Simple characters for principal orders, part II

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Introduction

In this paper, we extend some results on simple characters for central simple local algebras associated with a representation filter comming from a principal order. In particular, we give a general proof of rigidity and refinement of simple characters. We prove the "intertwining implies conjugacy" property in the tame case. We construct a list of hypothetical simple types and show for the case of $Gl_l(D)$, where l is a prime number, that these types are types in the sense of [BK4]. Finally, we explain a well definied map between certain conjugacy classes of extended types for a division algebra $\mathcal{D}$ of index $l \cdot Ind(D)$ ($Ind(D)$ the index of D) and conjugacy classes of simple types for tame $Gl_l(D)$, i.e. we assume that $l \cdot Ind(D)$ is prime to the residual characteristic of our ground field F. This map preserves central character as well as Artin conductor of representations containing these types. It turns out that it maps finite packages of representations containing these types to finite packages of the same size. Modulo the evident conjecture that this map is injective, we can show that our construction of simple types are complete in the sense that each irreducible essentially square integrable representation of tame $Gl_l(D)$ contains a simple type. In particular, it would imply that all supercuspidal representations of tame $Gl_l(D)$ are induced from a compact modulo center subgroup. Our main reference is the first part of [G2]. In particular, we use the general notation of [G2](0).

We impose our description of simple characters by establishing an explicit product expansion with respect to a given special approximation sequence (see 3.1).

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We hope to get the finish of the sketched proof from generalizations of considerations as done in [BH] to the mixed case of tame $A = M_m(D)$. To get the type property for our hypothetical types, we mainly need to solve the problems concerning the Bruhat decomposition as concidered in the proof of 12.7, because the rest is mainly the same as in our special case or contained in a thesis of Vincent Sécherre [VS].

Contents

1 Rigidity and refinement of simple characters

Before studying the Heisenberg representations attached to simple characters, we start by asking for the parameters of a given character set. As we have already seen in [G2](5), the character set $\mathcal{C}(\beta,\mathfrak{A})$ only depends on the equivalence class of a given simple stratum $[\mathfrak{A},n,0,\beta]$. Conversely, we may recover the stratum $[\mathfrak{A},n,[\frac{n}{2}],\beta]$ by restriction of simple characters. In this section, we generalize some further results of [BK1] to the case of an arbitrary simple algebra. First of all, we show that a character set can be recovered from a single simple character and we call this the rigidity property of a character set. The second property is the possibility to refine a simple character w.r.t. an enlarged approximation sequence. Conversely, the third property is the fact that a simple character, given at a certain level by a defining sequence depending on an approximation sequence, cannot intertwine with another simple character given by a shorter one and we call this the rigidity of the simple character itself. Before showing these properties, we need the corresponding properties for the simple strata. Finally, in the tame case we deduce from the three first properties a forth property, called "intertwining implies conjugacy", a property that has already been proved for simple strata (see [BG]). The proofs follow the presentation of [BK1] in the split case, but in a crucial point we use a trick of Corwin's [Co].

(1.1) Proposition ([BK1](3.5.1),(3.5.4)) *For i=1,2, let $[\mathfrak{A},n,\nu,\beta_i]$ be simple strata such that $\mathcal{C}(\beta_1,\mathfrak{A})_{|U^{\nu+1}(\mathfrak{A})\cap H^1(\beta_1,\mathfrak{A})} \cap \mathcal{C}(\beta_2,\mathfrak{A})_{|U^{\nu+1}(\mathfrak{A})\cap H^1(\beta_2,\mathfrak{A})} \neq \emptyset$ (in particular $U^{\nu+1}(\mathfrak{A})\cap H^1(\beta_1,\mathfrak{A}) = U^{\nu+1}(\mathfrak{A})\cap H^1(\beta_2,\mathfrak{A})$)), then $[F[\beta_1]:F]=[F[\beta_2]:F]$ and both strata share the same sequence of jumps.*

proof:
We may argue as in the split case and use the formula [G2](5.11) for the intertwining of the restriction of a simple character θ of the above intersection of character sets and obtain the equality $\mathcal{S}(\beta_1,\mathfrak{A})C_A(\beta_1)^\times\mathcal{S}(\beta_1,\mathfrak{A}) = \mathcal{S}(\beta_2,\mathfrak{A})C_A(\beta_2)^\times\mathcal{S}(\beta_2,\mathfrak{A})$, where $\mathcal{S}(\beta_i,\mathfrak{A}) = (1+\mathfrak{P}^{q-\nu}\cap\mathfrak{N}_{-\nu}(\beta_i,\mathfrak{A})+\mathfrak{P}^{[\frac{q+1}{2}]}\cap\mathfrak{J}^1(\beta_i,\mathfrak{A}))$.

Intersecting both sides by $\mathfrak{A}^\times$, multiplying and factoring by $(1+\mathfrak{P})$, we obtain $(Gl_{\bar{s}}(\mathfrak{k}))^{\bar{r}} \simeq (\mathfrak{A}\cap C_A(\beta_1))^\times/(1+\mathfrak{P}\cap C_A(\beta_1)) \simeq (\mathfrak{A}\cap C_A(\beta_2))^\times/(1+\mathfrak{P}\cap C_A(\beta_2))$. The assertion now follows from $\frac{N}{[F[\beta_1]:F]} = \bar{r}\bar{s}[\mathfrak{k}:\mathfrak{k}_F] = \frac{N}{[F[\beta_2]:F]}$, where $N = m\cdot d$ is the reduced degree of A.

The second assertion follows now from the first one by varying the power ν, because the equality of jumps must follow from the equality of field degrees for the simple approximations. In particular, we have $k_0(\beta_1,\mathfrak{A}) = k_0(\beta_2,\mathfrak{A})$. The choice of simple approximations and a further restriction of θ yields the equality of all the other jumps.

□

(1.2) Lemma *Let A/F be a split central simple algebra, let $E = F[\beta]$ and $F[\gamma]/F$ be totally ramified extensions in A and let $[\mathfrak{A},n,\nu,\beta]$ be a pure stratum with simple approximation $[\mathfrak{A},n,\nu,\gamma]$, where $\nu = -k_0(\beta,\mathfrak{A})$. Then $[\mathfrak{B}_\gamma,\nu,\nu-1,s_\gamma(\beta-\gamma)]$ is equivalent to a simple stratum $[\mathfrak{B}_\gamma,\nu,\nu-1,\delta]$, where $F[\gamma,\delta]$ is totally ramified over $F[\gamma]$ and $e(F[\gamma,\delta]/F)\mid e(E/F)$. Here $\mathfrak{B}_\gamma := \mathfrak{A}\cap C_A(\gamma)$ and $s_\gamma : A\to C_A(\gamma)$ is a tame corestriction with respect to the characters ψ_F and $\psi_{F[\gamma]}$.*

Proof:

The first assertion is a special case of [BK1](2.4.1)(iii). For the assertion about the invariants we follow the proof of [BK1](2.4.1)(iii) in detail and first observe that the assertion underlying [BK1](2.4.1)(i)(ii) is independent of the special choice of the approximation γ.

Consequently, let $B := C_A(E)$, $\mathfrak{B} := \mathfrak{A}\cap B$, $A(E) := End_F(E)$ and $\mathfrak{A}(E) := End_{o_F}\{\mathfrak{p}_E^j\}_{j\in\mathbb{Z}}$, then the choice of a '$(W,E)$-decomposition' according to [BK1](1.2.6) leads to an isomorphism $A(E)\otimes_E B \simeq A$ of $(A(E),B)$-bimodules and an isomorphism $\mathfrak{A}(E)\otimes_{\mathfrak{o}_E}\mathfrak{B} \simeq \mathfrak{A}$ of $(\mathfrak{A}(E),\mathfrak{B})$-bimodules. Now we choose a simple stratum $[\mathfrak{A}(E),\bar{n},\bar{\nu},\gamma]$ that is equivalent to $[\mathfrak{A}(E),\bar{n},\bar{\nu},\beta]$, where we identify the elements β,γ with the preimages in A, $\bar{n} := -\nu_E(\beta)$, $\bar{\nu} := [\frac{\nu+e-1}{e}]$ and $e := \frac{r(\mathfrak{A})}{e(E/F)}$. The tame corestriction $s_\gamma : A\to C_A(\gamma)$ can be written in the form $s_\gamma = \bar{s}\otimes 1$, where $\bar{s} : A(E)\to C_{A(E)}(\gamma)$ is a tame corestriction attached to the same additive characters ψ_F and $\psi_{F[\gamma]}$ as s_γ is.

Now we find a simple stratum $[\mathfrak{B}(E)_\gamma,\bar{\nu},\bar{\nu}-1,\delta]$ that is equivalent to the stratum $[\mathfrak{B}(E)_\gamma,\bar{\nu},\bar{\nu}-1,\bar{s}(\beta-\gamma)]$, where $\mathfrak{B}(E)_\gamma := \mathfrak{A}(E)\cap C_{A(E)}(\gamma)$.

The preimage of δ now fulfils the assumptions of the proposition, where we have to remark that $\mathfrak{A}(E)$ is a minimal order in $A(E)$ of period $e(E/F)$, and $e(F[\gamma,\delta]/F[\gamma])$ must be a divisor of the period of $\mathfrak{B}(E)_\gamma$, which is $\frac{e(E/F)}{e(F[\gamma]/F)}$. Also the degree of the unramified part of the extension must divide the invariant of $\mathfrak{B}(E)_\gamma$, which is 1; hence, $F[\gamma,\delta]/F[\gamma]$ is totally ramified.

□

(1.3) Proposition (Rigidity for simple strata) *Let A be an arbitrary simple algebra and $[\mathfrak{A},n,\nu,\beta]$ be a sound stratum with special simple approximation $[\mathfrak{A},n,\nu,\gamma]$, where $\nu = -k_0(\beta,\mathfrak{A})$, then $[\mathfrak{B}_\gamma,\bar{\nu},\bar{\nu}-1,s_\gamma(\beta-\gamma)]$ is equivalent to a simple stratum $[\mathfrak{B}_\gamma,\bar{\nu},\bar{\nu}-1,\delta]$ such that $F[\gamma,\delta]^{ur}\subset K$ and $e(F[\gamma,\delta]/F)\mid e(E/F)$. Here $\mathfrak{B}_\gamma := \mathfrak{A}\cap C_A(\gamma)$, $s_\gamma : A\to C_A(\gamma)$ is a tame corestriction with respect to the two additive characters ψ_F, $\psi_{F[\gamma]}$, $\bar{\nu} := \sigma_{\mathfrak{A}/\mathfrak{B}_\gamma}(\nu)$, $E := F[\beta]$ and $K := E^{ur}$.*

Proof:

As in the proof of the preceding lemma we first remark that the assertion is independent of the special choice of the special approximation sequence for $[\mathfrak{A},n,\nu,\gamma]$. To see this we generalize [BK1](2.4.1)(ii) by using the canonical splitting $\tilde{A} = A\otimes_F D^{op}$. For this purpose we observe that each tame corestriction $s_\gamma : A\to C_A(\gamma)$ leads to a tame corestriction $s_\gamma\otimes 1 : A\otimes_F D^{op}\to C_A(\gamma)\otimes_F D^{op}$.

Finally, we also want to find a reduction to the totally ramified case for the extension E/F. We set $\psi_K := \psi_F\circ tr_{K/F}$, $\psi_{K[\gamma]} := \psi_{F[\gamma]}\circ tr_{K[\gamma]/F[\gamma]}$, $C := C_A(K)$, $C_\gamma := C_A(K[\gamma])$. Let $s_K : A\to C$ be the tame corestriction attached to ψ_F and ψ_K, $s_{K[\gamma]}: C_A(\gamma) \to C_\gamma$ be the tame corestriction attached to $\psi_{F[\gamma]}$ and $\psi_{K[\gamma]}$. Finally, let $s_\gamma^K : C \to C_\gamma$ be the tame corestriction attached to ψ_K and $\psi_{K[\gamma]}$. By the definition of tame corestrictions, we have a commutative diagram:

$$\begin{array}{ccccc} & A & \xrightarrow{s_\gamma} & C_A(\gamma) & \\ s_K & \downarrow & & \downarrow & s_{K[\gamma]} \\ & C & \xrightarrow{s_\gamma^K} & C_\gamma . & \end{array}$$

A simple calculation shows (cf. [BK1](1.3.8)(ii)) that the vertical mappings are projectors. Since $s_\gamma(\beta-\gamma)$ commutes with γ, s_γ is a $(C_A(\gamma), C_A(\gamma))$-bimodule homomorphism, K commutes with $\beta-\gamma$ and also $K[\gamma]$ commutes with $s_\gamma(\beta-\gamma)$, we obtain $s_K^\gamma(s_\gamma(\beta-\gamma)) = s_\gamma(\beta-\gamma)$. As $\beta-\gamma$ commutes with K, we obtain $s_{K[\gamma]}(s_K(\beta-\gamma)) = s_{K[\gamma]}(\beta-\gamma)$. From the commutativity of the diagram above we obtain $s_\gamma(\beta-\gamma) = s_{K[\gamma]}(\beta-\gamma)$.

If $[\mathfrak{A}, n, \nu, \gamma]$ is a special approximation of $[\mathfrak{A}, n, \nu, \beta]$, then $[\mathfrak{C}, \tilde{n}, \tilde{\nu}, \gamma]$ is an approximation of $[\mathfrak{C}, \tilde{n}, \tilde{\nu}, \beta]$, where $\mathfrak{C} := \mathfrak{A} \cap C_A(K)$, $\tilde{n} := \sigma_{\mathfrak{A}/\mathfrak{C}}(n)$ and $\tilde{\nu} := \sigma_{\mathfrak{A}/\mathfrak{C}}(\nu)$. If we showed that the stratum $[\mathfrak{C}_\gamma, \nu', \nu'-1, s_{K[\gamma]}(\beta-\gamma)]$ to be equivalent to a simple stratum $[\mathfrak{C}_\gamma, \nu', \nu'-1, \delta']$, with $e(K[\delta', \gamma]/K) \mid e(E/K)$, where $\nu' := \sigma_{\mathfrak{A}/\mathfrak{C}_\gamma}(\nu)$, then of course for any special simple approximation $[\mathfrak{B}_\gamma, \bar{\nu}, \bar{\nu}-1, \delta]$ of the stratum $[\mathfrak{B}_\gamma, \bar{\nu}, \bar{\nu}-1, \delta']$, this stratum would also be equivalent to $[\mathfrak{B}_\gamma, \bar{\nu}, \bar{\nu}-1, s_\gamma(\beta-\gamma)]$, $F[\delta,\gamma]^{ur} \subset K$ and $e(F[\gamma,\delta]/F) \mid e(E/F)$.

Therefore, we restrict ourselves to the totally ramified case for the extension E/F. Consequently we assume that $K = F$ and $C = A$. For uniformity of notation we write for the uniformity of the notation $B_\gamma := C_A(\gamma) = C_\gamma$. We also remark that in our case $\bar{n} = n$ and $\bar{\nu} = \nu$.

We also choose an unramified splitting field L/F in D of degree d, where D is a division algebra of the same Brauer class of A; moreover, $A \simeq M_m(D)$, and therefore we have $A \otimes_F L \simeq M_{m\cdot d}(L)$. The stratum $[\mathfrak{A} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L, n, \nu, \beta \otimes 1]$ is pure and has the simple approximation $[\mathfrak{A} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L, n, \nu, \gamma \otimes 1]$. In the same manner we may consider the split algebra $\tilde{A} := A \otimes_F D^{op}$, which contains $A \otimes_F L$. As described in [G2](3) we form the canonical lift $\tilde{\mathfrak{A}}$ of our principal order and obtain $A \otimes_F L = C_{\tilde{A}}(L)$, $\mathfrak{A} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L = \tilde{\mathfrak{A}} \cap A \otimes_F L$. Again the stratum $[\tilde{\mathfrak{A}}, n, \nu, \beta \otimes 1]$ is pure and has the simple approximation $[\tilde{\mathfrak{A}}, n, \nu, \gamma \otimes 1]$. According to 1.2 the stratum $[\mathfrak{B}_\gamma \otimes_{\mathfrak{o}_F} \mathfrak{o}_L, \nu, \nu-1, s_\gamma(\beta-\gamma) \otimes 1]$ is equivalent to a simple stratum $[\mathfrak{B}_\gamma \otimes_{\mathfrak{o}_F} \mathfrak{o}_L, \nu, \nu-1, \delta_0]$, where $L[\delta_0, \gamma]/L$ is a totally ramified extension with $e(L[\delta_0, \gamma]/L) \mid e(E \otimes_F L/L)$.

We may also consider $\tilde{\mathfrak{B}}_\gamma := \tilde{\mathfrak{A}} \cap C_{\tilde{A}}(F[\gamma])$ and, of course, the stratum $[\tilde{\mathfrak{B}}_\gamma, \nu, \nu-1, s_\gamma(\beta-\gamma)]$ is then equivalent to $[\tilde{\mathfrak{B}}_\gamma, \nu, \nu-1, \delta_0]$. We may choose a lattice chain for $\mathfrak{B}_\gamma \otimes_{\mathfrak{o}_F} \mathfrak{o}_L$ in a simple $\tilde{B}_\gamma$-module which is also for $\tilde{\mathfrak{B}}_\gamma$ (note $\tilde{B}_\gamma := C_{\tilde{A}}(\gamma)$), the latter stratum is still pure. Consequently by [BH](4.3) there exists an element $\bar{\delta}$ of $B_\gamma \otimes_F L$, so that the stratum $[\mathfrak{B}_\gamma \otimes_{\mathfrak{o}_F} \mathfrak{o}_L, \nu, \nu-1, \bar{\delta}]$ is simple and equivalent to $[\mathfrak{B}_\gamma \otimes_{\mathfrak{o}_F} \mathfrak{o}_L, \nu, \nu-1, \delta_0]$ and so that the stratum $[\tilde{\mathfrak{B}}_\gamma, \nu, \nu-1, \bar{\delta}]$ remains simple. Now we know, because of [BG](5.1)(ii) and 1.2 that $\bar{\delta}$ generates a totally ramified extension over F and that $e(F[\bar{\delta}, \gamma]/F) \mid e(E/F)$. Because $E^\times$ can be embedded in $\mathcal{K}_{\mathfrak{A}}$ in a sound way, we deduce from [G1](2.2)(iii), that this is also the case for the field $F[\bar{\delta}, \gamma]$.

Since, in the totally ramified case, there is only one type of embedding, we may use [BG](4.1.2) to choose this embedding also over $F[\gamma]$. We find a simple stratum $[\mathfrak{B}_\gamma, \nu, \nu-1, \bar{\delta}]$ such that the strata $[\mathfrak{B}_\gamma \otimes_{\mathfrak{o}_F} \mathfrak{o}_L, \nu, \nu-1, \bar{\delta} \otimes 1]$ and $[\mathfrak{B}_\gamma \otimes_{\mathfrak{o}_F} \mathfrak{o}_L, \nu, \nu-1, \delta_0]$ intertwine. Therefore, by [BG](4.1.2) and the assumptions on δ_0, the strata $[\mathfrak{B}_\gamma \otimes_{\mathfrak{o}_F} \mathfrak{o}_L, \nu, \nu-1, \bar{\delta} \otimes 1]$ and $[\mathfrak{B}_\gamma \otimes_{\mathfrak{o}_F} \mathfrak{o}_L, \nu, \nu-1, s_\gamma(\beta-\gamma) \otimes 1]$ intertwine in $\mathcal{K}_{\mathfrak{B}_\gamma \otimes_{\mathfrak{o}_F} \mathfrak{o}_L}$. By [BG](4.3.9) we may replace $\bar{\delta}$ by a $\mathcal{K}_{\mathfrak{B}_\gamma}$-conjugate element δ such that the strata $[\mathfrak{B}_\gamma, \nu, \nu-1, \delta]$ and $[\mathfrak{B}_\gamma, \nu, \nu-1, s_\gamma(\beta-\gamma)]$ are equivalent.

The stratum $[\mathfrak{B}_\gamma, \nu, \nu-1, \delta]$ has the desired properties.

□

Using our canonical splitting $\tilde{A}$ we easily obtain:

(1.4) Lemma ([BK1](2.2.1)) *Let $[\mathfrak{A}, n, \nu, \beta]$ be a simple stratum and let $[\mathfrak{A}, n, \nu-1, \beta+b]$, $[\mathfrak{A}, n, \nu-1, \beta+b']$ be refinements with $b, b' \in \mathfrak{P}^{-\nu}$. Let $q = -k_0(\beta, \mathfrak{A})$ and assume that the strata $[\mathfrak{B}, \sigma_{\mathfrak{A}/\mathfrak{B}}(\nu), \sigma_{\mathfrak{A}/\mathfrak{B}}(\nu)-1, s_\beta(b)]$, $[\mathfrak{B}, \sigma_{\mathfrak{A}/\mathfrak{B}}(\nu), \sigma_{\mathfrak{A}/\mathfrak{B}}(\nu)-1, s_\beta(b')]$ are equivalent, where $\mathfrak{B} := \mathfrak{A} \cap B$, $B := C_A(\beta)$ and $s_\beta : A \to B$ is a tame corestriction. Then there exists an $y \in \mathfrak{P}^{q-\nu} \cap \mathfrak{N}_{-\nu}(\beta, \mathfrak{A})$ such that $[\mathfrak{A}, n, \nu-1, (1+y)^{-1}(\beta+b)(1+y)]$ is equivalent to $[\mathfrak{A}, n, \nu-1, \beta+b']$. We remark that $q-\nu \geq 1$.*

□

(1.5) Proposition (Refinement of strata) *Let $[\mathfrak{A}, n, \nu, \beta]$ be a simple stratum, $E = F[\beta]$, $B := C_A(E)$, $s_\beta : A \to B$ be a tame corestriction and $b \in \mathfrak{P}^{-\nu}$ such that there exists a simple stratum $[\mathfrak{B}, \sigma_{\mathfrak{A}/\mathfrak{B}}(\nu), \sigma_{\mathfrak{A}/\mathfrak{B}}(\nu)-1, c]$ that is equivalent to the stratum $[\mathfrak{B}, \sigma_{\mathfrak{A}/\mathfrak{B}}(\nu), \sigma_{\mathfrak{A}/\mathfrak{B}}(\nu)-1, s_\beta(b)]$, then there exists a simple stratum $[\mathfrak{A}, n, \nu-1, \beta_1]$ that is equivalent to $[\mathfrak{A}, n, \nu-1, \beta+b]$ such that, with $M := F[\beta_1]$, $F[c, \beta] := E_1$, we have:*

$$e(M/F) = e(E_1/F),\ f(M/F) = f(E_1/F) \text{ and } k_0(\beta_1, \mathfrak{A}) = \begin{cases} k_0(\beta, \mathfrak{A}) & \text{if } s_\beta(b) \in E \\ -\nu & \text{else.} \end{cases}$$

Proof:

As in the proof of 1.3, we want to restrict ourselves to the totally ramified case for the extension E_1/F. For this purpose we let $K := E_1^{ur}$, $\psi_K := \psi_F \circ tr_{K/F}$, $\psi_{K[\beta]} := \psi_{F[\beta]} \circ tr_{K[\beta]/F[\beta]}$, $C := C_A(K)$, $C_\beta := C_A(K[\beta])$, $s_K : A \to C$ the tame corestriction given by ψ_F and ψ_K, $s_{K[\beta]} : C_A(\beta) \to C_\beta$ the tame corestriction attached to $\psi_{F[\beta]}$ and $\psi_{K[\beta]}$. Moreover, let $s_\beta^K : C \to C_\beta$ be the tame corestriction given by ψ_K and $\psi_{K[\beta]}$. By the definition of a tame corestriction we have a commutative diagram:

$$\begin{array}{cccc} & A & \xrightarrow{s_\beta} & C_A(\beta) & \\ s_K & \downarrow & & \downarrow & s_{K[\beta]} \\ & C & \xrightarrow{s_\beta^K} & C_\beta. & \end{array}$$

The stratum $[\mathfrak{C}_\beta, \sigma_{\mathfrak{A}/\mathfrak{C}_\beta}(\nu), \sigma_{\mathfrak{A}/\mathfrak{C}_\beta}(\nu)-1, s_{K[\beta]}(s_\beta(b))] = [\mathfrak{C}_\beta, \sigma_{\mathfrak{A}/\mathfrak{C}_\beta}(\nu), \sigma_{\mathfrak{A}/\mathfrak{C}_\beta}(\nu)-1, s_\beta^K(s_K(b))]$ is again equivalent to the simple stratum $[\mathfrak{C}_\beta, \sigma_{\mathfrak{A}/\mathfrak{C}_\beta}(\nu), \sigma_{\mathfrak{A}/\mathfrak{C}_\beta}(\nu)-1, c]$.

Assume for a moment that the stratum $[\mathfrak{C}, \sigma_{\mathfrak{A}/\mathfrak{C}}(n), \sigma_{\mathfrak{A}/\mathfrak{C}}(\nu)-1, \beta+s_K(b)]$ is equivalent to a simple stratum $[\mathfrak{C}, \sigma_{\mathfrak{A}/\mathfrak{C}}(n), \sigma_{\mathfrak{A}/\mathfrak{C}}(\nu)-1, \bar{\beta}_1]$. Then the stratum $[\mathfrak{A}, n, \nu-1, \bar{\beta}_1]$ is still pure and equivalent to the stratum $[\mathfrak{A}, n, \nu-1, \beta+s_K(b)]$. We may replace $\bar{\beta}_1$ by a simple representative β_1', which still commutes with K, and the stratum $[\mathfrak{C}, \sigma_{\mathfrak{A}/\mathfrak{C}}(n), \sigma_{\mathfrak{A}/\mathfrak{C}}(\nu)-1, \beta_1]$ is simple and equivalent to $[\mathfrak{C}, \sigma_{\mathfrak{A}/\mathfrak{C}}(n), \sigma_{\mathfrak{A}/\mathfrak{C}}(\nu)-1, \beta+s_K(b)]$.

As c commutes with K, $[\mathfrak{C}_\beta, \sigma_{\mathfrak{A}/\mathfrak{C}_\beta}(\nu), \sigma_{\mathfrak{A}/\mathfrak{C}_\beta}(\nu)-1, c]$ is equivalent to the stratum $[\mathfrak{C}_\beta, \sigma_{\mathfrak{A}/\mathfrak{C}_\beta}(\nu), \sigma_{\mathfrak{A}/\mathfrak{C}_\beta}(\nu)-1, s_{K[\beta]}(s_\beta(b))]$. However, $s_{K[\beta]}(s_\beta(b)) = s_\beta^K(s_K(b)) = s_\beta^K(s_K \circ s_K(b)) = s_{K[\beta]}(s_\beta(s_K(b))) = s_\beta(s_K(b))$, where we use the same argument again (see [BK1](1.3.8)(ii)) to show that the vertical maps are projectors.

As, consequently, the strata $[\mathfrak{B}, \sigma_{\mathfrak{A}/\mathfrak{B}}(\nu), \sigma_{\mathfrak{A}/\mathfrak{B}}(\nu)-1, s_\beta(b)]$ and $[\mathfrak{B}, \sigma_{\mathfrak{A}/\mathfrak{B}}(\nu), \sigma_{\mathfrak{A}/\mathfrak{B}}(\nu)-1, s_\beta(s_K(b))]$ are equivalent, we may also apply 1.4 to restrict ourselves, up to $\mathcal{K}_{\mathfrak{A}}$-conjugacy, to the case where $[\mathfrak{A}, n, \nu-1, \beta+b]$ is equivalent to a simple stratum $[\mathfrak{A}, n, \nu-1, \beta_1]$. To show the relation between invariants, we consider $[\tilde{\mathfrak{A}}, n, \nu-1, \beta+b]$ and $[\tilde{\mathfrak{A}}, n, \nu-1, \beta_1]$ in a canonical splitting $\tilde{A} \simeq A \otimes D^{op}$, where $\tilde{\mathfrak{A}}$ is a canonical splitting of $\mathfrak{A}$ in $\tilde{A}$ (see [G2](3)). Here we notice that

$s_\beta \otimes 1 : A \otimes_F D^{op} \to B \otimes_F D^{op}$ is a suitable tame corestriction. The stratum $[\tilde{\mathfrak{B}}, \nu, \nu-1, s_\beta(b)]$ is equivalent to the simple stratum $[\tilde{\mathfrak{B}}, \nu, \nu-1, c]$, if $s_\beta(b) \notin E$. The latter is simple, because the stratum $[\mathfrak{B}, \sigma_{\mathfrak{A}/\mathfrak{B}}(\nu), \sigma_{\mathfrak{A}/\mathfrak{B}}(\nu)-1, c]$ is minimal (cf. [BK1](1.4.14)(1.4.15),[Br2]). However, then also the the stratum $[\tilde{\mathfrak{B}}, \nu, \nu-1, c]$ must be minimal, hence simple. Finaly, we observe that $k_0(\beta_1, \mathfrak{A}) = k_0(\beta_1, \tilde{\mathfrak{A}})$, $k_0(\beta, \tilde{\mathfrak{A}}) = k_0(\beta, \mathfrak{A})$ and $k_0(c, \tilde{\mathfrak{B}}) = -\nu$ and obtain the desired relation between invariants from [BK1](2.2.3).

Hence, we have reduced to the totally ramified case for the extension E_1/F, where we do not have to prove the relation of invariants. We assume now $K = F$, $A = C$, $\mathfrak{C} = \mathfrak{A}$, and $\mathfrak{C}_\beta = \mathfrak{B}$, we also denote $B = C_\beta$.

The rest of the proof is similar to that of 1.3 after the last reduction step and we just write down the arguments for completness.

We choose an unramified splitting field L/F in D of degree d, where D is the division algebra of the same Brauer class as A, so $A \simeq M_m(D)$, and we have $A \otimes_F L \simeq M_{m\cdot d}(L)$. The stratum $[\mathfrak{B} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L, \nu, \nu-1, c \otimes 1]$ is simple and equivalent to the stratum $[\mathfrak{B} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L, \nu, \nu-1, s_\beta(b) \otimes 1]$. In the same way we can consider the canonical splitting $\tilde{A} := A \otimes_F D^{op}$, which contains our first splitting by construction. We also consider the canonical splitting $\tilde{\mathfrak{A}}$ of our principal order as in [G2](3) and obtain $A \otimes_F L = C_{\tilde{A}}(L)$ as well as $\mathfrak{A} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L = \tilde{\mathfrak{A}} \cap A \otimes_F L$.

By [BK1](2.2.8) there exists a simple stratum $[\mathfrak{A} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L, n, \nu-1, \bar{\beta}_1]$ that is equivalent to $[\mathfrak{A} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L, n, \nu-1, (\beta+b) \otimes 1]$. The stratum $[\tilde{\mathfrak{A}}, n, \nu-1, \bar{\beta}_1]$ is still pure (see the proof of 1.3). By [BH](4.3) there also exists an element β_1' of $A \otimes_F L$ such that the stratum $[\mathfrak{A} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L, n, \nu-1, \beta_1']$ is simple and equivalent to $[\mathfrak{A} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L, n, \nu-1, \bar{\beta}_1]$ and, therefore, the stratum $[\tilde{\mathfrak{A}}, n, \nu-1, \beta_1]$ remains simple. By [BG](5.1)(ii) and [BK1](2.2.8) we know that β_1' also generates a totally ramified extension over F and that $e(M/F) = e(E_1'/F)$, where $E_1' := F[\beta_1']$. On the other hand, $M^\times$ can be embedded in $\mathcal{K}_\mathfrak{A}$ in a sound way. We deduce by [G1](2.2)(iii) that this is also possible for the field E_1'.

Hence, we find a simple stratum $[\mathfrak{A}, n, \nu-1, \tilde{\beta}_1]$ such that the strata $[\mathfrak{A} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L, n, \nu-1, \tilde{\beta}_1 \otimes 1]$ and $[\mathfrak{A} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L, n, \nu-1, (\beta+b) \otimes 1]$ intertwine. By [BG](4.1.2) and the assumptions on $\tilde{\beta}_1$ the strata $[\mathfrak{A} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L, n, \nu-1, \tilde{\beta}_1 \otimes 1]$ and $[\mathfrak{A} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L, n, \nu-1, (\beta+b) \otimes 1]$ intertwine in $\mathcal{K}_{\mathfrak{A} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L}$. According to [BG](4.3.9), up to $\mathcal{K}_\mathfrak{A}$-conjugacy, we may replace $\tilde{\beta}_1$ by an element β_1 such that the strata $[\mathfrak{A}, n, \nu-1, \beta_1]$ and $[\mathfrak{A}, n, \nu-1, \beta+b]$ are equivalent.

The stratum $[\mathfrak{A}, n, \nu-1, \beta_1]$ has all desired properties.

□

(1.6) Definition *Let $[\mathfrak{A}, n, \nu, \beta]$ be a simple stratum with $\nu < [\frac{n}{2}]$. Further, let q^+ be the smallest jump of the stratum $[\mathfrak{A}, n, 0, \beta]$ such that $\nu < [\frac{q^+}{2}]$ and let q^- be the previous jump, then each simple stratum $[\mathfrak{A}, n, q^-, \delta]$ that is equivalent to $[\mathfrak{A}, n, q^-, \beta]$ is called a core approximation of $[\mathfrak{A}, n, \nu, \beta]$.*

With the notation of 1.6 we obtain $-k_0(\delta, \mathfrak{A}) = q^+$, $\mathfrak{P}^{\nu+1} \cap \mathfrak{H}^1(\beta, \mathfrak{A}) = \mathfrak{P}^{\nu+2} \cap \mathfrak{H}^1(\beta, \mathfrak{A}) + \mathfrak{P}^{\nu+1} \cap C_A(\delta)$ and by [G2](5.14) we obtain $\mathcal{C}(\mathfrak{A}, \beta)_{|U^{\nu+1}(\mathfrak{A}) \cap H^1(\beta, \mathfrak{A})} = \mathcal{C}(\mathfrak{A}, \delta)_{|U^{\nu+1}(\mathfrak{A}) \cap H^1(\beta, \mathfrak{A})} \psi^\times_{\beta-\delta}$.

(1.7) Proposition *Let $[\mathfrak{A}, n, \nu, \beta]$ be a simple stratum with $\nu \geq 0$ and $b \in \mathfrak{P}^{-(\nu+1)}$. We consider $\psi_b^\times$ to be a character of $U^{\nu+1}(\mathfrak{A}) \cap H^1(\beta, \mathfrak{A})$ that is trivial on $U^{\nu+2}(\mathfrak{A}) \cap H^1(\beta, \mathfrak{A})$.*

(i) Suppose that there exists $\theta \in \mathcal{C}(\beta, \mathfrak{A})_{|U^{\nu+1}(\mathfrak{A}) \cap H^1(\beta, \mathfrak{A})}$ such that $\theta \cdot \psi_b^\times \in \mathcal{C}(\beta, \mathfrak{A})_{|U^{\nu+1}(\mathfrak{A}) \cap H^1(\beta, \mathfrak{A})}$. Then we have $\theta \psi_b^\times \in \mathcal{C}(\beta, \mathfrak{A})_{|U^{\nu+1}(\mathfrak{A}) \cap H^1(\beta, \mathfrak{A})}$ for all $\theta \in \mathcal{C}(\beta, \mathfrak{A})_{|U^{\nu+1}(\mathfrak{A}) \cap H^1(\beta, \mathfrak{A})}$.

(ii) The intersection $\mathcal{C}(\beta, \mathfrak{A})_{|U^{\nu+1}(\mathfrak{A}) \cap H^1(\beta, \mathfrak{A})} \cap \mathcal{C}(\beta, \mathfrak{A})_{|U^{\nu+1}(\mathfrak{A}) \cap H^1(\beta, \mathfrak{A})} \cdot \psi_b^\times$ is non empty if and only if, for any core approximation $[\mathfrak{A}, n, q^-, \delta]$ of $[\mathfrak{A}, n, \nu, \beta]$ and any tame corestriction $s_\delta : A \to C_A(\delta)$, the image $s_\delta(b)$ lies in the set $F[\delta] + \mathfrak{P}^{-\nu} \cap C_A(\delta)$.

Proof:

To prove (i) we follow the proof of [BK1](3.5.2) and use [G2](1.9) instead of [BK1](2.4.11), [G2](5.3) instead of [BK1](3.2.1) and [G2](5.14) instead of [BK1](3.3.18). To prove (ii) we follow the proof of [BK1](3.5.6) and use [G2](1.9); [G2](5.3) instead of [BK1](3.2.5).

□

(1.8) Proposition *Let $[\mathfrak{A}, n, \nu, \beta]$ be a simple stratum with $0 \leq k < [\frac{n}{2}]$ and let $[\mathfrak{A}, n, q^-, \delta]$ be an associated core approximation. Furthermore, let $\theta_1, \theta_2 \in \mathcal{C}(\beta, \mathfrak{A})_{|U^{\nu+1}(\mathfrak{A}) \cap H^1(\beta,\mathfrak{A})}$. Then the character θ_1/θ_2 of $U^{\nu+1}(\mathfrak{A}) \cap H^1(\beta, \mathfrak{A}) = U^{\nu+1}(\mathfrak{A}) \cap H^1(\delta, \mathfrak{A})$ is intertwined by all of $\mathcal{S}(\delta) C_A(\delta)^\times \mathcal{S}(\delta)$, where $\mathcal{S}(\delta) := 1 + \mathfrak{P}^{q^+ - q^-} \cap \mathfrak{N}_{-q^-}(\beta, \mathfrak{A}) + \mathfrak{P}^{[\frac{q^+ + 1}{2}]} \cap \mathfrak{J}^1(\beta, \mathfrak{A})$.*

Proof:

We follow the proof of [BK1](3.5.7) and use [G2](5.12) instead of [BK1](3.3.9).

□

(1.9) Proposition (Rigidity of a character set) *For $i = 1, 2$ let $[\mathfrak{A}, n, \nu, \beta_i]$ be simple strata. Suppose that $\mathcal{C}(\beta_1, \mathfrak{A})_{|U^{\nu+1}(\mathfrak{A}) \cap H^1(\beta_1,\mathfrak{A})} \cap \mathcal{C}(\beta_2, \mathfrak{A})_{|U^{\nu+1}(\mathfrak{A}) \cap H^1(\beta_2,\mathfrak{A})} \neq \emptyset$. Then we have $\mathcal{C}(\beta_1, \mathfrak{A})_{|U^{\nu+1}(\mathfrak{A}) \cap H^1(\beta_1,\mathfrak{A})} = \mathcal{C}(\beta_2, \mathfrak{A})_{|U^{\nu+1}(\mathfrak{A}) \cap H^1(\beta_2,\mathfrak{A})}$.*

Proof:

We follow the proof of [BK1](3.5.8) and use [G2](5.11) instead of [BK1](3.3.2), [G2](1.9) instead of [BK1](2.4.11), 1.8 instead of [BK1](3.5.7), 1.1 instead of [BK1](3.5.4) and, finally, 1.7 instead of [BK1](3.5.6).

□

(1.10) Proposition (Refinement of simple characters) *For $i = 1, 2$, let $[\mathfrak{A}, n, \nu, \beta_i]$ be simple strata with $\nu > 1$ and $\mathcal{C}(\beta_1, \mathfrak{A})_{|U^{\nu+1}(\mathfrak{A}) \cap H^1(\beta_1,\mathfrak{A})} = \mathcal{C}(\beta_2, \mathfrak{A})_{|U^{\nu+1}(\mathfrak{A}) \cap H^1(\beta_2,\mathfrak{A})}$. Then we have $U^\nu(\mathfrak{A}) \cap H^1(\beta_1, \mathfrak{A}) = U^\nu(\mathfrak{A}) \cap H^1(\beta_2, \mathfrak{A}) =: H^\nu$ and there exists a simple stratum $[\mathfrak{A}, n, \nu, \beta_1']$ that is equivalent to $[\mathfrak{A}, n, \nu, \beta_1]$ such that $\mathcal{C}(\beta_1', \mathfrak{A})_{|H^\nu} = \mathcal{C}(\beta_2, \mathfrak{A})_{|H^\nu}$.*

Proof:

We follow the proof of [BK1](3.5.9) step by step and use 1.1 instead of [BK1](3.5.4), 1.11 (see below) instead of [BK1](3.5.10), [G2](1.9) instead of [BK1](2.4.11), 1.5 instead of [BK1](2.2.8), [G2](5.13) instead of [BK1](3.3.20), 1.8 instead of [BK1](3.5.7), [G2](5.10) instead of [BK1](3.3.1), [G2](4.2); [G2](5.3); [G2](5.11); [G2](5.12) instead of [BK1](3.3.17) and [G2](5.12) instead of [BK1](3.3.9). As a corollary of the proof we obtain:

(1.11) Lemma *Let $[\mathfrak{A}, n, \nu, \beta]$ be a simple stratum with $\nu \geq 1$. Let $\theta \in \mathcal{C}(\beta, \mathfrak{A})_{|U^\nu(\mathfrak{A}) \cap H^1(\beta,\mathfrak{A})}$ and $x \in A^\times$ be an element of the normalizer $\theta_{|U^{\nu+1}(\mathfrak{A}) \cap H^1(\beta,\mathfrak{A})}$, then we have $\theta^x_{|U^{\sigma_{\mathfrak{A}/\mathfrak{B}}(\nu)}(\mathfrak{B})} = \theta_{|U^{\sigma_{\mathfrak{A}/\mathfrak{B}}(\nu)}(\mathfrak{B})}$, where $\mathfrak{B} = \mathfrak{A} \cap C_A(\beta)$.*

□

□

(1.12) Proposition (Rigidity of a simple character) *Let $[\mathfrak{A}, n, q-1, \beta]$ be a simple stratum in A with $-k_0(\beta, \mathfrak{A}) = q$, $n > q \geq 1$, and $[\mathfrak{A}, n, q, \gamma]$ be a simple special approximation of $[\mathfrak{A}, n, q, \beta]$. Further, let $\theta \in \mathcal{C}(\mathfrak{A}, \beta)_{|U^q(\mathfrak{A}) \cap H^1(\beta,\mathfrak{A})}$, $\phi \in \mathcal{C}(\mathfrak{A}, \gamma)_{|U^q(\mathfrak{A}) \cap H^1(\gamma,\mathfrak{A})}$ with $\theta_{|H^{q+1}} = \phi_{|H^{q+1}}$, where $H^{q+1} := U^{q+1}(\mathfrak{A}) \cap H^1(\beta, \mathfrak{A}) = U^{q+1}(\mathfrak{A}) \cap H^1(\gamma, \mathfrak{A})$, then θ and ϕ do not intertwine in $A^\times$.*

Proof:

We follow the proof of Lemma [BK1](3.5.12) step by step and take the ramification of continuations of principal orders into account. Moreover, we use [G2](1.9) instead of [BK1](2.4.11), 1.3 instead of [BK1](2.4.1)(iii), [G2](5.11) instead of [BK1](3.3.2) and 1.11 instead of [BK1](3.5.10). At the end of the proof, using a canonical splitting, we may also apply [BK1](2.6.2).

□

We recall the Definition [G2](2.4), which leads to a generalization of the notion of a sound embedding and, of course, we also need the notion of the invariant of such an embedding:

(1.13) Notation *An embedding $(E,\mathfrak{A})$ is called sound if it fulfils the equivalent conditions from [G2](2.3). The invariant d_1 from [G2](2.3) is also called the invariant of the sound embedding.*

(1.14) Corollary Let $(E,\mathfrak{A})$ be a sound embedding with the invariant d_1 and $(E',\mathfrak{A})$ a sound embedding with $E'^{ur}\subset E^{ur}$. Now if, $f:=\langle f(E/F),d\rangle$ and $f':=\langle f(E'/F),d\rangle$, then the invariant d_1' of the latter embedding is given by $d_1'=\langle f',d_1\rangle$.

Proof:

Let L/F and L'/F be the unique unramified subextensions of degree f and f' of E/F and E'/F, respectively, then, by assumption, $L'\subset L$ and the invariants d_1 and d_1' are the invariants of the embeddings $(L,\mathfrak{A})$ and $(L',\mathfrak{A})$ (see [BG] for the notation). If s is the invariant of $\mathfrak{A}$ and if s' and s'' are the invariants of $\mathfrak{A}\cap C_A(L)$ and $\mathfrak{A}\cap C_A(L')$, respectively, then we have $d_1=\frac{f}{(\frac{s}{s'})}$ and $d_1'=\frac{f'}{(\frac{s}{s''})}$ by [G2](2.2).

On the other hand, by [G1](1.9) $s''=\langle s'\cdot\frac{f}{f'},m\rangle$ and, by [G2](2.2), $s'=\frac{s\cdot d_1}{f}$. Consequently, $d_1'=\frac{f'}{(\frac{s}{\langle\frac{sd_1}{f'},m\rangle})}=\frac{f'\langle\frac{sd_1}{f'},m\rangle}{s}=\langle d_1,f'\cdot r\rangle=\langle d_1,f'\rangle$ as desired. We use that r, the period of $\mathfrak{A}$, is prime to d_1.

□

(1.15) Proposition ([BK1](3.5.11) "intertwining implies conjugacy") *Let A/F be a tame algebra, i.e., the residual characteristic of F is prime to N. For $i=1,2$, let $[\mathfrak{A},n,\nu,\beta_i]$ be simple strata with $\nu\geq 0$, which are all purely embedded with maximally possible embedding invariants for the embeddings $(F[\beta_i],\mathfrak{A})$ in the sense of [BG](3.1),1.13. For $i=1,2$, assume that $\theta_i\in\mathcal{C}(\beta_i,\mathfrak{A})_{|U^{\nu+1}(\mathfrak{A})\cap H^1(\beta_i,\mathfrak{A})}$ intertwine in $A^\times$, then there exists $x\in\mathcal{K}_{\mathfrak{A}}$ such that $\mathcal{C}(\beta_2,\mathfrak{A})_{|U^{\nu+1}(\mathfrak{A})\cap H^1(\beta_2,\mathfrak{A})}=\mathcal{C}(x^{-1}\beta_1x,\mathfrak{A})_{|U^{\nu+1}(\mathfrak{A})\cap H^1(\beta_1,\mathfrak{A})}$ and $\theta_2=\theta_1^x$.*

Proof:

By 1.9 the first assertion follows immediately from the second assertion. We consider the conjugate θ_1^x by $x\in\mathcal{K}_{\mathfrak{A}}$ and first observe that $H^1(x^{-1}\beta_1x,\mathfrak{A})=x^{-1}H^1(\beta_1,\mathfrak{A})x$. For $\nu\geq[\frac{n}{2}]$ the result directly follows from [BG](4.1.2). Indeed, in this case $\theta_i=\psi^\times_{\beta_i}$ and these characters correspond to the strata $[\mathfrak{A},n,\nu,\beta_i]$ with simple special approximations $[\mathfrak{A},n,\nu,\gamma_i]$ for $i=1,2$. Since the latter approximations intertwine by $\beta_i-\gamma_i\in\mathfrak{P}^{-\nu}$; $i=1,2$, using a canonical splitting $\tilde{\mathfrak{A}}$ and [BK1](2.4.1), we see that the invariants of the fields $F[\gamma_1]$ and $F[\gamma_2]$ are the same. By the assumptions on the invariants of the embeddings and as by 1.14, special approximations also have a maximal embedding invariant, the strata $[\mathfrak{A},n,\nu,\gamma_i]$ intertwine and share a common type of embedding with $i=1,2$. By [BG](4.1.2) the latter strata are conjugated in $\mathcal{K}_{\mathfrak{A}}$. The equality $\theta_2=\theta_1^x$ is equivalent to the equivalence of the strata $[\mathfrak{A},n,\nu,\beta_2]$ and $[\mathfrak{A},n,\nu,x^{-1}\beta_1x]$. If for $i=1,2$ the strata $[\mathfrak{A},n,\nu,\gamma_i]$ are simple special approximations and we have $x \in \mathcal{K}_{\mathfrak{A}}$ so that $[\mathfrak{A}, n, \nu, \gamma_2]$ is equivalent to $[\mathfrak{A}, n, \nu, x^{-1}\gamma_1 x]$, then also $[\mathfrak{A}, n, \nu, \beta_2]$ and $[\mathfrak{A}, n, \nu, x^{-1}\beta_1 x]$ are equivalent.

We deal with the general case by induction on ν and on the approximation sequence. The case $\nu \geq [\frac{n}{2}]$ has been done. All together for the next step we may assume that $U^{\nu+2}(\mathfrak{A}) \cap H^1(\beta_1, \mathfrak{A}) = U^{\nu+2}(\mathfrak{A}) \cap H^1(\beta_2, \mathfrak{A}) =: H^{\nu+2}$, $\theta_{2|H^{\nu+2}} = \theta_{1|H^{\nu+2}}$, $\nu < [\frac{n}{2}]$ and that $[\mathfrak{A}, n, \nu+1, \gamma_i]$ is a special approximation of the stratum $[\mathfrak{A}, n, \nu+1, \beta_i]$ for $i = 1, 2$. With respect to the minimal case, one could assume that the latter strata are equivalent. On the other hand, if we consider the case of $\nu \geq \hat{q}$, where $\hat{q}$ is the second last jump, then the last proper approximations $[\mathfrak{A}, n, \hat{q}, \gamma_{1,\mu-1}]$ and $[\mathfrak{A}, n, \hat{q}, \gamma_{2,\mu-1}]$, respectively, intertwine and are conjugate by [BG](4.1.2). Therefore, since, by [G2](5.14), replacement of equivalent approximations does not change the simple character, we could inductively assume, up to conjugacy, and in view of the minimal case that $\gamma_{1,\mu-1} = \gamma_{2,\mu-1}$. We decide for the medium strong assumption $F[\gamma_{1,\mu-1}] = F[\gamma_{2,\mu-1}]$. With respect to our induction, we assume that there exists simple strata $[\mathfrak{A}, n, \nu+1, \gamma_i]$ equivalent to the strata $[\mathfrak{A}, n, \nu+1, \beta_i]$ for $i = 1, 2$ such that $F[\gamma_1] = F[\gamma_2]$.

With respect to the final assertion and 1.1 we want to show first that the strata $[\mathfrak{A}, n, \nu+1, \beta_i]$ for $i = 1, 2$ are either already both simple, or both strata $[\mathfrak{A}, n, \nu, \beta_i]$ for $i = 1, 2$ have a common jump $\nu + 1 = -k_0(\beta_i, \mathfrak{A})$ for $i = 1, 2$.

If not, let us assume that $[\mathfrak{A}, n, \nu+1, \beta_1]$ is simple and $[\mathfrak{A}, n, \nu, \beta_2]$ has an entire approximation $[\mathfrak{A}, n, \nu+1, \gamma_2]$. In particular $\mathcal{C}(\beta_2, \mathfrak{A})_{|H^{\nu+2}} = \mathcal{C}(\gamma_2, \mathfrak{A})_{|H^{\nu+2}}$. By 1.10 we may reduce to the case of $\mathcal{C}(\beta_1, \mathfrak{A})_{|H^{\nu+1}} = \mathcal{C}(\gamma_2, \mathfrak{A})_{|H^{\nu+1}}$. But this contradicts 1.12.

We let $B := C_A(\gamma_1)$, $\mathfrak{B} := \mathfrak{A} \cap B$, $\mathfrak{Q} := \mathfrak{P} \cap B$, $\bar{\nu} := \sigma_{\mathfrak{A}/\mathfrak{B}}(\nu)$, s be a tame corestriction from A to B and $\epsilon := s(1)$, where, in the tame case, it easily follows as in [BK1](1.3) that $\epsilon \in \mathfrak{o}_{F[\gamma_1]}{}^{\times}$, because the trace $tr_{F[\gamma_1]/F}$ maps $\mathfrak{o}_{F[\gamma_1]}$ surjective to $\mathfrak{o}_F$ and $(Trd_{A/F})_{|C_A(\gamma_1)} = tr_{F[\gamma_1]/F} \circ Trd_{C_A(\gamma_1)/F[\gamma_1]}$.

We start with the case where the strata $[\mathfrak{A}, n, \nu+1, \beta_i]$ for $i = 1, 2$ are both simple. By 1.9 together with 1.10, we may assume that $U^{\nu+1}(\mathfrak{A}) \cap H^1(\beta_1, \mathfrak{A}) = U^{\nu+1}(\mathfrak{A}) \cap H^1(\beta_2, \mathfrak{A}) =: H^{\nu+1}$ and have shown the first assertion for the induction step. For $i = 1, 2$ the equivalent strata $[\mathfrak{A}, n, \nu+1, \gamma_i]$ fulfil $F[\gamma_1] = F[\gamma_2] =: M$ by induction and, in particular, s is a tame corestriction for the centralizers with respect to both elements. Using a canonical splitting $\tilde{\mathfrak{A}}$ of $\mathfrak{A}$, we may generalize [BK1](2.4.1)(ii) and find $\delta_i \in M$ such that $s(\beta_i - \gamma_i) - \delta_i \in \mathfrak{P}^{-\nu}$. In particular, we obtain refinements $[\mathfrak{A}, n, \nu, \beta_i] = [\mathfrak{A}, n, \nu, \gamma_i + (\beta_i - \gamma_i)]$ and $[\mathfrak{A}, n, \nu, \gamma_i + \delta_i \cdot \epsilon^{-1}]$ of the strata $[\mathfrak{A}, n, \nu+1, \gamma_i]$ $i = 1, 2$, and, by 1.4, they are conjugated in $\Gamma := (1 + \mathfrak{P}^{q-\nu-1} \cap \mathfrak{N}_{-\nu-1}(\gamma_1, \mathfrak{A})) = (1 + \mathfrak{P}^{q-\nu-1} \cap \mathfrak{N}_{-\nu-1}(\gamma_2, \mathfrak{A}))$, where $q = -k_0(\gamma_1, \mathfrak{A}) = -k_0(\gamma_2, \mathfrak{A})$ and where, by [G2](4.2)(4.4)(5.11) or [G2](5.12), the character $(\theta_1)_{|H^{m+2}} = (\theta_2)_{|H^{m+2}}$ is invariant under this conjugation. So, up to conjugacy, we may fulfil the third assertion of the induction step: The strata $[\mathfrak{A}, n, \nu, \beta_i]$ are equivalent to simple strata $[\mathfrak{A}, n, \nu, \beta'_i]$ for $i = 1, 2$ with $F[\beta'_1] = F[\beta'_2]$. In our special situation the latter field is again M and s is again a corestriction with respect to the centralizers of the latter elements.

By 1.10, we may find a simple stratum $[\mathfrak{A}, n, \nu, \beta''_1]$ equivalent to $[\mathfrak{A}, n, \nu, \beta'_1]$, such that $\mathcal{C}(\beta''_1, \mathfrak{A})_{|H^{\nu+1}} = \mathcal{C}(\beta'_2, \mathfrak{A})_{|H^{\nu+1}}$. As also $[\mathfrak{A}, n, \nu+1, \beta''_1]$ remains simple, we may again use the generalization of [BK1](2.4.1)(ii) to obtain $\mu \in F[\beta_1] \cap \mathfrak{P}^{-\nu-1}$ such that the strata $[\mathfrak{B}, -\sigma_{\mathfrak{A}/\mathfrak{B}}(-\nu-1), -\sigma_{\mathfrak{A}/\mathfrak{B}}(-\nu), \mu]$ and $[\mathfrak{B}, -\sigma_{\mathfrak{A}/\mathfrak{B}}(-\nu-1), -\sigma_{\mathfrak{A}/\mathfrak{B}}(-\nu), s(\beta''_1 - \beta'_1)]$ are equivalent. Consequently, by 1.4, the refinements $[\mathfrak{A}, n, \nu, \beta''_1] = [\mathfrak{A}, n, \nu, \beta'_1 + (\beta''_1 - \beta'_1)]$ and $[\mathfrak{A}, n, \nu, \beta'_1 + \mu \cdot \epsilon^{-1}]$ of the stratum $[\mathfrak{A}, n, \nu+1, \beta'_1]$ are conjugated by an element of $\Gamma = (1 + \mathfrak{P}^{q-\nu-1} \cap \mathfrak{N}_{-\nu-1}(\beta'_1, \mathfrak{A})) = (1 + \mathfrak{P}^{q-\nu-1} \cap \mathfrak{N}_{-\nu-1}(\beta''_1, \mathfrak{A}))$. Since the group Γ normalizes the character $\theta_{1|H^{\nu+2}}$ by [G2](4.2)(4.4)(5.11) or [G2](5.12) and preserves the equivalence of the strata $[\mathfrak{A}, n, \nu+1, \beta'_1]$ and $[\mathfrak{A}, n, \nu+1, \beta_1]$, we may conjugate θ_1 and its whole approximation sequence to obtain $\mathcal{C}(\beta'_1 + b, \mathfrak{A})_{|H^{\nu+1}} = \mathcal{C}(\beta'_2, \mathfrak{A})_{|H^{\nu+1}}$, where we replace β'_1 by its conjugate and where b is the corresponding conjugate of $\mu \cdot \epsilon^{-1} \in M$. By [G2](5.14) we have $\mathcal{C}(\beta_1', \mathfrak{A})_{|H^{\nu+1}}\psi_b^\times = \mathcal{C}(\beta_1' + b, \mathfrak{A})_{|H^{\nu+1}}$.

Let $c \in \mathfrak{P}^{-\nu-1}$ such that $\theta_1\psi_{b+c}^\times = \theta_2$. Then $\theta_1\psi_b^\times$ and $\theta_2 = \theta_1\psi_{b+c}^\times$ are both in $\mathcal{C}(\beta_1' + b, \mathfrak{A})$. Since $F[\beta_1' + b] = F[\beta_1'] = M$ and the restriction of θ_1 as well as the restriction of θ_2 to $U^{\nu+1}(\mathfrak{A}) \cap B$ factor through the reduced norm $Nrd_{B/M}$, we conclude from [G2](1.9) that $s(b+c) \in M \cap \mathfrak{P}^{-\nu-1} + \mathfrak{P}^{-\nu} \cap B$. We want to show that, in fact, $s(b + c) \in \mathfrak{P}^{-\nu} \cap B$ and therefore, assume for the next step that $-\sigma_{\mathfrak{A}/\mathfrak{B}}(-\nu) \neq -\sigma_{\mathfrak{A}/\mathfrak{B}}(-\nu - 1)$. We deduce that the stratum $[\mathfrak{B}, \bar{\nu} + 1, \bar{\nu}, s(b + c)]$ is equivalent to a simple stratum, where we also consider the case of the zero-stratum. We show that the latter case is valid because the stratum $[\mathfrak{B}, \bar{\nu} + 1, \bar{\nu}, s(b + c)]$ intertwines with the zero-stratum. For this purpose, we use the arguments of the proof of 1.12, which we repeat here because we have left the translation from the split case to the reader.

If g intertwines the characters θ_1 and θ_2, then, by [G2](5.11) and by induction, we have $g = x \cdot t \cdot y$ with $x, y \in (1 + \mathfrak{P}^{q-\nu-1} \cap \mathfrak{N}_{-\nu-1}(\beta_1', \mathfrak{A}) + \mathfrak{P}^{[\frac{q+1}{2}]} \cap \mathfrak{J}^1(\beta_1', \mathfrak{A}))$, where $t \in B^\times$. Since t now intertwines θ_1^x with $\theta_2^{y^{-1}}$, by 1.11 we may restrict ourselves to the case $g = t$. As again by 1.11, these characters coincide with θ_1 and θ_2 on $U^{\bar{\nu}+1}(\mathfrak{B})$, the restriction of θ_1 to B factors through $Nrd_{B/M}$ and because $\theta_2 = \theta_1 \cdot \psi_{b+c}^\times$, g intertwines $\psi_{s(b+c)}^\times$ and $\psi_0^\times$.

Since, the stratum $[\mathfrak{B}, \bar{\nu} + 1, \bar{\nu}, s(b + c)]$ intertwines the zero-stratum as required, $s(b + c)$ must lie in $\mathfrak{Q}^{-\bar{\nu}}$ according to [BG](4.1.2). Also θ_1 and θ_2 coincide on $U^{\bar{\nu}+1}(\mathfrak{B})$, too. For $\nu < [\frac{q}{2}]$ the induction step is reached. For the rest of the proof of the first case we assume $\nu \geq [\frac{q}{2}]$. Since $b + c$ modulo $\mathfrak{P}^{-\nu}$ is in the kernel of s, by [G2](2.10), we find a $d \in (\mathfrak{H}^1(\beta_1, \mathfrak{A}) \cap \mathfrak{P}^{\nu+2})^* \cap a_\beta(A)$ such that $\theta_2 = \theta_1 \cdot \psi_d^\times$. By [G2](4.5), we find an $x \in \mathfrak{P}^{q-\nu-1} \cap \mathfrak{N}_{-\nu-1}(\beta_1, \mathfrak{A}) + \mathfrak{J}^1(\beta_1, \mathfrak{A}) \cap \mathfrak{P}^{[\frac{q+1}{2}]}$ with $d = a_\beta(x)$ and by [G2](5.12) it follows that $\theta_2 = \theta_1^{1+x}$ and that $1 + x$ normalize $\theta_{1|H^{\nu+2}}$. Therefore we have handled the first case for both strata and our assertion is valid especialy in the minimal case.

So let $\nu < [\frac{n}{2}]$ and for $i = 1, 2$ assume that the strata $[\mathfrak{A}, n, \nu + 1, \beta_i]$ have proper special approximations $[\mathfrak{A}, n, \nu + 1, \gamma_i]$. Since, we are at the jump $\nu + 1 = k_0(\beta_i, \mathfrak{A})$ $(i = 1, 2)$, by [G2](5.14), we may replace the strata by equivalent ones and assume that $F[\gamma_1] = F[\gamma_2] =: M$. In the above notion we have $B = C_A(\gamma_1) = C_A(M)$, $\mathfrak{B} = \mathfrak{A} \cap C_A(M)$, $\bar{\nu} = \sigma_{\mathfrak{A}/\mathfrak{B}}(\nu)$ and s is a tame corestriction for M/F. Clearly, we have $U^{\nu+2}(\mathfrak{A}) \cap H^1(\beta_1, \mathfrak{A}) = U^{\nu+2} \cap H^1(\beta_2, \mathfrak{A}) = U^{\nu+2}(\mathfrak{A}) \cap H^1(\gamma_1, \mathfrak{A}) = U^{\nu+2}(\mathfrak{A}) \cap H^1(\gamma_2, \mathfrak{A}) =: H^{\nu+2}$ and $(\theta_1)_{H^{\nu+2}} = (\theta_2)_{H^{\nu+2}}$.

In principle the proof runs as in the previous case, i.e., first, we reduce to the case where θ_1 and θ_2 coincide on $U^{\bar{\nu}+1}(\mathfrak{B})$ and, second, we use the above argument including [G2](4.5) to find a conjugating element for the characters.

As in the case above, we show that we may change γ_1 up to equivalence of strata and preserving the assumption $F[\gamma_1] = F[\gamma_2]$ in such a way that $\mathcal{C}(\gamma_1, \mathfrak{A})_{|U^{\nu+1}(\mathfrak{A}) \cap H^1(\gamma_1, \mathfrak{A})} = \mathcal{C}(\gamma_2, \mathfrak{A})_{|U^{\nu+1}(\mathfrak{A}) \cap H^1(\gamma_2, \mathfrak{A})} =: \mathcal{C}(\nu)$. We now choose $\phi \in \mathcal{C}(\nu)$ such that $\phi_{|H^{\nu+2}} = (\theta_1)_{|H^{\nu+2}} = (\theta_2)_{|H^{\nu+2}}$. For $i = 1, 2$ by 1.3, the strata $[\mathfrak{B}, \bar{\nu} + 1, \bar{\nu}, s(\beta_i - \gamma_i)]$ are equivalent to simple strata $[\mathfrak{B}, \bar{\nu} + 1, \bar{\nu}, \delta_i]$ with $F[\delta_i, \gamma_i]^{ur} \subset F[\beta_i]^{ur}$ and $e(F[\delta_i, \gamma_i]/F) \mid e(F[\beta_i]/F)$. By 1.4, we may reduce to $\beta_i = \gamma_i + \delta_i \cdot \epsilon^{-1}$ without restriction of generality. On the other hand, obviously $F[\beta_i] = F[\gamma_i + \delta_i \cdot \epsilon^{-1}] \subset F[\gamma_i, \delta_i]$ and therefore, by the above use of 1.3, $F[\gamma_i, \delta_i] = F[\gamma_i + \delta_i \cdot \epsilon^{-1}] = F[\beta_i]$.

By [G2](5.13) we have $\mathcal{C}(\beta_i, \mathfrak{A})_{|U^{\nu+1}(\mathfrak{A}) \cap H^1(\beta_i, \mathfrak{A})} = \mathcal{C}(\gamma_i, \mathfrak{A})_{|U^{\nu+1}(\mathfrak{A}) \cap H^1(\beta_i, \mathfrak{A})} \cdot \psi_{\beta_i - \gamma_i}^\times$, $\phi \cdot \psi_{\beta_i - \gamma_i}^\times \in \mathcal{C}(\beta_i, \mathfrak{A})_{|U^{\nu+1}(\mathfrak{A}) \cap H^1(\beta_i, \mathfrak{A})}$ and $\theta_i = \phi' \cdot \psi_{\beta_i - \gamma_i}^\times$ with $\phi' \in \mathcal{C}(\nu)$. By the already used standard argument, for $i = 1, 2$ there exists $c_i \in \mathfrak{P}^{-\nu-1}$ with $s(c_i) \in M \cap \mathfrak{P}^{-\nu-1} + \mathfrak{Q}^{-\bar{\nu}}$ such that $\phi' = \phi \cdot \psi_{c_i}^\times$ and therefore $\theta_i = \phi \cdot \psi_{\beta_i - \gamma_i + c_i}^\times$.

Since, for $i = 1, 2$ the strata $[\mathfrak{B}, \bar{\nu} + 1, \bar{\nu}, \delta_i]$ are minimal (see [BK](1.4.14)(1.4.15) eventual applied to a canonical splitting) and non-scalar, the strata $[\mathfrak{B}, \bar{\nu} + 1, \bar{\nu}, s(\beta_i - \gamma_i + c_i)]$ are equivalent to the simple strata $[\mathfrak{B}, \bar{\nu} + 1, \bar{\nu}, \delta_i + \mu_i]$ with $\mu_i \in M \cap \mathfrak{P}^{-\nu-1}$ and $F[\delta_i + \mu_i, \gamma_i] = F[\delta_i, \gamma_i]$.

Now, if g is an element that intertwines θ_1 and θ_2, then $g = x \cdot t \cdot y$ with $x, y \in (1 + \mathfrak{P}^{q-\nu-1} \cap \mathfrak{N}_{-\nu-1}(\gamma_1, \mathfrak{A}) + \mathfrak{P}^{[\frac{\nu+1}{2}]} \cap \mathfrak{J}^1(\gamma_1, \mathfrak{A}))$, where $t \in B^\times$. Again t intertwines θ_1^x and $\theta_2^{y^{-1}}$ and by 1.11 these coincide with θ_1 and θ_2 on $U^{\bar{\nu}+1}(\mathfrak{B})$. We obtain that t intertwines $\psi^\times_{s(\beta_1-\gamma_1+\mu_1\cdot\epsilon^{-1})} = \psi^\times_{\delta_1+\mu_1}$ and $\psi^\times_{s(\beta_2-\gamma_2+\mu_2\epsilon^{-1})} = \psi^\times_{\delta_2+\mu_2}$.

Therefore, t intertwines the strata $[\mathfrak{B}, \bar{\nu}+1, \bar{\nu}, \delta_i + \mu_i]$, $i = 1, 2$, and in particular the fields $F[\gamma_1, \delta_1]$ and $F[\gamma_2, \delta_2]$ must have the same invariants. Again, using the above reduction to $F[\gamma_i, \delta_i] = F[\gamma_i + \delta_i \cdot \epsilon^{-1}] = F[\beta_i]$ and the assumption on the maximal embedding invariant, it is easy to see that, for $i = 1, 2$, the embeddings $(M[\delta_i], \mathfrak{A}) = (F[\gamma_i, \delta_i], \mathfrak{A}) = (F[\beta_i], \mathfrak{A})$ have the same embedding invariant. However, by [G2](2.2), we may recover the embedding invariant from the inertial degree of the field and the invariants of orders by taking certain centralizers, in particular, equality of embedding invariants holds share, if we take the centralizer with respect to M. Consequently, the strata $[\mathfrak{B}, \bar{\nu}+1, \bar{\nu}, \delta_i + \mu_i]$, $i = 1, 2$ share a common type of embedding. Since, they have the same type of embedding and intertwine, by [BG](4.1.2) they are conjugated by an element $z \in \mathcal{K}_{\mathfrak{B}}$. Since z commutes with all the elements ϵ, γ_i and μ_i $(i = 1, 2)$, the strata $[\mathfrak{A}, n, \nu, \beta_i + \mu_i \cdot \epsilon^{-1}]$, $i = 1, 2$, are conjugated by z. This leads up to conjugacy to the induction step for the assertion that there exists strata $[\mathfrak{A}, n, \nu, \beta_i']$ equivalent to $[\mathfrak{A}, n, \nu, \beta_i]$ such that $F[\beta_1'] = F[\beta_2']$, in particular $U^{\nu+1}(\mathfrak{A}) \cap H^1(\beta_1, \mathfrak{A}) = U^{\nu+1}(\mathfrak{A}) \cap H^1(\beta_2, \mathfrak{A})$. Explicitly, we may set $\beta_1' := \beta_2 + (\mu_2 - \mu_1)\epsilon^{-1}$, $\beta_2' = \beta_2$ for instance and we obtain $F[\beta_1'] = F[\beta_2'] = F[\beta_2]$, because by construction, $\mu_1, \mu_2 \in M \subset F[\beta_2]$. We may also assume that θ_1 and θ_2 agree on $H^{\nu+2}$ as well as on $U^{\bar{\nu}+1}(\mathfrak{B})$.

Now, we attain the induction step as before. By [G2](2.10), we find $d \in (\mathfrak{H}^1(\gamma_1, \mathfrak{A}) \cap \mathfrak{P}^{\nu+2})^* \cap a_{\gamma_1}(A)$ (i.e., with $s(d) = 0$ and $\psi_d^\times$ being trivial on $H^{\nu+2}$) such that $\theta_2 = \theta_1 \cdot \psi_d^\times$. By [G2](4.5) we find $x \in \mathfrak{P}^{q-\nu-1} \cap \mathfrak{N}_{-\nu-1}(\gamma_1, \mathfrak{A}) + \mathfrak{J}^1(\gamma_1, \mathfrak{A}) \cap \mathfrak{P}^{[\frac{q+1}{2}]}$ with $d = a_{\beta_1}(x)$ and with [G2](5.12) we obtain $\theta_2 = \theta_1^{1+x}$ and $1 + x$ normalize $\theta_{1|H^{\nu+2}}$ as required.

□

(1.16) Conjecture *Let A be a tame algebra, then the ps-characters defined in [G2](6.6) preserve intertwining of its simple character realizations (i.e. of its values on triples $(A', \iota, \mathfrak{A}')$) if restricted to all triples $(A', \iota, \mathfrak{A}')$, where ι is an embedding of an element β from a simple pair $[0, \beta]$ with maximal possible embedding invariant for $(F[\iota(\beta)], \mathfrak{A}')$ and A' is an algebra of the same reduced degree N as A.*

The evidence of this conjecture is given by the results of [BH] on local tame base change if we assume that these generalize from the split case to the mixed tame case $A = M_m(D)$.

2 Mixed groups and some group identities

Here we use the notation of [G2](0). Let $\mathfrak{A}$ be a principal $\mathfrak{o}_F$-order in A with Jacobson radical $\mathfrak{P}$, β be an $\mathfrak{A}$-sound element in the sense of Definition [G2](2.4)(iii) and $[0, \beta]$ be a simple pair in the sense of Definition [G2](3.7). In particular the stratum $[\mathfrak{A}, n, 0, \beta]$ is simple in the sense of [G2](3.1)(iii). As in Section [G2](5) we may associate the set $\mathcal{C}(\beta, \mathfrak{A})$ of simple characters with the simple stratum $[\mathfrak{A}, n, 0, \beta]$. With any fixed character $\theta_{\mathfrak{A}} \in \mathcal{C}(\beta, \mathfrak{A})$ we may associate a unique ps-character (θ, β) as in Definition [G2](6.6) characterized by $\theta(A, id_A, \mathfrak{A}) = \theta_{\mathfrak{A}}$. If we apply $\theta_{\mathfrak{A}}$ to commutators of the group $J^1(\beta, \mathfrak{A})$, by [G2](5.6) and [G2](4.1)(4.2), it induces a non-degenerate form $J^1(\beta, \mathfrak{A})/H^1(\beta, \mathfrak{A}) \times J^1(\beta, \mathfrak{A})/H^1(\beta, \mathfrak{A}) \to \mathbb{C}^\times$. Therefore, $\theta_{\mathfrak{A}}$ is a Heisenberg character in the sense of [Z2](4.2) and determines an irreducible Heisenberg representation $\eta_{\mathfrak{A}}$ of $J^1(\beta, \mathfrak{A})$. As $\theta_{\mathfrak{A}}$ is $\mathcal{K}_{\mathfrak{B}} J^1(\beta, \mathfrak{A})$-stable so is $\eta_{\mathfrak{A}}$, where $\mathcal{K}_{\mathfrak{B}}$ denotes the $B^\times$-normalizer of $\mathfrak{B}$.

Let $E := F[\beta]$, $B := C_A(E)$ and $\mathfrak{B} := \mathfrak{A} \cap B$, then $\mathfrak{B}$ is a principal order and, to each principal $\mathfrak{o}_E$-order $\mathfrak{B}_m \subset \mathfrak{B}$, we may assign a simple stratum $[\mathfrak{A}_m, n_m, 0, \beta]$, where $\mathfrak{A}_m$ is the unique continuation of $\mathfrak{B}_m$ to A and $n_m := -\nu_{\mathfrak{A}_m}(\beta)$.

We want to extend the representation $\eta_{\mathfrak{A}}$ to a representation $\eta_{\mathfrak{A},\mathfrak{B}_m}$ of $U^1(\mathfrak{B}_m)J^1(\beta, \mathfrak{A})$. If $\mathfrak{B}_m$ is a minimal principal order, then the latter group is a pro p-Sylow subgroup of $J(\beta, \mathfrak{A}) := \mathfrak{B}^\times J^1(\beta, \mathfrak{A})$. For that purpose we construct mixed characters of mixed groups to generalize the construction of Heisenberg representations with respect to the groups $H^1(\beta, \mathfrak{A})$ and $J^1(\beta, \mathfrak{A})$ to the groups $U^1(\mathfrak{B}_m)H^1(\beta, \mathfrak{A})$ and $U^1(\mathfrak{B}_m)J^1(\beta, \mathfrak{A})$. If $\mathfrak{B}_m$ is a minimal principal order, then, by a standard method for extending a representation to from a p-Sylow subgroup of a group one can extend $\eta_{\mathfrak{A},\mathfrak{B}_m}$ to a representation $\kappa_{\mathfrak{A}}$ of $J(\beta, \mathfrak{A})$. The representation $\kappa_{\mathfrak{A}}$ will then be used as a tensor factor of our conjectural simple type.

To simplify notation we set $H^1 := H^1(\beta, \mathfrak{A})$, $J^1 := J^1(\beta, \mathfrak{A})$, and $H^1_m := H^1(\beta, \mathfrak{A}_m)$ and $J^1_m := J^1(\beta, \mathfrak{A}_m)$.

We define mixed groups $J^1_{\mathfrak{A},\mathfrak{B}_m} := U^1(\mathfrak{B}_m)(J^1 \cap J^1_m)H^1$ and $H^1_{\mathfrak{A},\mathfrak{B}_m} := U^1(\mathfrak{B}_m)(J^1 \cap H^1_m)H^1$. We have to check the following lemma:

(2.1) Lemma *The sets $J^1_{\mathfrak{A},\mathfrak{B}_m}$ and $H^1_{\mathfrak{A},\mathfrak{B}_m}$ are groups and $H^1_{\mathfrak{A},\mathfrak{B}_m}$ is a normal subgroup of $J^1_{\mathfrak{A},\mathfrak{B}_m}$. The factor group is abelian and isomorphic to $J^1_{\mathfrak{A},\mathfrak{B}_m} \cap J^1 / H^1_{\mathfrak{A},\mathfrak{B}_m} \cap J^1$.*

<u>Proof:</u>

The fact that the sets $J^1_{\mathfrak{A},\mathfrak{B}_m}$ and $H^1_{\mathfrak{A},\mathfrak{B}_m}$ are groups follows from the properties of the groups H^1, J^1, H^1_m and J^1_m described in Section [G2](4).

We have $H^1_{\mathfrak{A},\mathfrak{B}_m} = U^1(\mathfrak{B}_m)(J^1 \cap H^1_m)H^1$ and $J^1_{\mathfrak{A},\mathfrak{B}_m} = U^1(\mathfrak{B}_m)(J^1 \cap J^1_m)H^1$. Clearly, H^1 normalizes itself and the commutator of $(J^1 \cap J^1_m)$ and H^1 is contained in H^1; hence, H^1 normalizes $J^1_{\mathfrak{A},\mathfrak{B}_m} \cap J^1$. Again $(J^1 \cap H^1_m)$ normalizes H^1 and the commutator of $(J^1 \cap J^1_m)$ and $(J^1 \cap H^1_m)$ is contained in H^1; hence, $(J^1 \cap J^1_m)$ normalizes $H^1_{\mathfrak{A},\mathfrak{B}_m} \cap J^1$. Finally, $U^1(\mathfrak{B}_m) \subset (\mathcal{K}_{\mathfrak{B}} \cap \mathcal{K}_{\mathfrak{B}_m})$; hence, $U^1(\mathfrak{B}_m)$ normalizes $H^1_{\mathfrak{A},\mathfrak{B}_m}$. Altogether we have proved that $H^1_{\mathfrak{A},\mathfrak{B}_m}$ is a normal subgroup of $J^1_{\mathfrak{A},\mathfrak{B}_m}$.

Since J^1/H^1 is abelian and $J^1_{\mathfrak{A},\mathfrak{B}_m} = (J^1_{\mathfrak{A},\mathfrak{B}_m} \cap J^1)H^1_{\mathfrak{A},\mathfrak{B}_m}$, we also deduce the above statement.

□

(2.2) Lemma *Let $A = M_m(D)$ be an arbitrary central simple algebra over F and K/F an unramified extension in D. Let $\mathfrak{B}$ be a principal order in $B = C_A(K) = M_m(C_D(K))$, which is given in the normal form and has period r', then the unique continuation $\mathfrak{A}$ of $\mathfrak{B}$ has period $r = \frac{r'}{\langle [K:F], r' \rangle}$ and there exists an $r' \times r'$-$(s' \times s')$ block monomial matrix g (i.e., $g = PM$, where P is an invertible $r' \times r'$-$(s' \times s')$ block monomial matrix where the entries are either 0 or the $s' \times s'$-unit matrix $I_{s'}$ and $M = diag(d_1 I_{s'}, \ldots, d_{r'} I_{s'})$, where $d_1, \ldots, d_{r'}$ are powers of a fixed prime element π of $\mathfrak{o}_D$) such that $g\mathfrak{A}g^{-1}$ is given in normal form and where $s' = \frac{m}{r'}$ is the invariant of $\mathfrak{B}$.*

<u>Proof:</u>

The proof is contained in [G1](3.2, 3.3).

□

(2.3) Lemma *Let E/F be an extension in A, $B := C_A(E)$ and $\mathfrak{B}$ be a principal order in B with period r'. Moreover, let $\mathfrak{A}$ be the unique continuation of $\mathfrak{B}$ to A, then we have $e_{\mathfrak{A}/\mathfrak{B}} = \frac{\langle [E:F], d \rangle}{\langle \langle [E:F], d \rangle, e(E/F) r' \rangle}$ in the notation of ([G2](2.5)). In particular, the continuation of $\mathfrak{B}$ by $\mathfrak{A}$ is unramified if and only if $\langle [E:F], d \rangle \mid e(E/F) r'$.*

Proof:
This is left to the reader or can be found in [G0](2.1.5).
□

(2.4) Lemma *Let E/F be an extension in A, $B = C_A(E)$ and $\mathfrak{B}$, $\mathfrak{B}'$ be principal orders in B with periods r' and $\tilde{r}'$, respectively, such that $\mathfrak{B}' \subset \mathfrak{B}$. Moreover, let $\mathfrak{A}$ and $\tilde{\mathfrak{A}}$ be the continuation of $\mathfrak{B}$ and $\mathfrak{B}'$, respectively, and assume that $\langle [E:F], d\rangle \mid e(E/F)r'$, i.e., $e_{\mathfrak{A}/\mathfrak{B}} = 1$, then we also have $\tilde{\mathfrak{A}} \subset \mathfrak{A}$.*

Proof:
Let K/F be the maximal unramified subextension of E/F such that $[K:F] \mid Ind(A) = d$, then we have $\langle f(E/K), \bar{d}\rangle = 1$, where $\bar{d} := \frac{d}{[K:F]}$ is the index of $C := C_A(K)$. Therefore, if we choose an unramified splitting field L/K of C, the algebra $E \otimes_K L$ is a field. Moreover, if $\mathfrak{C}$ and $\tilde{\mathfrak{C}}$ are the unique continuations of $\mathfrak{B}$ and $\mathfrak{B}'$, respectively, then $\mathfrak{C} \otimes_{\mathfrak{o}_K} \mathfrak{o}_L$, $\tilde{\mathfrak{C}} \otimes_{\mathfrak{o}_K} \mathfrak{o}_L$ are the unique continuations of $\mathfrak{B} \otimes_{\mathfrak{o}_K} \mathfrak{o}_L$ and $\mathfrak{B}' \otimes_{\mathfrak{o}_K} \mathfrak{o}_L$, respectively, from $B \otimes_K L$ to $C \otimes_K L$. As, by [BK1](1.2.1), the lattice chain associated with a hereditary order in a simple $B \otimes_K L$- and $C \otimes_K L$-module, respectively, is invariant under continuation, we deduce from $\mathfrak{B}' \otimes_{\mathfrak{o}_K} \mathfrak{o}_L \subset \mathfrak{B} \otimes_{\mathfrak{o}_K} \mathfrak{o}_L$ (which is valid by assumption) that $\tilde{\mathfrak{C}} \otimes_{\mathfrak{o}_K} \mathfrak{o}_L \subset \mathfrak{C} \otimes_{\mathfrak{o}_K} \mathfrak{o}_L$; hence, we have $\tilde{\mathfrak{C}} \subset \mathfrak{C}$.

Since $[K:F] \mid d$ and Skolem-Noether theorem, we may assume up to an inner automorphism of $A = M_m(D)$ that $K \subset D$ and we get $C = M_m(\Delta)$ with $\Delta := C_D(K)$. Moreover, up to conjugacy by an element of $C^\times$ we may assume that the orders $\mathfrak{C}$ and $\tilde{\mathfrak{C}}$ are given in simultaneous normal form. By [G1](2.2) the period of $\mathfrak{C}$ is $r'\frac{e(E/K)}{\langle e(E/K), \bar{d}\rangle}$; hence, by assumption the situation of [G1](3.2) and its proof apply with $t' = 1$ and for the continuation of $\mathfrak{C}$ by $\mathfrak{A}$. As the period of $\mathfrak{C}$ divides the period of $\mathfrak{C}'$, the same applies to the continuation of $\mathfrak{C}'$ by $\tilde{\mathfrak{A}}$. So let $\{\tilde{Y}_j\}_{j\in\mathbb{Z}}$ be the lattice chain of $\tilde{\mathfrak{C}}$ in the simple C-module Δ^m (column-vectors) and π a prime element of $\mathfrak{o}_D$ that normalizes K. Moreover, let $\{\tilde{X}_j\}_{j\in\mathbb{Z}}$ be the lattice chain of $\tilde{\mathfrak{A}}$ in the simple A-module D^m (column-vectors) and set $f := [K:F]$, then, by equation (7) in [G1] and with an appropriate numbering of the chains for all $j \in \mathbb{Z}$, we have $\tilde{X}_j = \oplus_{k=0}^{f-1} Y_{j-\frac{\hat{r}}{f}k}\pi^k$, where $\hat{r}$ is the period of $\tilde{\mathfrak{C}}$.

Let h be a uniformizer for $\tilde{\mathfrak{C}}$ and let μ be the quotient of the period $\hat{r}$ of $\tilde{\mathfrak{C}}$ and of the period of $\mathfrak{C}$, then h^μ is a uniformizer of $\mathfrak{C}$ and, if we choose the numbering of the chain $\{\tilde{Y}_j\}_{j\in\mathbb{Z}}$ in such a way that $\tilde{Y}_0$ is an indecomposable $\mathfrak{C}$-module, then $\{\tilde{Y}_{j\mu}\}_{j\in\mathbb{Z}}$ is the lattice chain of $\mathfrak{C}$ in Δ^m. Applying [G1](equation (7)) again, we see that $\mathfrak{A}$ has the lattice chain $\{\tilde{X}_{j\mu}\}_{j\in\mathbb{Z}}$; hence, $\tilde{\mathfrak{A}} \subset \mathfrak{A}$ as required.
□

(2.5) Lemma *Let $[\mathfrak{A}_1, n_1, 0, \beta]$, $[\mathfrak{A}_2, n_2, 0, \beta]$ be simple strata in the sense of [G2](3.1)(iii), $B = C_A(\beta)$, $\mathfrak{B}_1 = \mathfrak{A}_1 \cap B$ and $\mathfrak{B}_2 = \mathfrak{A}_2 \cap B$. Moreover, assume that there exists a minimal principal order $\mathfrak{B}_m$ contained in $\mathfrak{B}_1$ and $\mathfrak{B}_2$. Let $\mathfrak{A}_m$ be the unique continuation of $\mathfrak{B}_m$ to A, then there exist integers $q_1^{(1)}, \ldots, q_\mu^{(1)}$, $q_1^{(2)}, \ldots, q_\mu^{(2)}$ attached to each special defining sequence $[\mathfrak{A}_m, n_m, q_i^m, \gamma_i]$, $i = 1, \ldots, \mu$, such that $[\mathfrak{A}_1, n_1, q_i^{(1)}, \gamma_i]$, $i = 1, \ldots, \mu$, $[\mathfrak{A}_2, n_2, q_i^{(2)}, \gamma_i]$, $i = 1, \ldots, \mu$, are special approximation sequences for $[\mathfrak{A}_1, n_1, 0, \beta]$ and $[\mathfrak{A}_2, n_2, 0, \beta]$, respectively.*

Proof:
If h_m, h_1, h_2 are uniformizers for $\mathfrak{B}_m$, $\mathfrak{B}_1$, $\mathfrak{B}_2$, (i.e., $\mathcal{K}_{\mathfrak{B}_m} = \langle h_m\rangle \mathfrak{B}_m^\times$, $\mathcal{K}_{\mathfrak{B}_1} = \langle h_1\rangle \mathfrak{B}_1^\times$, $\mathcal{K}_{\mathfrak{B}_2} = \langle h_2\rangle \mathfrak{B}_2^\times$), then there exist natural numbers t_1, t_2 such that $h_m^{t_1} = h_1 b_1$ and $h_m^{t_2} = h_2 b_2$, where $b_1 \in \mathfrak{B}_1^\times$ and $b_2 \in \mathfrak{B}_2^\times$.

Let $\mathfrak{C}_m$, $\mathfrak{C}_1$, $\mathfrak{C}_2$ be the unique continuations of $\mathfrak{B}_m$, $\mathfrak{B}_1$, $\mathfrak{B}_2$ to $C := C_A(K)$, where $K = E^{ur}$ is the unramified part of the extension E/F and $E := F[\beta]$. By [G](2.2) these continuations are unramified; hence, h_m, h_1, h_2 are also uniformizers for $\mathfrak{C}_m$, $\mathfrak{C}_1$, $\mathfrak{C}_2$. By the given containments of orders, (2.4) and [BG](5.1)(ii), the strata $[\mathfrak{C}_1, -\nu_{\mathfrak{C}_1}(\beta), 0, \beta]$, $[\mathfrak{C}_2, -\nu_{\mathfrak{C}_2}(\beta), 0, \beta]$ and $[\mathfrak{C}_m, -\nu_{\mathfrak{C}_m}(\beta), 0, \beta]$ are simple.

Note that $\mathfrak{A}_m$ is the unique continuation of $\mathfrak{C}_m$ to A and let $[\mathfrak{A}_m, n_m, q_i^m, \gamma_i]$, $i = 1, \dots, \mu$, be a special approximation sequence as in [G2](3.11) for $[\mathfrak{A}_m, -\nu_{\mathfrak{A}_m}(\beta), 0, \beta]$, then there exist integers $k_1^m, \dots, k_\mu^m$ such that

(2.6) $\gamma_i = h_m^{k_i^m} c_i$,

where $c_i \in \mathfrak{C}_m^\times$ for $i = 1, \dots, \mu$. Finally, $\nu_{\mathfrak{C}_m}(\beta) = \nu_{\mathfrak{C}_m}(\gamma_i)$ for $i = 1, \dots, \mu$, we actually have $k_1^m = \dots = k_\mu^m =: k^m$. Moreover, since the representation (2.6) is unique, $\mathfrak{C}_m^\times \subset \mathfrak{C}_1^\times, \mathfrak{C}_2^\times$ and $\beta \in \mathcal{K}_{\mathfrak{C}_1}, \mathcal{K}_{\mathfrak{C}_2}$, we conclude that $t_1, t_2 \mid k^m$; hence, $\gamma_1, \dots, \gamma_\mu$ must be $\mathfrak{C}_1$- and $\mathfrak{C}_2$-pure. From the remarks following [G2](3.9) we deduce that, with $q_1^{(1)} = q_1^{(2)} = 0$, $q_\mu^{(1)} = -\nu_{\mathfrak{A}_1}(\beta)$, $q_\mu^{(2)} = -\nu_{\mathfrak{A}_2}(\beta)$ and $q_{i+1}^{(1)} = -k_0(\gamma_i, \mathfrak{A}_1)$, $q_{i+1}^{(2)} = -k_0(\gamma_i, \mathfrak{A}_2)$ for $i = 1, \dots, \mu - 1$, we obtain defining sequences $[\mathfrak{A}_1, -\nu_{\mathfrak{A}_1}(\beta), q_i^{(1)}, \gamma_i]$, $[\mathfrak{A}_2, -\nu_{\mathfrak{A}_2}(\beta), q_i^{(2)}, \gamma_i]$, $i = 1, \dots, \mu$, for the simple strata $[\mathfrak{A}_1, -\nu_{\mathfrak{A}_1}(\beta), 0, \beta]$, $[\mathfrak{A}_2, -\nu_{\mathfrak{A}_2}(\beta), 0, \beta]$ as required, because they are common realizations of the sequence of simple pairs $[0, \beta]$, $[-k_0(\gamma_i, \mathfrak{A}(F[\gamma_i])), \gamma_i]$ for $i = 1, \dots, \mu - 1$, and by straightforward arguments using [BG](5.1)(ii) these approximation sequences are special.

□

(2.7) Lemma *Let $\nu, \bar\nu, \tilde\nu$ be non-negative integers, let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum and $\mathfrak{B}_m$ a principal order contained in $\mathfrak{B} := \mathfrak{A} \cap C_A(\beta)$. Moreover, let $\mathfrak{P}$ be the Jacobson radical of $\mathfrak{A}$, $\mathfrak{A}_m$ be the unique continuation of $\mathfrak{B}_m$, $\mathfrak{P}_m$ be the Jacobson radical of $\mathfrak{A}_m$, and let $[\mathfrak{A}, n, q_i, \gamma_i]$, $[\mathfrak{A}_m, n', q_i', \gamma_i]$ with $i = 1, \dots, \mu$ be simultaneous special approximation sequences for the strata $[\mathfrak{A}, n, 0, \beta]$, $[\mathfrak{A}_m, n', 0, \beta]$ as in (2.5), where $n' := \nu_{\mathfrak{A}_m}(\beta)$. Then we have*

(i) $(\mathfrak{P}^\nu + \mathfrak{P}_m^{\bar\nu}) \cap C_A(\gamma_i) = (\mathfrak{P}^\nu \cap C_A(\gamma_i) + \mathfrak{P}_m^{\bar\nu} \cap C_A(\gamma_i))$ and

(ii) $(\mathfrak{P}^\nu \cap \mathfrak{P}_m^{\bar\nu} + \mathfrak{P}^{\tilde\nu}) \cap C_A(\gamma_i) = \mathfrak{P}^\nu \cap \mathfrak{P}_m^{\bar\nu} \cap C_A(\gamma_i) + \mathfrak{P}^{\tilde\nu} \cap C_A(\gamma_i)$

for $i = 1, \dots, \mu$.

Proof:

Let K_k/F be the maximal unramified subextension of $F[\gamma_k]/F$ with respect to the property that $[K_k : F] \mid d$ for $k = 1, \dots, \mu$. Let us fix an index k. Set $\mathfrak{C} := \mathfrak{A} \cap C_A(K_k)$, $\mathfrak{C}_m := \mathfrak{A}_m \cap C_A(K_k)$, $\mathfrak{B} := \mathfrak{A} \cap C_A(\gamma_k)$ and $\mathfrak{A} \cap C_A(\gamma_k)$.

The algebra $C_A(K_k)/K_k$ has index $\frac{d}{[K_k:F]}$ and therefore, by (2.3),(2.4), the continuations of $\mathfrak{B}, \mathfrak{B}_m$ by $\mathfrak{C}$,$\mathfrak{C}_m$ are unramified and $\mathfrak{C} \supset \mathfrak{C}_m$. Moreover, up to conjugacy, we may assume that $K_k \subset D$ and that $\mathfrak{C}, \mathfrak{C}_m$ are in simultaneous normal form. By (2.2) we may find monomial matrices with entries powers of the same prime element π of $\mathfrak{o}_D$ such that $\mathfrak{A}$ and $\mathfrak{A}_m$ are conjugated by them to their normal forms in $A = M_m(D)$. Consequently, we have $\mathfrak{P}^\nu = (\mathfrak{p}_D^{\nu_{i,j}})_{i,j=1,\dots,m}$, $\mathfrak{P}_m^{\bar\nu} = (\mathfrak{p}_D^{\bar\nu_{i,j}})_{i,j=1,\dots,m}$ and $\mathfrak{P}^{\tilde\nu} = (\mathfrak{p}_D^{\tilde\nu_{i,j}})_{i,j=1,\dots,m}$, where these representations mean $m \times m$-matrices where the (i,j) entry lies in $\mathfrak{p}_D^{\nu_{i,j}}$, $\mathfrak{p}_D^{\bar\nu_{i,j}}$ and $\mathfrak{p}_D^{\tilde\nu_{i,j}}$, respectively. If we set $D_k := C_D(K_k)$, we obtain $(\mathfrak{P}^\nu + \mathfrak{P}_m^{\bar\nu}) \cap C_A(K_k) = (\mathfrak{p}_{D_k}^{\eta_{i,j}})_{i,j=1,\dots,m} = \mathfrak{P}^\nu \cap C_A(K_k) + \mathfrak{P}_m^{\bar\nu} \cap C_A(K_k)$, where $\eta_{i,j} = \sigma_{\mathfrak{o}_D/\mathfrak{o}_{D_k}}(max(\nu_{i,j}, \bar\nu_{i,j})) = max(\sigma_{\mathfrak{o}_D/\mathfrak{o}_{D_k}}(\nu_{i,j}), \sigma_{\mathfrak{o}_D/\mathfrak{o}_{D_k}}(\bar\nu_{i,j}))$ and $(\mathfrak{P}^\nu \cap \mathfrak{P}_m^{\bar\nu} + \mathfrak{P}^{\tilde\nu}) \cap C_A(K_k) = (\mathfrak{p}_{D_k}^{\hat\eta_{i,j}})_{i,j=1,\dots,m} = (\mathfrak{P}^\nu \cap \mathfrak{P}_m^{\bar\nu} \cap C_A(K_k) + \mathfrak{P}^{\tilde\nu} \cap C_A(K_k))$, where $\hat\eta_{i,j} = \sigma_{\mathfrak{o}_D/\mathfrak{o}_{D_k}}(max(min(\nu_{i,j}, \bar\nu_{i,j}), \tilde\nu_{i,j})) = max(min(\sigma_{\mathfrak{o}_D/\mathfrak{o}_{D_k}}(\nu_{i,j}), \sigma_{\mathfrak{o}_D/\mathfrak{o}_{D_k}}(\bar\nu_{i,j})), \sigma_{\mathfrak{o}_D/\mathfrak{o}_{D_k}}(\tilde\nu_{i,j}))$.

Therefore, we restrict ourselves to the case of $K_k = F$ and where $F[\gamma_i] \otimes_F L$ is a field for the unramified extension L/F of degree d which splits A. The continuations of $\mathfrak{B}$ and $\mathfrak{B}_m$ by $\mathfrak{A}$ and $\mathfrak{A}_m$ are unramified.

With $\bar{A} := A \otimes_F L$, $\bar{B} := C_{\bar{A}}(\gamma_k \otimes 1)$, $\bar{\mathfrak{A}} := \mathfrak{A} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L$, $\bar{\mathfrak{A}}_m := \mathfrak{A}_m \otimes_{\mathfrak{o}_F} \mathfrak{o}_L$, $\bar{\mathfrak{B}} := \mathfrak{B} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L$ and $\bar{\mathfrak{B}}_m := \mathfrak{B}_m \otimes_{\mathfrak{o}_F} \mathfrak{o}_L$, according to [BK1](1.2.6)(1.2.8), we obtain isomorphisms $\bar{A} \simeq A(F[\gamma_k] \otimes_L \bar{B}$, $\bar{\mathfrak{A}} \simeq \mathfrak{A}(F[\gamma_k]) \otimes_{\mathfrak{o}_L} \bar{\mathfrak{B}}$ and $\bar{\mathfrak{A}}_m \simeq \mathfrak{A}(F[\gamma_k]) \otimes \bar{\mathfrak{B}}_m$, where $A(F[\gamma_k) := End_F(F[\gamma_k])$ and $\mathfrak{A}(\gamma_k) := End_{\mathfrak{o}_F}(\{\mathfrak{p}^j_{F[\gamma_k]}\}_{j\in\mathbb{Z}})$.

By [BK1](1.2.10) the powers of the Jacobson radicals $\bar{\mathfrak{P}} := \mathfrak{P} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L$, $\bar{\mathfrak{P}}_m := \mathfrak{P}_m \otimes_{\mathfrak{o}_F} \mathfrak{o}_L$ of $\bar{\mathfrak{A}}$ and $\bar{\mathfrak{A}}_m$ fulfil corresponding isomorphisms and are, by [BK1](1.3.12), $F[\gamma_k \otimes 1]$-exact in the sense of Definition [BK1](1.3.11) for the tame corestriction $s := s_{\gamma_k} \otimes 1 : \bar{A} \to \bar{B}$, defined via a tame corestriction $s_{\gamma_k} : A \to B$.

From [BK1](1.3.12)(1.3.10) we deduce $(\bar{\mathfrak{P}}^\nu + \bar{\mathfrak{P}}_m^{\bar{\nu}}) \cap \bar{B} = s(\bar{\mathfrak{P}}^\nu + \bar{\mathfrak{P}}_m^{\bar{\nu}}) = s(\bar{\mathfrak{P}}^\nu) + s(\bar{\mathfrak{P}}_m^{\bar{\nu}}) = (\bar{\mathfrak{P}}^\nu \cap \bar{B} + \bar{\mathfrak{P}}_m^{\bar{\nu}} \cap \bar{B})$ and $(\bar{\mathfrak{P}}^\nu \cap \bar{\mathfrak{P}}_m^{\bar{\nu}} + \bar{\mathfrak{P}}^{\tilde{\nu}}) \cap \bar{B} = s(\bar{\mathfrak{P}}^\nu \cap \bar{\mathfrak{P}}_m^{\bar{\nu}} + \bar{\mathfrak{P}}^{\tilde{\nu}}) = s(\bar{\mathfrak{P}}^\nu \cap \bar{\mathfrak{P}}_m^{\bar{\nu}}) + s(\bar{\mathfrak{P}}^{\tilde{\nu}}) = (\bar{\mathfrak{P}}^\nu \cap \bar{\mathfrak{P}}_m^{\bar{\nu}} \cap \bar{B} + \bar{\mathfrak{P}}^{\tilde{\nu}} \cap \bar{B})$. The assertion of our Lemma now follows by taking, on both sides of the equations, the sets of Galois invariant elements with respect to the action of $Gal(L/F)$ on the second tensor factor of $A \otimes_F L$.

□

(2.8) Lemma *Let $\mathfrak{A}$, $\mathfrak{A}_m$ and $\mathfrak{B}_m$ be as above. Moreover, choose a sequence of elements $\gamma_1, ..., \gamma_\mu$ such that the sequences $[\mathfrak{A}, n, q_i, \gamma_i]$; $[\mathfrak{A}_m, n_m, q_i', \gamma_i]$, $i = 1, \ldots, \mu$, are special approximation sequences for the strata $[\mathfrak{A}, n, 0, \beta]$ and $[\mathfrak{A}_m, n_m, 0, \beta]$, respectively, where $n_m := -\nu_{\mathfrak{A}_m}(\beta)$, $n := -\nu_{\mathfrak{A}}(\beta)$ and $\gamma_1, \ldots, \gamma_\mu$; $q_1, \ldots, q_\mu$; $q_1', \ldots, q_\mu'$ are suitably chosen according to (2.5).*

To simplify the notation set $J^1 =: 1 + \mathcal{J}^1$, $J_m^1 =: 1 + \mathcal{J}_m^1$, $H^1 =: 1 + \mathcal{H}^1$ and $H_m^1 =: 1 + \mathcal{H}_m^1$. Then we have

$$\mathfrak{P}^{[\frac{q_i+1}{2}]} \cap \mathcal{J}^1 \cap \mathfrak{P}_m^{[\frac{q_i'+1}{2}]} \cap \mathcal{J}_m^1 = \sum_{j=i}^{\mu} \mathfrak{P}^{[\frac{q_j+1}{2}]} \cap \mathfrak{P}_m^{[\frac{q_j'+1}{2}]} \cap C_A(\gamma_j),$$

$$\mathfrak{P}^{[\frac{q_i+1}{2}]} \cap \mathcal{J}^1 \cap \mathfrak{P}_m^{[\frac{q_i'}{2}]+1} \cap \mathcal{H}_m^1 = \sum_{j=i}^{\mu} \mathfrak{P}^{[\frac{q_j+1}{2}]} \cap \mathfrak{P}_m^{[\frac{q_j'}{2}]+1} \cap C_A(\gamma_j)$$

and

$$\mathfrak{P}^{[\frac{q_i}{2}]+1} \cap \mathcal{H}^1 \cap \mathfrak{P}_m^{[\frac{q_i'}{2}]+1} \cap \mathcal{H}_m^1 = \sum_{j=i}^{\mu} \mathfrak{P}^{[\frac{q_j}{2}]+1} \cap \mathfrak{P}_m^{[\frac{q_j'}{2}]+1} \cap C_A(\gamma_j)$$

for $i = 1, \ldots, \mu$. In particular, for $i = 1$ we obtain the left-hand sides $\mathfrak{J}^1 \cap \mathfrak{J}_m^1$, $\mathfrak{J}^1 \cap \mathfrak{H}_m^1$ and $\mathfrak{H}^1 \cap \mathfrak{H}_m^1$.

<u>Proof:</u>

As all these assertions are similar to prove, we restrict ourselves to the first case. With the notation of ([G2](4.1)), we have $\mathcal{J}^1 = \mathcal{J}^1(\beta, \mathfrak{A})$ and $\mathcal{J}_m^1 = \mathcal{J}^1(\beta, \mathfrak{A}_m)$. Moreover, from ([G2](4.3)), we easily deduce

$$\mathfrak{P}^{[\frac{q_i+1}{2}]} \cap \mathcal{J}^1 = \mathfrak{P}^{[\frac{q_i+1}{2}]} \cap \mathcal{J}^1(\gamma_i, \mathfrak{A}) = \sum_{j=i}^{\mu} \mathfrak{P}^{[\frac{q_j+1}{2}]} \cap C_A(\gamma_j) = \mathfrak{P}^{[\frac{q_i+1}{2}]} \cap C_A(\gamma_i) + \mathfrak{P}^{[\frac{q_{i+1}+1}{2}]} \cap \mathcal{J}^1 \quad (1)$$

and

$$\mathfrak{P}_m^{[\frac{q_i'+1}{2}]} \cap \mathcal{J}_m^1 = \mathfrak{P}_m^{[\frac{q_i'+1}{2}]} \cap \mathcal{J}^1(\gamma_i, \mathfrak{A}_m) = \sum_{j=i}^{\mu} \mathfrak{P}_m^{[\frac{q_j'+1}{2}]} \cap C_A(\gamma_j) = \mathfrak{P}_m^{[\frac{q_i'+1}{2}]} \cap C_A(\gamma_i) + \mathfrak{P}_m^{[\frac{q_{i+1}'+1}{2}]} \cap \mathcal{J}_m^1. \quad (2)$$

Using induction over the approximation sequence, it is sufficient to show that

$$(\mathfrak{P}^{[\frac{q_i+1}{2}]}\cap C_A(\gamma_i)+\mathfrak{P}^{[\frac{q_{i+1}+1}{2}]}\cap\mathcal{J}^1)\cap(\mathfrak{P}_m^{[\frac{q_i'+1}{2}]}\cap C_A(\gamma_i)+\mathfrak{P}_m^{[\frac{q_{i+1}'+1}{2}]}\cap\mathcal{J}_m^1)=$$
$$\mathfrak{P}^{[\frac{q_i+1}{2}]}\cap\mathfrak{P}_m^{[\frac{q_i'+1}{2}]}\cap C_A(\gamma_i)+\mathfrak{P}^{[\frac{q_{i+1}+1}{2}]}\cap\mathcal{J}^1\cap\mathfrak{P}_m^{[\frac{q_{i+1}'+1}{2}]}\cap\mathcal{J}_m^1. \tag{3}$$

The proof is now the same as that of Lemma [BK1](3.6.10)(p. 140,141) in the split case and we give a more detailed proof for our different situation.

We also consider the adjoint map $a_{\gamma_i}\in End_F(V)$ of [G2](2) or [BK](1.4) and note that $ker(a_{\gamma_i})=C_A(\gamma_i)$. Let L_l be the $\mathfrak{o}_F$-lattice on the left-hand side of the equation (3) and L_r be the $\mathfrak{o}_F$-lattice on the right-hand side of equation (3), then we obviously have $L_l\supset L_r$ and $a_{\gamma_i}(L_l)\simeq L_l/L_l\cap ker(a_{\gamma_i})\supset L_r/L_r\cap ker(a_{\gamma_i})\simeq a_{\gamma_i}(L_r)$. Since $L_l\cap ker(a_{\gamma_i})=L_l\cap C_A(\gamma_i)=\mathfrak{P}^{[\frac{q_i+1}{2}]}\cap\mathfrak{P}_m^{[\frac{q_i'+1}{2}]}\cap C_A(\gamma_i)=L_r\cap C_A(\gamma_i)=L_r\cap ker(a_{\gamma_i})$, it is sufficient to show that $a_{\gamma_i}(L_l)=a_{\gamma_i}(L_r)$.

We have

$$a_{\gamma_i}(\mathfrak{P}^{[\frac{q_{i+1}+1}{2}]}\cap\mathcal{J}^1\cap\mathfrak{P}_m^{[\frac{q_{i+1}'+1}{2}]}\cap\mathcal{J}_m^1)=a_{\gamma_i}(L_r)\subseteq$$

$$a_{\gamma_i}(L_l)\subseteq a_{\gamma_i}(\mathfrak{P}^{[\frac{q_i+1}{2}]}\cap C_A(\gamma_i)+\mathfrak{P}^{[\frac{q_{i+1}+1}{2}]}\cap\mathcal{J}^1)\cap a_{\gamma_i}(\mathfrak{P}_m^{[\frac{q_i'+1}{2}]}\cap C_A(\gamma_i)+\mathfrak{P}_m^{[\frac{q_{i+1}'+1}{2}]}\cap\mathcal{J}_m^1)=$$

$$a_{\gamma_i}(\mathfrak{P}^{[\frac{q_{i+1}+1}{2}]}\cap\mathcal{J}^1)\cap a_{\gamma_i}(\mathfrak{P}_m^{[\frac{q_{i+1}'+1}{2}]}\cap\mathcal{J}_m^1).$$

Hence, it only remains to prove

$$a_{\gamma_i}(\mathfrak{P}^{[\frac{q_{i+1}+1}{2}]}\cap\mathcal{J}^1\cap\mathfrak{P}_m^{[\frac{q_{i+1}'+1}{2}]}\cap\mathcal{J}_m^1)=a_{\gamma_i}(\mathfrak{P}^{[\frac{q_{i+1}+1}{2}]}\cap\mathcal{J}^1)\cap a_{\gamma_i}(\mathfrak{P}_m^{[\frac{q_{i+1}'+1}{2}]}\cap\mathcal{J}_m^1).$$

We use the general fact following [BK1](3.6.17), which yields

$$a_{\gamma_i}(\mathfrak{P}^{[\frac{q_{i+1}+1}{2}]}\cap\mathcal{J}^1)\cap a_{\gamma_i}(\mathfrak{P}_m^{[\frac{q_{i+1}'+1}{2}]}\cap\mathcal{J}_m^1)/a_{\gamma_i}(\mathfrak{P}^{[\frac{q_{i+1}+1}{2}]}\cap\mathcal{J}^1\cap\mathfrak{P}_m^{[\frac{q_{i+1}'+1}{2}]}\cap\mathcal{J}_m^1)\simeq$$

$$(\mathfrak{P}^{[\frac{q_{i+1}+1}{2}]}\cap\mathcal{J}^1+\mathfrak{P}_m^{[\frac{q_{i+1}'+1}{2}]}\cap\mathcal{J}_m^1)\cap ker(a_{\gamma_i})/(\mathfrak{P}^{[\frac{q_{i+1}+1}{2}]}\cap\mathcal{J}^1\cap ker(a_{\gamma_i})+\mathfrak{P}_m^{[\frac{q_{i+1}'+1}{2}]}\cap\mathcal{J}_m^1\cap ker(a_{\gamma_i}))$$

and we are reduced to showing that

$$(\mathfrak{P}^{[\frac{q_{i+1}+1}{2}]}\cap\mathcal{J}^1+\mathfrak{P}_m^{[\frac{q_{i+1}'+1}{2}]}\cap\mathcal{J}_m^1)\cap C_A(\gamma_i)=\mathfrak{P}^{[\frac{q_{i+1}+1}{2}]}\cap\mathcal{J}^1\cap C_A(\gamma_i)+\mathfrak{P}_m^{[\frac{q_{i+1}'+1}{2}]}\cap\mathcal{J}_m^1\cap C_A(\gamma_i).$$

On the other hand, we have

$$\mathfrak{P}^{[\frac{q_{i+1}+1}{2}]}\cap C_A(\gamma_i)+\mathfrak{P}_m^{[\frac{q_{i+1}'+1}{2}]}\cap C_A(\gamma_i)\subseteq$$

$$\mathfrak{P}^{[\frac{q_{i+1}+1}{2}]}\cap\mathcal{J}^1\cap C_A(\gamma_i)+\mathfrak{P}_m^{[\frac{q_{i+1}'+1}{2}]}\cap\mathcal{J}_m^1\cap C_A(\gamma_i)\subseteq$$

$$(\mathfrak{P}^{[\frac{q_{i+1}+1}{2}]}\cap\mathcal{J}^1+\mathfrak{P}_m^{[\frac{q_{i+1}'+1}{2}]}\cap\mathcal{J}_m^1)\cap C_A(\gamma_i)\subseteq$$

$$(\mathfrak{P}^{[\frac{q_{i+1}+1}{2}]}+\mathfrak{P}_m^{[\frac{q_{i+1}'+1}{2}]})\cap C_A(\gamma_i),$$

where the first and the last term are equal by (2.7)(i), and we are done.

□

(2.9) Lemma *With the assumptions and notations of (2.8) and $i \in \{1, \ldots, \mu\}$ we have*

$$J^1 \cap J^1_m = (\prod_{\nu=1}^{i-1}(1+\mathfrak{P}^{[\frac{q_\nu+1}{2}]} \cap \mathfrak{P}_m^{[\frac{q'_\nu+1}{2}]} \cap C_A(\gamma_\nu)))(U^{[\frac{q_i+1}{2}]}(\mathfrak{A}) \cap J^1 \cap U^{[\frac{q'_i+1}{2}]}(\mathfrak{A}_m) \cap J^1_m),$$

$$H^1 \cap H^1_m = (\prod_{\nu=1}^{i-1}(1+\mathfrak{P}^{[\frac{q_\nu}{2}]+1} \cap \mathfrak{P}_m^{[\frac{q'_\nu}{2}]+1} \cap C_A(\gamma_\nu)))(U^{[\frac{q_i}{2}]+1}(\mathfrak{A}) \cap H^1 \cap U^{[\frac{q'_i}{2}]+1}(\mathfrak{A}_m) \cap H^1_m)$$

and

$$J^1 \cap H^1_m = (\prod_{\nu=1}^{i-1}(1+\mathfrak{P}^{[\frac{q_\nu+1}{2}]} \cap \mathfrak{P}_m^{[\frac{q'_\nu}{2}]+1} \cap C_A(\gamma_\nu)))(U^{[\frac{q_i+1}{2}]}(\mathfrak{A}) \cap J^1 \cap U^{[\frac{q'_i}{2}]+1}(\mathfrak{A}_m) \cap H^1_m).$$

Proof:

The proofs of the first and of the other assertions are essentially the same, so that we restrict ourselves to the first one. For $i = 1$ there is nothing to prove. Assuming the assertion for a fixed $i \in \{1, \ldots, \mu-1\}$, we have to show for the proof of the assertion for $i+1$ that $U^{[\frac{q_i+1}{2}]}(\mathfrak{A}) \cap J^1 \cap U^{[\frac{q'_i+1}{2}]}(\mathfrak{A}_m) \cap J^1_m = (1+\mathfrak{P}^{[\frac{q_i+1}{2}]} \cap \mathfrak{P}_m^{[\frac{q'_i+1}{2}]} \cap C_A(\gamma_i)))(U^{[\frac{q_{i+1}+1}{2}]}(\mathfrak{A}) \cap J^1 \cap U^{[\frac{q'_{i+1}+1}{2}]}(\mathfrak{A}_m) \cap J^1_m)$.

Using (2.8) and [G2](4.3), we have $U^{[\frac{q_i+1}{2}]}(\mathfrak{A}) \cap J^1 \cap U^{[\frac{q'_i+1}{2}]}(\mathfrak{A}_m) \cap J^1_m = U^{[\frac{q_i+1}{2}]}(\mathfrak{A}) \cap J^1(\gamma_i, \mathfrak{A}) \cap U^{[\frac{q'_i+1}{2}]}(\mathfrak{A}_m) \cap J^1(\gamma_i, \mathfrak{A}_m) = 1 + \sum_{\nu=i}^{\mu} \mathfrak{P}^{[\frac{q_\nu+1}{2}]} \cap \mathfrak{P}_m^{[\frac{q'_\nu+1}{2}]} \cap C_A(\gamma_\nu) = 1 + \mathfrak{P}^{[\frac{q_i+1}{2}]} \cap \mathfrak{P}_m^{[\frac{q'_i+1}{2}]} \cap C_A(\gamma_i) + \mathfrak{P}^{[\frac{q_{i+1}+1}{2}]} \cap \mathcal{J}^1 \cap \mathfrak{P}_m^{[\frac{q'_{i+1}+1}{2}]} \cap \mathcal{J}^1_m$.

Therefore, it remains to show that $1 + \mathfrak{P}^{[\frac{q_i+1}{2}]} \cap \mathfrak{P}_m^{[\frac{q'_i+1}{2}]} \cap C_A(\gamma_i) + \mathfrak{P}^{[\frac{q_{i+1}+1}{2}]} \cap \mathcal{J}^1 \cap \mathfrak{P}_m^{[\frac{q'_{i+1}+1}{2}]} \cap \mathcal{J}^1_m = (1 + \mathfrak{P}^{[\frac{q_i+1}{2}]} \cap \mathfrak{P}_m^{[\frac{q'_i+1}{2}]} \cap C_A(\gamma_i))(1 + \mathfrak{P}^{[\frac{q_{i+1}+1}{2}]} \cap \mathcal{J}^1 \cap \mathfrak{P}_m^{[\frac{q'_{i+1}+1}{2}]} \cap \mathcal{J}^1_m)$. By [G2](4.3), $\mathfrak{P}^{[\frac{q_{i+1}+1}{2}]} \cap \mathcal{J}^1 = \mathfrak{P}^{[\frac{q_{i+1}+1}{2}]} \cap \mathcal{J}^1(\gamma_i, \mathfrak{A})$ and $\mathfrak{P}_m^{[\frac{q'_i+1}{2}]} \cap \mathcal{J}^1_m = \mathfrak{P}_m^{[\frac{q'_i+1}{2}]} \cap \mathcal{J}^1(\gamma_i, \mathfrak{A}_m)$. We deduce that $\mathfrak{P}^{[\frac{q_{i+1}+1}{2}]} \cap \mathcal{J}^1 \cap \mathfrak{P}_m^{[\frac{q'_{i+1}+1}{2}]} \cap \mathcal{J}^1_m$ is a $(\mathfrak{P}^{[\frac{q_i+1}{2}]} \cap \mathfrak{P}_m^{[\frac{q'_i+1}{2}]} \cap C_A(\gamma_i), \mathfrak{P}^{[\frac{q_i+1}{2}]} \cap \mathfrak{P}_m^{[\frac{q'_i+1}{2}]} \cap C_A(\gamma_i))$-bimodule; hence, the containment $\supseteq$ is clear.

For the reverse containment, let $x \in \mathfrak{P}^{[\frac{q_i+1}{2}]} \cap \mathfrak{P}_m^{[\frac{q'_i+1}{2}]} \cap C_A(\gamma_i)$ and $y \in \mathfrak{P}^{[\frac{q_{i+1}+1}{2}]} \cap \mathcal{J}^1 \cap \mathfrak{P}_m^{[\frac{q'_{i+1}+1}{2}]} \cap \mathcal{J}^1_m$. Then $1 + x + y = (1+x)(1+(1+x)^{-1}y) = (1+x)(1+\sum_{\nu=0}^{\infty}(-x)^\nu y) \in (1 + \mathfrak{P}^{[\frac{q_i+1}{2}]} \cap \mathfrak{P}_m^{[\frac{q'_i+1}{2}]} \cap C_A(\gamma_i))(1 + \mathfrak{P}^{[\frac{q_{i+1}+1}{2}]} \cap \mathcal{J}^1 \cap \mathfrak{P}_m^{[\frac{q'_{i+1}+1}{2}]} \cap \mathcal{J}^1_m)$ as required.

□

3 A property of matching in a special situation

(3.1) Lemma *Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum and $\theta_{\mathfrak{A}} \in \mathcal{C}(\beta, \mathfrak{A})$ be a simple character. Then, for each defining sequence $\{(\gamma_\nu, \lambda_{\gamma_\nu})\}_{\nu=1,\ldots,\mu}$, with underlying special approximation sequence, we obtain an explicit formula for $\theta_{\mathfrak{A}}$ in the following sense:*

If $1 + x \in H^1(\beta, \mathfrak{A})$, $i \in \{1, \ldots, \mu\}$ and $1 + x = (\prod_{\nu=1}^{i-1} 1 + x_\nu)(1 + x'_i)$, where $x_\nu \in \mathfrak{P}^{[\frac{q_\nu}{2}]+1} \cap C_A(\gamma_\nu)$ for $\nu = 1, \ldots, i-1$ and $x'_i \in \mathfrak{P}^{[\frac{q_i}{2}]+1} \cap \mathfrak{H}^1(\gamma_i, \mathfrak{A})$, then we have

$$\theta_{\mathfrak{A}}(1+x) = (\prod_{\nu=1}^{i-1} \lambda_{\gamma_\nu} \circ Nrd_{C_A(\gamma_\nu)/F[\gamma_\nu]}(1+x_\nu)\psi^+_{\beta-\gamma_\nu}(x_\nu))\psi^+_{\beta-\gamma_i}(x'_i)\theta_{\gamma_i}(1+x'_i),$$

where $\theta_{\gamma_i} \in \mathcal{C}(\gamma_i, \mathfrak{A})$ is the unique simple character defined by the sequence $\{(\gamma_\nu, \lambda_{\gamma_\nu})\}_{\nu=i,\ldots,\mu}$ (in particular, we set $\theta_{\gamma_\mu} \equiv 1$).

<u>Proof:</u>

Our assertion is trivial for $i = 1$. So let us assume that the assertion holds for $\mu > i \geq 1$ and write $(1 + x_i') = (1 + x_i)(1 + x_{i+1}')$ according to a product representation of $H^1(\beta, \mathfrak{A})$, which we may easily deduce as in (2.9), where $x_i \in \mathfrak{P}^{[\frac{q_i}{2}]+1} \cap C_A(\gamma_i)$ and $x_{i+1}' \in \mathfrak{P}^{[\frac{q_{i+1}}{2}]+1} \cap \mathfrak{H}^1(\gamma_{i+1}, \mathfrak{A})$.

Since $([\frac{q_i}{2}] + 1) + ([\frac{q_{i+1}}{2}] + 1) \geq q_i + 1$ and $\beta - \gamma_i \in \mathfrak{P}^{-q_i}$, we obtain that $\psi^+_{\beta-\gamma_i}(x_i') = \psi^+_{\beta-\gamma_i}(x_i + x_i x_{i+1}' + x_{i+1}') = \psi^+_{\beta-\gamma_i}(x_i + x_{i+1}')$. By definition of simple characters, we also know that $\theta_{\gamma_i}(1 + x_i') = \lambda_{\gamma_i} \circ Nrd_{C_A(\gamma_i)/F[\gamma_i]}(1 + x_i)\psi^+_{\gamma_i - \gamma_{i+1}}(x_{i+1}')\theta_{\mathfrak{A}}(1 + x_{i+1}')$ and the induction step follows from $\psi^+_{\beta-\gamma_i}(x_{i+1}')\psi^+_{\gamma_i-\gamma_{i+1}}(x_{i+1}') = \psi^+_{\beta-\gamma_{i+1}}(x_{i+1}')$.
□

The property of matching as introduced in [G2](6.6) is a generalization of the existence part of [BK1](3.6.1) in a special case:

(3.2) Corollary *With the assumptions and notations of (2.5) let $\theta_1 \in \mathcal{C}(\beta, \mathfrak{A}_1)$ and $\theta_2 \in \mathcal{C}(\beta, \mathfrak{A}_2)$ such that $\theta_1 = \theta(A, id_A, \mathfrak{A}_1)$ and $\theta_2 = \theta(A, id_A, \mathfrak{A}_2)$, where (θ, β) is a ps-character. Then $(\theta_1)_{|H^1(\beta,\mathfrak{A}_1)\cap H^1(\beta,\mathfrak{A}_2)} = (\theta_2)_{|H^1(\beta,\mathfrak{A}_1)\cap H^1(\beta,\mathfrak{A}_2)}$.*

<u>Proof:</u>

Choose simultaneous special approximation sequences for $[\mathfrak{A}(E), -\nu_E(\beta), 0, \beta]$ and $[\mathfrak{A}_m, n_m, 0, \beta]$ that fulfil the assumptions of Theorem ([G2](6.6)). By 2.5 we may find common special approximation sequences for $[\mathfrak{A}_1, -\nu_{\mathfrak{A}_1}(\beta), 0, \beta]$ and $[\mathfrak{A}_2, -\nu_{\mathfrak{A}_2}(\beta), 0, \beta]$ that fulfil the assumptions of Theorem ([G2](6.6)) with respect to the same special approximation sequence of $[\mathfrak{A}(E), -\nu_E(\beta), 0, \beta]$. Applying Corollary ([G2](6.7)) twice to $\theta(A(E), id_{A(E)}, \mathfrak{A}(E))$ it turns out that θ_1 and θ_2 have a common defining sequence $\{(\gamma_i, \lambda_{\gamma_i})\}_{i=0,\ldots,\mu}$. Thus, our assertion follows directly from expanding $H^1(\beta, \mathfrak{A}_1) \cap H^1(\beta, \mathfrak{A}_2)$ as in (2.9) and (3.1) applied to the case of $i = \mu$.
□

4 Simple characters and their extensions by transferring between orders

(4.1) Proposition *Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum and use the notations of Section (2). Let $\mathfrak{B}_m \subset \mathfrak{B}$ and $H^1_{\mathfrak{A},\mathfrak{B}_m}$ be the mixed group as in Lemma (2.1). Then there exists a unique character $\theta_{\mathfrak{A},\mathfrak{B}_m}$ of $H^1_{\mathfrak{A},\mathfrak{B}_m}$ that concides with $\theta_{\mathfrak{A}}$ on H^1 and with $\theta_{\mathfrak{A}_m} := \theta(A, id, \mathfrak{A}_m)$ on $H^1_{\mathfrak{A},\mathfrak{B}_m} \cap H^1_m$, where θ is the ps-character characterized by $\theta(A, id, \mathfrak{A}) = \theta_{\mathfrak{A}}$. Moreover, the group $J^1_{\mathfrak{A},\mathfrak{B}_m}$ normalizes the character $\theta_{\mathfrak{A},\mathfrak{B}_m}$.*

<u>Proof:</u>

The uniqueness of the character is clear. That the set $H^1_{\mathfrak{A},\mathfrak{B}_m}$ is a group follows from (2.1). As, by ([G2](5.11)), $\mathcal{K}_{\mathfrak{B}} J^1$ normalizes $\theta_{\mathfrak{A}}$ and $\mathcal{K}_{\mathfrak{B}_m} J^1_m$ normalizes $\theta_{\mathfrak{A}_m}$ one only needs the equality $(\theta_{\mathfrak{A}})_{|H^1 \cap H^1_m} = (\theta_{\mathfrak{A}_m})_{|H^1 \cap H^1_m}$ for the existence of the character; hence, it follows from (3.2).

Moreover, we have to note that $(J^1 \cap J^1_m)$ normalizes $\theta_{\mathfrak{A}}$ and $\theta_{\mathfrak{A}_m}$, hence $\theta_{\mathfrak{A},\mathfrak{B}_m}$, and since $J^1_{\mathfrak{A},\mathfrak{B}_m} = (J^1 \cap J^1_m) H^1_{\mathfrak{A},\mathfrak{B}_m}$, we are also done with the last statement.
□

For the construction of our ‘extended Heisenberg representations’ we need the following notion:

(4.2) Definition *Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum and $\mathfrak{B}_m \subset \mathfrak{B} = \mathfrak{A} \cap C_A(\beta)$. Moreover, keep the notation as above, then the pair $([\mathfrak{A}, n, 0, \beta], \mathfrak{B}_m)$ is called non-degenerate if*

$$(J^1 : J^1_{\mathfrak{A},\mathfrak{B}_m} \cap J^1) = (H^1_{\mathfrak{A},\mathfrak{B}_m} \cap J^1 : H^1).$$

5 Reimann's method to extend to a pro-p-Sylow subgroup (cf. [HR])

(5.1) Lemma *Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum in A and $\theta_{\mathfrak{A}} \in \mathcal{C}(\beta, \mathfrak{A})$. Then there is a unique irreducible Heisenberg representation $\eta_{\mathfrak{A}}$ of $J^1(\beta, \mathfrak{A}) = J^1$ which is, up to isomorphism, determined by the property that it contains $\theta_{\mathfrak{A}}$. Moreover, $(\eta_{\mathfrak{A}})_{|H^1(\beta,\mathfrak{A})}$ is a multiple of $\theta_{\mathfrak{A}}$ and $dim(\eta_{\mathfrak{A}}) = (J^1 : H^1)^{\frac{1}{2}}$. The $A^\times$-intertwining of $\eta_{\mathfrak{A}}$ is $J^1 B^\times J^1$.*

Proof:
The proof is the same as that of [BK1](5.1.1).
□

(5.2) Proposition *Under the assumption that the underlying pair $([\mathfrak{A}, n, 0, \beta], \mathfrak{B}_m)$ is non-degenerate, there exists a unique extension $\eta_{\mathfrak{A},\mathfrak{B}_m}$ of $\eta_{\mathfrak{A}}$ to $U^1(\mathfrak{B}_m)J^1$ such that*

$$Ind_{H^1_{\mathfrak{A},\mathfrak{B}_m}}^{U^1(\mathfrak{B}_m)J^1}(\theta_{\mathfrak{A},\mathfrak{B}_m}) \simeq (J^1_{\mathfrak{A},\mathfrak{B}_m} : H^1_{\mathfrak{A},\mathfrak{B}_m})^{\frac{1}{2}} \eta_{\mathfrak{A},\mathfrak{B}_m}.$$

Proof:
The uniqueness of $\eta_{\mathfrak{A},\mathfrak{B}_m}$ is clear. For the existence let us consider the non-degenerate form $\chi_{\theta_{\mathfrak{A}}} : J^1/H^1 \times J^1/H^1 \to \mathbb{C}^\times$, which is induced by the application of $\theta_{\mathfrak{A}}$ to commutators of J^1. Since $(\theta_{\mathfrak{A}})_{|H^1 \cap H^1_m} = (\theta_{\mathfrak{A}_m})_{|H^1 \cap H^1_m}$, it is easy to see that the subquotient $(J^1 \cap H^1_m)H^1/H^1$ is isotropic with respect to $\chi_{\theta_{\mathfrak{A}}}$. By assumption (4.2) $J^1_{\mathfrak{A},\mathfrak{B}_m} \cap J^1/H^1$ is the orthogonal of $H^1_{\mathfrak{A},\mathfrak{B}_m} \cap J^1/H^1 = (J^1 \cap H^1_m)H^1/H^1$ with respect to $\chi_{\theta_{\mathfrak{A}}}$.

Clearly, $\theta_{\mathfrak{A}}$ extends to a character $\theta^\star_{\mathfrak{A}}$ of $H^1_{\mathfrak{A},\mathfrak{B}_m} \cap J^1$ and, similarly, we obtain a non-degenerate form $\chi_{\theta^\star_{\mathfrak{A}}} : J^1_{\mathfrak{A},\mathfrak{B}_m} \cap J^1/H^1_{\mathfrak{A},\mathfrak{B}_m} \cap J^1 \times J^1_{\mathfrak{A},\mathfrak{B}_m} \cap J^1/H^1_{\mathfrak{A},\mathfrak{B}_m} \cap J^1 \to \mathbb{C}^\times$. This leads to a Heisenberg representation $\hat{\eta}_{\mathfrak{A}}$ of $J^1_{\mathfrak{A},\mathfrak{B}_m} \cap J^1$. Due to (2.1) and (4.1) we may identify the form $\chi_{\theta^\star_{\mathfrak{A}}}$ with the form

$$\chi_{\theta_{\mathfrak{A},\mathfrak{B}_m}} : J^1_{\mathfrak{A},\mathfrak{B}_m}/H^1_{\mathfrak{A},\mathfrak{B}_m} \times J^1_{\mathfrak{A},\mathfrak{B}_m}/H^1_{\mathfrak{A},\mathfrak{B}_m} \to \mathbb{C}^\times,$$

which comes from $\theta_{\mathfrak{A},\mathfrak{B}_m}$. Clearly, this form is non-degenerate.

So we have shown that $\theta_{\mathfrak{A},\mathfrak{B}_m}$ is a Heisenberg character in the sense of [Z2](4.2) with respect to $J^1_{\mathfrak{A},\mathfrak{B}_m}$ and that it leads to an irreducible Heisenberg representation $\hat{\eta}_{\mathfrak{A},\mathfrak{B}_m}$ of $J^1_{\mathfrak{A},\mathfrak{B}_m}$ with $Ind_{H^1_{\mathfrak{A},\mathfrak{B}_m}}^{J^1_{\mathfrak{A},\mathfrak{B}_m}}(\theta_{\mathfrak{A},\mathfrak{B}_m}) \simeq (J^1_{\mathfrak{A},\mathfrak{B}_m} : H^1_{\mathfrak{A},\mathfrak{B}_m})^{\frac{1}{2}} \hat{\eta}_{\mathfrak{A},\mathfrak{B}_m}$. By construction and Mackey's formula we have $(\hat{\eta}_{\mathfrak{A},\mathfrak{B}_m})_{|J^1_{\mathfrak{A},\mathfrak{B}_m} \cap J^1} \simeq \hat{\eta}_{\mathfrak{A}}$.

Again by Mackey's formula and by construction, we have $\eta_{\mathfrak{A}} \simeq Ind_{J^1_{\mathfrak{A},\mathfrak{B}_m} \cap J^1}^{J^1}(\hat{\eta}_{\mathfrak{A}}) \simeq (Ind_{J^1_{\mathfrak{A},\mathfrak{B}_m}}^{U^1(\mathfrak{B}_m)J^1}(\hat{\eta}_{\mathfrak{A},\mathfrak{B}_m}))_{|J^1}$ so that $\eta_{\mathfrak{A},\mathfrak{B}_m} := Ind_{J^1_{\mathfrak{A},\mathfrak{B}_m}}^{U^1(\mathfrak{B}_m)J^1}(\hat{\eta}_{\mathfrak{A},\mathfrak{B}_m})$ is irreducible and a continuation of $\eta_{\mathfrak{A}}$. Finally, by transitivity of induction, we have $Ind_{H^1_{\mathfrak{A},\mathfrak{B}_m}}^{U^1(\mathfrak{B}_m)J^1}(\theta_{\mathfrak{A},\mathfrak{B}_m}) \simeq (J^1_{\mathfrak{A},\mathfrak{B}_m} : H^1_{\mathfrak{A},\mathfrak{B}_m})^{\frac{1}{2}} \eta_{\mathfrak{A},\mathfrak{B}_m}$, as required.
□

6 A simplification of the problem of non-degeneracy of pairs

(6.1) Lemma *Let M be an abstract abelian group and $\{M_i\}_{i=1,\dots,n}$, $\{N_i\}_{i=1,\dots,n}$ be two families of subgroups such that $M_i \supset N_i$ for $i = 1, \dots, n$. Then, under the assumption that all group indices involved are finite, we have*

$$(\sum_{i=1}^n M_i : \sum_{i=1}^n N_i) = \frac{\prod_{i=1}^n (M_i : N_i)}{\prod_{i=1}^{n-1} (M_i \cap (\sum_{j=i+1}^n M_j) : N_i \cap (\sum_{j=i+1}^n N_j))}.$$

Proof:

Lets us prove our lemma by induction on n. The case of $n = 1$ is trivial. So let us assume that the lemma holds true for $n \geq 1$.

Let $M_2' := \sum_{i=2}^{n+1} M_i$, $N_2' := \sum_{i=2}^{n+1} N_i$ and assume that the lemma holds true for the case where $n = 2$. Then $(M_1 + M_2' : N_1 + N_2') = \frac{(M_1:N_1)(M_2':N_2')}{(M_1 \cap M_2' : N_1 \cap N_2')}$ and we obtain the desired assertion by including the definitions of M_2', N_2' and applying the induction hypothesis to the index $(M_2' : N_2')$.

Thus it remains to deal with the case that $n = 2$. Here we have $(M_1 + M_2 : N_1 + N_2) = (M_1 + M_2 : N_1 + M_2)(N_1 + M_2 : N_1 + N_2)$ and our assertions follows from the indentities $(M_1 + M_2 : N_1 + M_2) = \frac{(M_1:N_1)}{(M_1 \cap M_2 : N_1 \cap M_2)}$, $(N_1 + M_2 : N_1 + N_2) = \frac{(M_2:N_2)}{(N_1 \cap M_2 : N_1 \cap N_2)}$ and $(M_1 \cap M_2 : N_1 \cap M_2)(N_1 \cap M_2 : N_1 \cap N_2) = (M_1 \cap M_2 : N_1 \cap N_2)$.

□

(6.2) Proposition *Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum and $\mathfrak{B}_m$ be a principal order contained in $\mathfrak{B} := \mathfrak{A} \cap C_A(\beta)$. Let $\mathfrak{A}_m$ be the unique continuation of $\mathfrak{B}_m$ to A and $\gamma_1, ..., \gamma_\mu$ be elements of A as in (2.5) such that the sequences $[\mathfrak{A}, n, q_i, \gamma_i]$, $[\mathfrak{A}_m, n_m, q_i', \gamma_i]$, $i = 1, \dots, \mu$, are approximation sequences for the strata $[\mathfrak{A}, n, 0, \beta]$ and $[\mathfrak{A}_m, n_m, 0, \beta]$, respectively, where $n_m := -\nu_{\mathfrak{A}_m}(\beta)$.*

With the notation of Section (2) above we have:

$$(J^1 : J^1_{\mathfrak{A},\mathfrak{B}_m} \cap J^1) = \frac{\prod_{i=1}^{\mu} (\mathfrak{P}^{[\frac{q_i+1}{2}]} \cap C_A(\gamma_i) : \mathfrak{P}^{[\frac{q_i+1}{2}]} \cap \mathfrak{P}_m^{[\frac{q_i'+1}{2}]} \cap C_A(\gamma_i) + \mathfrak{P}^{[\frac{q_i}{2}]+1} \cap C_A(\gamma_i))}{\prod_{i=1}^{\mu-1} (\mathfrak{P}^{[\frac{q_{i+1}+1}{2}]} \cap C_A(\gamma_i) : \mathfrak{P}^{[\frac{q_{i+1}+1}{2}]} \cap \mathfrak{P}_m^{[\frac{q_{i+1}'+1}{2}]} \cap C_A(\gamma_i) + \mathfrak{P}^{[\frac{q_{i+1}}{2}]+1} \cap C_A(\gamma_i))}$$

and

$$(H^1_{\mathfrak{A},\mathfrak{B}_m} \cap J^1 : H^1) = \frac{\prod_{i=1}^{\mu} (\mathfrak{P}^{[\frac{q_i+1}{2}]} \cap \mathfrak{P}_m^{[\frac{q_i'}{2}]+1} \cap C_A(\gamma_i) + \mathfrak{P}^{[\frac{q_i}{2}]+1} \cap C_A(\gamma_i) : \mathfrak{P}^{[\frac{q_i}{2}]+1} \cap C_A(\gamma_i))}{\prod_{i=1}^{\mu-1} (\mathfrak{P}^{[\frac{q_{i+1}+1}{2}]} \cap \mathfrak{P}_m^{[\frac{q_{i+1}'}{2}]+1} \cap C_A(\gamma_i) + \mathfrak{P}^{[\frac{q_{i+1}}{2}]+1} \cap C_A(\gamma_i) : \mathfrak{P}^{[\frac{q_{i+1}}{2}]+1} \cap C_A(\gamma_i))}.$$

Proof:

Again we only prove the first identity. It is easy to see that $(J^1 : J^1_{\mathfrak{A},\mathfrak{B}_m} \cap J^1) = (\mathcal{J}^1 : \mathcal{J}^1 \cap \mathcal{J}^1_m + \mathcal{H}^1)$. By (6.1) and (2.8) the last Lemma (6.1) applies with $M_i = \mathfrak{P}^{[\frac{q_i+1}{2}]} \cap C_A(\gamma_i)$ and $N_i = \mathfrak{P}^{[\frac{q_i+1}{2}]} \cap \mathfrak{P}_m^{[\frac{q_i'+1}{2}]} \cap C_A(\gamma_i) + \mathfrak{P}^{[\frac{q_i}{2}]+1} \cap C_A(\gamma_i)$ for $i = 1, \dots, \mu$. Then it is sufficient to show that, for $i = 1, ..., \mu - 1$, we have

$$\mathfrak{P}^{[\frac{q_i+1}{2}]} \cap C_A(\gamma_i) \cap (\textstyle\sum_{\nu=i+1}^{\mu} \mathfrak{P}^{[\frac{q_\nu+1}{2}]} \cap C_A(\gamma_\nu)) = \mathfrak{P}^{[\frac{q_{i+1}+1}{2}]} \cap C_A(\gamma_i) \text{ and} \tag{4}$$

$$\begin{aligned}&(\mathfrak{P}^{[\frac{q_i+1}{2}]} \cap \mathfrak{P}_m^{[\frac{q_i'+1}{2}]} \cap C_A(\gamma_i) + \mathfrak{P}^{[\frac{q_i}{2}]+1} \cap C_A(\gamma_i)) \cap \\ &(\textstyle\sum_{\nu=i+1}^{\mu} \mathfrak{P}^{[\frac{q_\nu+1}{2}]} \cap \mathfrak{P}_m^{[\frac{q_\nu'+1}{2}]} \cap C_A(\gamma_\nu) + \mathfrak{P}^{[\frac{q_\nu}{2}]+1} \cap C_A(\gamma_\nu)) = \\ &\mathfrak{P}^{[\frac{q_{i+1}+1}{2}]} \cap \mathfrak{P}_m^{[\frac{q_{i+1}'+1}{2}]} \cap C_A(\gamma_i) + \mathfrak{P}^{[\frac{q_{i+1}}{2}]+1} \cap C_A(\gamma_i).\end{aligned} \tag{5}$$

As (4) is a special case of (5) in some sense, we can restrict ourselves to prove (5).
By (2.8) we have

$$\mathfrak{P}^{[\frac{q_{i+1}+1}{2}]}\cap\mathfrak{P}_m^{[\frac{q'_{i+1}+1}{2}]}\cap C_A(\gamma_i)+\mathfrak{P}^{[\frac{q_{i+1}}{2}]+1}\cap C_A(\gamma_i)\subseteq$$

$$(\mathfrak{P}^{[\frac{q_i+1}{2}]}\cap\mathfrak{P}_m^{[\frac{q'_i+1}{2}]}\cap C_A(\gamma_i)+\mathfrak{P}^{[\frac{q_i}{2}]+1}\cap C_A(\gamma_i))\cap$$

$$(\mathfrak{P}^{[\frac{q_{i+1}+1}{2}]}\cap\mathcal{J}^1\cap\mathfrak{P}_m^{[\frac{q'_{i+1}+1}{2}]}\cap\mathcal{J}_m^1+\mathfrak{P}^{[\frac{q_{i+1}}{2}]+1}\cap\mathcal{H}^1)=$$

$$(\mathfrak{P}^{[\frac{q_i+1}{2}]}\cap\mathfrak{P}_m^{[\frac{q'_i+1}{2}]}\cap C_A(\gamma_i)+\mathfrak{P}^{[\frac{q_i}{2}]+1}\cap C_A(\gamma_i))\cap$$

$$(\sum_{\nu=i+1}^{\mu}\mathfrak{P}^{[\frac{q_\nu+1}{2}]}\cap\mathfrak{P}_m^{[\frac{q'_\nu+1}{2}]}\cap C_A(\gamma_\nu)+\mathfrak{P}^{[\frac{q_\nu}{2}]+1}\cap C_A(\gamma_\nu)).$$

On the other hand, (2.7)(ii) leads to

$$(\mathfrak{P}^{[\frac{q_i+1}{2}]}\cap\mathfrak{P}_m^{[\frac{q'_i+1}{2}]}\cap C_A(\gamma_i)+\mathfrak{P}^{[\frac{q_i}{2}]+1}\cap C_A(\gamma_i))\cap$$

$$(\mathfrak{P}^{[\frac{q_{i+1}+1}{2}]}\cap\mathcal{J}^1\cap\mathfrak{P}_m^{[\frac{q'_{i+1}+1}{2}]}\cap\mathcal{J}_m^1+\mathfrak{P}^{[\frac{q_{i+1}}{2}]+1}\cap\mathcal{H}^1)=$$

$$(\mathfrak{P}^{[\frac{q_i+1}{2}]}\cap\mathfrak{P}_m^{[\frac{q'_i+1}{2}]}+\mathfrak{P}^{[\frac{q_i}{2}]+1})\cap$$

$$C_A(\gamma_i)\cap(\mathfrak{P}^{[\frac{q_{i+1}+1}{2}]}\cap\mathcal{J}^1\cap\mathfrak{P}_m^{[\frac{q'_{i+1}+1}{2}]}\cap\mathcal{J}_m^1+\mathfrak{P}^{[\frac{q_{i+1}}{2}]+1}\cap\mathcal{H}^1)=$$

$$C_A(\gamma_i)\cap(\mathfrak{P}^{[\frac{q_{i+1}+1}{2}]}\cap\mathcal{J}^1\cap\mathfrak{P}_m^{[\frac{q'_{i+1}+1}{2}]}\cap\mathcal{J}_m^1+\mathfrak{P}^{[\frac{q_{i+1}}{2}]+1}\cap\mathcal{H}^1)\subseteq$$

$$C_A(\gamma_i)\cap(\mathfrak{P}^{[\frac{q_{i+1}+1}{2}]}\cap\mathfrak{P}_m^{[\frac{q'_{i+1}+1}{2}]}+\mathfrak{P}^{[\frac{q_{i+1}}{2}]+1})=$$

$$\mathfrak{P}^{[\frac{q_{i+1}+1}{2}]}\cap\mathfrak{P}_m^{[\frac{q'_{i+1}+1}{2}]}\cap C_A(\gamma_i)+\mathfrak{P}^{[\frac{q_{i+1}}{2}]+1}\cap C_A(\gamma_i).$$

□

(6.3) Proposition *Let* $[\mathfrak{A},n,0,\beta]$ *be a simple stratum and let* $\mathfrak{B}_m$ *be a principal order contained in* $\mathfrak{B}:=\mathfrak{A}\cap C_A(\beta)$. *Choose simultaneous special approximation sequences* $[\mathfrak{A},n,q_i,\gamma_i]$, $[\mathfrak{A}_m,n_m,q'_i,\gamma_i]$, $i=1,\ldots,\mu$, *for* $[\mathfrak{A},n,0,\beta]$ *and* $[\mathfrak{A}_m,n_m,0,\beta]$, *respectively, where* $\mathfrak{A}_m$ *is the unique continuation of* $\mathfrak{B}_m$ *to* A. *Let* $E:=F[\beta]$, K/F *be the maximal unramified part of the extension* E/F *and set* $A_i:=C_A(\gamma_i)$, $C_i:=C_A(K[\gamma_i])$, *for* $i=1,\ldots,\mu$. *Then, in order to show the non-degeneracy of the pair* $([\mathfrak{A},n,0,\beta],\mathfrak{B}_m)$, *it is now sufficient to show, for all* $i\in\{1,\ldots,\mu\}$ *such that* $2e_{\mathfrak{A}/\mathfrak{A}\cap A_i}\mid q_i$ *but* $2e_{\mathfrak{A}/\mathfrak{A}\cap C_i}\nmid q_i$, *as well as for the* $i\in\{1,\ldots,\mu-1\}$ *such that* $2e_{\mathfrak{A}/\mathfrak{A}\cap A_i}\mid q_{i+1}$ *but* $2e_{\mathfrak{A}/\mathfrak{A}\cap C_i}\nmid q_{i+1}$, *that we have*
$(\mathfrak{P}_i^{\frac{-e_i}{2}}:\mathfrak{P}_i^{\frac{-e_i}{2}}\cap\mathfrak{P}_{i,m}^{-\frac{\bar r_m}{\bar r}\frac{e_i}{2}}+\mathfrak{P}_i^{\frac{-e_i}{2}+1})=$ *where*
$(\mathfrak{P}_i^{\frac{-e_i}{2}}\cap\mathfrak{P}_{i,m}^{-\frac{\bar r_m}{\bar r}\frac{e_i}{2}+1}+\mathfrak{P}_i^{\frac{-e_i}{2}+1}:\mathfrak{P}_i^{\frac{-e_i}{2}+1})$,
$\mathfrak{P}_i$, $\mathfrak{P}_{i,m}$ *are the Jacobson radical of* $\mathfrak{A}\cap A_i$ *and* $\mathfrak{A}_m\cap A_i$, *respectively, and* $\bar r$, $\bar r_m$ *are the period of* $\mathfrak{A}\cap A_i$ *and* $\mathfrak{A}_m\cap A_i$ *respectively,* $e_i:=e_{\mathfrak{A}\cap A_i/\mathfrak{A}\cap C_i}$.

Proof:

By (6.2) it is sufficient to show that $(\mathfrak{P}^{[\frac{q_i+1}{2}]}\cap A_i : \mathfrak{P}^{[\frac{q_i+1}{2}]}\cap \mathfrak{P}_m^{[\frac{q_i'+1}{2}]}\cap A_i + \mathfrak{P}^{[\frac{q_i}{2}]+1}\cap A_i) = (\mathfrak{P}^{[\frac{q_i+1}{2}]}\cap \mathfrak{P}_m^{[\frac{q_i'}{2}]+1}\cap A_i + \mathfrak{P}^{[\frac{q_i}{2}]+1}\cap A_i : \mathfrak{P}^{[\frac{q_i}{2}]+1}\cap A_i)$

for all $i=1,\ldots,\mu$, and $(\mathfrak{P}^{[\frac{q_{i+1}+1}{2}]}\cap A_i : \mathfrak{P}^{[\frac{q_{i+1}+1}{2}]}\cap \mathfrak{P}_m^{[\frac{q_{i+1}'+1}{2}]}\cap A_i + \mathfrak{P}^{[\frac{q_{i+1}}{2}]+1}\cap A_i) = (\mathfrak{P}^{[\frac{q_{i+1}+1}{2}]}\cap \mathfrak{P}_m^{[\frac{q_{i+1}'}{2}]+1}\cap A_i + \mathfrak{P}^{[\frac{q_{i+1}}{2}]+1}\cap A_i : \mathfrak{P}^{[\frac{q_{i+1}}{2}]+1}\cap A_i)$
for all $i=1,\ldots,\mu-1$. If $2e_{\mathfrak{A}/\mathfrak{A}\cap A_i}\nmid q_i$, the first equation we have to show is trivial, because then we have $\mathfrak{P}^{[\frac{q_i+1}{2}]}\cap A_i = \mathfrak{P}^{[\frac{q_i}{2}]+1}\cap A_i$. Similarly, for $i<\mu$, the case of $2e_{\mathfrak{A}/\mathfrak{A}\cap A_i}\nmid q_{i+1}$ leads trivially to the second equation we have to show. Therefore, we exclude these cases and restrict ourselves to show that

$$\begin{aligned}(\mathfrak{P}_i^{\frac{q_i}{2e_{\mathfrak{A}/\mathfrak{A}\cap A_i}}} : \mathfrak{P}_i^{\frac{q_i}{2e_{\mathfrak{A}/\mathfrak{A}\cap A_i}}}\cap \mathfrak{P}_{i,m}^{\frac{q_i'}{2e_{\mathfrak{A}/\mathfrak{A}\cap A_i}}} + \mathfrak{P}_i^{\frac{q_i}{2e_{\mathfrak{A}/\mathfrak{A}\cap A_i}}+1}) = \\ (\mathfrak{P}_i^{\frac{q_i}{2e_{\mathfrak{A}/\mathfrak{A}\cap A_i}}}\cap \mathfrak{P}_{i,m}^{\frac{q_i'}{2e_{\mathfrak{A}_m/\mathfrak{A}_m\cap A_i}}+1} + \mathfrak{P}_i^{\frac{q_i}{2e_{\mathfrak{A}/\mathfrak{A}\cap A_i}}+1} : \mathfrak{P}_i^{\frac{q_i}{2e_{\mathfrak{A}/\mathfrak{A}\cap A_i}}+1})\end{aligned} \tag{6}$$

for $i\in\{1,\ldots,\mu\}$ such that $2e_{\mathfrak{A}/\mathfrak{A}\cap A_i}\mid q_i$, and

$$\begin{aligned}(\mathfrak{P}_i^{\frac{q_{i+1}}{2e_{\mathfrak{A}/\mathfrak{A}\cap A_i}}} : \mathfrak{P}_i^{\frac{q_{i+1}}{2e_{\mathfrak{A}/\mathfrak{A}\cap A_i}}}\cap \mathfrak{P}_{i,m}^{\frac{q_{i+1}'}{2e_{\mathfrak{A}/\mathfrak{A}\cap A_i}}} + \mathfrak{P}_i^{\frac{q_{i+1}}{2e_{\mathfrak{A}/\mathfrak{A}\cap A_i}}+1}) = \\ (\mathfrak{P}_i^{\frac{q_{i+1}}{2e_{\mathfrak{A}/\mathfrak{A}\cap A_i}}}\cap \mathfrak{P}_{i,m}^{\frac{q_{i+1}'}{2e_{\mathfrak{A}_m/\mathfrak{A}_m\cap A_i}}+1} + \mathfrak{P}_i^{\frac{q_{i+1}}{2e_{\mathfrak{A}/\mathfrak{A}\cap A_i}}+1} : \mathfrak{P}_i^{\frac{q_{i+1}}{2e_{\mathfrak{A}/\mathfrak{A}\cap A_i}}+1})\end{aligned} \tag{7}$$

for $i\in\{1,\ldots,\mu-1\}$ such that $2e_{\mathfrak{A}/\mathfrak{A}\cap A_i}\mid q_{i+1}$. Our aim is to reduce the latter assertions to the assertion of our proposition.

For that purpose let us fix an arbitrary but fixed element $i\in\{1,\ldots,\mu\}$. Then $\mathfrak{A}\cap A_i$ and $\mathfrak{A}_m\cap A_i$ are continuations of $\mathfrak{A}\cap C_i$ and $\mathfrak{A}_m\cap C_i$, respectively.

Since the extension E/K is totally ramified, $\mathfrak{A}\cap C$ is an unramified continuation of $\mathfrak{B}$ to $C := C_A(K)$, and $\mathfrak{A}_m\cap C$ is an unramified continuation of $\mathfrak{B}_m$ to C. From (2.4) we conclude that $\mathfrak{A}\cap C\supset \mathfrak{A}_m\cap C$. Moreover, we obtain $\mathfrak{A}\cap C_i = \mathfrak{A}\cap C\cap C_i \supset \mathfrak{A}_m\cap C\cap C_i = \mathfrak{A}_m\cap C_i$.

Let $M/F[\gamma_i]$ be the minimal subextension of $K[\gamma_i]/F[\gamma_i]$ such that the continuation $\mathfrak{D}$ of $\mathfrak{A}\cap C_i$ to $C_A(M)$ is unramified. As the period of $\mathfrak{A}\cap C_i$ is a divisor of the period of $\mathfrak{A}_m\cap C_i$, we conclude from (2.3) that the unique continuation $\mathfrak{D}_m$ of $\mathfrak{A}_m\cap C_i$ to $C_A(M)$ is unramified. Consequently we have $e_{\mathfrak{A}\cap A_i/\mathfrak{A}\cap C_i} = e_{\mathfrak{A}\cap A_i/\mathfrak{D}}$ and $e_{\mathfrak{A}_m\cap A_i/\mathfrak{A}_m\cap C_i} = e_{\mathfrak{A}_m\cap A_i/\mathfrak{D}_m}$. Moreover, $e_{\mathfrak{A}/\mathfrak{D}} = e_{\mathfrak{A}/\mathfrak{A}\cap A_i}e_{\mathfrak{A}\cap A_i/\mathfrak{D}}$ and $e_{\mathfrak{A}_m/\mathfrak{D}_m} = e_{\mathfrak{A}_m/\mathfrak{A}_m\cap A_i}e_{\mathfrak{A}_m\cap A_i/\mathfrak{D}_m}$.

Now let $\bar r$, $\bar r_m$, $\hat r$ and $\hat r_m$ be the periods of $\mathfrak{A}\cap A_i$, $\mathfrak{A}_m\cap A_i$, $\mathfrak{D}$ and $\mathfrak{D}_m$, respectively. Let h_m be a uniformizer of $\mathfrak{D}_m$. Then $h := h_m^{\frac{\hat r_m}{\hat r}}$ is a uniformizer for $\mathfrak{D}$ and $\nu_{\mathfrak{A}\cap A_i}(h) = e_{\mathfrak{A}\cap A_i/\mathfrak{D}}$.

As the extension $M/F[\gamma_i]$ is unramified by ([G2](2.6))(iii), $\frac{\hat r_m}{\hat r}e_{\mathfrak{A}_m/\mathfrak{A}_m\cap A_i} = \frac{\hat r_m}{\hat r}\frac{\bar r_m\bar d}{\hat r_m\hat d} = \frac{\bar r_m}{\bar r}\frac{\bar r\bar d}{\hat r\hat d} = \frac{\bar r_m}{\bar r}e_{\mathfrak{A}/\mathfrak{A}\cap A_i}$, where $\bar d$ is the index of A_i and $\hat d$ is the index of $C_A(M)$. Moreover, we have $\frac{\bar r_m}{\bar r}\frac{q_i}{2e_{\mathfrak{A}/\mathfrak{A}\cap A_i}} = \frac{q_i'}{2e_{\mathfrak{A}_m/\mathfrak{A}_m\cap A_i}}$.

First assume that $2e_{\mathfrak{A}/\mathfrak{A}\cap C_i}\nmid q_i$. Then, we may write $\frac{q_i}{e_{\mathfrak{A}/\mathfrak{A}\cap C_i}} = 2\nu+1$ for a non-negative integer ν since $e_{\mathfrak{A}/\mathfrak{A}\cap C_i}\mid q_i$ and, thus $\frac{q_i}{2e_{\mathfrak{A}/\mathfrak{A}\cap A_i}} = \nu e_{\mathfrak{A}\cap A_i/\mathfrak{D}} + \frac{e_{\mathfrak{A}\cap A_i/\mathfrak{D}}}{2}$.

Multiplying equation (6) by $h^{-\nu-1}$ and using the equality $\nu_{\mathfrak{A}_m\cap A_i}(h) = \frac{\hat r_m}{\hat r}\nu_{\mathfrak{A}_m\cap A_i}(h_m) = \frac{\hat r_m}{\hat r}e_{\mathfrak{A}_m/\mathfrak{A}_m\cap A_i} = \frac{\bar r_m}{\bar r}e_{\mathfrak{A}/\mathfrak{A}\cap A_i}$, we reduce (6) to the equation $(\mathfrak{P}_i^{\frac{-e_{\mathfrak{A}\cap A_i/\mathfrak{A}\cap C_i}}{2}} : \mathfrak{P}_i^{\frac{-e_{\mathfrak{A}\cap A_i/\mathfrak{A}\cap C_i}}{2}}\cap \mathfrak{P}_{i,m}^{-\frac{\bar r_m}{\bar r}\frac{e_{\mathfrak{A}\cap A_i/\mathfrak{A}\cap C_i}}{2}} + \mathfrak{P}^{\frac{-e_{\mathfrak{A}\cap A_i/\mathfrak{A}\cap C_i}}{2}+1}) = (\mathfrak{P}_i^{\frac{-e_{\mathfrak{A}\cap A_i/\mathfrak{A}\cap C_i}}{2}}\cap \mathfrak{P}_{i,m}^{-\frac{\bar r_m}{\bar r}\frac{e_{\mathfrak{A}\cap A_i/\mathfrak{A}\cap C_i}}{2}+1} + \mathfrak{P}_i^{\frac{-e_{\mathfrak{A}\cap A_i/\mathfrak{A}\cap C_i}}{2}+1} : \mathfrak{P}_i^{\frac{-e_{\mathfrak{A}\cap A_i/\mathfrak{A}\cap C_i}}{2}+1})$, as required. Similarly for $i < \mu$, we may reduce the second equation (7) to the same statement if we assume that $2e_{\mathfrak{A}\cap A_i/\mathfrak{A}\cap C_i} \nmid q_{i+1}$.

Thus, we only have to handle the cases $2e_{\mathfrak{A}/\mathfrak{A}\cap C_i} \mid q_i$ and $2e_{\mathfrak{A}/\mathfrak{A}\cap C_i} \mid q_{i+1}$ for $i < \mu$. However in this case, we may multiply our equations by $h^{-\frac{q_i}{2e_{\mathfrak{A}/\mathfrak{A}\cap C_i}}}$ and $h^{-\frac{q_{i+1}}{2e_{\mathfrak{A}/\mathfrak{A}\cap C_i}}}$, respectively, and reduce both equations to

$$(\mathfrak{A}\cap A_i : \mathfrak{A}\cap\mathfrak{A}_m\cap A_i + \mathfrak{P}\cap A_i) = (\mathfrak{A}\cap\mathfrak{P}_m\cap A_i + \mathfrak{P}\cap A_i : \mathfrak{P}\cap A_i). \tag{8}$$

To show this, let us fix an isomorphism $A_i \simeq M_t(\bar{D})$, where $\bar{D}$ is a central division algebra over $F[\gamma_i]$ of index $\bar{d}$. Then up to conjugacy in $A_i^\times$, we may assume that $M \subset \bar{D}$ (note that $[M : F[\gamma_i]] = e_{\mathfrak{A}\cap A_i/\mathfrak{A}\cap C_i} \mid \bar{d}$). Doing so, we identify $C_A(M)$ with $M_t(\hat{D})$, where $\hat{D} = C_{\bar{D}}(M)$. Up to conjugacy in $C_A(M)^\times$ we may assume that $\mathfrak{D}$ and $\mathfrak{D}_m$ are given in simultaneous normal form. Moreover, we note that, as the continuation of $\mathfrak{D}$ by $\mathfrak{A}\cap A_i$ is totally ramified, we have $\bar{s} = \hat{s} \mid \hat{s}_m \mid \bar{s}_m$, where $\bar{s}$, $\bar{s}_m$, $\hat{s}$ and $\hat{s}_m$ are the invariants of $\mathfrak{A}\cap A_i$, $\mathfrak{A}_m\cap A_i$, $\mathfrak{D}$ and $\mathfrak{D}_m$, respectively. By (2.2) $\mathfrak{A}\cap A_i$ is conjugated by an $\hat{s}\times\hat{s}$-monomial matrix g with its normal form in $M_t(\hat{D})$ and $\mathfrak{A}_m\cap A_i$ is conjugated by an $\hat{s}_m\times\hat{s}_m$-monomial matrix with its normal form in the same algebra.

From $\mathfrak{A}_m\cap A_i\cap C_A(M) = \mathfrak{D}_m$, we deduce that the $\hat{s}_m\times\hat{s}_m$-blocks above the block diagonal of $\mathfrak{A}_m\cap A_i$ must be in a power of $\mathfrak{p}_{\bar{D}}$ of the exponent ≥ 1, the diagonal blocks are in $\mathfrak{o}_{\bar{D}}$ and the $\hat{s}_m\times\hat{s}_m$-blocks below the diagonal blocks are in a power of $\mathfrak{p}_{\bar{D}}$ of the exponent ≤ 0. The same holds true if we restrict this consideration to the diagonal blocks of an $\hat{s}\times\hat{s}$-block representation $\mathfrak{A}_m\cap A_i = (A^m_{i,j})_{i,j=1,\ldots,\hat{r}}$ and to the diagonal blocks of an $\hat{s}\times\hat{s}$-block representation $g\mathfrak{A}_m\cap A_i g^{-1} = (B^m_{i,j})_{i,j=1,\ldots,\hat{r}_\nu}$, as conjugation by the $\hat{s}\times\hat{s}$-block monomial matrix g just permutes the diagonal blocks and conjugates them by a power of a prime element $\bar{\pi}$ of $\mathfrak{o}_{\bar{D}}$.

Further, consider an $\hat{s}\times\hat{s}$-block representation $\mathfrak{A}\cap A_i = (A_{i,j})_{i,j=1,\ldots,\hat{r}}$, and for an $\hat{s}\times\hat{s}$-block representation $g\mathfrak{A}\cap A_i g^{-1} = (B_{i,j})_{i,j=1,\ldots,\hat{r}_\nu}$, the $B_{i,i} = M_{\hat{s}}(\mathfrak{o}_{\bar{D}})$ are again a permutation of the $A_{i,i}$ for $i = 1,\ldots,\mu$, $B_{i,j} = M_{\hat{s}}(\mathfrak{o}_{\bar{D}})$ for $i > j$ and $B_{i,j} = M_{\hat{s}}(\mathfrak{p}_{\bar{D}})$ for $i < j$.

Similarly, we have $\mathfrak{P}\cap A_i = (C_{i,j})_{i,j=1,\ldots,\hat{r}}$ and $g\mathfrak{P}\cap A_i g^{-1} = (D_{i,j})_{i,j=1,\ldots,\hat{r}}$, where the $D_{i,i}$ are permutations of the blocks $C_{i,i}$ for $i = 1,\ldots,\hat{r}$, and the blocks represent sets of $\hat{s}\times\hat{s}$-matrices with entries from powers of $\mathfrak{p}_{\bar{D}}$, where the exponent depends only on ν, i, j. For the blocks $D_{i,j}$ we just have $D_{i,j} = M_{\hat{s}}(\mathfrak{o}_{\bar{D}})$ for $i > j$ and $B_{i,j} = M_{\hat{s}}(\mathfrak{p}_{\bar{D}})$ for $i \leq j$.

Altogether we obtain $g\mathfrak{A}\cap\mathfrak{A}_m\cap A_i g^{-1} + g\mathfrak{P}\cap A_i g^{-1} = (B_{i,j}\cap B^m_{i,j} + D_{i,j})_{i,j=1,\ldots,\hat{r}}$. In particular, for $i > j$ we have $(B_{i,j}\cap B^m_{i,j} + D_{i,j})_{i,j=1,\ldots,\hat{r}} = (M_{\bar{s}}(\mathfrak{o}_{\bar{D}})\cap B^m_{i,j} + M_{\hat{s}}(\mathfrak{o}_{\bar{D}})) = M_{\hat{s}}(\mathfrak{o}_{\hat{D}})$ and for $i < j$ we have $(B_{i,j}\cap B^m_{i,j} + D_{i,j}) = (M_{\hat{s}}(\mathfrak{p}_{\bar{D}})\cap B^m_{i,j} + M_{\hat{s}}(\mathfrak{p}_{\bar{D}})) = M_{\hat{s}}(\mathfrak{p}_{\bar{D}})$. As $\mathfrak{A}_m\cap A_i$ is conjugated by an $\hat{s}_m\times\hat{s}_m$-block-monomial matrix with its normal form, the blocks $A^m_{i,i}$, resp. $B^m_{i,i}$ have an $\hat{s}_m\times\hat{s}_m$-block description such that the diagonal blocks have entries in $\mathfrak{o}_{\bar{D}}$. Since (as shown above) the $\hat{s}_m\times\hat{s}_m$-blocks below the block diagonal have entries from powers of $\mathfrak{p}_{\bar{D}}$ of the the exponent ≤ 0 and the $\hat{s}_m\times\hat{s}_m$-blocks above the block diagonal have entries in a power of $\mathfrak{p}_{\bar{D}}$ of the exponent ≥ 1, the blocks below the block diagonal in $B^m_{i,i}$ are intersected down to $\mathfrak{o}_{\bar{D}}$ in $M_{\hat{s}}(\mathfrak{o}_{\bar{D}})\cap B^m_{i,i}$ and the the blocks above the block diagonal in $M_{\hat{s}}(\mathfrak{o}_{\bar{D}})\cap B^m_{i,i}$ are filled up by $\mathfrak{p}_{\bar{D}}$ in $M_{\hat{s}}(\mathfrak{o}_{\bar{D}})\cap B^m_{i,i} + M_{\hat{s}}(\mathfrak{p}_{\bar{D}})$ for $i = 1,\ldots,\hat{r}$; hence, we deduce that $(B_{i,i}\cap B^m_{i,i} + D_{i,i}) = (M_{\hat{s}}(\mathfrak{o}_{\bar{D}})\cap B^m_{i,i} + M_{\hat{s}}(\mathfrak{p}_{\bar{D}}))$ is the principal order given in normal form in $M_{\hat{s}}(\bar{D})$ with invariant $\hat{s}_m$ and period $\frac{\hat{s}_m}{\hat{s}}$. Summarizing we conclude that $g\mathfrak{A}\cap\mathfrak{A}_m\cap A_i g^{-1} + g\mathfrak{P}\cap A_i g^{-1}$ is the principal order given in normal form in $M_t(\bar{D})$ with invariant $\hat{s}_m$ and period $\hat{r}_m$.

Similar, we may show that $g\mathfrak{A}\cap\mathfrak{P}_m\cap A_i g^{-1} + g\mathfrak{P}\cap A_i g^{-1}$ is the Jacobson radical of $g\mathfrak{A}\cap\mathfrak{A}_m\cap A_i g^{-1} + g\mathfrak{P}\cap A_i g^{-1}$; hence, by the block-descriptions of the normal forms, we deduce that

$$(g\mathfrak{A} \cap A_i g^{-1} : g\mathfrak{A} \cap \mathfrak{A}_m \cap A_i g^{-1} + g\mathfrak{P} \cap A_i g^{-1}) = (g\mathfrak{A} \cap \mathfrak{P}_m \cap A_i g^{-1} + g\mathfrak{P} \cap A_i g^{-1} : g\mathfrak{P} \cap A_i g^{-1}).$$

Conjugating the latter indices on both sides of the latter equation by g^{-1}, we obtain equation (8), as required.

□

7 Some special cases where all pairs are non-degenerate

(7.1) Corollary *All pairs* $([\mathfrak{A}, n, 0, \beta], \mathfrak{B}_m)$ *of simple strata* $[\mathfrak{A}, n, 0, \beta]$ *and principal orders* $\mathfrak{B}_m \subset \mathfrak{B} := \mathfrak{A} \cap C_A(\beta)$ *are sufficiently non-degenerate if*

(i) $A = D$ *is a central divison algebra over* F*.*

(ii) $A = M_l(D)$*, where* D *is a central division algebra over* F *and* l *is a prime number.*

(iii) The index d *of the algebra* A *is odd.*

Proof:

(i) In a central divison algebra $D = A$ over F of finite dimension, there is a unique principal order $\mathfrak{A} = \mathfrak{o}_D$ and the same holds true for the centralizer $B := C_A(\beta)$; hence, in this case we always have $\mathfrak{B}_m = \mathfrak{B}$ and our index equation always reduces to the trivial equation $1 = 1$.

(ii) If $A = M_l(D)$, where l is a prime number and D is a central division algebra over F, we have $[E : F] \mid d = Ind(A)$ (where $E := F[\beta]$) or $B := C_A(\beta)$ is a divison algebra. In the latter case we have $\mathfrak{B}_m = \mathfrak{B}$ again and the non-degeneracy of all pairs involving β is clear.

If $[E : F] \mid d$, then we have $B = M_l(D')$, where D' is a central division algebra over E of index $\frac{d}{[E:F]}$ and the case $\mathfrak{B}_m = \mathfrak{B}$ is trivial again. If $\mathfrak{B}_m \neq \mathfrak{B}$, then $\mathfrak{B}_m$ must be the minimal principal order in B, which has period l and invariant 1, and $\mathfrak{B}$ must be the maximal order with period 1 and invariant l. By [G1](1.9,1.8(iii)) we also conlude that $\mathfrak{A}$ is again a maximal order.

Now we use the arguments of the proof of (6.3) in a modified way. We use the notation and assumptions of the proof of (6.3) for $\gamma_0, \ldots, \gamma_\mu$, A_i, C_i, $M/F[\gamma_i]$, $\mathfrak{D}$ and $\mathfrak{D}_\mathfrak{m}$, where we fix an i such that $2e_{\mathfrak{A}/\mathfrak{A}\cap A_i} \mid q_i$ and $2e_{\mathfrak{A}/\mathfrak{A}\cap C_i} \nmid q_i$ or $2e_{\mathfrak{A}/\mathfrak{A}\cap A_i} \mid q_{i+1}$ and $2e_{\mathfrak{A}/\mathfrak{A}\cap C_i} \nmid q_{i+1}$ if $i < \mu$. Since $\mathfrak{D}$ as well as $\mathfrak{A} \cap A_i$ are maximal orders of invariant l, by the arguments in the proof of (6.3) we may restrict ourselves to the case where $\mathfrak{A} \cap A_i$ is conjugated by an $l \times l$-block monomial matrix with its normal form; hence, it has to be given itself in normal form in the sense that, if $C_A(M) = M_l(\hat{D})$, $A_i = M_l(\bar{D})$ and $\mathfrak{D} = M_l(\mathfrak{o}_{\hat{D}})$, then we have $\mathfrak{A} \cap A_i = M_l(\mathfrak{o}_{\bar{D}})$. Let us fix a prime element $\bar{\pi}$ of $\mathfrak{o}_{\bar{D}}$. We also assume $\mathfrak{A}_m \cap A_i$ to be conjugated with its normal form by an $\hat{s}_m \times \hat{s}_m$-block monomial matrix in certain powers of $\bar{\pi}$ (where $\hat{s}_m$ is the invariant of $\mathfrak{D}$). We deduce that $\bar{\pi} I_l$ is a uniformizer of $\mathfrak{A} \cap A_i$ that normalizes $\mathfrak{A}_m \cap A_i$. We may multiply the terms in the index equation

$$(\mathfrak{P}_i^{\frac{-e_{\mathfrak{A}\cap A_i/\mathfrak{A}\cap C_i}}{2}} : \mathfrak{P}_i^{\frac{-e_{\mathfrak{A}\cap A_i/\mathfrak{A}\cap C_i}}{2}} \cap \mathfrak{P}_{i,m}^{-\frac{\bar{r}_m}{\bar{r}}\frac{e_{\mathfrak{A}\cap A_i/\mathfrak{A}\cap C_i}}{2}} + \mathfrak{P}_i^{\frac{-e_{\mathfrak{A}\cap A_i/\mathfrak{A}\cap C_i}}{2}+1}) =$$

$$(\mathfrak{P}_i^{\frac{-e_{\mathfrak{A}\cap A_i/\mathfrak{A}\cap C_i}}{2}} \cap \mathfrak{P}_{i,m}^{-\frac{\bar{r}_m}{\bar{r}}\frac{e_{\mathfrak{A}\cap A_i/\mathfrak{A}\cap C_i}}{2}+1} + \mathfrak{P}_i^{\frac{-e_{\mathfrak{A}\cap A_i/\mathfrak{A}\cap C_i}}{2}+1} : \mathfrak{P}_i^{\frac{-e_{\mathfrak{A}\cap A_i/\mathfrak{A}\cap C_i}}{2}+1})$$

by $\bar{\pi}^{\frac{e_{\mathfrak{A}\cap A_i/\mathfrak{A}\cap C_i}}{2}}$ and restrict ourselves to show equation (8), which can be proved in the same way as in the proof of (6.3). Therefore, with these modified arguments of the proof of (6.3) we may show that (6.3) applies and that all pairs $([\mathfrak{A}, n, 0, \beta], \mathfrak{B}_m)$ where $\mathfrak{A}$ is a maximal order, $\mathfrak{B}_m$ is minimal and $[E : F] \mid d$, are non-degenerate, as required.

(iii) Again we want to apply (6.3) and we remark that our conditions from (6.3) will become empty if we may show that

$$e_{\mathfrak{A}/\mathfrak{A}\cap C_i} \mid d, q_i, q_{i+1}, \tag{9}$$

as, by assumption on d, $e_{\mathfrak{A}/\mathfrak{A}\cap C_i}$ must be odd and, therefore, we may deduce from $2e_{\mathfrak{A}/\mathfrak{A}\cap A_i} \mid q_i$, $2e_{\mathfrak{A}/\mathfrak{A}\cap A_i} \mid q_{i+1}$ that $2e_{\mathfrak{A}/\mathfrak{A}\cap C_i} \mid q_i$, $2e_{\mathfrak{A}/\mathfrak{A}\cap C_i} \mid q_{i+1}$.

By [G1](1.9)(b) $e_{\mathfrak{A}/\mathfrak{A}\cap C_i}$ is a divisor of $f(K[\gamma_i]/F) = [K : F]$. If now K_d/F is the unique subextension of K/F with $[K_d : F] =< [K : F], d >$, then $C_A(K_d)$ has index $\frac{d}{[K_d:F]}$ (see for example [Z1](Proposition 1)) and by [G1](2.2)(b) the continuation of $\mathfrak{A} \cap C_i$ by $\mathfrak{A} \cap C_A(K_d)$ is unramified. Thus, we have $e_{\mathfrak{A}/\mathfrak{A}\cap C_i} = e_{\mathfrak{A}/\mathfrak{A}\cap C_A(K_d)} \mid [K_d : F] \mid d$ as well.

By ([G2](3.9))(C) we have $\frac{rd}{e(F[\gamma_{i-1}]/F)} \mid q_i$ and $\frac{rd}{e(F[\gamma_i]/F)} \mid q_{i+1}$. By ([G2](2.6)) we have $e_{\mathfrak{A}/\mathfrak{A}\cap C_i} = \frac{rd}{\bar{r}\bar{d}e(K[\gamma_i]/F)}$, where $\bar{r}$ is the period of $\mathfrak{A} \cap C_i$ and $\bar{d}$ is the index of C_i. It follows immediately that $e_{\mathfrak{A}/\mathfrak{A}\cap C_i} \mid q_{i+1}$. Now, if we denote the index of $C_A(K)$ by d' and the period of $\mathfrak{A} \cap C_A(K)$ by r', then [G2](2.2)(i)(a) implies $\bar{r} = \frac{r'<e(K[\gamma_i]/K),d'>}{e(K[\gamma_i]/K)}$ and, again by [Z1](Proposition 1), we have $\bar{d} = \frac{d'}{<e(K[\gamma_i]/K),d'>}$. Finally, we obtain $\frac{\frac{rd}{e(F[\gamma_{i-1}]/F)}}{e_{\mathfrak{A}/\mathfrak{A}\cap C_i}} = \frac{\bar{r}\bar{d}e(K[\gamma_i]/F)}{e(F[\gamma_{i-1}]/F)} = \frac{r'd'}{e(K[\gamma_{i-1}]/K)}$. However, by [G1](1.9)(iii), the last number is integral, since γ_{i-1} is $\mathfrak{A} \cap C_A(K)$-sound. Hence, we also obtain the relation $e_{\mathfrak{A}/\mathfrak{A}\cap C_i} \mid q_{i+1}$.

Altogether we have shown (9) and deduced (iii) as described above.

□

(7.2) Corollary *All pairs $([\mathfrak{A}, n, 0, \beta], \mathfrak{B}_m)$ of simple strata $[\mathfrak{A}, n, 0, \beta]$ and principal orders $\mathfrak{B}_m \subset \mathfrak{B} := \mathfrak{A} \cap C_A(\beta)$ with $\langle [F[\beta] : F], d\rangle \mid e(F[\beta]/F)r'$, where r' is the period of the principal order $\mathfrak{B} := \mathfrak{A} \cap C_A(\beta)$, are non-degenerate.*

Proof:

By (2.3) and in the notation of (6.3) we have $e_{\mathfrak{A}/\mathfrak{A}\cap A_i} = e_{\mathfrak{A}/\mathfrak{A}\cap C_i} = 1$ for $i = 0, \ldots, \mu$; hence, the non-degeneracy conditions of (6.3) are empty.

□

To compare our construction in Section (5) with the construction of [BK1] we use the following lemma:

(7.3) Lemma *Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum and $\mathfrak{B}_m \subset \mathfrak{B} := \mathfrak{A}\cap C_A(\beta)$ a principal order. Let $\mathfrak{A}_m$ be the unique continuation of $\mathfrak{B}_m$ to A, let $\theta_{\mathfrak{A}}$, $\theta_{\mathfrak{A}_m}$ be realizations of the same ps-characters for the groups H^1, H^1_m, and let $\eta_{\mathfrak{A}}$, $\eta_{\mathfrak{A}_m}$ be the associated Heisenberg representations. Moreover, use the assumptions of 7.2 (the unramified case) and let $\theta^\star_{\mathfrak{A},\mathfrak{B}_m}$ be a character living on a subgroup $H^{1\star}_{\mathfrak{A},\mathfrak{B}_m}$ of $U^1(\mathfrak{B}_m)J^1$ such that $(\theta^\star_{\mathfrak{A},\mathfrak{B}_m})_{|H^{1\star}_{\mathfrak{A},\mathfrak{B}_m}\cap H^1_m} = (\theta_{\mathfrak{A}_m})_{|H^{1\star}_{\mathfrak{A},\mathfrak{B}_m}\cap H^1_m}$. Finally, assume that $Ind_{H^{1\star}_{\mathfrak{A},\mathfrak{B}_m}}^{U^1(\mathfrak{B}_m)J^1}(\theta^\star_{\mathfrak{A},\mathfrak{B}_m})$ is a multiple of an irreducible extension $\tilde{\eta}_{\mathfrak{A},\mathfrak{B}_m}$ of $\eta_{\mathfrak{A}}$ to $U^1(\mathfrak{B}_m)J^1$. Then $\tilde{\eta}_{\mathfrak{A},\mathfrak{B}_m}$ and $\eta_{\mathfrak{A}_m}$ induce the same irreducible representation of $U^1(\mathfrak{A}_m)$.*

Proof:

First we note that, in the unramified case, by 2.4, we have $U^1(\mathfrak{A}_m) \supset U^1(\mathfrak{A})$ and $U^1(\mathfrak{A}) \supset J^1$. Therefore, we also have $U^1(\mathfrak{A}_m) \supset U^1(\mathfrak{B}_m)J^1$.

The $A^\times$-intertwinig of $\tilde{\eta}_{\mathfrak{A},\mathfrak{B}_m}$ is clearly contained in the $A^\times$-intertwining of $\eta_{\mathfrak{A}}$, which is $J^1B^\times J^1$.

As $J^1B^\times J^1 \cap U^1(\mathfrak{A}_m) = U^1(\mathfrak{B}_m)J^1$ Mackey's Theorem applies and the induction of $\tilde{\eta}_{\mathfrak{A},\mathfrak{B}_m}$ to $U^1(\mathfrak{A}_m)$ is irreducible. Since the intertwining of $\eta_{\mathfrak{A}_m}$ is $J^1_mB^\times J^1_m$ (see (5.1)) and $J^1_mB^\times J^1_m \cap U^1(\mathfrak{A}_m) = J^1_m$, the induction of $\eta_{\mathfrak{A}_m}$ to $U^1(\mathfrak{A}_m)$ is also irreducible. One has to show that these induced representations are equivalent. As $Ind_{H^1}^{J^1}(\theta_{\mathfrak{A}})$ is a multiple of $\eta_{\mathfrak{A}}$ and $Ind_{H^{1\star}_{\mathfrak{A},\mathfrak{B}_m}}^{U^1(\mathfrak{B}_m)J^1}(\theta^\star_{\mathfrak{A},\mathfrak{B}_m})$ is a multiple of

$\tilde{\eta}_{\mathfrak{A},\mathfrak{B}_m}$, it suffices to show that

$$\emptyset \neq Hom_{U^1(\mathfrak{A}_m)}(Ind_{H^1_m}^{U^1(\mathfrak{A}_m)}(\theta_{\mathfrak{A}_m}), Ind_{H^{1\star}_{\mathfrak{A},\mathfrak{B}_m}}^{U^1(\mathfrak{A}_m)}(\theta^\star_{\mathfrak{A},\mathfrak{B}_m})) \simeq$$

$$Hom_{H^1_m}(\theta_{\mathfrak{A}_m}, Res_{H^1_m}^{U^1(\mathfrak{A}_m)} \circ Ind_{H^{1\star}_{\mathfrak{A},\mathfrak{B}_m}}^{U^1(\mathfrak{A}_m)}(\theta^\star_{\mathfrak{A},\mathfrak{B}_m})) \simeq$$

$$Hom_{H^1_m}(\theta_{\mathfrak{A}_m}, \bigoplus_{x\in H^{1\star}_{\mathfrak{A},\mathfrak{B}_m}\backslash U^1(\mathfrak{A}_m)/H^{1\star}_{\mathfrak{A},\mathfrak{B}_m}} Ind_{(H^{1\star}_{\mathfrak{A},\mathfrak{B}_m})^x\cap H^1_m}^{H^1_m}((\theta^\star_{\mathfrak{A},\mathfrak{B}_m})^x)) \simeq$$

$$\bigoplus_{x\in H^{1\star}_{\mathfrak{A},\mathfrak{B}_m}\backslash U^1(\mathfrak{A}_m)/H^{1\star}_{\mathfrak{A},\mathfrak{B}_m}} Hom_{H^1_m}(\theta_{\mathfrak{A}_m}, Ind_{(H^{1\star}_{\mathfrak{A},\mathfrak{B}_m})^x\cap H^1_m}^{H^1_m}((\theta^\star_{\mathfrak{A},\mathfrak{B}_m})^x)).$$

Taking the component $x = 1$ we see that, by assumption, $Hom_{H^1_m}(\theta_{\mathfrak{A}_m}, Ind_{H^{1\star}_{\mathfrak{A},\mathfrak{B}_m}\cap H^1_m}^{H^1_m}(\theta^\star_{\mathfrak{A},\mathfrak{B}_m})) \simeq$ $Hom_{H^{1\star}_{\mathfrak{A},\mathfrak{B}_m}\cap H^1_m}(Res_{H^{1\star}_{\mathfrak{A},\mathfrak{B}_m}\cap H^1_m}^{H^1_m}(\theta_{\mathfrak{A}_m}), Res_{H^{1\star}_{\mathfrak{A},\mathfrak{B}_m}\cap H^1_m}^{H^1_{\mathfrak{A},\mathfrak{B}_m}}(\theta^\star_{\mathfrak{A},\mathfrak{B}_m})) = \mathbb{C}$ as required.
□

Since, in a split algebra, we are always in the unramified case (2.3), the combination of (7.1), (5.2), (4.1) and (7.3) provides:

(7.4) Corollary *Let $A \simeq M_N(F)$ be a split algebra, let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum and $\mathfrak{B}_m \subset \mathfrak{B} := \mathfrak{A}\cap B \subset \mathfrak{B}_M$, where $\mathfrak{B}_m$ is a minimal principal order and $\mathfrak{B}_M$ is a maximal order in $B := C_A(\beta)$. Moreover, let $\mathfrak{A}_m$ and $\mathfrak{A}_M$ be the unique continuations of $\mathfrak{B}_m$ and $\mathfrak{B}_M$, respectively, and let $\theta_{\mathfrak{A}}$, $\theta_{\mathfrak{A}_m}$, $\theta_{\mathfrak{A}_M}$ be the corresponding realizations of a fixed ps-character θ for the simple strata $[\mathfrak{A}, n, 0, \beta]$, $[\mathfrak{A}_m, -\nu_{\mathfrak{A}_m}(\beta), 0, \beta]$, $[\mathfrak{A}_M, -\nu_{\mathfrak{A}_M}(\beta), 0, \beta]$. Then there exist representations $\eta_{\mathfrak{A},\mathfrak{B}_m}$, $\eta_{\mathfrak{A}_M,\mathfrak{B}}$, $\eta_{\mathfrak{A}_M,\mathfrak{B}_m}$ of $U^1(\mathfrak{B}_m)J^1$, $U^1(\mathfrak{B})J^1_M$ and $U^1(\mathfrak{B}_m)J^1_M$, respectively, as in 5.2, and we have:*

(i) $\eta_{\mathfrak{A},\mathfrak{B}_m}$ is equivalent to the representation $\tilde{\eta}$ of [BK1](5.1.15),

(ii) $\eta_{\mathfrak{A}_M,\mathfrak{B}}$ is equivalent to the representation $\hat{\eta}_M$ of [BK1](5.1.16) and

(iii) $\eta_{\mathfrak{A}_M,\mathfrak{B}_m}$ is equivalent to the representation $\tilde{\eta}_M$ of [BK1](5.1.14).

In particular, in the split case all of $B^\times$ intertwines the representations $\eta_{\mathfrak{A},\mathfrak{B}_m}$, $\eta_{\mathfrak{A}_M,\mathfrak{B}}$ and $\eta_{\mathfrak{A}_M,\mathfrak{B}_m}$ with themselves.

□

8 Extending to level 0

The main result of this section is obtained by a combination of ideas of [HR](2.4) and [BK1](5.2.4).

(8.1) Proposition *In the following, let all groups be subgroups of a locally profinite group G. Let $\bar{J}$ be a compact group with pro-p-Sylow subgroup $\bar{J}_p$ and $\bar{\eta}$ be an irreducible representation of $\bar{J}^1$ with representation space V so that its isomorphy class is stable under the adjoint action of $\bar{J}$. Here $\bar{J}^1 \subset \bar{J}_p$ is a normal subgroup of $\bar{J}$ and $\bar{\eta}$ has an extension to $\bar{J}_p$. We also assume that $dim_{\mathbb{C}}(V)$ is a power of p and $\bar{J}/\bar{J}^1 \simeq \bar{J}\cap\bar{H}/\bar{J}^1\cap\bar{H}$ is isomorphic to a parabolic subgroup of $Gl_{n_1}(\mathfrak{k}) \times \ldots \times Gl_{n_l}(\mathfrak{k})$, where $\mathfrak{k}$ is a finite field and $\mathfrak{k} \neq \mathbb{F}_2$. In the case of $\mathfrak{k} = \mathbb{F}_2$, we do not allow proper parabolic subgroups and assume that $n_i \neq 2$ for $i = 1, \ldots l$. Finally, let $\bar{H}$ be a compact subgroup such that there lives a linear character $\theta^\star$ with $(\bar{J}\cap\bar{H})\bar{J}_p = \bar{J}_p(\bar{J}\cap\bar{H}) = \bar{J}$. We assume that $\bar{\eta}_{|\bar{J}^1\cap\bar{H}}$ is a multiple of $\theta^\star_{|\bar{J}^1\cap\bar{H}}$.*

Under these assumptions there exists a unique extension κ of $\bar{\eta}$ to $\bar{J}$ such that $\kappa_{|\bar{J}\cap\bar{H}} = \theta^\star_{|\bar{J}\cap\bar{H}} \otimes W$ and W is a representation of $\bar{J}\cap\bar{H}/\bar{J}^1\cap\bar{H}$ with $det_V(W) \equiv 1$.

In the extreme case where $\mathfrak{k} = \mathbb{F}_2$ and $n_1 = \ldots = n_l = 2$ for any extension η' of $\bar{\eta}$ to $\bar{J}_p$, there exists a unique extension κ of η' to $\bar{J}$.

Proof:

Since $\bar{\eta}$ is, up to isomorphy, stable under the adjoint action of $\bar{J}$ and is irreducible, there exists a mapping $\zeta : \bar{J} \to Aut_{\mathbb{C}}(V)$ such that we have $\zeta(x)^{-1}\bar{\eta}(y)\zeta(x) = \bar{\eta}(x^{-1}yx)$ for all $x \in \bar{J}$ and all $y \in \bar{J}^1$ and by Schurs Lemma the operators $\zeta(x)$ (for fixed x but varying y) are uniquely determined up to a scalar. Since $\bar{\eta}$ has an extension η' to $\bar{J}_p$, we may set $\zeta(x) := \eta'(x)$ for all $x \in \bar{J}_p$.

We may find a system of representatives R of the double cosets $\bar{J}_p\backslash\bar{J}/\bar{J}_p$ contained in $\bar{J} \cap \bar{H}$ and such that $x \in R$ and $x \in \bar{J}_p$ implies $x = 1$. With this choice we may arange for all $j_1, j_2 \in \bar{J}_p$ and $x \in R$ that $\zeta(j_1xj_2) = \eta'(j_1)\zeta(x)\eta'(j_2)$. Since replacing $\zeta(x)$ by non-zero scalars preserves the assumptions, we may assume that the equation $\zeta(x) = \theta^\star(x) \otimes W'(x)$ holds for $x \in R \subset \bar{J} \cap \bar{H}$, where

$$det_V(W'(x)) = 1. \tag{10}$$

For all $x \in \bar{J}_p$, we set $W'(x) := \theta^\star(x)^{-1}\eta'(x)$ and so W' extents to a function $W' : \bar{J} \cap \bar{H} \to Aut_{\mathbb{C}}(V)$, which is the identity on $\bar{J}^1 \cap \bar{H}$. Since $W'_{|\bar{J}_p\cap\bar{H}}$ is a representation of $\bar{J}_p \cap \bar{H}/\bar{J}^1 \cap \bar{H}$, we deduce that

$$det_V(W')_{|\bar{J}_p\cap\bar{H}} \text{ is a linear character with order dividing } (\bar{J}_p \cap \bar{H} : \bar{J}^1 \cap \bar{H}). \tag{11}$$

For all $x_1, x_2 \in \bar{J}$, we have $\zeta(x_1)\zeta(x_2) = \alpha(x_1, x_2)\zeta(x_1x_2)$ with a 2-cocycle α of $\bar{J}$ with values in $\mathbb{C}^\times$. For all $x_1, x_2 \in \bar{J} \cap \bar{H}$ we also have $W'(x_1)W'(x_2) = \alpha(x_1, x_2)W'(x_1x_2)$. By construction, we have

$$\alpha_{|\bar{J}_p\times\bar{J}_p} \equiv 1. \tag{12}$$

Moreover, we may find elements $j, j' \in \bar{J}_p$ and $h, h' \in \bar{J} \cap \bar{H}$ for $x_1, x_2 \in \bar{J}$ such that $x_1 = jh$ and $x_2 = h'j'$. We have

$$\alpha(x_1, x_2)^{dim_{\mathbb{C}}(V)(\bar{J}_p\cap\bar{H}:\bar{J}^1\cap\bar{H})} = det_V(\zeta(x_1)\zeta(x_2)\zeta(x_1x_2)^{-1})^{(\bar{J}_p\cap\bar{H}:\bar{J}^1\cap\bar{H})} =$$
$$det_V(\eta'(j)\zeta(h)\zeta(h')\eta'(j')(\eta'(j)\zeta(hh')\eta'(j'))^{-1})^{(\bar{J}_p\cap\bar{H}:\bar{J}^1\cap\bar{H})} =$$
$$det_V(\zeta(h)\zeta(h')\zeta(hh')^{-1})^{(\bar{J}_p\cap\bar{H}:\bar{J}^1\cap\bar{H})} =$$
$$det_V(\theta^\star(h)W'(h)\theta^\star(h')W'(h')\theta^\star(hh')^{-1}W'(hh')^{-1})^{(\bar{J}_p\cap\bar{H}:\bar{J}^1\cap\bar{H})} =$$
$$det_V(W'(h)W'(h')W'(hh')^{-1})^{(\bar{J}_p\cap\bar{H}:\bar{J}^1\cap\bar{H})}.$$

The above and the properties (10)(11) imply

$$\alpha^{dim_{\mathbb{C}}(V)(\bar{J}_p\cap\bar{H}:\bar{J}^1\cap\bar{H})} \equiv 1. \tag{13}$$

As $\bar{J}_p$ is a pro-p-Sylow subgroup of $\bar{J}$, we know from the cohomology theory of finite groups that the mapping $H^2(\bar{J}/\bar{J}^1, \mathbb{C}^\times)_p \overset{res}{\to} H^2(\bar{J}_p/\bar{J}^1, \mathbb{C}^\times)$ is injective, where the subscript p means the subgroup of all cohomology classes of p-power order. By (13) and the assumtions, the class of α is from the first group and by (12) the restriction of α is trivial. Therefore, α must be a 2-coboundary and we find a function $\chi : \bar{J}/\bar{J}^1 \to \mathbb{C}^\times$ such that we have $\alpha(x_1, x_2) = \chi(x_1)\chi(x_1x_2)^{-1}\chi(x_2)$ for all $x_1, x_2 \in \bar{J}$. Of course, $\kappa' := \chi\zeta$ is a representation of $\bar{J}$ and an extension of $\bar{\eta}$.

Considering the function χ as a complex function of $\bar{J} \cap \bar{H}/\bar{J}^1 \cap \bar{H}$, equation (12) says that $\chi_{|\bar{J}_p\cap\bar{H}}$ is a linear character of $\bar{J}_p \cap \bar{H}/\bar{J}^1 \cap \bar{H}$ and equation (13) indicates that $\chi^{dim_{\mathbb{C}}(V)(\bar{J}_p\cap\bar{H}:\bar{J}^1\cap\bar{H})}$ is a linear character of $\bar{J} \cap \bar{H}/\bar{J}^1 \cap \bar{H}$. Of course, the character $\chi_{|\bar{J}_p\cap\bar{H}}$ has order dividing $(\bar{J}_p \cap \bar{H} : \bar{J}^1 \cap \bar{H})$

and therefore $\chi^{dim_{\mathbb{C}}(V)(\bar{J}_p\cap\bar{H}:\bar{J}^1\cap\bar{H})}$ is a character of $\bar{J}\cap\bar{H}$ that is trivial on $\bar{J}_p\cap\bar{H}$. Let $N := ker(\chi^{dim_{\mathbb{C}}(V)(\bar{J}_p\cap\bar{H}:\bar{J}^1\cap\bar{H})})$. Then we have $\#(\bar{J}_p\cap\bar{H}/\bar{J}^1\cap\bar{H}) \mid \#(N/\bar{J}^1\cap\bar{H})$ and therefore $\#(\bar{J}\cap\bar{H}/N) \mid \#(\bar{J}\cap\bar{H}/\bar{J}_p\cap\bar{H})$, where the last number is prime to p.

We deduce that $\chi^{dim_{\mathbb{C}}(V)(\bar{J}_p\cap\bar{H}:\bar{J}^1\cap\bar{H})}$ has roots of unity as values with order prime to p. But since $dim_{\mathbb{C}}(V)(\bar{J}_p\cap\bar{H}:\bar{J}^1\cap\bar{H})$ is a power of p, the k-th power of the inverse of this character, where k is the multiplicative inverse of the this p power mod the order of this character, is a character χ' of $\bar{J}\cap\bar{H}/\bar{J}^1\cap\bar{H}$ such that $(\chi')^{dim_{\mathbb{C}}(V)(\bar{J}_p\cap\bar{H}:\bar{J}^1\cap\bar{H})} = \chi^{-dim_{\mathbb{C}}(V)(\bar{J}_p\cap\bar{H}:\bar{J}^1\cap\bar{H})}$. We conclude that

$$det(W')(\chi\chi')^{dim_{\mathbb{C}}(V)} \text{ is a character of } p\text{-power order.} \tag{14}$$

Let us assume first that $\mathfrak{k}\neq\mathbb{F}_2$ and consider the parabolic subgroup $P\simeq\bar{J}\cap\bar{H}/\bar{J}^1\cap\bar{H}$ from $Gl_{n_1}(\mathfrak{k})\times\ldots\times Gl_{n_l}(\mathfrak{k})$. Without restriction of generality, we may assume that P is a direct product of standard parabolic subgroups and has a decomposition $P=LU$, where U is the unipotent radical consisting of upper triangular unipotent block matrices and $L\simeq Gl_{k_1}(\mathfrak{k})\times\ldots Gl_{k_m}(\mathfrak{k})$ is the corresponding standard Levi subgroup consisting of block diagonal matrices. The partition $k_1,\ldots,k_m$ is a refinement of the partition $n_1,\ldots,n_l$. By [HR](2.9) a linear character of P of p-power order would be trivial on L and on the commutator subgroup of U, but, therefore, it has to be supported on the upper beside diagonal elements of matrices from U and should be invariant under conjugation by diagonal matrices from L there. Since $\mathfrak{k}$ has more than two elements and L contains all diagonal matrices, any character trivial on L must be trivial on this upper beside diagonal elements as well. Hence, the character from (14) must also be trivial if $\mathfrak{k}\neq\mathbb{F}_2$. Assuming that $\bar{J}\cap\bar{H}/\bar{J}^1\cap\bar{H}\simeq Gl_{n_1}(\mathfrak{k})\times\ldots\times Gl_{n_l}(\mathfrak{k})$ has no factor isomorphic to $Gl_2(\mathbb{F}_2)$, the triviality of the character (14) again follows from [HR](2.9).

In these cases we may set $\kappa := \chi'\kappa'$, where χ' is considered to be a character of $\bar{J}/\bar{J}^1 \simeq \bar{J}\cap\bar{H}/\bar{J}^1\cap\bar{H}$ and we deduce that κ is again a continuation of $\bar{\eta}$. We obtain $(\kappa)_{|\bar{J}\cap\bar{H}} = \theta^\star\otimes\chi\chi'W'$ and, for $W := \chi\chi'W'$, we obtain that $det_V(W)\equiv 1$ by (14). The uniqueness of such an extension κ follows again from the same arguments and the fact that another extension $\bar{\eta}$ fulfilling (i) must be a twist of κ by a linear character of $\bar{J}/\bar{J}^1$ of order $dim_{\mathbb{C}}(V)$. Hence, as the order is a power of p, such a character must be trivial again.

Finally, we want to consider the extreme case where $\bar{J}/\bar{J}^1$ is isomorphic to $\times^l Gl_2(\mathbb{F}_2)$. In this case, each character of $\bar{J}_p/\bar{J}^1$ must be a product of l characters of a subgroup of $Gl_2(\mathbb{F}_2)$, isomorphic to $\mathbb{F}_2^+$ and, in fact, all these characters are restrictions of characters of $Gl_2(\mathbb{F}_2)\simeq S_3$, namely, the trivial character or the signum of a permutation of three letters. Any two extensions of $\bar{\eta}$ to $\bar{J}_p$ must differ by such a character, we may twist κ' in such a way that it is an extension of η'. The uniqueness follows from the facts that $\#(\bar{J}/\bar{J}_p)=3^l$, each other extension of η' must be a twist by a linear character of $\bar{J}/\bar{J}_p$ and such that S_3 has no factor group of order 3.

□

Using this more general result, we obtain extensions of our Heisenberg representations.

(8.2) Corollary *Let $([\mathfrak{A},n,0,\beta],\mathfrak{B}_m)$ be a non-degenerate pair in the sense of (4.2), where $\mathfrak{B}_m$ is a minimal principal order in $B := C_A(\beta)$ and let $\theta^\star_{\mathfrak{A}}$ be an extension of a simple character $\theta_{\mathfrak{A}}\in\mathcal{C}(\beta,\mathfrak{A})$ to $H := \mathfrak{B}^\times H^1$ factorizing on $\mathfrak{B}^\times$ through the reduced norm $Nrd_{B|F[\beta]}$ (i.e., $\theta^\star_{\mathfrak{A}}$ is given in terms of defining sequences by an extension of λ_β from $U^1(\mathfrak{o}_{F[\beta]})$ to $\mathfrak{o}^\times_{F[\beta]}$). Then there exists a unique extension $\kappa_{\mathfrak{A}}$ of the Heisenberg representation $\eta_{\mathfrak{A}}$ of $\theta_{\mathfrak{A}}$ to $J := \mathfrak{B}^\times J^1$ fulfilling one of the following properties:*

(i) If V is the representation space of $\kappa_{\mathfrak{A}}$ and $\mathfrak{B}^\times/U^1(\mathfrak{B})$ has no factor isomorphic to $Gl_2(\mathbb{F}_2)$, then $(\kappa_{\mathfrak{A}})_{|H} = \theta^\star_{\mathfrak{A}}\otimes W$, where W is a representation of H/H^1 with $det_V(W)\equiv 1$.

(ii) If $\mathfrak{B}^\times/U^1(\mathfrak{B})$ has a factor isomorphic to $Gl_2(\mathbb{F}_2)$ (in particular $\mathfrak{B} \neq \mathfrak{B}_m$), then $\kappa_{\mathfrak{A}}$ is an extension of $\eta_{\mathfrak{A},\mathfrak{B}_m}$.

Proof:

First, note that in the case where $\dim_{\mathbb{C}}(\eta_{\mathfrak{A}}) = (J^1 : H^1)^{\frac{1}{2}} = 1$, we clearly have $\kappa_{\mathfrak{A}} = \theta^\star_{\mathfrak{A}}$; hence, we assume in the following that $\dim_{\mathbb{C}}(V) > 1$. In the case where $\mathfrak{B} = \mathfrak{B}_m$, we clearly have $\eta_{\mathfrak{A}} = \eta_{\mathfrak{A},\mathfrak{B}_m}$ and $U^1(\mathfrak{B}_m)J^1 = J^1$; hence our following arguments also work in that case.

We will apply Proposition (8.1) with $\bar{J} = J$, $\bar{J}^1 = J^1$, $\bar{\eta} = \eta_{\mathfrak{A}}$, $\bar{J}_p = U^1(\mathfrak{B}_m)J^1$, $\bar{H} = H$ and $\theta^\star = \theta^\star_{\mathfrak{A}}$. Indeed, $\eta_{\mathfrak{A}}$ is stable under the action of J and it is irreducible, $\eta_{\mathfrak{A},\mathfrak{B}_m}$ is an extension of $\eta_{\mathfrak{A}}$ to $U^1(\mathfrak{B}_m)J^1$, the dimension of $\eta_{\mathfrak{A}}$ is $(J^1 : H^1)^{\frac{1}{2}}$ and therefore a power of p. We have $J \supset H$, $J^1 \cap H = H^1$, $J = HJ^1 = J^1H$, the restriction of $\eta_{\mathfrak{A}}$ to H^1 is a multiple of $(\theta^\star_{\mathfrak{A}})_{|H^1}$, $J/J^1 \simeq H/H^1 \simeq (Gl_{s'}(\mathfrak{k}_{D'}))^{r'}$, where s', r' are the invariants of $\mathfrak{B}$ and D' is a division algebra of the same Brauer class as B.

Consequently, all assumptions of 8.1 hold and the resulting representation $\kappa_{\mathfrak{A}} = \kappa$ obviously fulfils the conditions (i), (ii), which are equivalent to the conditions (i), (ii) from (8.1) under our assuptions.
□

Remark: The diagrams of containments of groups

$$\begin{array}{ccccccc}
J & & & \supset & & & H \\
\cup & & & & & & \cup \\
U^1(\mathfrak{B}_m)J^1 & \supset & J^1_{\mathfrak{A},\mathfrak{B}_m} & \supset & H^1_{\mathfrak{A},\mathfrak{B}_m} & \supset & U^1(\mathfrak{B}_m)H^1 \\
\cup & & \cup & & \cup & & \cup \\
J^1 & \supset & J^1_{\mathfrak{A},\mathfrak{B}_m} \cap J^1 & \supset & H^1_{\mathfrak{A},\mathfrak{B}_m} \cap J^1 & \supset & H^1
\end{array}$$

and of associated representations

$$\begin{array}{ccccccc}
\kappa_{\mathfrak{A}} & & & \supset & & & \theta^\star_{\mathfrak{A}} \\
\cup & & & & & & \cup \\
\eta_{\mathfrak{A},\mathfrak{B}_m} & \supset & \hat{\eta}_{\mathfrak{A},\mathfrak{B}_m} & \supset & \theta_{\mathfrak{A},\mathfrak{B}_m} & \supset & (\theta_{\mathfrak{A},\mathfrak{B}_m})_{|U^1(\mathfrak{B}_m)H^1} \\
\cup & & \cup & & \cup & & \cup \\
\eta_{\mathfrak{A}} & \supset & \hat{\eta}_{\mathfrak{A}} & \supset & (\theta_{\mathfrak{A},\mathfrak{B}_m})_{|H^1_{\mathfrak{A},\mathfrak{B}_m} \cap J^1} & \supset & \theta_{\mathfrak{A}}
\end{array}$$

show the strategy for constructing $\kappa_{\mathfrak{A}}$.

We have $J = HJ^1$, $H^1 = J^1 \cap H$, H^1 is normal in J^1 and J^1/H^1 is a finite abelian p-group. The character $\theta_{\mathfrak{A}}$ applied to the commutators of J^1 induces a nondegenerate alternating form $J^1/H^1 \times J^1/H^1 \to \mathbb{C}^\times$. The characters $\theta_{\mathfrak{A}}$ and $\theta_{\mathfrak{A},\mathfrak{B}_m}$ concide on $H^1 \subset H \cap H^1_{\mathfrak{A},\mathfrak{B}_m}$. Moreover, $U^1(\mathfrak{B}_m)H^1/H^1 = H \cap H^1_{\mathfrak{A},\mathfrak{B}_m}/H^1$ is a p-Sylow group in H/H^1. Also $J^1_{\mathfrak{A},\mathfrak{B}_m} \cap J^1$ is a subgroup of J^1 containing $H^1_{\mathfrak{A},\mathfrak{B}_m} \cap J^1$ and normalizing the pair $(H^1_{\mathfrak{A},\mathfrak{B}_m}, \theta_{\mathfrak{A},\mathfrak{B}_m})$. Finally, if the stratum $[\mathfrak{A}, n, 0, \beta]$ is non-degenerate and J/J^1 has no direct factor isomorphic to $Gl_2(\mathbb{F}_2)$, then $(J^1 : J^1_{\mathfrak{A},\mathfrak{B}_m} \cap J^1) = (H^1_{\mathfrak{A},\mathfrak{B}_m} \cap J^1 : H^1)$ and $H^1(J/J^1, \mathbb{F}_p) = 0$, where p is the residual characteristic of F. Hence, [HR](2.4) applies in this case and, in fact, in the case of (8.2(i)) $\kappa_{\mathfrak{A}}$ is just the τ constructed in [HR](2.4).

9 Another remark on the split case

(9.1) Lemma *Use the previous notations for $\theta_{\mathfrak{A}}$ and $\eta_{\mathfrak{A}}$. Let $\mu_{\mathfrak{A},\mathfrak{B}_m}$ be an extension of $\eta_{\mathfrak{A}}$ to $U^1(\mathfrak{B}_m)J^1$ that is intertwined by the whole of $\mathfrak{B}^\times$. Then $\mu_{\mathfrak{A},\mathfrak{B}_m}$ is uniquely determined, except for* *the case where $\mathfrak{B}^\times/U^1(\mathfrak{B})$ has a direct factor that is isomorphic to $Gl_2(\mathbb{F}_2)$. In the latter case, $\mathfrak{B}$ must have the invariant 2 and we have $\mathfrak{B}^\times/U^1(\mathfrak{B}) \simeq (Gl_2(\mathbb{F}_2))^{r'} \simeq {S_3}^{r'}$, where r' is the period of $\mathfrak{B}$. All other extensions fulfilling the properties of $\mu_{\mathfrak{A},\mathfrak{B}_m}$ differ from $\mu_{\mathfrak{A},\mathfrak{B}_m}$ by a twist by a character of $U^1(\mathfrak{B}_m)J^1/J^1 \simeq U^1(\mathfrak{B}_m)/U^1(\mathfrak{B}) \hookrightarrow \mathfrak{B}^\times/U^1(\mathfrak{B}) \simeq (S_3)^{r'}$ that has values in $\{+1,-1\}$ and extends to $(S_3)^{r'}$.*

Proof:

We have $\mathfrak{B}^\times/U^1(\mathfrak{B}) \simeq (Gl_{s'}(\mathfrak{k}_{D'}))^{r'}$, where s' is the invariant of $\mathfrak{B}$, r' is the period of $\mathfrak{B}$ and D' is a division algebra of the same Brauer class as B. Moreover, $U^1(\mathfrak{B}_m)/U^1(\mathfrak{B}) \simeq N^{r'}$, where N is the set of lower triangular unipotent matrices in $Gl_{s'}(\mathfrak{k}_{D'})$. If $\mu'_{\mathfrak{A},\mathfrak{B}_m}$ is an extension of $\eta_{\mathfrak{A}}$ that is intertwined by the whole of $\mathfrak{B}^\times$, then it differs from $\mu_{\mathfrak{A},\mathfrak{B}_m}$ by a character ϕ of $U^1(\mathfrak{B}_m)J^1/J^1 \simeq U^1(\mathfrak{B}_m)/U^1(\mathfrak{B}) \simeq N^{r'}$ that is intertwined by the whole of $\mathfrak{B}^\times$; ϕ is the inflation of $\phi_1 \otimes \ldots \otimes \phi_{r'}$, where ϕ_ν is a character of N for $\nu = 1,\ldots,r'$.

Fix an arbitrary $\nu \in \{1,\ldots,r'\}$. The commutator group of N consists of all matrices $n = (n_{l,m}) \in N$ such that all coefficients are zero except the lower beside diagonal elements $n_{l+1,l}$ for $1 \leq l \leq s'-1$. Thus ϕ_ν is of the form $n \mapsto \psi(\sum_{l=1}^{s'-1} a_l n_{l+1,l})$, where ψ is an additive character of $\mathfrak{k}_{D'}$. Suppose first that $\mathfrak{k}_{D'}$ has at least three elements. The character ϕ_ν is stable under conjugation by any diagonal matrix of $Gl_{s'}(\mathfrak{k}_{D'})$, and this implies that all a_i are zero. In this case ϕ_ν is trivial. Now assume that $\mathfrak{k}_{D'} = \mathbb{F}_2$. The case of $s' = 1$ is trivial and we assume first that $s' > 2$. If we take into account that ϕ_ν is intertwined by the permutation matrix that corresponds to the cycle $(1,\ldots,s')$, we deduce that, in this case, we must have $a_1 = a_2 = \ldots = a_{s'-1}$. On the other hand, the fact that the permutation matrix corresponding to the transposition $(1,2)$ intertwines ϕ_ν implies $a_1 = 0$; hence, ϕ_ν is trivial again. As already remarked above, $Gl_2(\mathbb{F}_2) \simeq S_3$ (the permutation group of three elements) and $N \simeq \mathbb{F}_2^+$, too. Hence, N has only the trivial representation or the character that has the value -1 on the preimage of 1. The first lifts to the trivial representation of S_3 and the latter to the signum character of a permutation. This proves the last assertion and the case of $s' = 2$.

□

(9.2) Corollary *If $A = M_N(F)$ is a split algebra and using the notations, assumptions and conclusions of (8.2), the representation $\kappa_{\mathfrak{A}}$ is an extension of $\eta_{\mathfrak{A},\mathfrak{B}_m}$. Further, assuming that $\mathfrak{B}$ is a maximal order in $C_A(\beta)$, $\kappa_{\mathfrak{A}}$ is a β-extension of $\eta_{\mathfrak{A}}$ in the sense of [BK1](5.2.1).*

Proof:

By (7.4) and [BK1](5.1.9) (resp. [BK1](5.1.1) for $\mathfrak{B} = \mathfrak{B}_m$) $\eta_{\mathfrak{A},\mathfrak{B}_m}$ (where we set $\eta_{\mathfrak{A}_m,\mathfrak{B}_m} := \eta_{\mathfrak{A}_m}$) is intertwined by the whole of $B^\times$; hence, $(\kappa_{\mathfrak{A}})_{|U^1(\mathfrak{B}_m)J^1}$ and $\eta_{\mathfrak{A},\mathfrak{B}_m}$ are extensions of $\eta_{\mathfrak{A}}$ that are intertwined by the whole of $\mathfrak{B}^\times$, and (9.1) applies. We deduce that even if $\mathfrak{B}^\times/U^1(\mathfrak{B})$ does not contain any factor isomorphic to $Gl_2(\mathbb{F}_2)$, $\kappa_{\mathfrak{A}}$ is an extension of $\eta_{\mathfrak{A},\mathfrak{B}_m}$ and, thus, by Corollary (7.4) and the remark before [BK1](5.2.7), it is one of the extensions obtained as in [BK1](5.2.5). Following the arguments of [BK1](Chapter 5) we obtain the desired reference to [BK1](5.2.1).

□

10 A hypothetical list of simple types and the "level-0-case"

In section (1) we have seen that for suitable embedded strata, from the intertwining of two simple characters, follows their conjugacy by an element of the normalizer of the associated principal order. Now, just one simple character should be chosen from each conjugacy class, because we wish to have exactly one conjugacy class of simple characters in each essentially square integrable representation. We follow the plan sketched by [Z4].

For each possible period r of a principal order in A, we fix such an order $\mathfrak{A}$ in the corresponding conjugacy class. For a fixed order $\mathfrak{A}$ let us take the union $\hat{\mathcal{C}}(\mathfrak{A})$ of all character sets $\mathcal{C}(\beta,\mathfrak{A})$, where we vary over all simple strata $[\mathfrak{A},n,0,\beta]$ with $r=\frac{e(F[\beta]/F)}{\langle d,e(F[\beta]/F)\rangle}$ and maximally possible invariants d_1 in the sense of 1.13 for the sound embeddings $(F[\beta],\mathfrak{A})$. Let $\mathcal{C}(\mathfrak{A})$ be the set of all possible extensions of the characters $\theta_{\mathfrak{A}}$ with underlying stratum $[\mathfrak{A},n,0,\beta]$ to $H=\mathfrak{B}^\times H^1(\beta,\mathfrak{A})$ of a representation system of the Orbits in $\hat{\mathcal{C}}(\mathfrak{A})$ under the adjoint action of the normalizer of $\mathfrak{A}$, obtained by extending the associated defining character λ_β from $U^1(\mathfrak{o}_{F[\beta]})$ to $\mathfrak{o}_{F[\beta]}^\times$. For the latter set, we also have to fix a field $F[\beta]$ for each representative $\theta_{\mathfrak{A}}$ such that we obtain a unique leading term λ_β for its defining sequence with respect to an approximation sequence starting with $[\mathfrak{A},n,0,\beta]$. The choice is arbitrary but can be taken, for example, from a fundamental domain as in [Z2][Z7] for the division algebra case, which can be transferred to any local central simple algebra of the same reduced degree by [BG]. We call the elements of $\mathcal{C}(\mathfrak{A})$ for varying $\mathfrak{A}$ *fundamental characters* for short.

Let A be a central simple algebra of index d and reduced degree $N=md$ and $\mathcal{D}$ be a central simple division algebra of index N with a ring of integers $\mathcal{O}$. To a fundamental character $\theta_{\mathfrak{A}}$ with underlying stratum $[\mathfrak{A},n,0,\beta]$, field $F[\beta]$ and defining character λ_β, we may fix an embedding $\beta_{\mathcal{D}}$ of β into the algebra $\mathcal{D}$. Fixing this embedding, we may find a unique character $\theta_{\mathcal{O}}$ such that $\theta_{\mathfrak{A}}$ and $\theta_{\mathcal{O}}$ are realizations of a common "PS-character" in the sense of [G2](6.6). Of course, we may extend all such realizations $\theta_{\mathcal{O}}$ in the same way by extending $\lambda_{\beta_{\mathcal{D}}}=\lambda_\beta$ and obtain a fundamental character set $\mathcal{C}(\mathcal{O})$. The character set contains all extended realizations from all character sets $\mathcal{C}(\mathfrak{A})$ for varying $\mathfrak{A}$.

To each fundamental character $\theta\in\mathcal{C}(\mathfrak{A})$, we may assign a hypothetical unique extension $\kappa_{\mathfrak{A}}$ of the associated Heisenberg representation $\eta_{\mathfrak{A}}$ to $J_A:=J(\beta,\mathfrak{A})$.

(10.1) Proposition *Let E/F be a field extension in A and let $\mathfrak{A}$ be a principal order with period $\frac{e(E/F)}{\langle d,e(E/F)\rangle}$. Then there exists a sound embedding $(E,\mathfrak{A})$. Moreover, assume that the latter embedding $(E,\mathfrak{A})$ has maximally possible embedding invariants d_1 in the sense of 1.13. Then $\mathfrak{A}\cap C_A(E)$ is a maximal order.*

Proof:

The first assertion follows from [G1](1.9)(iii)(b) for example. If $d^\star$ is the largest divisor of d that is prime to the period r of $\mathfrak{A}$, then, by the above choice, $\frac{d}{\langle e(E/F),d\rangle}\mid d^\star$. The maximally possible embedding invariant for E/F is $d_1=\langle f(E/F),d^\star\rangle$ and, by [G2](2.2), the period of $\mathfrak{A}\cap C_A(L)$, where L/F is unramified in E/F of degree $f:=\langle f(E/F),d\rangle$, is $\frac{r\cdot f}{d_1}$.

Consequently, $C_A(L)$ has index $\frac{d}{f}$ and $\langle f(E/L),\frac{d}{f}\rangle=1$, [G1](2.2) applies, and $\mathfrak{A}\cap C_A(E)$ has period $\frac{r\cdot f\cdot\langle e(E/F),\frac{d}{f}\rangle}{d_1\cdot e(E/F)}=\frac{e(E/F)\cdot\langle e(E/F)f,d\rangle}{e(E/F)\cdot d_1\cdot\langle e(E/F),d\rangle}=\frac{\langle f,\frac{d}{\langle e(E/F),d\rangle}\rangle}{d_1}\mid\frac{\langle f,d^\star\rangle}{d_1}=\frac{d_1}{d_1}=1$, i.e., $\mathfrak{A}\cap C_A(E)$ must be maximal.

□

By the choice of the period of $\mathfrak{A}$ and the embedding invariant for $(F[\beta],\mathfrak{A})$, 10.1 applies and $\mathfrak{B}_M:=\mathfrak{A}\cap C_A(\beta)$ is a maximal order. If now $\kappa_{\mathcal{O}}$ is the corresponding representation of $J_{\mathcal{D}}:=J(\beta_{\mathcal{D}},\mathcal{O})$, we obtain from [Z2], [Z7] or [Br3] that each representation $\lambda_{\mathcal{D}}:=\kappa_{\mathcal{O}}\otimes\chi$, where χ is a linear character of the cyclic group $\mathcal{M}:=J_{\mathcal{D}}/J^1(\beta_{\mathcal{D}},\mathcal{O})$, is contained in exactly one unramified twist class of irreducible representations of $\mathcal{D}^\times$. Conversely, we will show below that each irreducible representation of $\mathcal{D}^\times$ can be twisted (by a general twist) in such a way that the twisted representation is contained in an unramified twist class of representations of a minimal Artin conductor in its twist class for arbitrary twist and each such representation of a minimal Artin conductor contains such a representation $\lambda_{\mathcal{D}}$.

If χ is given, then $\mathcal{M}$ is the multiplicative group of a finite extension field $\mathfrak{k}$ of $\mathfrak{k}_{F[\beta]}$ and the Galois group $\mathrm{Gal}(\mathfrak{k}/\mathfrak{k}_{F[\beta]})$ acts on χ. If f is the length of this Galois orbit, m' the invariant of $\mathfrak{B}_M$, d' the index $B := C_A(\beta)$, $f' := \langle f, d'\rangle$ and $e' := \langle \frac{N}{f}, m'\rangle$, then, by [SZ](1.1), there exists a supercuspidal representation σ_0 of the finite group $Gl_{f'}(\mathfrak{k}')$, where $\mathfrak{B}_M/\mathfrak{Q}_M \simeq M_{m'}(\mathfrak{k}')$ ($\mathfrak{k}'/\mathfrak{k}_{F[\beta]}$ is a finite extension of degree d' and $\mathfrak{Q}_M$ the Jacobson radical of $\mathfrak{B}_M$) and where χ is the so-called Green parameter of the representation. Finally, we choose a principal order $\mathfrak{B} \supset \mathfrak{B}_M$ with period e' such that $\mathfrak{B}/\mathfrak{Q}_M$ is isomorphic to the set of lower $f' \times f'$-block triangular matrices in $M_{m'}(\mathfrak{k}')$. We may inflate $\sigma_0^{e'}$ to a representation τ of $J := \mathfrak{B}^\times J^1(\beta, \mathfrak{A})$. We obtain a representation $\lambda := (\kappa_{\mathfrak{A}})_{|J} \otimes \tau$ of J. We choose from each Galois orbit only one χ and this choice also fixes a Galois representative σ_0 in its orbit under the action of $Gal(\mathfrak{k}'/\mathfrak{k}_{F[\beta]})$. As in [BK1](5.5.10), we define a list of so-called simple types:

(10.2) Definition *A simple type is either*

(i) a twisted representation $\Lambda := \lambda \otimes \mu$ with $\mu = \lambda_\alpha \circ Nrd_{A/F}$ and λ_α a character of $\mathfrak{o}_F^\times$, where λ is constructed as above from a fundamental character and from a character χ of the corresponding finite cyclic group $\mathcal{M}$ or

(ii) an irreducible representation of $\mathfrak{A}^\times$ that is in fact inflated from the quotient $\mathfrak{A}^\times/(1+\mathfrak{P}) \simeq Gl_s(k_D)^r$ and is given there by a tensor power $\sigma_0^{\otimes r}$ of a supercuspidal representation σ_0 of $Gl_s(k_D)$, where we choose only one σ_0 from each orbit under the action of $Gal(\mathfrak{k}_D/\mathfrak{k}_F)$.

The case (ii) can be considered to be the special case $\kappa_{\mathfrak{A}} \equiv 1$ and it is the situation of the level-0-case of the work of [SZ]. In [GSZ](5.3) it is shown that the types defined in (10.2)(ii) are indeed simple types of a single Bernstein component of the category of smooth representations of $A^\times$ in the sense of [BK4](4.1)(4.2). We conjecture that, in general, the above definition is a list of simple types of a corresponding single Bernstein component containing a single unramified twist class of essentially square integrable irreducible representations.

(10.3) Conjecture *(i) All pairs $([\mathfrak{A}, n, 0, \beta], \mathfrak{B}_m)$ are non-degenerate.*

(ii) All extensions $\kappa_{\mathfrak{A}}$ are intertwined by the whole of $B^\times$.

(iii) For any type of representation Λ, with irreducible factor τ, we have an isomorphism of Hecke algebras $\mathcal{H}(A^\times, \Lambda) \simeq \mathcal{H}(B^\times, \tau_{|\mathfrak{B}^\times})$.

(iv) For any representation Λ with $\mathfrak{B} = \mathfrak{B}_M$ maximal and τ being inflated from a single factor σ_0, there exists an extension to the stabelizer of Λ that induces irreducibly to $A^\times$.

(v) Each representation Λ for $\mathfrak{B} \neq \mathfrak{B}_M$ is a so-called covering type in the sense of [BK4](8) of a Levi component with e' factors isomorphic to $Gl_{\frac{m}{e'}}(D)$.

In the split case of $A = M_N(F)$, we have already seen in (9.2) that the whole of $B^\times$ intertwines all extensions $\kappa_{\mathfrak{A}}$ with themselves. Our definition differs from the one of [BK1](5.5.10) in two points. First we use types λ' as defined in [BK1](5.5.10) and used later to calculate the Hecke algebra and, second, we first construct the types for the irreducible essentially square integrable representations of a minimal Artin conductor in its (general) twist class and obtain the others by the fact that induction commutes with twist. By a thesis of Vincent Sécherre [VS] the properties (i) and (ii) can be replaced by another construction that is less close to the ideas of Reimann [HR], but more close to those of Bushnell and Kutzko [BK1].

(10.4) Definition In the following we call a type with maximal order $\mathfrak{B} = \mathfrak{B}_M$ as in [BK1](6) a maximal simple type.

We have shown the property (i) in case of $m = l$, l an arbitrary prime number, and would like to show (ii) later, independently of the method of [VS]. Finally, we want to show that the list is also complete in the sense that each single Bernstein component containing an irreducible essentially square integrable representation is parametrized by such a type. Moreover, this will imply that all supercuspidal representations of $Gl_l(D)$ are induced by a "compact modulo center subgroup".

In the following, we need two lemmas, which are generalizations of similar lemmas from [BK1].

(10.5) Lemma *Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum in A and $\theta_{\mathfrak{A}} \in \mathcal{C}(\beta, \mathfrak{A})$. Let $\eta_{\mathfrak{A}}$ be the Heisenberg representation of $J^1(\beta, \mathfrak{A}) = J^1$ associated with $\theta_{\mathfrak{A}}$ as in (5.1). Then, for $x \in A^\times$, we have the equality*

$$dim_{\mathbb{C}}(I_x(\eta_{\mathfrak{A}})) = \begin{cases} 1 & \textit{if } x \in J^1 B^\times J^1 \\ 0 & \textit{else,} \end{cases}$$

where we set $I_x(\rho) := Hom_{K^x \cap K}(\rho, \rho^x)$ as in [BK1](5.1.7) if ρ is a representation of an open compact subgroup K of $A^\times$.

Proof:

The proof is the same as in [BK1](5.1.8)(5.1.9) if we may prove the following generalization of [BK1](5.1.10):

(10.6) Lemma *For $y \in B^\times$ we have $(J^1 : J^1 \cap (J^1)^y) = (H^1 : H^1 \cap (H^1)^y)$.*

Proof:

We follow the proof of [BK1](5.1.10) and have to take into account the following changes. As above we write $H^1 = 1 + \mathfrak{H}^1$, $J^1 = 1 + \mathfrak{J}^1$ and choose Haar maesures $\mu^\times$ for $A^\times$ and μ for A^+ such that $\mu^\times(H^1) = \mu(\mathfrak{H}^1)$ and $\mu^\times(J^1) = \mu(\mathfrak{J}^1)$. Since $ker(a_\beta) = B$ and by ([G2](4.5)), there is an exact sequence

$$0 \to \mathfrak{Q} \to \mathfrak{J}^1 \xrightarrow{a_\beta} (\mathfrak{H}^1)^* \xrightarrow{\mathfrak{s}_\beta} \mathfrak{B} \to 0.$$

This sequence remains exact if we conjugate it by y, and we assert that the sequence

$$0 \to \mathfrak{Q} + \mathfrak{Q}^y \to \mathfrak{J}^1 + (\mathfrak{J}^1)^y \xrightarrow{a_\beta} (\mathfrak{H}^1)^* + ((\mathfrak{H}^1)^*)^y \xrightarrow{\mathfrak{s}_\beta} \mathfrak{B} + \mathfrak{B}^y \to 0$$

remains exact. By ([G2](4.6)) we have $(\mathfrak{H}^1)^* = a_\beta(\mathfrak{J}^1) + \mathfrak{A}$ and thus the last arrow induced by $\mathfrak{s}_\beta$ is surjective. Furthermore,

$$(\mathfrak{H}^1)^* + ((\mathfrak{H}^1)^*)^y = a_\beta(\mathfrak{J}^1 + (\mathfrak{J}^1)^y) + \mathfrak{A} + \mathfrak{A}^y. \tag{15}$$

Next we want to examine the exactness at $(\mathfrak{H}^1)^* + ((\mathfrak{H}^1)^*)^y$.

First we may embed into the canonical splitting $\tilde{A}$ of A and, as in ([G2](3.3)), we may consider the splitting $\tilde{\mathfrak{A}}$ of $\mathfrak{A}$. We set $\tilde{B} := C_{\tilde{A}}(E)$ and $\tilde{\mathfrak{B}} := \tilde{\mathfrak{A}} \cap \tilde{B}$. Applying [BK](1.4.16) for $h_1 = h_2 = y$, a sufficiently small i, sufficiently big n and $j = m = 0$, we deduce that, for $k = k_0(\beta, \mathfrak{A})$, we have the equality $ya_\beta(\tilde{\mathfrak{Q}}^{-k}\mathfrak{N}_k(\beta, \tilde{\mathfrak{A}})) + a_\beta(\tilde{\mathfrak{Q}}^{-k}\mathfrak{N}_k(\beta, \tilde{\mathfrak{A}}))y = a_\beta(y\tilde{\mathfrak{Q}}^{-k}\mathfrak{N}_k(\beta, \tilde{\mathfrak{A}}) + \tilde{\mathfrak{Q}}^{-k}\mathfrak{N}_k(\beta, \tilde{\mathfrak{A}})y) = (y\tilde{\mathfrak{A}} + \tilde{\mathfrak{A}}y) \cap ker(\mathfrak{s}_\beta) = (y\tilde{\mathfrak{A}} + \tilde{\mathfrak{A}}y) \cap a_\beta(\tilde{A})$. From [BK](1.4.10) we also obtain $a_\beta(\tilde{\mathfrak{Q}}^{-k}\mathfrak{N}_k(\beta, \tilde{\mathfrak{A}})) = \tilde{\mathfrak{A}} \cap a_\beta(\tilde{A})$, which implies $(y\tilde{\mathfrak{A}} + \tilde{\mathfrak{A}}y) \cap a_\beta(\tilde{A}) = y(\tilde{\mathfrak{A}} \cap a_\beta(\tilde{A})) + (\tilde{\mathfrak{A}} \cap a_\beta(\tilde{A}))y$.

Taking the intersection with A on both sides and including this into equation (15), we obtain $((\mathfrak{H}^1)^* + ((\mathfrak{H}^1)^*)^y) \cap ker(\mathfrak{s}_\beta) = a_\beta(\mathfrak{J}^1 + (\mathfrak{J}^1)^y) + \mathfrak{A} \cap a_\beta(A) + (\mathfrak{A} \cap a_\beta(A))^y$. Since, by [BK1](3.1.10) the equalities $\mathfrak{A} \cap a_\beta(A) = (\tilde{\mathfrak{A}} \cap a_\beta(\tilde{A})) \cap A = a_\beta(\tilde{\mathfrak{Q}}^{-k}\mathfrak{N}_k(\beta, \tilde{\mathfrak{A}})) \cap A \subset a_\beta(U^{[\frac{k+1}{2}]}(\tilde{\mathfrak{A}})\mathfrak{J}^1(\beta, \tilde{\mathfrak{A}})) \cap A = a_\beta(\mathfrak{J}^1)$ are true, where $k = -k_0(\beta, \tilde{\mathfrak{A}})$, we deduce that the sequence is exact at $(\mathfrak{H}^1)^* +$

$((\mathfrak{H}^1)^*)^y$. By [BK1](1.3.16) we also have $\mathfrak{Q}+\mathfrak{Q}^y \subset (\mathfrak{J}^1+(\mathfrak{J}^1)^y)\cap ker(a_\beta) \subset (\mathfrak{P}+\mathfrak{P}^y)\cap ker(a_\beta) = ((\tilde{\mathfrak{P}}+\tilde{\mathfrak{P}^y})\cap ker(a_\beta))\cap A = (\tilde{\mathfrak{Q}}+\tilde{\mathfrak{Q}}^y)\cap A = \mathfrak{Q}+\mathfrak{Q}^y$ and, therefore, also exactness at the other positions.

The rest of the proof is exactly the same as for [BK1](5.1.10), where we have to note that [BK1](5.1.5) holds in our more general case of an arbitrary local central simple algebra.

□

□

(10.7) Lemma *Let $([\mathfrak{A},n,0,\beta],\mathfrak{B}_m)$ be a non-degenerate pair and $\kappa_{\mathfrak{A}}$ be a representation as in Theorem (8.2). Let ξ be an irreducible representation of $J/J^1=\mathfrak{B}^\times/U^1(\mathfrak{B})$. Assuming that the whole of $B^\times$ intertwines $\kappa_{\mathfrak{A}}$ with itself, we have:*

(i) $I_{A^\times}(\kappa_{\mathfrak{A}}\otimes\xi)=JI_{B^\times}(\xi_{|\mathfrak{B}^\times})J$,

(ii) for all $y\in B^\times$, the spaces $I_y(\kappa_{\mathfrak{A}}\otimes\xi)$ and $I_y(\xi)$ have the same dimension and

(iii) the representation $\kappa_{\mathfrak{A}}\otimes\xi$ is irreducible.

Proof:

The proof is the same as in [BK1](5.3.2) if we use Lemma (5.1) instead of [BK1](5.1.1), Lemma (10.5) instead of [BK1](5.1.8) and our assumption on the intertwining of $\kappa_{\mathfrak{A}}$ instead of [BK1](5.2.7).

□

11 Maximal simple types

(11.1) Proposition *Let $\lambda=\kappa_{\mathfrak{A}}\otimes\sigma$ be a maximal simple type associated with a simple stratum $[\mathfrak{A},n,0,\beta]$ and assume that the whole of $B^\times$ intertwines $\kappa_{\mathfrak{A}}$ with itself, then we have*

(i) λ is a simple type for a single Bernstein component of the category of smooth representations, where the irreducible elements of this component just vary over a single inertial class of supercuspidal representations of $A^\times$.

(ii) λ and $\lambda'=\kappa_{\mathfrak{A}}\otimes\sigma'$ are types for the same Bernstein component iff the supercuspidal representations σ_0 and σ_0' of $Gl_{s'}(\mathfrak{k}_{D'})$ belong to the same Γ-orbit, where $\Gamma:=Gal(\mathfrak{k}_{D'}/\mathfrak{k}_{F[\beta]})$ acts on the entries of a matrix of $Gl_{s'}(\mathfrak{k}_{D'})$, hence, on the group and on the representations of the group.

Proof:

(i): By assumption on $\kappa_{\mathfrak{A}}$, Lemma (10.7)(iii) applies and we deduce that λ is irreducible. Further, the combination of (10.7)(i), [GSZ](2.1) and the assumption that $\mathfrak{B}$ is a maximal order, lead to the equality $I_{A^\times}(\lambda)=JI_{B^\times}(\sigma_{|\mathfrak{B}^\times})J=<h^l>J=:\tilde{J}$, where $h\in\mathcal{K}_{\mathfrak{B}}$ is an arbitrary but fixed uniformizer of $\mathfrak{B}$ and l is the length of the Galois orbit of $\sigma_{|\mathfrak{B}^\times}$ or σ_0, as in [GSZ](Section 2). As h^l normalizes the group J, we may extend the representation λ to a representation $\tilde{\lambda}$ of $\tilde{J}$.

By [Ca](1.5) the compactly induced representation $\Pi:=\text{c-Ind}_{\tilde{J}}^{A^\times}(\tilde{\lambda})$ is irreducible and supercuspidal.

Using arguments as in the proof of [GSZ](5.1) or [BK4](5.4) we conclude that λ is a type for a single Bernstein component of the category of smooth representations of $A^\times$, where the irreducible elements all belong to the inertial class of Π.

(ii) First we use arguments analogous to those in the beginning of the proof of [GSZ](5.2) to show that λ and λ' parametrize the same Bernstein component if σ and σ' resp. σ_0 and σ_0' are conjugated under the action of Γ. For this purpose we consider an extension $\tilde{\lambda}$ of λ to $\tilde{J}$ as in the proof of (i). As the adjoint action of h (as above) induces the action of a suitable generator of Γ, $\tilde{J}$ is a normal subgroup of index l in $< h > J = \mathcal{K}_{\mathfrak{B}} J$ and the induction of $\tilde{\lambda}$ to $\mathcal{K}_{\mathfrak{B}} J^1$ is irreducible, all h-conjugates must appear in its restriction to J. Therefore $\Pi = \text{c-Ind}_J^{A^\times}(\tilde{\lambda})$ also contains λ' and hence, by [BK4](5.2) λ and λ' parametrize the same Bernstein component.

Conversely, assume that λ and λ' parametrize the same Bernstein component. Then, by [BK4](5.2), they appear in the same irreducible representation; hence, by [BK5](6.5) they must intertwine. Now we use arguments as in the proof of [BK4](5.3.2) and (10.7), respectively.

Assume that $y \in A^\times$ intertwines λ and λ', then it also intertwines $\lambda_{|J^1}$ and $\lambda'_{|J^1}$, which are both multiples of $\eta_{\mathfrak{A}}$. By (5.1) we must have $y \in J^1 B^\times J^1$ and, as J^1 normalizes λ as well as λ', we may reduce to the case where $y \in B^\times$.

Let X be the representation space of $\kappa_{\mathfrak{A}}$, Y be the representation space of σ and Y' be the representation space of σ'. Then an intertwining operator $\phi \in Hom_{J^1 \cap (J^1)^y}(\lambda', \lambda^y)$ associated with y can be considered as an element of $End_{\mathbb{C}}(X) \otimes Hom_{\mathbb{C}}(Y', Y) \simeq Hom_{\mathbb{C}}(X \otimes Y', X \otimes Y)$. So let us write $\phi = \sum_j S_j \otimes T_j$ with $S_j \in End_{\mathbb{C}}(X)$ such that $\{T_j\}$ is a basis of $Hom_{\mathbb{C}}(Y', Y)$. Then $\sum_j (\kappa_{\mathfrak{A}}(h) \circ S_j - S_j \circ \kappa_{\mathfrak{A}}^y(h)) \otimes T_j = 0$ for all $h \in J^1 \cap (J^1)^y$ by the triviality of σ and σ' on arguments from J^1. By the linear independence of the T_j we conclude that all S_j lie in $I_y(\eta_{\mathfrak{A}})$. As, by assumption, $\kappa_{\mathfrak{A}}$ is intertwined by the whole of $B^\times$, all spaces $I_y(\eta_{\mathfrak{A}})$ are one-dimensional by (10.5), and $\kappa_{\mathfrak{A}}$ is an extension of $\eta_{\mathfrak{A}}$, we conclude that all S_j lie in fact in $I_y(\kappa_{\mathfrak{A}})$. Again, as the spaces $I_y(\kappa_{\mathfrak{A}})$ are one-dimensional, we have $\phi = S \otimes T$ for suitable $S \in I_y(\kappa_{\mathfrak{A}})$ and $T \in Hom_{\mathbb{C}}(Y', Y)$.

Now choose $h \in J \cap J^y$. Then the intertwining condition takes the form $S \circ \kappa_{\mathfrak{A}}^y \otimes T \circ \sigma^y = \kappa_{\mathfrak{A}} \circ S \otimes T \circ \sigma^y = \kappa_{\mathfrak{A}} \circ S \otimes \sigma' \circ T$ and, therefore, $T \in Hom_{J \cap J^y}(\sigma', \sigma^y)$. We deduce that $\sigma_{|\mathfrak{B}^\times}$ and $\sigma'_{|\mathfrak{B}^\times}$ intertwine in $B^\times$ and we have reduced to the level-0-case in the algebra B. Finally, we may follow the second part of the proof of [GSZ](5.2) to show that σ and σ' resp. σ_0 and σ'_0 belong to the same orbit under the action of Γ.

□

12 The case of tame $Gl_l(D)$ with l prime

We first remark that in the case of $G_l(D)$ all pairs $([\mathfrak{A}, n, 0, \beta], \mathfrak{B}_m)$ are non-degenerate by (7.1)(ii).

(12.1) Corollary *Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum such that $B = C_A(\beta)$ is a division algebra. Then the pair $([\mathfrak{A}, n, 0, \beta], \mathfrak{B}_m)$ is non-degenerate, the associated representation $\kappa_{\mathfrak{A}}$ from (8.2) is intertwined by the whole of $B^\times$ and the types from (10.2) are maximal types that fulfil the conclusions of (11.1).*

Proof:

If B is a division algebra, then we have $\mathfrak{B} = \mathfrak{o}_B = \mathfrak{B}_m$ and each such pair $([\mathfrak{A}, n, 0, \beta], \mathfrak{B}_m)$ is non-degenerate. The group $\mathcal{K}_{\mathfrak{B}}$ normalizes the group J and each representation $\eta_{\mathfrak{A}}$ so that each representation $\kappa_{\mathfrak{A}}$ is uniquely determined by properties that are stable under the action of $\mathcal{K}_{\mathfrak{B}}$. Therefore, the equivalence class of $\kappa_{\mathfrak{A}}$ is normalized by $\mathcal{K}_{\mathfrak{B}}$ and, in particular, $\kappa_{\mathfrak{A}}$ is intertwined by the whole of $B^\times = \mathcal{K}_{\mathfrak{B}}$. As $\mathfrak{B} = \mathfrak{o}_B$ is a maximal order in B, (11.1) can be applied.

□

(12.2) Corollary *Let $A = M_l(D)$, where l is prime and $[\mathfrak{A}, n, 0, \beta]$ is a simple stratum such that $\mathfrak{B} = \mathfrak{B}_M := \mathfrak{A} \cap B$ is a maximal order in $B := C_A(\beta) = M_l(D')$. If $\mathfrak{k}_{D'} \neq \mathbb{F}_2$, then all associated representations $\kappa_{\mathfrak{A}}$ from (8.2) are intertwined by the whole of $B^\times$ and the types from (10.2) are maximal types, satisfying the conclusions of (11.1).*

Proof:

As above let d be the index of D and $E := F[\beta]$. Of course, $[E:F]$ must be a divisor of d, because otherwise B is a division algebra and we are in the case (12.1).

Therefore, we may assume without restriction of generality that $E \subset D$ and, thus, $B = M_l(D')$, where $D' := C_D(\beta)$. Moreover, let $\mathfrak{B}$ be given in normal form, i.e., $\mathfrak{B} = M_l(\mathfrak{o}_{D'})$ and $\mathfrak{A} = M_l(\mathfrak{o}_D)$.

Let $\mathcal{J}_0^1 := \mathcal{J}^1(\beta, \mathfrak{o}_D)$ and $\mathcal{H}_0^1 := \mathcal{H}^1(\beta, \mathfrak{o}_D)$, then we see easily that $J^1 = I_l + M_l(\mathcal{J}_0^1)$ and $H^1 = I_l + M_l(\mathcal{H}_0^1)$.

For the proof of the proposition, we have to show that $\kappa := \kappa_{\mathfrak{A}}$ is intertwined by the whole of $B^\times$. Since $J = \mathfrak{B}^\times J^1$, we just need to consider representives y of the double cosets $\mathfrak{B}^\times \backslash B^\times / \mathfrak{B}^\times$ and assume that $y = diag(\pi'^{\nu_1} I_{n_1}, \ldots, \pi'^{\nu_r} I_r)$ with $\nu_1 > \ldots > \nu_r$, where π' is a prime element of $\mathfrak{o}_{D'}$ and $n = (n_1, \ldots, n_r)$ is a partition of l.

Let $M := diag(M_{n_1}(D) \times \ldots \times M_{n_r}(D))$ and let U and U^- be the groups of upper and lower unipotent blockmatrices, where we work with the block decomposition $A = (M_{n_i \times n_j}(D))_{i,j=1,\ldots,r}$. To the partition $n := (n_1, \ldots, n_r)$ we find a hereditary principal standard order $\mathfrak{B}_n \subset \mathfrak{B}$ with invariants $n_1, \ldots, n_r$. If we consider the orders in the transposed normal form ([G2](0.3)), i.e., in upper block triangular form modulo $\pi' M_l(\mathfrak{o}_{D'}) = \pi'\mathfrak{B} =: \mathfrak{Q}$, then the following facts, similar to [BK](5.2.9), hold:

(12.3) [BK1](5.2.9)

(i) y normalizes $\mathfrak{B}_n \cap M$ and

(ii) $\mathfrak{B} \cap y^{-1}\mathfrak{B}y \subset \pi'\mathfrak{B} + \mathfrak{B}_n \cap y^{-1}\mathfrak{B}_n y$.

We have $(\mathfrak{B}_n^\times J^1) \cap (\mathfrak{B}_n^\times J^1)^y = (\mathfrak{B}_n \cap M)((U^1(\mathfrak{B}_n)) \cap (U^1(\mathfrak{B}_n))^y)(J^1 \cap (J^1)^y)$ and $(J^1 \cap (J^1)^y) = ((U^1(\mathfrak{B})) \cap (U^1(\mathfrak{B}))^y)(J^1 \cap (J^1)^y)$.

First we show the

(12.4) intermediate assertion *that $\kappa_{|\mathfrak{B}_n^\times} J^1$ is intertwined by y.*

Proof of the intermediate assertion:

Let (Θ, β) be the PS-character with $\theta := \theta_{\mathfrak{A}} = \Theta(A, id_A, \mathfrak{A})$. We set $\theta_i := \Theta(M_{n_i}(D), p_i, \beta)$, $J_i^1 := p_i(J^1 \cap M)$ and $H_i^1 := p_i(H^1 \cap M)$ for $i = 1, \ldots, r$, where p_i is the projector to the i-th diagonal component of M. Then due to the product decomposition (2.9) and (3.1), we have the relations $H^1 \cap M = \prod_{i=1}^r H_i^1$ and $\theta_{|H^1 \cap M} = \bigotimes_{i=1}^r \theta_i$. It is easy to show that for the groups $J^1 \cap U$, $J^1 \cap U^-$, $H^1 \cap U$ and $H^1 \cap U^-$ the corresponding product decompositions (2.9) hold and, by (3.1), θ vanishes on $H^1 \cap U$ and $H^1 \cap U^-$.

As in the split case we obtain:

(12.5) [BK1](7.1.19), [BK1](7.2.3) *The form χ_θ of J^1/H^1 which is defined by application of θ to the commutators of J^1, fulfils the following conditions:*

(i) The restriction of χ_θ to $J^1 \cap M / H^1 \cap M$ is non-degenerate and is the direct orthogonal sum of the corresponding forms χ_{θ_i} of J_i^1/H_i^1 for $i = 1, \ldots, r$;

(ii) the spaces $J^1 \cap U / H^1 \cap U$ and $J^1 \cap U^- / H^1 \cap U^-$ are isotropic and orthogonal to $J^1 \cap M / H^1 \cap M$ with respect to χ_θ;

(iii) with respect to χ_θ we have an orthogonal decomposition $J^1/H^1 = J^1 \cap M / H^1 \cap M \perp J^1 \cap U / H^1 \cap U \times J^1 \cap U^- / H^1 \cap U^-$.

As in the split case we prove:

(12.6) [BK1](7.2.4) *Let $\eta = \eta_{\mathfrak{A}}$ (η_i respectively) be the Heisenberg representation of J^1 (J_i^1 respectively) attached to the Heisenberg character θ (resp. θ_i) and Φ be a character of the abelian group $J^1 \cap U/H^1 \cap U$. Then there exists a unique representation η_Φ of $J^1_{MU} := (J^1 \cap M)(J^1 \cap U)H^1$ up to isomorphism such that $(\eta_\Phi)_{|J^1 \cap M} \simeq \eta_1 \otimes \ldots \otimes \eta_r$, $(\eta_\Phi)_{|H^1}$ is a multiple of θ and $(\eta_\Phi)_{|J^1 \cap U}$ is a multiple of Φ.*

With the notations as above we also obtain
(i) $\eta_{|J^1_{MU}} \simeq \bigoplus_{\Phi \in J^1 \cap \widehat{U/H^1} \cap U} \eta_\Phi$;
(ii) $\forall \Phi \in J^1 \cap \widehat{U/H^1} \cap U : Ind_{J^1_{MU}}^{J^1}(\eta_\Phi) \simeq \eta$;
(iii) $\forall \Phi_1, \Phi_2 \in J^1 \cap \widehat{U/H^1} \cap U \exists x \in J^1 \cap U^-/H^1 \cap U^- : \eta_{\Phi_2} \simeq \eta^x_{\Phi_1}$.

Let us consider the representations $\eta_P := \eta_1$ and $\eta_U := (\eta_P)_{|J^1 \cap M} \simeq \eta_1 \otimes \ldots \otimes \eta_r$, where η_P can be considered as a representation of J^1_{MU} on the space of $J^1 \cap U$ fixed vectors in η, and η_U as a representation of $(J^1 \cap M)(J^1 \cap U)/(J^1 \cap U)$.

The representation $\kappa_{|\mathfrak{B}_n^\times J^1}$ fulfils the conditions of proposition (8.1) with respect to the characters $\theta^\star_{|\mathfrak{B}_n^\times H^1}$ with $\bar{J} := \mathfrak{B}_n^\times J^1$, $\bar{H} := \mathfrak{B}_n^\times H^1$, $\bar{J}^1 := J^1$, $\bar{J}_p := U^1(\mathfrak{B}_m)J^1$, $\bar{\eta} = \eta$ and $\eta' = \eta_{\mathfrak{A},\mathfrak{B}_m}$, where $\mathfrak{B}_m$ is a minimal order contained in $\mathfrak{B}_n$ and $\mathfrak{B}_n^\times J^1/J^1 \simeq P_n(\mathfrak{k}_{D'}) \subset Gl_l(\mathfrak{k}_{D'})$ the parabolic standard subgroup attached to the partition n.

Let us consider the representation $\eta_U \simeq \eta_1 \otimes \ldots \otimes \eta_r$ of $J^1 \cap M$. We want to apply proposition (8.2) to the character $\theta^\star_U = \theta^\star_1 \otimes \ldots \otimes \theta^\star_r$ and to η_U, where $\theta^\star_i$ is an extension of θ_i. For each i we obtain this extension from the same extension of the corresponding character λ_β from the defining sequence of θ. We obtain an extension κ_U of η_U from $J^1 \cap M$ to $(\mathfrak{B}_n^\times J^1) \cap M$, which is uniquely determined by the properties $(\kappa_U)_{|J^1 \cap M} = \eta_U$ and $(\kappa_U)_{|\mathfrak{B}_n^\times H^1 \cap M} \simeq \theta^\star_{|\mathfrak{B}_n^\times H^1 \cap M} \otimes W'$, where the determinant character of W' is trivial and $\theta^\star_{|\mathfrak{B}_n^\times H^1 \cap M} = \theta^\star_U$.

Now we extend κ_U trivially on $U^1(\mathfrak{B}_n)J^1 \cap U$ and $H^1 \cap U^-$. We obtain an extension κ_P of η_P to $(\mathfrak{B}_n^\times \cap M)(U^1(\mathfrak{B}_n) \cap U)J^1_{MU}$, which is intertwind by y due to the choice of the partition n, because y normalizes the group $(\mathfrak{B}_n^\times J^1) \cap M$, the representation η_U and the character $\theta^\star_U$ and does not violate the properties that characterize κ_U.

Now, we consider $\kappa' \simeq Ind^{\mathfrak{B}_n^\times J^1}_{(\mathfrak{B}_n^\times \cap M)(U^1(\mathfrak{B}_n) \cap U)J^1_{MU}}(\kappa_P)$, $\mathfrak{B}_n^\times J^1 \cap J^1 = J^1$ and $(\mathfrak{B}_n^\times \cap M)(U^1(\mathfrak{B}_n) \cap U)J^1_{MU} \cap J^1 = J^1_{MU}$. We obtain $\kappa'_{|J^1} \simeq Ind^{J^1}_{J^1_{MU}}((\kappa_P)_{|J^1_{MU}}) \simeq Ind^{J^1}_{J^1_{MU}}(\eta_P) \simeq \eta$. Since $\mathfrak{B}_n^\times J^1 \supset (\mathfrak{B}_n^\times \cap M)(U^1(\mathfrak{B}_n) \cap U)J^1_{MU}$ and since $\mathfrak{B}_n^\times J^1$ normalize the group $\mathfrak{B}_n^\times H^1$ and the character $\theta^\star$, we obtain $\kappa'_{|\mathfrak{B}_n^\times H^1} \simeq \bigoplus_{x \in \mathfrak{B}_n^\times J^1/\mathfrak{B}_n^\times H^1} \theta^\star_{|\mathfrak{B}_n^\times H^1} \otimes W'^x$, where we consider W' to be a representation of $\mathfrak{B}_n^\times H^1/H^1$.

Of course, the representation $\bigoplus_{x \in \mathfrak{B}_n^\times J^1/\mathfrak{B}_n^\times H^1} W'^x$ has a trivial determinant character and therefore, by (8.2), we have $\kappa_{|\mathfrak{B}_n^\times J^1} \simeq \kappa'$. However, since y intertwines κ_P and κ' is induced by κ_P, [BK1](4.1.3) leads to the proof of the intermediate assertion.

□

On account of the above properties (12.3)(i)(ii) of $\mathfrak{B}_n$ we have $J \cap J^y = (\mathfrak{B}^\times \cap (\mathfrak{B}^\times)^y)(J^1 \cap (J^1)^y) = (U^1(\mathfrak{B}) \cap (\mathfrak{B}^\times)^y)(U^1(\mathfrak{B}_n)J^1 \cap (U^1(\mathfrak{B}_n)J^1)^y)$. As in the split case, $(U^1(\mathfrak{B}) \cap (\mathfrak{B}^\times)^y) \subset (U^1(\mathfrak{B}) \cap (U^1(\mathfrak{B}))^y)(\mathfrak{B}'^\times_n \cap (\mathfrak{B}'^\times_n)^y)$, where $\mathfrak{B}'_n$ is the corresponding order in opposite normal form to the partition of n, i.e., $\mathfrak{B}'_n$ is of upper block traingular form modulo $\mathfrak{Q}$.

Now κ on $U^1(\mathfrak{B})$ is given by $\theta = \theta_{\mathfrak{A}}$ and intertwined by y. Also by the above argument, y intertwines the restriction to $U^1(\mathfrak{B}_n)J^1$. Replacing y by y^{-1}, $\mathfrak{B}_n$ by $\mathfrak{B}'_n$, U by U^- and so on, and observing that the intertwining is invariant under taking inverse elements, we conclude that y intertwines the restriction of κ to $\mathfrak{B}'^\times_n J^1$. Thus, the assertion follows from the relation $(\mathfrak{B}^\times \cap (\mathfrak{B}^\times)^y)(J^1 \cap (J^1)^y) \subset$

$(U^1(\mathfrak{B}) \cap (U^1(\mathfrak{B}))^y)(\mathfrak{B}_n'^\times \cap (\mathfrak{B}_n'^\times)^y)(U^1(\mathfrak{B}_n)J^1 \cap (U^1(\mathfrak{B}_n)J^1)^y)$ and because, by (10.5), all involved representations can be assumed to be intertwined by the same intertwining operator that finally turns out to be an intertwining operator for κ.

$\square$

(12.7) Proposition *Let $A = M_l(D)$ (l a prime number) and Λ be a type representation with irreducible factor τ as in Definition 10.2. Then we have an isomorphism of Hecke algebras $\mathcal{H}(A^\times, \Lambda) \simeq \mathcal{H}(B^\times, \tau_{|\mathfrak{B}^\times})$.*

Proof:

First, by Lemma 10.7, it is clear that the supports of the Hecke algebras $\mathcal{H}(A^\times, \Lambda)$ and that, $\mathcal{H}(B^\times, \tau_{|\mathfrak{B}^\times})$ have the same intersection with $B^\times$ and to any element x of the intersection, there exists a constant multiple of exactly one entire function f, which lives on a double coset JxJ and $\mathfrak{B}^\times x\mathfrak{B}^\times$, respectively. We obtain an isomorphism of $\mathbb{C}$-vector spaces between the algebras.

If $\Lambda = (\kappa_{\mathfrak{A}})_{|J} \otimes \mu \otimes \tau$ and τ is a lift of an entire supercuspidal representation σ_0 of $\mathfrak{B}_M^\times/(1 + \mathfrak{Q}_M) \simeq Gl_l(k)$ (also $\mathfrak{B} = \mathfrak{B}_M$), then the double cosets $J\backslash B^\times/J$, $\mathfrak{B}^\times\backslash B^\times/\mathfrak{B}^\times$ are represented by scalar matrices in $D^\times$ and there is a scalar matrix $h \in B^\times \cap D^\times I_l$, such that the cyclic group $\langle h \rangle$ forms a system of represetatives of the subsystems of double cosets that form the supports of the Hecke algebras. If f^A is then a function given by a non-trivial intertwining operator $\Phi \in I_h(\Lambda)$ (see [BK1](4.1.1)), then we map f^A to f^B by shifting the intertwining operator via the isomorphism of 10.7(ii). The functions f^A and f^B are invertible and the inverses are determined up to a multiple by any function supported on an entire double coset represented by h^{-1}. Indeed: Since h and h^{-1} normalize the groups J and $\mathfrak{B}^\times$, the products of the respective inverse elements are supported on J and $\mathfrak{B}^\times$ respectively. Moreover, the product has non-trivial value on 1, hence, a non-zero value, which can be normalized to be the identity. As $(f^A)^m$ and $(f^B)^m$ for $m \in \mathbb{Z}$ are functions with support on a single double coset represented by h^m, it easily follows that the $\mathbb{C}$-vector space isomorphism is a Hecke algebra isomorphism with respect to the convolutional product of functions.

In our special case of $A = M_l(D)$ we just have the other extreme case that $\mathfrak{B} \neq \mathfrak{B}_M$, where $\mathfrak{B}$ is a minimal principal order of period l. Then τ is the lifting of a tensor power $\otimes\sigma_0^l$, where σ_0 is the character of the multiplicative group of a finite field k, which is the residual field of the centralizer D' of D with respect to a pure element β, which can be embedded to D without restriction of generality.

Using the fact that the factor $(\kappa_{\mathfrak{A}})_{|J}$ of Λ extends to $\mathfrak{B}_M^\times J$, there is a canonical isomorphism $\mathcal{H}(\mathfrak{B}_M^\times, \tau_{|\mathfrak{B}^\times}) \simeq \mathcal{H}(\mathfrak{B}_M^\times J, \Lambda)$, given by tensoring a function by $\kappa_{\mathfrak{A}}$ (also see [BK](4)(5.6) and [GSZ](4.1)).

By [SZ] Lemma (4.4) we also have an isomorphy $\mathcal{H}(B^\times, \tau_{|\mathfrak{B}^\times}) \simeq \mathcal{H}(l, q^{d'})$ that extends the above isomorphism of the right-hand side and just depends on the choice of a function f^B, which lives on the double coset $\mathfrak{B}^\times h\mathfrak{B}^\times$, where

$$h := \begin{pmatrix} 0 & I_{l-1} \\ \pi'^f & 0 \end{pmatrix}.$$

Here q is the cardinality of the residual field $\mathfrak{k}_E$, where $E = F[\beta]$ and d' is the index of D'. We assume that $\mathfrak{B}$ is in standard normal form as in [G2](0.3). Let π' be a prime element of D' and f be the length of the galois orbit of χ under the action of $\mathrm{Gal}(k/k_E)$. To see the isomorphism on the left-hand side, we have to discuss the function f^A corrsponding to f^B by 10.7(ii) with support JhJ. Since the argumentation is the same as in the proof of [GSZ](4.4) and [BK1](5.6.6), respectively, we may reduce this question to an equality of supports. In detail, we have to show for arbitrary x from the support that the intersections of the triple classes $JxJhJ$ and $JxJh^{-1}J$ with the support of the Hecke algebra $\mathcal{H}(A^\times, \Lambda)$ are just $JxhJ$ and $Jxh^{-1}J$, respectively.

If we choose orders $\mathfrak{B}, \mathfrak{B}_M, \mathfrak{A}$ in simultaneous normal form, then for the corresponding minimal order $\mathfrak{A}_m$ in normal form we have $\mathfrak{A}_m^\times \supset J$. Note that $\mathfrak{A}_m$ is not assumed to be a continuation of $\mathfrak{B}$. Now it is obvious that it is sufficient to show that there is exactly one double coset from $\mathfrak{A}_m^\times \backslash B^\times / \mathfrak{A}_m^\times$ that appears in the Bruhat decomposition of the triple class $\mathfrak{A}_m^\times x \mathfrak{A}_m^\times h \mathfrak{A}_m^\times$ resp. $\mathfrak{A}_m^\times x \mathfrak{A}_m^\times h^{-1} \mathfrak{A}_m^\times$, which has a non-empty intersection with the support of the Hecke algebra, namely, $\mathfrak{A}_m^\times xh \mathfrak{A}_m^\times$ and $\mathfrak{A}_m^\times xh^{-1} \mathfrak{A}_m^\times$, respectively.

Indeed: Let $\nu_D(\pi') =: k$; then, if we multiply the support of the Hecke algebra from both sides by $\mathfrak{A}_m^\times$, the resulting double cosets are represented by $\tilde{W}_{A_{l,f\cdot k}}$ in the notation of [GSZ] and we may apply Lemma [GSZ](2.7) with $m = r = l$. It is easy to see that no representative can get lost, because the triple classes become larger and the system of representatives is respected.

Therefore, we obtain the desired result by the special structure of the possible choices of simultaneous normal forms for $\mathfrak{B}, \mathfrak{B}_M$ and $\mathfrak{A}$ in case of $A = M_l(D)$ and the result can be reduced to the level-0-case.

□

(12.8) Corollary *Let $A = M_l(D)$, l a prime number and Λ be a simple type in the sense of Definition 10.2. If $\mathfrak{B} \neq \mathfrak{B}_M$, then τ is lifted from a tensor power $\otimes \sigma_0^l$, where σ_0 is the character of the multiplicative group of a finite field. Then Λ is a covering type in the sense of [BK4] of a single Bernstein component with representations with supercuspidal support on a Levi group of the form $T(D) := \times D^l$.*

Let us be more precise. Let $\Lambda = (\kappa_{\mathfrak{A}})_{|J} \otimes \mu \otimes \tau$ and the fundamental character θ that generates $\kappa_{\mathfrak{A}}$, extended from $\mathcal{C}(\beta, \mathfrak{A})$, where β is without restriction of the generality embedded diagonal into the group $T(D)$ and the orders $\mathfrak{B}$, $\mathfrak{B}_M$ and $\mathfrak{A}$ are given in simultaneous normal form. Let $\mathcal{J}_D^1 := \mathcal{J}^1(\beta, \mathfrak{o}_D)$ and $\mathcal{J}_D := C_{\mathfrak{o}_D}(\beta) + \mathcal{J}^1(\beta, \mathfrak{o}_D)$, then J admit a decomposition $J = N_-(\mathcal{J}_D^1)T(\mathcal{J}_D^\times)N(\mathcal{J}_D)$, where N_- are the lower unipotent triangular matrices, N are the upper unipotent triangular matrices and T is the above Levi in the stated domain of coefficients. Furthermore, $\Lambda_{|T(\mathcal{J}_D^\times)}$ is a tensor power of a maximal simple type $\Lambda_D = \kappa_{\mathfrak{o}_D} \otimes \mu \otimes \sigma_0$, where we restrict μ to a diagonal component by letting the others trivial. The factor $\kappa_{\mathfrak{o}_D}$ corresponds to a fundamental character θ_D, which is, together with θ, a common extended value of a PS-character (see [G2](6.6)). The supercuspidal support of each irreducible subquotient of each representation generated by its Λ-isotypic component is determined up to an unramified twist by a tensor power $\otimes \Lambda_D^l$ (see [BK4]).

Proof:

The assertion of the corollary already contains some hints for the proof. According to [BK4](6)(8) we just have to find a totally $(T(D)N(D), J)$-positive element and an invertible function of the Hecke algebra $\mathcal{H}(A^\times, \Lambda)$ supported on the corresponding double coset.

Indeed, the element $t := \mathrm{diag}(\pi_F^l, \pi_F^{l-1}, \ldots, \pi_F)$ is totally $(T(D)N(D)), J)$-positive in the sense of [BK4](6.16) and because t is in $B^\times := C_A(\beta)$ and intertwines the representation τ, which is inflated from the diagonal, t is in the support of the Hecke algebra $\mathcal{H}(A^\times, \Lambda)$. There exists a non-trivial function f_t with support JtJ. Finally, by reduction to the level-0-case and using 12.7, the statement [GSZ](4.6) applies also in our case and f_t must be invertible. This completes the proof.

□

(12.9) Corollary *Let $A = M_l(D)$ (l a prime number) and $[\mathfrak{A}, n, 0, \beta]$ a simple stratum such that $\mathfrak{B} = \mathfrak{B}_M := \mathfrak{A} \cap B$ is a maximal order in $B := C_A(\beta) = M_l(D')$, where D' is a division algebra of the* *same Brauer type as* B*. If* $\mathfrak{k}_{D'} = \mathbb{F}_2$*, then the associated representation* $\kappa_{\mathfrak{A}}$ *from (8.2) is intertwined by the whole of* $B^\times$ *and the type representations from (10.2) are maximal types fulfilling the conclusions of (11.1).*

Proof:
In our special case we immediatly see that $\mathfrak{k}_{D'} = \mathfrak{k}_E = \mathfrak{k}_F = \mathbb{F}_2$ is the field with two elements and, in particular, $F[\beta] = E/F$ must be totally ramified and B must be split. Consequently the continuations of $\mathfrak{B}$ by $\mathfrak{A}$ and those of $\mathfrak{B}_m$ by $\mathfrak{A}_m$ are unramified and 7.3 applies. By the same argument, we may deduce in the manner of the proof of [BK1](5.1.19) that $\eta_{\mathfrak{A},\mathfrak{B}_m}$ must be intertwined by the whole of $B^\times$.

If $l \neq 2$, by 9.1 each extension of $\eta_{\mathfrak{A}}$ to J must be also an extension of $\eta_{\mathfrak{A},\mathfrak{B}_m}$. Moreover, another extension of $\eta_{\mathfrak{A}}$ to J as $\kappa_{\mathfrak{A}}$ must be a twist by a character of J trivial on $U^1(\mathfrak{B}_m)J^1$. Using the arguments of the proof of [BK1](5.2.4), we may show that the latter character factorizes through the determinant map of the finite linear group $\mathfrak{B}^\times/(1+\mathfrak{Q})$ and is lifted by the isomorphy $\mathfrak{B}^\times/(1+\mathfrak{Q}) \simeq J/J^1(\beta,\mathfrak{A})$ to J. But, as the elements of $\mathfrak{B}^\times/(1+\mathfrak{Q}) \simeq Gl_l(\mathbb{F}_2)$ have all determinant 1, there is no proper twist of $\kappa_{\mathfrak{A}}$.

If $l = 2$ and $J(\beta,\mathfrak{A})/J^1(\beta,\mathfrak{A}) \simeq Gl_2(\mathbb{F}_2) \simeq S_3$, by 9.1 we have exactly two extensions of $\eta_{\mathfrak{A}}$ to $J(\beta,\mathfrak{A})$, which are determined uniquely by their restrictions to $U^1(\mathfrak{B}_m)J^1(\beta,\mathfrak{A})$. By definition our extension $\kappa_{\mathfrak{A}}$ is now determined by $\eta_{\mathfrak{A},\mathfrak{B}_m}$ on that subgroup. The other extension is given by a twist with the sign character of S_3. However, as the restriction of the sign character to a transposition group corresponding to a group generated by an upper triangular unipotent matrix in $Gl_2(\mathbb{F}_2)$ is intertwined by a matrix of the form $\mathrm{diag}(\pi_E^{-1},1)$ with the trivial character, again, such a twist cannot lead to an extension as desired.

Therefore, the only possible extension of $\eta_{\mathfrak{A}}$ to J that is intertwined by the whole of $B^\times$ must be $\kappa_{\mathfrak{A}}$ and is an extension of $\eta_{\mathfrak{A},\mathfrak{B}_m}$. In our case this easily implies that the "Weil representation" W must be the identity and we are left to show that $\theta_{\mathfrak{A}}^\star$ is intertwined by the whole of $B^\times$. We leave this last fact to the reader or refer to [VS], where it is proved that a "β-extension" always exists.

□

(12.10) Corollary *In case* $A = M_l(D)$*, where* l *is a prime number, all hypothetical simple types are also types in the sense of [BK4](4.1) with Bernstein components containing irreducible essentially square integrable representations.*

Proof:
The Corollaries 12.1, 12.2, 12.9 and 12.8 cover all cases.

□

We may generalize Lemma [BK1](6.2.5) without a big change if we replace the determinant mapping by the reduced norm mapping and reformulate the modifications of invariants in the calculation within the proof. We obtain:

(12.11) Lemma *Let* A *be an arbitrary central simple algebra over* F *and* Π *be an irreducible supercuspidal representation containing the type* Λ *and fulfilling the assumptions of Proposition 11.1. If the tensor factor* $\kappa_{\mathfrak{A}}$ *is constructed from a simple character attached to a simple stratum* $[\mathfrak{A}, n, 0, \beta]$*, then we may determine the ramification index* $e(F[\beta]/F)$ *from the set of unramified characters* χ *satisfying* $\Pi \simeq \Pi \otimes \chi \circ Nrd_{A/F}$*. More precisely, these characters are exactly those of order dividing* $\frac{N}{e(F[\beta]/F)}$*, where* N *is the reduced degree of* A*.*

□

(12.12) Proposition *Let $A = M_l(D)$, where l is a prime number, be a tame algebra (see Introduction or 1.15). Let Π be an irreducible essentially square integrable representation of $Gl_l(D)$ of a minimal Artin conductor in its twist class, containing the types $\Lambda = \lambda \otimes \mu$ and $\Lambda' = \lambda' \otimes \mu'$, then, without restriction of generality, $\mu \equiv \mu' \equiv 1$, $\lambda = \kappa_{\mathfrak{A}} \otimes \tau$ and $\lambda' = \kappa_{\mathfrak{A}} \otimes \tau'$ have a common factor that comes from the same fundamental character, and finally the respective second factors τ and τ' are liftings of the tensor powers $\otimes\sigma_0^k$ and $\otimes(\sigma_0')^k$, respectively, with $k = 1$ or $k = l$, where σ_0 and σ_0' are Galois conjugated.*

Proof:

Indeed: A maximal type can only appear in a supercuspidal representation. But if Π is a discrete series representation, then it has a supercuspidal support and the associated type is the covering type for a supercuspidal representation of the Levi subgroup $T(D)$ as in 12.8. Therefore, we may apply Lemma 12.11 either to the type Λ or to the type Λ_D of a diagonal component as in 12.8.

Hence, we can recover the invariant $e(F[\beta]/F)$ and, by the choice of the period above (10.2), the invariant of the principal order $\mathfrak{A}$ from the representation Π. Therefore, the simple characters θ and θ' corresponding to the tensor factors of Λ and Λ', respectively, must be from a common fundamental character set $\mathcal{C}(\mathfrak{A})$. Also the underlying simple character must be from underlying simple strata $[\mathfrak{A}, n, 0, \beta]$ and $[\mathfrak{A}, n, 0, \beta']$, where n can be calculated from the Artin conductor of Π as calculated in [Z6] and [BF] as $s(\mathfrak{A})n$ ($s(\mathfrak{A})$ the invariant of $\mathfrak{A}$). As the characters $\theta_{|H^1(\beta,\mathfrak{A})}$ and $\theta'_{|H^1(\beta',\mathfrak{A})}$ appear in the same irreducible representation Π, they must intertwine. Consequently, by 1.15, they are conjugated in the normalizer of $\mathfrak{A}$.

We reduce to $\theta_{|H^1} = \theta'_{|H^1}$, where $H^1(\beta,\mathfrak{A}) = H^1(\beta',\mathfrak{A}) =: H^1$, and $J_A = \mathfrak{B}_M^\times J^1(\beta,\mathfrak{A}) = \mathfrak{B}_M^\times J^1(\beta',\mathfrak{A})$. Consequently, an intertwining element must be from $J_A B^\times J_A$ and J_A normalizes $\theta_{|H^1}$, we may choose it as a representative of a coset in $J_A \backslash B^\times / J_A$. Fixing an isomorphism $B^\times \simeq Gl_{m'}(D')$, we may choose the intertwining element as a monomial matrix in $D'^\times$. As it normalizes the set of diagonal matrices $\times(\mathfrak{o}_{D'}{}^\times)^{m'}$, the defining characters λ_β and $\lambda_{\beta'}$ of θ and θ', respectively, must coincide because the reduced norm mapping from D' to $F[\beta]$ maps $\mathfrak{o}_{D'}{}^\times$ surjective to $\mathfrak{o}_{F[\beta]}{}^\times = \mathfrak{o}_{F[\beta']}{}^\times$, because we assume that each representative of orbits in $\hat{\mathcal{C}}(\mathfrak{A})$ is extended from the same field $F[\beta]$ and also by the argument in [BF](1.4.5). We obtain $\theta = \theta'$.

The types λ and λ' have the same factor $(\kappa_{\mathfrak{A}})_{|J}$ and again, we may decide from the fact that the representation Π being supercuspidal or not whether the order $\mathfrak{B}$ is maximal or minimal. Applying 10.7, we deduce that the factors τ and τ' must intertwine in $B^\times$. The assertion on σ_0 and σ_0' can be deduced as in the prove of 11.1(ii).

Concerning the assertion on the minimal conductor, we must follow a part of the argument of the proof of 12.14 below. This is possible, because we avoid a cyclic argumentation. We use the abstract matching theorem [BDKV] and the fact that we already have greater knowledge on the side of the division algebra $\mathcal{D}$.

We first show in the general case that the representations containing the types λ (i.e., a type Λ with $\mu \equiv 1$) must have a minimal Artin conductor in its twist class with arbitrary twist. So let Π be chosen in such a way that they contain a λ and assume that $\Pi \otimes \bar{\mu}$ has a minimal Artin conductor in its twist class. If $[\mathfrak{A}, n, 0, \beta]$ is the stratum corresponding to $\kappa_{\mathfrak{A}}$, then $\Pi \otimes \bar{\mu}$ contains the character $(\psi_\beta^\times \bar{\mu})_{|U^n(\mathfrak{A})}$. If $\bar{\mu}$ is trivial on $U^{n+1}(\mathfrak{A})$, then there exists $\alpha \in F$ such that $\bar{\mu}_{|U^n(\mathfrak{A})} = \psi_\alpha^\times$. The representation $\Pi \otimes \bar{\mu}$ contains the pure stratum $[\mathfrak{A}, n, n-1, \beta + \alpha]$. Since the stratum $[\mathfrak{A}, n, n-1, \beta]$ is equivalent to a simple stratum $[\mathfrak{A}, n, n-1, \gamma]$ with $F[\gamma] \neq F$, by [BK1](1.4.14)(1.4.15)(eventual applied to a canonical splitting) also the stratum $[\mathfrak{A}, n, n-1, \gamma+\alpha]$ must be simple and equivalent to $[\mathfrak{A}, n, n-1, \beta+\alpha]$. Then in particular, $\psi_\beta^\times \bar{\mu}$ is on $U^n(\mathfrak{A})$ a non-trivial character, which is trivial on $U^{n+1}(\mathfrak{A})$ and given by $\psi_{\gamma+\alpha}^\times$. The latter character must be contained in an irreducible representation of $\mathcal{K}_\mathfrak{A}$, which, due to $\gamma+\alpha \in \mathcal{K}_\mathfrak{A}$, is non-degenerate in the sense of [BF](2.2.7)(2.2.8). By [BF](3.3.8)(iv) and the remark [BF](3.4)(C) or [Z6] the Artin conductor of $\Pi \otimes \bar{\mu}$ is equal to the Artin conductor of Π, which has the value $s(\mathfrak{A}) \cdot n$, where $s(\mathfrak{A})$ is the invariant of $\mathfrak{A}$. If $\bar{\mu}$ is now non-trivial on $U^{n+1}(\mathfrak{A})$, it is possible to show in the same way that $\Pi \otimes \bar{\mu}$ contains a minimal stratum $[\mathfrak{A}, n', n'-1, \alpha]$ with $\alpha \in F$ and $n' > n$. However, then the Artin conductor $s(\mathfrak{A})n'$ of $\Pi \otimes \bar{\mu}$ is greater than the Artin conductor of Π. As, the latter case cannot appear by assumption, Π was already minimal in its twist class.

As far as the division algebra case is concerned, also the statement with converse implication is valid. By [Z2] resp. [Z7] each irreducible representation of $\mathcal{D}^\times$ can be induced by an extension of a type representation with underlying minimal stratum $[\mathfrak{o}_\mathcal{D}, n, n-1, \gamma]$. Here, type representations like ours appear, i.e., $F[\gamma] \neq F$ or $\gamma \in F$. In the latter case the kernel of the representation can be enlarged by a character twist and because the induction commutes with character twist, we obtain the twisted representation by induction. But now the Artin conductor is given according to [Z6] resp. [BF] by the formula $-N\frac{\nu_F(\beta)}{e(F[\beta]/F)} = n$, i.e., the bigger the kernel of the type the smaller is the Artin conductor. If we now consider the corresponding types, then, we may assume inductively that they are from the first kind, where all twists make the Artin conductor smaller. Therefore, we obtain all representations of minimal conductor on the division algebra side, if we assume the first case, i.e., of type representations $\lambda_\mathcal{D}$ as constructed in this work. We will show later, within the proof of theorem 12.14, that the mapping at the beginning of 10 is, on the set of representations of minimal conductor, a permutation of the abstract matching theorem [BDKV], which is consistent with the character twist and in particular preserves Artin conductor and the central character. If we now assume Π to be of a minimal conductor in its twist class, then we may assume $\mu \equiv \mu' \equiv 1$.

We can use Lemma 10.7 to reduce the final assertion to the level-0-case in the algebra $B := C_A(\beta)$. The last assertion now follows from [GSZ]((5.2) resp. Theorem 1).

□

(12.13) Proposition *Let Λ be a type representation for $A := M_l(D)$, where l is a prime number. Let $\omega\Lambda$ be an extension of Λ to the group $F^\times J$. Then there exists a finite number of essentially square integrable representations with a central character ω containing $\omega\Lambda$. If $\Lambda = \lambda$ is a type representation corresponding to a type representation λ for a division algebra $\mathcal{D}$ of the same reduced degree N as A by the mapping described before 10.2 and supposed that the parameter χ has a certain Galois length f, then the last finite number is exactly $\frac{N}{f(F[\beta]/F)f}$, where β is a pure element that underlies the tensor factor of $\lambda = (\kappa_\mathfrak{A})_{|J} \otimes \chi$ as described above 10.2.*

<u>Proof:</u>

In the level-0-case we set $e(F[\beta]/F) = 1$ and refer to [SZ]. So let Λ be a type representation with non-trivial tensor factor κ and assume that Λ is a maximal type first.

If the centralizer $B = C_A(\beta)$ is a division algebra, then χ can also be considered to be a character of J and being of the same Galois length f. If π is a prime element of B, then the stabilizer of Λ is given by $\langle \pi^f \rangle J$ and each extension to this stabilizer induces an irreducible supercuspidal representation of $A^\times$. We obtain $(\langle \pi^f \rangle J : F^\times J)$ non-isomorphic representations containing $\omega\Lambda$. As $\pi^{\frac{N}{[F[\beta]:F]}}$ is up to a unit the prime element of $F[\beta]$, the assertion follows in this case.

If the centralizer is $B = C_A(\beta) = M_l(D')$, then D' has index $d' = \frac{N}{l[F[\beta]:F]}$. The representation σ_0 of $\mathfrak{B}^\times$ corresponding to χ has Galois length $\bar{f} := \langle f, d' \rangle$ by [SZ] and if π is a prime element of D', the assertion follows as in the first case by calculating the group index of $F^\times J$ in the stabilizer $\langle \pi^{\bar{f}} I_l \rangle J$ of Λ. As $\pi^{d'}$ is up to a unit a prime element of $F[\beta]$, we obtain the result by transforming the expression $\frac{d'}{f} e(F[\beta]/F)$. Here we note that l is prime to $\frac{N}{[F[\beta]:F]f}$ by our assumption.

Indeed: If l is a divisor of $\frac{N}{[F[\beta]:F]f}$, then Λ is a covering type and f is a divisor of the index d' of D'. The character χ factors via the norm through a field $\mathfrak{k}$ of degree f over the residual field $\mathfrak{k}_{F[\beta]}$. The factor σ_0 is now the uniquely determined character factoring through $\mathfrak{k}$, where $\mathfrak{k}$ is a subfield of $\mathfrak{k}_{D'}$ by assumption. If τ is the tensor factor of Λ, inflated from a tensor power $\otimes\sigma_0^l$ from $T(D) \cap J/T(D) \cap J^1(\beta_A, \mathfrak{A})$ to J and $\Omega\tau_{|\mathfrak{B}^\times}$ is an extension of $\tau_{|\mathfrak{B}^\times}$ to $F[\beta]^\times \mathfrak{B}^\times$, then, by [SZ], there are exactly $\frac{N}{[F[\beta]:F]f}$ irreducible representations of the discrete series of $B^\times$ containing $\Omega\tau_{|\mathfrak{B}^\times}$ and, thus, the above number contains $\omega\tau_{|\mathfrak{B}^\times}$. We use the Hecke algebra isomorphism 12.7, to show the corresponding assertion for our types Λ.

Since the arguments of proposition [BK1](7.7.4) and of Corollary [BK1](7.7.5) are founded on the fact that a representation Π containing a simple type Λ is generated by its Λ-isotypic component and uses a Morita equivalence, which was shown more generally in [BK4](2.12), the results of 12.8 and [BK4](4.2)(4.3) can be easily generalized to our case. If we fix a J invariant scalar product $\langle \ldots, \ldots \rangle_W$ on the representation space of Λ, then, for $\phi, \psi \in \mathcal{H}(A^\times, \omega\Lambda)$, a scalar product is given by $h_{\omega\Lambda}(\phi, \psi) := tr_W(\phi \star \bar{\psi}(1))$, where $\bar{\psi}(g)$ is as usual the adjoint element of $\psi(g^{-1})$ with respect to the form $\langle \ldots, \ldots \rangle_W$. By completion with respect to this scalar product on the algebra, we obtain a completed algebra $\mathcal{H}^2(A^\times, \omega\Lambda)$ that inherits the convolutional product and the unit element. The consequences of the considerations in [BK1] are now that a representation Π containing Λ is square integrable if and only if it is Morita equivalent to a $\mathcal{H}(A^\times, \omega\Lambda)$-module that can be embedded into $\mathcal{H}^2(A^\times, \omega\Lambda)$. Within this proof we have to reduce up to unramified character twist to the case that the central character ω of Π is unitary.

Similarly, we define the product $h_{\omega\tau}$ on the Hecke algebra $\mathcal{H}(B^\times, \omega\tau_{|\mathfrak{B}^\times})$, complete it to $\mathcal{H}^2(B^\times, \omega\tau_{|\mathfrak{B}^\times})$ and find a corresponding assertion also for the irreducible level-0-representations of $B^\times$ containing $\omega\tau$. We want to show that the number of square integrable irreducible representations of $A^\times$ containing $\omega\Lambda$ is as large as the number of irreducible square integrable representations of $B^\times$ containing τ. By the above arguments it is enough to show that the algebras $\mathcal{H}^2(A^\times, \omega\Lambda)$ and $\mathcal{H}^2(B^\times, \omega\tau)$ are isometrically isomorphic, i.e., that there is a Hecke algebra isomorphism that respects the forms $h_{\omega\Lambda}$ and $h_{\omega\tau}$.

Now the algebras $\mathcal{H}(F^\times \mathfrak{B}_M^\times, \omega\tau_{|\mathfrak{B}^\times}) \simeq \mathcal{H}(F^\times \mathfrak{B}_M^\times J, \omega\lambda)$ are isomorphic by the same argument as in 12.7, using the fact that the factor $\omega(\kappa_{\mathfrak{A}})_{|J}$ of $\omega\Lambda$ extends to $F^\times \mathfrak{B}_M^\times J$. The isomorphism preserves support of functions and works in the same manner as 12.7, where we use that ω is trivial on π_F, i.e., Λ, $\tau_{|\mathfrak{B}^\times}$ are extended trivially by ω to the infinite cyclic group generated by π_F. Therefore, the isomorphy only depends on the functions f^A and f^B, respectively, or more exactly on their images under a suitable projector. Indeed, the Hecke algebras of the extended representation by the central character ω are quotients of the Hecke algebras in 12.7. For the corresponding projections f_ω^A and f_ω^B we have the same relations $e_{\omega\Lambda} - (f_\omega^A)^{r(\mathfrak{B})d'e(F[\beta]/F)} = 0$, $e_{\omega\tau} - (f_\omega^B)^{r(\mathfrak{B})d'e(F[\beta]/F)} = 0$ respectively, where $r(\mathfrak{B})$ is the period of $\mathfrak{B}$, d' is the index of B and $e_{\omega\Lambda}$ and $e_{\omega\tau}$ are the unit elements of $\mathcal{H}(A^\times, \omega\Lambda)$ and $\mathcal{H}(B^\times, \tau_{|\mathfrak{B}^\times})$ respectively.

The functions f_ω^A and f_ω^B up to a scalar multiple can be described explicitly by $f_\omega^A(g) \sim \sum_{n=-\infty}^{+\infty} f(\pi_F^n \cdot g)$ resp. $f_\omega^B(b) \sim \sum_{n=-\infty}^{+\infty} f(\pi_F^n \cdot b)$ and it is clear that the mapping that sends f_ω^A to f_ω^B extends the above Hecke algebra isomorphism to an isomorphy $\mathcal{H}(A^\times, \omega\Lambda) \simeq \mathcal{H}(B^\times, \omega\tau_{|\mathfrak{B}^\times})$ preserving support of functions (see 12.7). On the other hand, the forms $h_{\omega\Lambda}$ and $h_{\omega\tau}$ are defined by evaluating a product of functions on the unit element and then taking the trace. Therefore, the two forms differ just by a constant factor depending on the dimensions of representations and the choice of measures if applied to functions on the left-hand side and images of isomorphism on the right-hand side (see [BK1] page 260). As the isomorphism preserve forms up to a harmless factor, we may extend it to the completions $\mathcal{H}^2(A^\times, \omega\Lambda) \simeq \mathcal{H}^2(B^\times, \omega\tau_{|\mathfrak{B}^\times})$.

Since we know that the number of irreducible essentially square integrable representations of $B^\times$ containing $\omega\tau_{|\mathfrak{B}^\times}$ is given by the formula of our proposition, the same formula is valid for the number of irreducible square integrable representations of $A^\times$ containing $\omega\Lambda$ as required.

□

(12.14) Theorem *Let $A = M_l(D)$ be a tame algebra and l a suitable prime number, then, assuming that Conjecture 1.16 holds true, the list 10.2 is a complete list of types for those Bernstein components containing an irreducible essentially square integrable representation. In particular, all irreducible supercuspidal representations of $Gl_l(D)$ are induced from a compact modulo center subgroup.*

Proof:

Again, we want to return to the construction in the beginning of Section 10. Let $\mathcal{D}$ be a division algebra of the same index N as A. To a type $\lambda_{\mathcal{D}} = \kappa \otimes \chi$ for an unramified twist class of irreducible representations of $\mathcal{D}^\times$ of a minimal conductor in its twist class with arbitrary twist, can be assigned via the mapping described at the beginning of Section 10 a type λ for a Bernstein component containing an unramified twistclass of essentially square integrable representations, which is of a minimal conductor in its twist class with arbitrary twist, too. By [BF] and [Z6], respectively, this mapping preserves Artin conductor and commutes with the character twist. Due to 12.12 the essentially square integrable representations containing λ have a minimal Artin conductor in there twist classes. Now we may extend $\lambda_{\mathcal{D}}$ and λ by a central character ω to $F^\times J_{\mathcal{D}}$ and $F^\times J$, respectively. By [G2](6.4) 8.2 and [SZ] the restrictions to ${\mathfrak{o}_F}^\times$ lead to the same central linear characters of $\lambda_{\mathcal{D}}$ and λ.

Indeed: As $\mathfrak{o}_F^\times / U^1(\mathfrak{o}_F)$ is of finite order prime to p and the Weil representation has the determinant character 1 and p-power dimension, the Weil representation must be trivial on this quotient and for the factors $\kappa_{\mathcal{O}}$ and $\kappa_{\mathfrak{A}}$ it is enough to compare the fundamental characters. Linkwise the case of $J/J^1(\beta, \mathfrak{A}) \simeq Gl_2(\mathbb{F}_2)$ does not cause any problem, because the quotient $\mathfrak{o}_F^\times / U^1(\mathfrak{o}_F)$ has only one element in this case. The level-0-factor can be compared successfully using [SZ]. By 12.13 the extensions $\omega\lambda_{\mathcal{D}}$ and $\omega\lambda$ parametrize the same finite number of essentially square integrable representations containing the same extended simple type.

Finally, from [Z2] and [Z7], we know that these finite packets for varying $\lambda_{\mathcal{D}}$ and ω exhaust, on the division algebra side, all irreducible smooth representations with minimal conductor. The packets are also pairwise disjoint and the mapping explained above leads, assuming 1.16 and using 12.12, to an injection of the irreducible representations of $\mathcal{D}^\times$ of a minimal conductor into the set of all irreducible essentially square integrable representations of a minimal conductor of $A^\times$. This injection preserves the central character and the Artin conductor of the parametrized representations. By [Kr], there are only a finite number of field extensions $F[\beta]/F$ of bounded finite degree and, hence, to a fixed n up to conjugacy, just a finite number of simple strata $[\mathfrak{A}, n, 0, \beta]$. There are only a finite number of fundamental characters to such a fixed stratum; hence, the number of corresponding extensions $\kappa_{\mathfrak{A}}$ is finite. Finally, there are only a finite number of level-0-components χ on the division algebra side and, therefore, a finite number of corresponding types $\omega\lambda$ for fixed ω and fixed Artin conductor.

Indeed: As by [Z6] the Artin conductor for representations containing λ are given by the expression $s(\mathfrak{A})n = -N\frac{\nu_{F[\beta]}(\beta)}{e(F[\beta]/F)}$, there are only a finite number of possiblities to choose strata $[\mathfrak{A}, n, 0, \beta]$ to get Bernstein components containing irreducible essentially square integrable representations with these

Artin conductor and if we fix the central character ω, we take just a finite number of representations. At least the corresponding assertion for the division algebra side is clear, but the mapping of the abstract matching theorem [BDKV] is surjective and also preserves the Artin conductor and the central character; hence, maps finite packets to finite packets. Therefore, the mapping of packets given by the above correspondence of extended types must be surjective, too.

The matching map [BDKV] is consistent with the character twist and on the division algebra part we obtain all unramified twist classes of irreducible representations by twist with varying arbitrary μ. Hence, also the types Λ parametrize all Bernstein components containing essentially square integrable representations of $A^{\times}$.

In fact, we have just handled the case of 10.2(i), but from [SZ] we know that the abstract matching theorem preserves the level-0-case 10.2(ii), which, for this case, leads to another proof of completeness of construction independent of [GSZ].

Now the Proposition 11.1 and its proof lead to the last assertion.

□

13 References

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