $a_{q,j}(u)$ are locally computable in the sense that they are obtained by integrating local densities which in any coordinate system depend in a universal fashion on the coefficients $g_{i,j}(x)$ and $h_{i,j}(x)$ of the metrics and a finite number of their partial derivatives. In the same sense, the coefficients $b_{q,j}(u)$ are locally computable in terms of the extrinsic geometry of the boundary, e.g., its induced metric and second fundamental form (cf. [G] for details). Set

$$c_{q,j}(u) = \begin{cases} b_{q,j}(u), & \text{if } j = 2l+1; \\ a_{q,j/2}(u) + b_{q,j}(u), & \text{if } j = 2l. \end{cases}$$

Furthermore, let $m > n/2$ and denote by $\mu_u(t)$ the difference of $\operatorname{Tr}\big(e^{-t\Delta_q(u)}\big)$ and its asymptotic expansion up to order m. Denote by $P_q(u)$ the harmonic projection. Then, using the definition of $\zeta_q(s; u)$ in terms of the heat kernel, we get

$$(3.6) \qquad \begin{aligned} \frac{\partial}{\partial s}\zeta_q(s; u)\big|_{s=0} &= \int_1^\infty t^{-1} \left(\operatorname{Tr}(e^{-t\Delta_q(u)}) - P_q(u)\right) dt + \int_0^1 t^{-1} \mu_u(t)\, dt \\ &\quad + \sum_{j \neq n} \frac{c_{q,j}(u)}{-n/2 + j/2} + \big(c_{q,n}(u) - \dim H^q(M_u; E_u)\big)\gamma \end{aligned}$$

where $\gamma = -\Gamma'(1)$ is Euler's constant. Now we have to investigate the behaviour of the individual terms as $u \to 0$. Let Δ_q be the Laplacian on E_0-valued q-forms on M_0. Then we have a corresponding asymptotic expansion as $t \to 0$ of the form

$$\operatorname{Tr}(e^{-t\Delta_q}) \sim \sum_{j=0}^{m} a_{q,j}\, t^{-n/2+j} + O(t^{-n/2+m+1}).$$

The nature of the coefficients $a_{q,j}(u)$ and $a_{q,j}$ implies immediately that

$$\lim_{u\to 0} a_{q,j}(u) = a_{q,j}, \quad q = 0, ..., m.$$

Furthermore, by assumption, the boundary terms occurring in (3.5) are the same for both M_u and M'_u. Hence the contribution to (3.4) made by the coefficients of the asymptotic expansion (3.5) and its analogues for M'_u, M_0 and M'_0 will cancel out in the limit.

Next consider $\dim H^q(M_u; E_u)$. Recall that $M_u \subset M_0$ and $E_u = E_0|M_u$. Hence there is the long exact cohomology sequence

(3.7)
$$\cdots \to H^q(M_0, M_u; E_0) \xrightarrow{j} H^q(M_0; E_0) \xrightarrow{l} H^q(M_u; E_u) \xrightarrow{\delta} H^{q+1}(M_0, M_u; E_0) \to \cdots$$

and, by excision,

$$H^q(M_0, M_u; E_0) \cong H^q(N(S^k), \partial N(S^k); E_0|N(S^k)).$$

Using (3.2), we obtain

(3.8)
$$H^q(M_0, M_u; E_0) \cong H^q(N(S^k), \partial N(S^k)) \otimes \mathbf{R}^N.$$

Since M'_1 is obtained from M'_0 by surgery on an embedded k-sphere $S^k \subset M_0$ with the same k, (3.8) also holds for (M'_0, M'_u). Let $k_q = \dim(\ker l_q)$. Then, by the exactness of (3.7), we get

(3.9)
$$\dim H^q(M_u; E_u) = \dim H^q(M_0; E_0) - k_q + \dim H^{q+1}(M_0, M_u; E_0) - k_{q+1}.$$

Let k'_q have the same meaning with respect to (M'_0, M'_u). Then the contribution to (3.4) of the last term in (3.6) and its analogues for M'_u, M_0, M'_0 is

(3.10)
$$\gamma \sum_{q=0}^{n} (-1)^q q((k_q + k_{q+1}) - (k'_q + k'_{q+1})).$$

This term will be further discussed below.

The essential part is to study the behaviour, as $u \to 0$, of the first integral on the right hand side of (3.6). For this purpose we need two estimates describing the behaviour of the heat kernel as $u \to 0$. Let $K_u(x, y, t)$ and $K_0(x, y, t)$ denote the kernel of $e^{-t\Delta_q(u)}$ and $e^{-t\Delta_q}$, respectively.

Theorem 3.11. *Let $k < n-1$. Given $T, m, u_0 > 0$, there exists $C_m(T, u_0)$ such that for $t \leq T$ and $x, y \in M_{u_0} \subset M_u$,*

$$\|K_0(x,y,t) - K_u(x,y,t)\| \leq C_m(T,u_0)\, t^m \begin{cases} u^{n-k-2}, & \text{if } n-k > 2; \\ \dfrac{1}{|\log u|}, & \text{if } n-k = 2. \end{cases}$$

This theorem states that away from $S^k \subset M_0$, the heat kernel of (M_u, E_u) converges uniformly to that of (M_0, E_0) as $u \to 0$.

The next result describes the behaviour of the heat kernel near S^k. For $u < u_0$, let A^{n-k}_{u,u_0} denote the annulus obtained by removing the $(n-k)$-ball D^{n-k}_u of radius u from the $(n-k)$-ball $D^{n-k}_{u_0}$ of radius u_0. Let $K_{u,1}(t)$ be the heat kernel on q-forms on $S^k \times A^{n-k}_{u,1}$ with coefficients in the trivial flat $\mathbf{R}^N$-bundle satisfying absolute boundary conditions. Introduce polar coordinates on $A^{n-k}_{u,1}$ and write $z \in S^k \times A^{n-k}_{u,1}$ as $z = (x, r)$ where $x \in S^k \times S^{n-k-1}$ and $r \in (u, 1)$. Fix $u_0 < 1$.

Theorem 3.12. *Let $k < n-1$. Given T, m, there exists a constant $C_m(T)$ such that*
(1) For $r_1, r_2 < u_0$, $x_1, x_2 \in S^k \times S^{n-k-1}$,

$$\begin{aligned} &\|K_{u,1}((x_1,r_1),(x_2,r_2),t) - K_u((x_1,r_1),(x_2,r_2),t)\| \\ &\leq C_m(T)t^m \begin{cases} (1 + u^{n-k-2} r_1^{-(n-k-1)})(1 + u^{n-k-2} r_2^{-(n-k-1)}), & \text{if } n-k > 2; \\ \left(1 + \dfrac{r_1^{-1}}{|\log u|}\right)\left(1 + \dfrac{r_2^{-1}}{|\log u|}\right), & \text{if } n-k = 2. \end{cases} \end{aligned}$$

(2)

$$\begin{aligned} &\left| \int_{S^k \times A^{n-k}_{u,u_0}} \left\{ \operatorname{tr}\big(K_{u,1}((x,r),(x,r),t)\big) - \operatorname{tr}\big(K_u((x,r),(x,r),t)\big) \right\} \right| \\ &\leq C_m(T)t^m \begin{cases} (u_0^{n-k-1} + u^{n-k-3})(u_0 - u), & \text{if } n-k > 2; \\ u_0(u_0 - u) + \dfrac{\log(u_0/u)}{\log^2 u}, & \text{if } n-k = 2. \end{cases} \end{aligned}$$

The proof of Theorem 3.11 and Theorem 3.12, which, for orthogonal flat bundles, is given in section 7 of [C], depends on two kinds of results. First of all, some standard estimates for the heat kernel are used (cf. section 5 of [C]). These estimates are derived from Duhamel's principle together with the Sobolev inequality. Both Duhamel's principle and the Sobolev inequality do not require any special assumption on the bundle E_ρ and therefore, the estimates obtained by Cheeger in section 5 of [C] are also valid in our setting. The other part which is important for the proof of Theorem 3.11 and Theorem 3.12 is the local analysis near S^k. One has to construct a parametrix for $\Delta_q(u)$ satisfying absolute boundary conditions which allows us to study its behaviour as $u \to 0$. The parametrix is obtained by patching an interior parametrix and a parametrix near the boundary. The interior parametrix is obtained by general constructions which again do not require special assumptions. To construct the parametrix near the boundary it suffices to construct the Green's operator for Δ on $\mathbf{R}^N$-valued differential forms on $S^k \times A^{n-k}_{u,1}$ satisfying absolute boundary conditions. By our assumption, the metrics on $S^k \times A^{n-k}_{u,1}$ and

$$E_u|(S^k \times A^{n-k}_{u,1}) \cong S^k \times A^{n-k}_{u,1} \times \mathbf{R}^N$$

are the standard product metrics. Therefore, all the local analysis done by Cheeger in section 6 and section 7 of [C] can be applied to our case without any change. This suffices to prove Theorem 3.11 and Theorem 3.12. We just follow the proof of these theorems in section 7 of [C] line by line.

In order to estimate $\mathrm{Tr}(e^{-t\Delta_q(u)})$ for large t one has to study the small eigenvalues, that is, those eigenvalues of $\Delta_q(u)$ which converge to zero as $u \to 0$. Set

$$r(q) = \dim H^{q-(n-k-1)}(S^k).$$

The behaviour of the small eigenvalues is described by

Proposition 3.13. *There exists $\lambda > 0$ with the following property. For all $0 < \varepsilon < \lambda$, there exists $\delta > 0$ such that, for $u < \delta$, $\Delta_q(u)$ has exactly*

$$\dim H^q(M_0; E_0) + N \cdot r(q)$$

eigenvalues $< \lambda$ and all of them are $< \varepsilon$.

The proof is exactly the same as the proof of Proposition 7.19 in [C].

Let $0 < \lambda_1^q(u) \le \lambda_2^q(u) \le \cdots \le \lambda_{s(q)}^q(u) < \lambda$ be the non-zero eigenvalues described by Proposition 3.13 . By Proposition 3.13, we have

$$\dim H^q(M_u; E_u) + s(q) = \dim H^q(M_0; E_0) + \dim H^{q+1}(M_0, M_u; E_0).$$

Comparing this with (3.9), we get

$$k_q + k_{q+1} = s(q). \tag{3.14}$$

Now let $\Delta_q'(u)$ denote the restriction of $\Delta_q(u)$ to the orthogonal complement in $L^2\Lambda^q(E_u)$ of the sum of the eigenspaces corresponding to the small eigenvalues $< \lambda$ described by Proposition 3.13. Note that $\lambda^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-\lambda t}\,dt$. Taking the derivative at $s = 0$, gives

$$\int_1^\infty t^{-1}e^{-\lambda t}\,dt = -\log\lambda - \int_0^1 t^{-1}(e^{-\lambda t} - 1)\,dt - \gamma$$

where $\gamma = -\Gamma'(1)$. Then the first integral on the right hand side of (3.6) equals

$$-\sum_{j=1}^{s(q)} \log\lambda_j^q(u) + \int_1^\infty t^{-1}\,\mathrm{Tr}(e^{-t\Delta_q'(u)})\,dt - \int_0^1 t^{-1}\Big(\sum_{j=1}^{s(q)} e^{-\lambda_j^q(u)t} - s(q)\Big)\,dt - s(q)\gamma.$$

By (3.14), the contribution of the last term to (3.4) cancels (3.10). Furthermore, by Proposition 3.13, the second integral tends to zero as $u \to 0$ and for any given $\varepsilon > 0$ there exists T' such that for $T > T'$, $u < \delta$,

$$\int_T^\infty t^{-1}\mathrm{Tr}\big(e^{-t\Delta_q'(u)}\big)\,dt < \varepsilon.$$

Fix $u_0, T > 0$. Then

$$\begin{aligned}\int_1^T t^{-1}\mathrm{Tr}\big(e^{-t\Delta_q'(u)}\big)\,dt &= \int_1^T t^{-1}\int_{M_{u_0}} \mathrm{tr}K_u^q(x,x,t)\,dx\,dt\\ &\quad+\int_1^T t^{-1}\int_{S^k\times A^{n-k}_{u,u_0}} \mathrm{tr}K_u^q(x,x,t)\,dx\,dt\\ &\quad-\sum_{j=1}^{s(q)}\int_1^T t^{-1}\,e^{-\lambda_j^q(u)t}\,dt - \log T\,\dim H^q(M_u;E_u).\end{aligned}\tag{3.15}$$

By Proposition 3.13, the last two terms tend to

$$-\log T\,\big(s(q)+\dim H^q(M_u;E_u)\big)$$

as $u \to 0$, and, by Proposition 3.13, this equals

$$-\log T\,\big(\dim H^q(M_0;E_0)+N\cdot r(q)\big).$$

The same holds if we replace M_u by M_u', so that the combined contribution to (3.3) of the last two terms in (3.15) and the analogous terms with respect to M_u' will cancel out in the limit $u \to 0$.

Finally, one uses Theorem 3.11 and Theorem 3.12 to tread the remaining integrals. Let $\underline{K}_{u,1}(t)$ denote the kernel obtained by projecting the heat kernel $K_{u,1}(t)$ defined above onto the orthogonal complement of the harmonic forms. Then the final result can be stated as

(3.16)

$$\begin{aligned}\lim_{u_0\to 0}\lim_{u\to 0}\Bigg|\int_1^\infty t^{-1}\,\mathrm{Tr}\big(e^{-t\Delta_q(u)}-P^q(u)\big)\,dt-\int_1^\infty t^{-1}\,\mathrm{Tr}\big(e^{-t\Delta_q}-P^q\big)\,dt\\ -\int_1^\infty t^{-1}\int_{S^k\times A^{n-k}_{u,u_0}}\mathrm{Tr}\big(\underline{K}_{u,1}(x,x,t)\big)\,dx\,dt+\sum_{j=1}^{s(q)}\log(\lambda_j^q(u))\Bigg| = 0.\end{aligned}$$

Next we have to investigate the integrals $\int_0^1 t^{-1}\mu_u(t)\,dt$ and $\int_0^1 t^{-1}\mu_0(t)\,dt$. Recall that $\mu_u(x,t)$, $\mu_0(x,t)$ and $\mu_{u,1}(x,t)$ are the differences between $\mathrm{tr}K_u^q(x,x,t)$, $\mathrm{tr}K_0^q(x,x,t)$ and $\mathrm{tr}K_{u,1}^q(x,x,t)$, respectively, and their asymptotic expansion up to order $m > n/2$. Then the argument which led to (3.16) can also be used to prove

$$\begin{aligned}\lim_{u_0\to 0}\lim_{u\to 0}\Bigg|\int_0^1 t^{-1}\mu_u(t)\,dt-\int_0^1 t^{-1}\int_{M_0}\mu_0(x,t)\,dx\,dt\\ -\int_0^1 t^{-1}\int_{S^k\times A^{n-k}_{u,u_0}}\mu_{u,1}(x,t)\,dx\,dt\Bigg| = 0.\end{aligned}\tag{3.17}$$

Now consider M_0', M_u', E_0', E_u'. Let $0 < \lambda_1'^q(u) \le \cdots \le \lambda_{s(q)}'^q(u) < \lambda$ be the non-zero eigenvalues described by Proposition 3.13 with respect to M_u', E_u'. Subtracting the corresponding versions of (3.16) and (3.17) for M_0', M_u', E_0', E_u' from (3.16) and (3.17), respectively, and using (3.6) we can summarize our results by

$$\begin{aligned}\lim_{u\to 0}\Big\{ \big(\log T_{M_u}(\rho; h_u) - \log T_{M_0}(\rho_0; h_0)\big) - \big(\log T_{M_u'}(\rho'; h_u') - \log T_{M_0'}(\rho_0'; h_0')\big) \\ + \sum_{q=0}^{n}(-1)^q q \sum_{j=1}^{s(q)} \log\big(\lambda_j^q(u)/\lambda_j'^q(u)\big)\Big\} = 0.\end{aligned} \tag{3.18}$$

It remains to investigate the corresponding expression for the R-torsion. First we consider

$$\log \tau_{M_u}(\rho; h_u) - \log \tau_{M_0}(\rho_0; h_0).$$

Let K_0 be a smooth triangulation of M_0 and $K' \subset K_0$ a subcomplex which induces a triangulation of the submanifold $\tilde{M} \subset M_0$. Recall that M_u denotes the manifold $\tilde{M}$ equipped with the metric g_u. Furthermore, $\tilde{E} = E_0|\tilde{M}$ and E_u is the flat bundle $\tilde{E}$ equipped with the metric h_u. Then we get a short exact sequence

$$0 \to C_\bullet(K'; \tilde{E}) \longrightarrow C_\bullet(K_0; E_0) \longrightarrow C_\bullet(K_0, K'; E_0) \to 0 \tag{3.19}$$

of chain complexes. Each of these chain complexes has a distinguished volume determined by preferred bases and the metrics h_0, g_0, h_u, g_u determine volumes $\mu_0 \in \det H_*(M_0; E_0)$, $\mu_u \in \det H_*(M_u; E_u)$. Since $M_0 - \tilde{M} \cong N(S^k)$, $E_0|N(S^k) \cong N(S^k) \times \mathbf{R}^N$ and the metrics are the standard product metrics, the torsion of the relative chain complex equals $\tau_{N(S^k)}(\mathbf{1})^N$ where $\mathbf{1}$ stands for the trivial 1-dimensional representation. Moreover, it is easy to verify that the torsion of each complex

$$0 \to C_q(K'; \tilde{E}) \longrightarrow C_q(K_0; E_0) \longrightarrow C_q(K_0, K'; E_0) \to 0$$

equals 1. Thus we can apply (1.12) to (3.19) and we obtain

$$\log \tau_{M_0}(\rho_0; h_0) = \log \tau_{M_u}(\rho, h_u) + N \cdot \log \tau_{N(S^k)}(\mathbf{1}) + \log \tau(\mathcal{H}_\bullet; \mu_u)$$

where $\mathcal{H}_\bullet$ denotes the long exact homology sequence of (3.19) regarded as an acyclic chain complex of length 3n equipped with the volumes μ_u, μ_0. Subtracting the corresponding equation for M_0', M_u', E_0', E_u', we get

$$\begin{aligned}\lim_{u\to 0}\Big\{ \big(\log \tau_{M_u}(\rho; h_u) - \log \tau_{M_0}(\rho_0; h_0)\big) - \big(\log \tau_{M_u'}(\rho'; h_u') - \log \tau_{M_0'}(\rho_0'; h_0')\big) \\ + \big(\log \tau(\mathcal{H}_\bullet; \mu_u) - \log \tau(\mathcal{H}_\bullet'; \mu_u')\big)\Big\} = 0.\end{aligned} \tag{3.20}$$

Now one has to study the behaviour of $\log \tau(\mathcal{H}_\bullet; \mu_u)$ and $\log \tau(\mathcal{H}_\bullet'; \mu_u')$ as $u \to 0$. This is completely analogous to [C] and we leave it to the reader (cf. p.316 of [C]). As result we obtain that in the limit $u \to 0$, $\log \tau(\mathcal{H}_\bullet; \mu_u) - \log \tau(\mathcal{H}_\bullet'; \mu_u')$ cancels the contribution of the small eigenvalues in (3.18). This completes the proof of Theorem 3.3.

Now we can proceed exactly in the same way as in [C]. Write $d_k = e_1 - e_0$ where $e_i = \log T_{M_i}(\rho_i; h_i) - \log \tau_{M_i}(\rho_i; h_i)$ and M_1 is obtained from M_0 by surgery on an embedded k-sphere.

Proposition 3.21. *We have $d_k = 0$ for $1 \leq k \leq n-2$. Moreover, d_0 is well defined.*

We just follow the proof of Propositon 8.20 and Proposition 8.21 of [C] line by line to obtain the proof of Proposition 3.21.

We can now prove Theorem 1 of the Introduction.To prove this theorem we proceed in exactly the same way as on p.318 of [C], using Proposition 1.25, Proposition 1.27, Proposition 2.15 and Proposition 2.19.

Remark. Everything that has been said in the previous sections works as well for finite dimensional complex representations of $\pi_1(M)$.

4.Some examples

In this section we shall discuss two examples where non-orthogonal or non-unitary representations of the fundamental group arise naturally and the torsion is of relevance in this context

4.1. Locally symmetric manifolds.

Let G be a connected semi-simple Lie group with finite center. We also assume that G has no compact factors. Let K be a maximal compact subgroup of G. The Lie algebras of G and K will be denoted by $\mathfrak{g}$ and $\mathfrak{k}$, respectively. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ with respect to $\mathfrak{k}$ and let θ be the Cartan involution of $(\mathfrak{g}, \mathfrak{k})$. The quotient space $X = G/K$ is then a symmetric Riemannian manifold and G is the identity component of the group of orientation preserving isometries of X. We shall denote by $x_0 \in X$ the coset eK of the identity $e \in G$.

Let $\Gamma \subset G$ be a discrete, torsion free, co-compact subgroup of G. Then $M = \Gamma\backslash X$ is a compact locally symmetric manifold covered by X with $\pi_1(M) = \Gamma$.

Let $\rho : G \to \mathrm{GL}(E)$ be a representation of G on a finite dimensional complex vector space E. If we restrict ρ to Γ, we obtain a representation $\rho_\Gamma : \Gamma \to \mathrm{GL}(E)$ with associated flat vector bundle E_ρ over $\Gamma\backslash X$. Since E is a Γ-module, the group cohomology $H^*(\Gamma; E)$ is defined and we have the equality

$$H^*(\Gamma\backslash X; E_\rho) = H^*(\Gamma; E).$$

There is a different way to describe the bundle E_ρ. Let $\rho_K : K \to \mathrm{GL}(E)$ be the restriction of ρ to K and consider the fibration

$$\Gamma\backslash G \longrightarrow \Gamma\backslash X \tag{4.1}$$

which is principal with structure group K. Then ρ_K defines the induced bundle $\tilde{E}_\rho$ over $\Gamma\backslash X$ whose global C^∞-sections are the C^∞-functions

$$f : \Gamma\backslash G \longrightarrow E$$

which satisfy

$$f(gk) = \rho(k)^{-1}\big(f(g)\big) \quad \text{for all} \quad g \in G, k \in K.$$

Lemma 4.2. *The bundles E_ρ and $\tilde{E}_\rho$ are naturally isomorphic. If $\tilde{f} : \Gamma\backslash G \to E$ is a section of $\tilde{E}_\rho$, set $f(x) = f(gx_0) = \rho(g)\tilde{f}(g)$ where $x = gx_0$. Then $f : X \to E$ defines a section of E_ρ and $\tilde{f} \mapsto f$ establishes an isomorphism of the corresponding spaces of C^∞-sections.*

For the proof see Proposition 3.1 in [MM].

Now observe that the flat bundles E_ρ defined by a representation $\rho : G \to \mathrm{GL}(E)$ fit into the setting of the previous sections. Namely, we have

Lemma 4.3. *Any finite dimensional representation $\rho : G \to \mathrm{GL}(E)$ satisfies* $\det \rho(g) = 1$ *for all* $g \in G$.

Proof. Let $N = \dim_{\mathbf{C}} E$. Then G acts on $\Lambda^N E$ via the character $\chi = \det \circ \rho$. Since G is semi-simple, we have $D\mathfrak{g} = \mathfrak{g}$ and therefore, $d\chi = 0$. But G is connected which implies $\chi \equiv 1$. Q.E.D.

Thus each of the flat bundles E_ρ constructed above is unimodular. Next we equip E_ρ with a canonical metric. Following Matsushima and Murakami in [MM], we call a hermitian inner product $\langle u, v\rangle$ on E *admissible* if

a) $\langle \rho(Y)u, v\rangle = -\langle u, \rho(Y)v\rangle$ for all $Y \in \mathfrak{k}$, $u, v \in E$.
b) $\langle \rho(Y)u, v\rangle = \langle u, \rho(Y)v\rangle$ for all $Y \in \mathfrak{p}$, $u, v \in E$.

The existence of an admissible inner product on any G-module E is proved in [MM], Lemma 3.1. Condition a) means that $\langle\cdot,\cdot\rangle$ is invariant under $\rho(K)$ and therefore, it defines a hermitian metric on E_ρ. This is the *canonical metric* on E_ρ defined by the admissible inner product on E.

This choice of an inner product on E_ρ allows us to use harmonic analysis to study the Laplacian Δ on E_ρ-valued differential forms on $\Gamma\backslash X$. The tangent space to X at x_0 can be identified with $\mathfrak{p}$ and the tangent bundle of $\Gamma\backslash X$ is the bundle induced from (4.1) by the adjoint representation $ad_{\mathfrak{p}}$: $K \to \mathrm{GL}(\mathfrak{p})$. Therefore, we have a natural identification

$$\Lambda^q(\Gamma\backslash X, E) = \{\varphi : \Gamma\backslash G \to \Lambda^q\mathfrak{p}^* \otimes E \mid \varphi \text{ is } C^\infty \text{ and}$$
$$\varphi(gk) = \left(\Lambda^q ad_{\mathfrak{p}}^*(k^{-1}) \otimes \rho(k^{-1})\right)(\varphi(g)),\ g \in G, k \in K\}.$$

Concerning the cohomology $H^*(\Gamma; E)$ there exist plenty of results , in particular, vanishing theorems (cf. chapter VII of [BW]) telling us which representations are acyclic. We mention one of them. Let $\mathfrak{h}^+ \subset \mathfrak{k}$ be a maximal abelian subalgebra of $\mathfrak{k}$. Let $\mathfrak{h}$ be the centralizer of $\mathfrak{h}^+$ in $\mathfrak{g}$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Let Φ be a root system of $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ and let Φ_k be a root system of $(\mathfrak{k}_{\mathbf{C}}, \mathfrak{h}^+_{\mathbf{C}})$. Fix a set Φ_k^+ of positive roots for Φ_k and let Φ^+ be a system of positive roots for Φ compatible with Φ_k^+ (see p.65 in [BW]). This means, in particular, that $\theta\alpha \in \Phi^+$ whenever $\alpha \in \Phi^+$. As usually set

$$2\underline{\rho} = \sum_{\alpha\in\Phi^+} \alpha.$$

Theorem 4.4.(Borel-Wallach). *Let E be an irreducible, finite dimensional G-module with highest weight $\Lambda - \underline{\rho}$. If $\theta\Lambda \neq \Lambda$, then $H^*(\Gamma; E) = 0$.*

For the proof see Theorem 6.7 of chapter VII in [BW].

Let $\mathfrak{h}^-_{\mathbf{C}} = \{H \in \mathfrak{h}_{\mathbf{C}} \mid \theta H = -H\}$. Then $\mathfrak{h}^*_{\mathbf{C}} = (\mathfrak{h}^+_{\mathbf{C}})^* \oplus (\mathfrak{h}^-_{\mathbf{C}})^*$. Moreover, assume that $\mathrm{rk}\, G > \mathrm{rk}\, K$. Then $\dim \mathfrak{h}^-_{\mathbf{C}} \geq 1$ and the highest weight of a generic representation satisfies $\theta\Lambda \neq \Lambda$. Note that the condition $\operatorname{rk} G > \operatorname{rk} K$ is satisfied whenever $\dim G/K$ is odd which is the case we are mainly interested in. Thus, for compact locally symmetric manifolds $\Gamma\backslash X$ of odd dimension, Theorem 4.4 produces a large class of acyclic unimodular representations of Γ.

As an example we shall discuss three dimensional hyperbolic manifolds. In this case we have $G = \mathrm{SL}(2,\mathbf{C})$, $K = \mathrm{SU}(2)$ and $H^3 = \mathrm{SL}(2,\mathbf{C})/\mathrm{SU}(2)$ is the three dimensional hyperbolic space. Let $\Gamma \subset \mathrm{SL}(2,\mathbf{C})$ be as above, that is, a discrete, torsion free, co-compact subgroup. Since $\mathrm{SL}(2,\mathbf{C})$ is simple, Γ is an irreducible discrete subgroup.

To describe the irreducible finite dimensional representations of $\mathrm{SL}(2,\mathbf{C})$, we have to consider $\mathfrak{sl}(2,\mathbf{C})$ as Lie algebra over $\mathbf{R}$ which we denote by $\mathfrak{sl}(2,\mathbf{C})^{\mathbf{R}}$. It is well known that

$$\mathfrak{sl}(2,\mathbf{C})^{\mathbf{R}} \cong \mathfrak{so}(3,1) \quad \text{and} \quad \mathfrak{su}(2) \cong \mathfrak{so}(3,\mathbf{R})$$

where $\mathfrak{so}(3,1)$ is the Lie algebra of $\mathrm{SO}(3,1) \subset \mathrm{GL}(4,\mathbf{R})$. Furthermore, we also have an isomorphism

$$\varphi : \mathfrak{so}(3,1) \underset{\mathbf{R}}{\otimes} \mathbf{C} \cong \mathfrak{so}(4,\mathbf{C}). \tag{4.5}$$

The Killing form of $\mathfrak{so}(3,1)$ is given by $B(X,Y) = 2\mathrm{Tr}(X \circ Y)$, $X, Y \in \mathfrak{so}(3,1)$. Let

$$H_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \text{and} \quad H_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then $H_1 \in \mathfrak{so}(3,\mathbf{R})$, $H_3 \in \mathfrak{so}(3,1)$, and $B(H_1, H_3) = 0$. Thus $H_3 \in \mathfrak{p}$. Since $[H_1, H_3] = 0$, it follows that $\mathfrak{h}_+ = \mathbf{R}H_1$ and $\mathfrak{h} = \mathbf{R}H_1 \oplus \mathbf{R}H_2$ are the Cartan subalgebras of $\mathfrak{so}(3)$ and $\mathfrak{so}(3,1)$, respectively, which we considered above. Furthermore, with respect to the isomorphism (4.5), $\mathfrak{h}_{\mathbf{C}} \cong \mathbf{C}H_1 \oplus \mathbf{C}H_2$. Therefore, the complexified Cartan involution θ acts on $\mathfrak{h}_{\mathbf{C}}$ by $\theta(aH_1 + bH_2) = aH_1 - bH_2$. Define $e_j \in \mathfrak{h}_{\mathbf{C}}^*$, $i = 1, 2$, by

$$e_j(H_k) = -i\delta_{jk}.$$

Then the roots of $(\mathfrak{so}(4,\mathbf{C}), \mathfrak{h}_{\mathbf{C}})$ are given by $\pm(e_1 - e_2)$, $\pm(e_1 + e_2)$ (cf. [H], p.188). We choose the positive roots Φ^+ to be $\{\alpha, \beta\}$ where $\alpha = e_1 - e_2$, $\beta = e_1 + e_2$. Then θ preserves Φ^+. Set

$$H_\alpha = i(H_1 - H_2) \quad \text{and} \quad H_\beta = i(H_1 + H_2).$$

Then we have $\alpha(H_\alpha) = \beta(H_\beta) = 2$ and $\alpha(H_\beta) = \beta(H_\alpha) = 0$. Thus H_α, H_β are the dual roots and $\omega_1 = \frac{1}{2}\alpha$, $\omega_2 = \frac{1}{2}\beta$ are the *fundamental weights*. Hence the heighest weight Λ of any irreducible representation $\rho : \mathfrak{so}(4,\mathbf{C}) \to \mathfrak{gl}(E)$ is of the form $\Lambda = \frac{1}{2}(p\alpha + q\beta)$, $p, q \in \mathbf{N}$. This shows that the irreducible representations of $\mathfrak{so}(3,1)$ on a complex finite dimensional vector space are parametrized by pairs $(p,q) \in \mathbf{N}^2$ and the Cartan involution acts by $\theta(p,q) = (q,p)$. Since $\mathrm{SL}(2,\mathbf{C})$ is simply-connected, the representations of the Lie algebra can be lifted to $\mathrm{SL}(2,\mathbf{C})$. Hence for each $(p,q) \in \mathbf{N}^2$, we get an irreducible representation

$$\rho_{p,q} : \mathrm{SL}(2,\mathbf{C}) \longrightarrow \mathrm{GL}(E_{p,q}).$$

Using Theorem 4.4, we get

Lemma 4.6. *Let Γ be a discrete, torsion free, co-compact subgroup of* $\mathrm{SL}(2,\mathbf{C})$. *For* $(p,q)\in\mathbf{N}^2$, *let $E_{p,q}$ be the irreducible* $\mathrm{SL}(2,\mathbf{C})$*-module described above. Then*

$$H^*(\Gamma\backslash H^3, E_{p,q}) = 0 \quad \text{if} \quad p\neq q.$$

We note that the complexification of the standard representation of $\mathfrak{so}(3,1)$ on $\mathbf{R}^4$ has heighest weight $\frac{1}{2}\alpha+\frac{1}{2}\beta$ and therefore, it is equivalent to the $\mathfrak{sl}(2,\mathbf{C})$-module $E_{1,1}$. Thus we can restate Lemma 4.6 as follows. If a given irreducible $\mathfrak{sl}(2,\mathbf{C})$-module E_Λ with heighest weight Λ has non-trivial cohomology, then Λ is a multiple of the heighest weight of the standard representation of $\mathfrak{so}(3,1)$. This agrees with Theorem 1 in [R].

There is a more explicit way to describe the $\mathfrak{sl}(2,\mathbf{C})$-modules $E_{p,q}$. First we observe that there is a natural isomorphism

$$\psi : \mathfrak{so}(4,\mathbf{C}) \cong \mathfrak{sl}(2,\mathbf{C})\times\mathfrak{sl}(2,\mathbf{C}). \tag{4.7}$$

Moreover, if φ is the isomorphism (4.5), then we have

$$\psi\circ\varphi(\mathfrak{so}(3,1)) = \left\{(X,\overline{X})\,|\,X\in\mathfrak{sl}(2,\mathbf{C})\right\}.$$

Next recall that the irreducible finite dimensional representations of the complex Lie algebra $\mathfrak{sl}(2,\mathbf{C})$ are parametrized by $p\in\mathbf{N}$. Given $p\in\mathbf{N}$, let W_p be the corresponding irreducible $\mathfrak{sl}(2,\mathbf{C})$-module. Then $W_1=\mathbf{C}^2$ and $\rho_1 : \mathfrak{sl}(2,\mathbf{C})\to\mathfrak{gl}(\mathbf{C}^2)$ is the standard representation. Furthermore for each $p\in\mathbf{N}$, W_p is the p-th symmetric power of the module $W_1=\mathbf{C}^2$.

Let E be a finite dimensional complex vector space. Then the complex conjugate $\overline{E}$ of E is defined to be the dual vector space of the vector space of all anti-linear forms on E. Every $x\in E$ determines a unique element $\overline{x}\in\overline{E}$. In particular, we can consider the complex conjugate $\overline{W}_q$ of the $\mathfrak{sl}(2,\mathbf{C})$-module W_q. Then $\mathfrak{sl}(2,\mathbf{C})$ acts on $\overline{W}_q$ by

$$\overline{\rho}_q(X)\overline{w} = \overline{\rho_q(X)w}, \quad X\in\mathfrak{sl}(2,\mathbf{C}),\ \overline{w}\in\overline{W}_q,$$

and we get an anti-holomorphic representation

$$\overline{\rho}_q : \mathrm{SL}(2,\mathbf{C}) \longrightarrow \mathrm{GL}(\overline{W}_q).$$

Given $(p,q)\in\mathbf{N}^2$, set $\tilde{\rho}_{p,q}=\rho_p\otimes\overline{\rho}_q$. Then

$$\tilde{\rho}_{p,q} : \mathrm{SL}(2,\mathbf{C}) \longrightarrow \mathrm{GL}(W_p\otimes\overline{W}_q)$$

is an irreducible representation which is equivalent to the representation $\rho_{p,q}$ above. This is a special case of a more general result (cf. Theorem 3.1.1.1 in [Wa]).

For all these acyclic representations ρ, the analytic torsion $T_M(\rho)$ and the R-torsion $\tau_M(\rho)$ are defined and independent of any choice of metrics.

A possible application will be the following. For acyclic unitary representations φ of Γ, Moscovici and Stanton introduced in [MS] a certain zeta function $Z_\varphi(s)$ which is defined in terms of the geodesic flow Φ on the unit tangent bundle of $\Gamma\backslash X$. The set of periods of closed orbits of Φ is discrete and the periodic set breaks up into connected components which are parametrized by the non-trivial conjugacy classes $\{\gamma\}$ in Γ. Each connected component X_γ is itself a compact locally symmetric manifold of non-positive sectional curvature and Φ restricts to a periodic flow on X_γ . Let $\hat{X}_\gamma = X_\gamma/\Phi$, let l_γ be the common length of the periodic orbits in X_γ and μ_γ the multiplicity of a generic orbit of $\Phi|X_\gamma$. Then

$$Z_\varphi(s) = \exp - \sum_{\{\gamma\}\neq 1} \mathrm{Tr}\varphi(\gamma)\,\chi(\hat{X}_\gamma)\,\frac{e^{-sl_\gamma}}{\mu_\gamma}.$$

It is proved in [MS] that $Z_\varphi(s)$ is analytic for $\mathrm{Re}(s) \gg 0$ and admits a meromorphic continuation to $\mathbf{C}$ which is holomorphic at $s = 0$. The main result of [MS] is then that

$$Z_\varphi(0) = \tau_M(\varphi)^2 \quad \text{where} \quad M = \Gamma\backslash X.$$

In view of Theorem 1, it seems to be reasonably to conjecture that this continues to hold for all finite dimensional unimodular representations $\rho : G \to \mathrm{GL}(E)$. We shall return to this point in a forthcoming paper. This may have interesting applications, because most of the representations of Γ are non-unitary.

4.2. Chern-Simons gauge theory with non-compact gauge group.

Chern-Simons gauge theory is a three dimensional gauge field theory with pure Chern-Simons action. It was used by Witten [W1] to introduce new 3-manifold invariants. The basic setting for Chern-Simons theory is a compact oriented three dimensional manifold M without boundary and a Lie group G. We start with the case where G is compact and for simplicity, we take G to be $\mathrm{SU}(N)$. Consider the space $\mathcal{A}$ of all G-connections on the trivial G-bundle over M. In fact, every principal G-bundle over M is trivial. The space $\mathcal{A}$ may be identified with the space $\Lambda^1(M, \mathfrak{g})$ of differential 1-forms on M with values in the Lie algebra $\mathfrak{g}$ of G. For a given connection $A \in \mathcal{A}$, the Chern-Simons action is defined to be

$$I(A) = \frac{1}{4\pi} \int_M \mathrm{Tr}(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A) \tag{4.8}$$

where Tr is the trace of $\mathfrak{su}(N)$ in the standard representation. This is a real valued non-linear functional on $\mathcal{A}$. The gauge group $\mathcal{G} = \mathrm{Map}(M, G)$ acts on $\mathcal{A}$ by the usual prescription $A^g = g^{-1}Ag + g^{-1}dg$, $g \in G$, $A \in \mathcal{A}$. The Chern-Simons functional I is not invariant under the action of $\mathcal{G}$, but it satisfies

$$I(A^g) = I(A) + 2\pi\, m$$

for some $m \in \mathbf{Z}$ depending on $g \in \mathcal{G}$. Let $k \in \mathbf{N}$. Then $e^{ikI(A)}$ is a $\mathcal{G}$-invariant function on $\mathcal{A}$ and Witten's invariant of M is defined as the path integral

$$Z_M(k) = \int e^{ikI(A)}\, \mathcal{D}A \tag{4.9}$$

where the integration is over all gauge equivalence classes of connections. This, however, has to be considered as a formal expression because no measure $\mathcal{D}A$ has been constructed up to now. Part of this theory can be made rigorous and Witten gave an explicit recipe for computing $Z_M(k)$. Moreover, using the theory of quantum groups, Reshetikhin and Turaev [RT] introduced invariants $\tilde{Z}_M(q)$ of a 3-manifold M depending on a root of unity $q = e^{2\pi i/r}$ and they suggest that $\tilde{Z}_M(e^{2\pi i/(k+2)})$ coincides with $Z_M(k)$ after normalization.

A standard way to study functional integrals like (4.9) is to use the method of stationary phase approximation which predicts the behaviour of $Z_M(k)$ for large k. In the present context this method is again not based on solid ground, but it gives very interesting results. By the method of stationary phase, the leading order contribution to $Z_M(k)$ comes from the critical points of the action (4.8). The Euler-Lagrange equation for (4.8) is

$$dA + A \wedge A$$

which shows that the critical points of (4.8) are precisely the connections with vanishing curvature, that is, the flat connections on the bundle $P = M \times G$. A flat connection A is determined up to gauge equivalence by a representation

$$\alpha : \pi_1(M) \to G \tag{4.10}$$

up to conjugacy. Hence the space of gauge equivalence classes of flat connections on P can be identified with $\mathrm{Hom}(\pi_1(M), G)/G$ where G acts on $\mathrm{Hom}(\pi_1(M), G)$ by conjugation. Assume that the topology of M is such that there exists only a finite number of gauge equivalence classes of flat connections on P, say $A_1, ..., A_m$ and let $\alpha_1, ..., \alpha_m$ denote the corresponding homomorphisms (4.10).

Let A be a flat connection and for simplicity, assume that A is irreducible. To work out the leading perturbative approximation to the contribution of the critical point A to the path integral $Z_M(k)$, gauge fixing is needed. This is achieved by picking a Riemannian metric g on M. Let $\alpha : \pi_1(M) \to G$ be the holonomy representation of A and $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ the adjoint representation. Then $\mathrm{Ad} \circ \alpha$ is a representation

$$\rho_\alpha : \pi_1(M) \longrightarrow \mathrm{GL}(\mathfrak{g}) \tag{4.11}$$

and we denote the associated flat bundle by $\mathfrak{g}_\alpha$. Since $\mathfrak{g}$ is compact, the Killing form is negative definite on $\mathfrak{g}$. Therefore, the negative of the Killing form defines a G-invariant inner product on $\mathfrak{g}$ and with respect to this choice of an inner product, the representation (4.11) is orthogonal. Finally, we note that the De Rham complex $\Lambda^*(M; \mathfrak{g}_\alpha)$ is equivalent to the complex of $\mathfrak{g}$-valued differential forms

$$0 \to \Lambda^0(M; \mathfrak{g}) \xrightarrow{d_A} \Lambda^1(M; \mathfrak{g}) \xrightarrow{d_A} \Lambda^2(M; \mathfrak{g}) \xrightarrow{d_A} \Lambda^3(M; \mathfrak{g}) \to 0$$

where $\Lambda^q(M; \mathfrak{g}) = \Lambda^q(M) \otimes \mathfrak{g}$ and d_A is the covariant derivative with respect to the connection A. Since A is flat, we have $d_A^2 = 0$. By assumption, A is irreducible and isolated modulo gauge equivalence. Therefore, we have $H^0(M; \mathfrak{g}_\alpha) = H^1(M; \mathfrak{g}_\alpha) = 0$ and Poincaré duality implies $H^*(M; \mathfrak{g}_\alpha) = 0$, so that ρ_α is acyclic.

To describe the final result we need some more notation. Let $c_2(G)$ be the value of the Casimir operator of G in the adjoint representation, normalized so that $c_2(\mathrm{SU}(N)) = 2N$. Furthermore, let $I(g)$ be the Chern-Simons invariant of the Levi-Civita connection of g with respect to a given trivialization of the tangent bundle of M and let $\eta(g)$ be the η-invariant of the metric g.

Assume now that the representatives $A_1, ..., A_m$ of the gauge equivalence classes of flat connections are all irreducible. Then Witten's formula for the stationary phase approximation (or one loop approximation) of the path integral (4.9) is

$$Z_M(k) \sim \frac{1}{\#Z(G)} e^{i\pi d\left(\frac{1}{2}\eta(g) + \frac{1}{24\pi} I(g)\right)} \sum_{j=1}^{m} e^{i(k+c_2(G)/2)I(A_j)} \sqrt{T_M(\rho_{\alpha_j})} \tag{4.12}$$

where $Z(G)$ is the center of G, d is the dimension of G and $T_M(\rho_{\alpha_j})$ is the analytic torsion of the flat connection ρ_{α_j}. In fact, formula (2.23) in [W1] has to be slightly corrected (cf. also (1.32) in [FG]).

D.Freed and R.Gompf [FG] have done explicit computations in a number of cases supporting the believe that (4.12) gives the correct asymptotic behaviour of Wittten's invariant.

Since each ρ_{α_j} is acyclic, $T_M(\rho_{\alpha_j})$ is independent of the choice of the metric g on M and, by [C], [Mü], it coincides with the R-torsion $\tau_M(\rho_{\alpha_j})$. Furthermore, by the Atiyah-Patodi-Singer theorem,

$$\tfrac{1}{2}\eta(g) + \tfrac{1}{24\pi} I(g)$$

is also independent of the metric. It depends only on the trivialization of the tangent bundle. As we know, the R-torsion $\tau_M(\rho_{\alpha_j})$ can be computed from a triangulation K of M in a pure combinatorial way. This suggests that one may be able to develop a rigorous treatment of the path integral (4.9) on the combinatorial level and derive the asymptotic behaviour (4.12) in this way.

There exist also conjectures how (4.12) has to be modified if we give up the assumption that the gauge equivalence classes of flat connections are isolated and irreducible (cf. (1.36) in [FG]).

So far we considered the case of a compact gauge group. Witten has also started to investigate Chern-Simons theory with non-compact gauge group [W3]. There exist several motivations to develop such a theory. For example, $2+1$ dimensional gravity is related to Chern-Simons gauge theory with gauge group $\mathrm{SL}(2,\mathbf{C})$, $ISO(2,1)$ or $\mathrm{SL}(2,\mathbf{R})\times \mathrm{SL}(2,\mathbf{R})$ depending on wether the cosmological constant is positive, zero, or negative [W2]. For a general non-compact Lie group G, the quantization of Chern-Simons gauge theory with gauge group G is not yet understood. Nevertheless, one can study the perturbative expansion of the corresponding path integral [BNW].

The perturbative treatment of Chern-Simons gauge theory with non-compact gauge group requires again gauge fixing. Since the Killing form is indefinite there exists no obvious gauge fixing as in the compact case and different approaches are possible [BNW].For a semi-simple Lie group G, the most natural gauge fixing seems to be the unitary gauge fixing described in section 4 of [BNW]. Let A be a flat connection on the trivial G-bundle over M with holonomy representation $\alpha : \pi_1(M) \to G$. As above, let $\mathfrak{g}_\alpha$ be the flat bundle defined by $\rho_\alpha = \mathrm{Ad} \circ \alpha$. Then the unitary gauge fixing amounts to the choice of a Riemannian metric g on M and a Hermitian metric h on $\mathfrak{g}_\alpha$. We observe that $\rho_\alpha : \pi_1(M) \to \mathrm{GL}(\mathfrak{g})$ is unimodular. In fact, since $\mathfrak{g}$ is semi-simple, the Killing form is non-degenerate. Hence, for each $g \in G$, $\mathrm{Ad}(g)$ preserves a non-degenerate symmetric bilinear form on $\mathfrak{g}$ which implies that $|\det \mathrm{Ad}(g)| = 1$. This is precisely the setting of section 2.

Under the same assumption as above, one gets a formula for the one loop approximation of the path integral which is similar to (4.12). The analytic torsion $T_M(\rho_{\alpha_j})$ is now defined by (2.2). For the discussion of the phase factor see section 4 of [BNW]. By assumption, each representation ρ_{α_j} is acyclic and therefore, by Corollary 2.13, $T_M(\rho_{\alpha_j})$ is independent of the choice of the metric on M and $\mathfrak{g}_\alpha$. Moreover, by Theorem 1, $T_M(\rho_{\alpha_j})$ equals the R-torsion $\tau_M(\rho_{\alpha_j})$ which has again a pure combinatorial describtion. This suggests that Chern-Simons gauge theory with a non-compact but semi-simple gauge group should also be accessible to a combinatorial treatment.

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