# ON ISOTROPY OF QUADRATIC PAIR

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Abstract. Let $F$ be an arbitrary field (of arbitrary characteristic). Let $A$ be a central simple $F$-algebra endowed with a quadratic pair $\sigma$ (if $\operatorname{char} F \neq 2$ then $\sigma$ is simply an orthogonal involution on $A$). We show that the Witt index of $\sigma$ over the function field of the Severi-Brauer variety of $A$ is divisible by the Schur index of the algebra $A$.

## 1. Introduction

Let $F$ be a field (of arbitrary characteristic). Let $A$ be a central simple $F$-algebra endowed with a quadratic pair $\sigma$ (cf. §2.1, a reader without interest in characteristic 2 may replace $\sigma$ by an orthogonal involution).

Let $X$ be the Severi-Brauer variety of the algebra $A$ (cf. §2.5) and let $F(X)$ stands for the function field of $X$. We show (Theorem 3.3) that the Witt index $\operatorname{ind} \sigma_{F(X)}$ of the quadratic pair $\sigma_{F(X)}$ (cf. §2.4) is divisible by the Schur index $\operatorname{ind} A$ of the algebra $A$.

This result generalizes [6, Theorem 5.3] stating that $\sigma_{F(X)}$ is anisotropic provided that $A$ is a division algebra.

Besides, this result supports the affirmative answer to the following

**Question 1.1.** *Assume that the quadratic pair $\sigma_{F(X)}$ is isotropic. Does the $F$-variety $Y_d$ of right $\sigma$-isotropic ideals in $A$ of reduced dimension $d = \operatorname{ind} A$ possess a 0-dimensional cycle of degree 1?*

Indeed, $Y_d(F(X)) \neq \emptyset$ by Theorem 3.3.

If $\operatorname{ind} A = 2$, then Question 1.1 is answered in the affirmative for $A$ and moreover $Y_2(F) \neq \emptyset$ by [8, Corollary 3.4]. We recall that in general it is not known whether $Y_d(F) \neq \emptyset$ provided that the variety $Y_d$ has a 0-dimensional cycle of degree 1 (cf. [2, Question after Proposition 4.1]).

## 2. Preliminaries

A *variety* is a separated scheme of finite type over a field.

2.1. **Quadratic pairs.** Let $A$ be a central simple $F$-algebra. A *quadratic pair* $\sigma$ on $A$ is given by an involution of the first kind $\tilde{\sigma}$ on $A$ together with a linear map $\sigma'$ of the space of the $\tilde{\sigma}$-symmetric elements of $A$ to $F$, subject to certain conditions (cf. [7, Definition (5.4) of Chapter I]).

If $\operatorname{char} F \neq 2$, then $\tilde{\sigma}$ is an arbitrary orthogonal involution on $A$ and the map $\sigma'$ is determined by $\tilde{\sigma}$. Therefore the notion of quadratic pair is equivalent to the notion of orthogonal involution in characteristic $\neq 2$.

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If $\operatorname{char} F = 2$, then the algebra $A$ is of even degree and the involution $\tilde{\sigma}$ is of symplectic type.

In arbitrary characteristic, any quadratic form on a finite-dimensional vector space $V$ over $F$ such that its polar bilinear form is non-degenerate, produces a quadratic pair on the endomorphisms algebra $\operatorname{End}(V)$, called the quadratic pair *adjoint* to the quadratic form (the involution of the adjoint quadratic pair is the involution adjoint to the polar symmetric bilinear form of the quadratic form). This way one gets a bijection of the set of quadratic forms on $V$ (up to a factor in $F^{\times}$) having non-degenerate polar forms onto the set of quadratic pairs on $\operatorname{End}(V)$.

A right ideal $I$ of a central simple $F$-algebra $A$ endowed with a quadratic pair $\sigma$ is called *isotropic* or *$\sigma$-isotropic*, if $\tilde{\sigma}(I) \cdot I = 0$ and $\sigma'$ is 0 on the part of $I$ where $\sigma'$ is defined (we mean that $\sigma'$ is 0 on the set of $\tilde{\sigma}$-symmetric elements of $I$).

Dimension over $F$ of any right ideal of $A$ is divisible by the degree $\deg A$ of $A$; the quotient is called the *reduced dimension* of the ideal.

Let $r \geq 0$ be an integer. The variety $Y_r$ of the right $\sigma$-isotropic ideals in $A$ of reduced dimension $r$ is empty if $r > (\deg A)/2$. For $r \leq (\deg A)/2$, $Y_r$ is a projective homogeneous variety under the action of the linear algebraic group $\operatorname{Aut}(A, \sigma)$. In particular, $Y_0$ is $\operatorname{Spec} F$ with the trivial action. If $r < (\deg A)/2$ then $Y_r$ is a projective homogeneous variety under the action of the connected linear algebraic group $\operatorname{Aut}(A, \sigma)^{\mathrm{o}}$ (connected component of $\operatorname{Aut}(A, \sigma)$); in particular, $Y_r$ is integral for such $r$. If $r = (\deg A)/2$ (for even $\deg A$) and the discriminant of the quadratic pair $\sigma$ (cf. [7, §7B of Chapter II]) is trivial, then the variety $Y_r$ has two connected components each of which is a projective homogeneous variety under $\operatorname{Aut}(A, \sigma)^{\mathrm{o}}$; these components are isomorphic to each other if and only if the algebra $A$ is split.

If $A = \operatorname{End}(V)$ and $\sigma$ is adjoint to a quadratic form $\varphi$ on $V$, then for any $r$, Morita equivalence identifies the variety $Y_r$ with the variety of $r$-dimensional totally isotropic subspaces of $V$. In particular, $Y_1$ is the projective quadric of $\varphi$.

2.2. **Chow groups.** Let $X$ be a variety over $F$. A *splitting field* of $X$ is a field extension $E/F$ such that the Chow motive of $X_E$ is a direct sum of twists of the motive of the point $\operatorname{Spec} E$. Any projective homogeneous (under an action of a linear algebraic group) variety (in particular, each variety $Y_r$ of §2.1) has a splitting field.

Given a variety $X$ over $F$, we write $\operatorname{Ch}(X)$ for the Chow group modulo 2 (i.e., with coefficients $\mathbb{Z}/2\mathbb{Z}$) of $X$. As in [4, §72], we write $\operatorname{Ch}(\bar{X})$ for the colimit $\operatorname{colim}_L \operatorname{Ch}(X_L)$ over all field extensions $L$ of $F$. Note that for any splitting field $E/F$ of $X$ the canonical homomorphism $\operatorname{Ch}(X_E) \to \operatorname{Ch}(\bar{X})$ is an isomorphism.

We write $\bar{\operatorname{Ch}}(X)$ for the image of the homomorphism $\operatorname{Ch}(X) \to \operatorname{Ch}(\bar{X})$. An element of $\operatorname{Ch}(\bar{X})$ is called *rational* or *$F$-rational*, if it is inside of $\bar{\operatorname{Ch}}(X)$.

2.3. **Varieties of isotropic ideals.** Let $A$ be a central simple $F$-algebra endowed with a quadratic pair $\sigma$. Let $Y = Y_1$ be the variety of right $\sigma$-isotropic ideals in $A$ of reduced dimension 1. Note that for any splitting field $L/F$ of the algebra $A$, the variety $Y_L$ is isomorphic to a projective quadric. Therefore $\operatorname{Ch}(\bar{Y})$ has an $\mathbb{F}_2$-basis given by the elements $h^i, l_i$, $i = 0, \ldots, [\dim Y/2]$, introduced in [4, §68].

For any $i \geq 0$, the element $h^i$ is the $i$th power of the hyperplane section $h$ (in particular, $h^0 = [Y]$ and $h^i = 0$ for $i > \dim Y$). The element $h$ can be non-rational. It is $L$-rational, where $L$ is a splitting field of the algebra $A$.

The element $l_i$ is the class of an $i$-dimensional linear subspace lying inside of $Y_E$ (where $E$ is a splitting field of $Y$). If $i \neq \dim Y/2$, then this class does not depend on the choice of the linear subspace. However in the case of even $\dim Y$ and $i = \dim Y/2$, there are (exactly) two different classes of $i$-dimensional linear subspaces on $Y_E$ (and their sum is equal to $h^i$). An *orientation* of the variety $Y$ is a choice of one of these two classes.

The basis elements of $\mathrm{Ch}(\bar{Y})$ introduced above satisfy the formula $hl_i = l_{i-1}$ for $i = 1, \dots, [\dim Y/2]$.

2.4. **Witt index.** Let $\sigma$ be a quadratic pair on a central simple $F$-algebra $A$. There exists an integer $\operatorname{ind}\sigma$ such that $\{0, \operatorname{ind} A, 2\operatorname{ind} A, \dots, (\operatorname{ind}\sigma)(\operatorname{ind} A)\}$ is the set of the reduced dimensions of the right $\sigma$-isotropic ideals in $A$. We call $\operatorname{ind}\sigma$ the (Witt) index of the quadratic pair $\sigma$. It satisfies the inequalities $0 \leq \operatorname{ind}\sigma \leq (\deg A)/(2\operatorname{ind} A)$.

Now we assume that the algebra $A$ is split (this is in fact the only case where we use the definition of the Witt index of a quadratic pair). Then the quadratic pair $\sigma$ is adjoint with respect to some non-degenerate quadratic form $\varphi$ (whose similarity class is uniquely determined by $\sigma$) and $\operatorname{ind}\sigma$ is the Witt index of $\varphi$. If $Y$ is the variety of right $\sigma$-isotropic ideals in $A$ of reduced dimension 1 and $i$ is a nonnegative integer, then the element $l_i \in \mathrm{Ch}(\bar{Y})$ is rational if and only if $i < \operatorname{ind}\sigma$, cf. [4, Corollary 72.6].

2.5. **Severi-Brauer varieties.** Let $A$ be a central simple $F$-algebra. Let $X$ be the Severi-Brauer variety of $A$, that is, the variety of all right ideals in $A$ of reduced dimension 1. Over any splitting field (of $A$ or, equivalently, of $X$), the variety $X$ is isomorphic to a projective space of dimension $(\deg A) - 1$. For any $i \geq 0$, we write $h^i \in \mathrm{Ch}^i(\bar{X})$ for the $i$th power of the hyperplane class $h \in \mathrm{Ch}^1(\bar{X})$. Therefore, for any $i$ with $0 \leq i \leq \dim X$, $h^i$ is the only nonzero element of the group $\mathrm{Ch}^i(\bar{X})$. Note that $h^i$ is rational if $i$ is divisible by $\operatorname{ind} A$ (cf. [1]).

Now we assume that the Schur index of $A$ is a power of 2. Then by [5, Proposition 2.1.1], we have $\bar{\mathrm{Ch}}^i(X) = 0$ for all $i$ not divisible by $\operatorname{ind} A$. Let us additionally assume that $A$ is a division algebra. Since the (say, first) projection $X^2 \to X$ is a projective bundle, we have a (natural with respect to the base field change) isomorphism $\mathrm{Ch}_{\dim X}(X^2) \simeq \mathrm{Ch}(X)$ Passing to $\bar{\mathrm{Ch}}$, we get an isomorphism $\bar{\mathrm{Ch}}_{\dim X}(X^2) \simeq \bar{\mathrm{Ch}}(X) = \bar{\mathrm{Ch}}^0(X)$ showing that $\dim_{\mathbb{F}_2} \bar{\mathrm{Ch}}_{\dim X}(X^2) = 1$. Since the diagonal class

$$h^0 \times h^{(\deg A)-1} + h^1 \times h^{(\deg A)-2} + \cdots + h^{(\deg A)-1} \times h^0 \in \bar{\mathrm{Ch}}_{\dim X}(X^2)$$

is nonzero, it follows that this is the *only* nonzero element of the group. This result is generalized in Lemma 3.1 below.

## 3. Witt index

Let $F$ be a field.

**Lemma 3.1.** *Let $A$ be a central simple $F$-algebra such that the Schur index of $A$ is a power of 2. Let $d = \operatorname{ind} A$ and $n = (\deg A)/(\operatorname{ind} A)$. Let $X_A$ be the Severi-Brauer variety* *of $A$. Let $X$ be the Severi-Brauer variety of a central division $F$-algebra Brauer-equivalent to $A$. Then for any $r = 1, \dots, n$, the element*

$$h^0 \times h^{rd-1} + h^1 \times h^{rd-2} + \dots + h^{d-1} \times h^{(r-1)d}$$

*is the only nonzero element of the group $\bar{\mathrm{Ch}}^{rd-1}(X \times X_A)$.*

*Proof.* Let $r$ be an integer satisfying $1 \le r \le n$.

Since the projection $X \times X_A \to X$ is a projective bundle and $\bar{\mathrm{Ch}}^i(X) = 0$ for $i \neq 0$ (cf. §2.5), we have an isomorphism $\bar{\mathrm{Ch}}^j(X \times X_A) \simeq \bar{\mathrm{Ch}}^0(X)$ for any $j = 0, \dots, \dim X_A$ (and, in particular, for $j = rd - 1$). Therefore the group $\bar{\mathrm{Ch}}^j(X \times X_A)$ has only one nonzero element for such $j$ (for $j > \dim X_A$ this group is zero).

We write $D$ for a central division $F$-algebra Brauer equivalent to $A$. We fix an isomorphism of $A$ with the tensor product $D \otimes M_n(F)$ where $M_n(F)$ is the algebra of square $n$-matrices over $F$. Tensor product of ideals produces a closed embedding $X \times \mathbb{P}^{n-1} \hookrightarrow X_A$ (which is a twisted form of the Segre embedding). Picking up a rational point of $\mathbb{P}^{n-1}$ we get a closed embedding $in : X \hookrightarrow X_A$ such that for any splitting field $E$ of $X$ the image of $X_E$ is a linear subspace of the projective space $(X_A)_E$. The image of the diagonal class under the push-forward with respect to the closed embedding $\mathrm{id}_X \times in : X^2 \hookrightarrow X \times X_A$ is equal to

$$\alpha_n = h^0 \times h^{nd-1} + h^1 \times h^{kd-2} + \dots + h^{d-1} \times h^{(n-1)d}\ .$$

It follows that $\alpha_n$ is the only nonzero element of the group $\bar{\mathrm{Ch}}^{nd-1}(X \times X_A)$.

A basis of $\mathrm{Ch}^{rd-1}(\bar{X} \times \bar{X}_A)$ is given by the elements

$$h^0 \times h^{rd-1}, h^1 \times h^{rd-2}, \dots, h^{d-1} \times h^{(r-1)d}.$$

Let

$$\alpha_r = a_0 h^0 \times h^{rd-1} + a_1 h^1 \times h^{rd-2} + \dots + a_{d-1} h^{d-1} \times h^{(r-1)d}$$

with some $a_0, a_1, \dots, a_{d-1} \in \mathbb{Z}/2\mathbb{Z}$ be the nonzero element of the subgroup

$$\bar{\mathrm{Ch}}^{rd-1}(X \times X_A) \subset \mathrm{Ch}^{rd-1}(\bar{X} \times \bar{X}_A).$$

Since

$$\alpha_r \cdot (h^0 \times h^{(n-r)d}) = a_0 h^0 \times h^{nd-1} + a_1 h^1 \times h^{nd-2} + \dots + a_{d-1} h^{d-1} \times h^{(n-1)d}$$

is a nonzero element of the group $\bar{\mathrm{Ch}}^{nd-1}(X \times X_A)$ (here we use the fact that the element $h^{(n-r)d} \in \mathrm{Ch}(\bar{X}_A)$ is rational, mentioned in §2.5), it follows that $\alpha_r \cdot (h^0 \times h^{(n-r)d}) = \alpha_n$, i.e., $a_0 = a_1 = \dots = a_{d-1} = 1$. □

**Proposition 3.2.** *Let $A$ be a central simple $F$-algebra. Let $d = \mathrm{ind}\, A$. Let $\sigma$ be a quadratic pair on $A$. Let $X$ be the Severi-Brauer variety of a central simple $F$-algebra Brauer-equivalent to $A$. Let $Y$ be the variety of $\sigma$-isotropic ideals of reduced dimension 1 in $A$. If for some integer $r \ge 0$ the cycle $l_{rd} \in \mathrm{Ch}(\bar{Y})$ is $F(X)$-rational, then (for an appropriately chosen orientation of $Y$) the cycle $l_{(r+1)d-1} \in \mathrm{Ch}(\bar{Y})$ is $F$-rational.*

*Proof.* Let $D$ be a central division $F$-algebra Brauer-equivalent to $A$. We may assume that $X$ is the Severi-Brauer variety of $D$.

Since the algebra $A$ possesses an involution of the first type, the index $d$ of $A$ is a power of 2.

If $d = 1$, then there is nothing to prove. We assume that $d \geq 2$ in the sequel.

For any field extension $L/F$, the pull-back homomorphism $\mathrm{Ch}(X_L \times Y_L) \to \mathrm{Ch}(Y_{L(X)})$ with respect to the morphism $Y_{L(X)} = (\operatorname{Spec} L(X)) \times Y \to X_L \times Y_L$ given by the generic point of $X$ is surjective by [4, Corollary 57.11]. These pull-backs give a surjection

$$f : \mathrm{Ch}(\bar{X} \times \bar{Y}) \to \mathrm{Ch}(\bar{Y})$$

such that the image of the subgroup of rational cycles in $\mathrm{Ch}(\bar{X} \times \bar{Y})$ is the subgroup of $F(X)$-rational cycles in $\mathrm{Ch}(\bar{Y})$. Since the external product $\mathrm{Ch}(\bar{X}) \otimes \mathrm{Ch}(\bar{Y}) \to \mathrm{Ch}(\bar{X} \times \bar{Y})$ is an isomorphism (cf. [4, Proposition 64.3]), the external products of the basis elements of $\mathrm{Ch}(\bar{X})$ with the basis elements of $\mathrm{Ch}(\bar{Y})$ form a basis of $\mathrm{Ch}(\bar{X} \times \bar{Y})$. The homomorphism $\mathrm{Ch}(\bar{X} \times \bar{Y}) \to \mathrm{Ch}(\bar{Y})$ is easily computed in terms of this basis: for any basis element $\beta \in \mathrm{Ch}(\bar{Y})$, the image of $h^0 \times \beta$ is $\beta$ and the image of any other basis element of $\mathrm{Ch}(\bar{X} \times \bar{Y})$ is 0.

We fix an integer $r \geq 0$ such that the cycle $l_{rd} \in \mathrm{Ch}(\bar{Y})$ is $F(X)$-rational. Let $s$ stands for the integer $(\deg A)/2 - 1$. Note that $rd \leq s$ (otherwise, the cycle $l_{rd}$ is not defined at all). Let $\alpha \in \mathrm{Ch}(\bar{X} \times \bar{Y})$ be a rational cycle whose image in $\mathrm{Ch}(\bar{Y})$ under the surjection $f$ is $l_{rd}$. We have

$$\alpha = h^0 \times l_{rd} + a_1 h^1 \times l_{rd+1} + \cdots + a_{s-rd} h^{s-rd} \times l_s \\ + b_s h^{s-rd} \times h^s + b_{s-1} h^{s-rd+1} \times h^{s-1} + \cdots + b_0 h^{2s-rd} \times h^0$$

for some $a_1, \ldots, a_{s-rd}, b_s, \ldots, b_0 \in \mathbb{Z}/2\mathbb{Z}$ and for an arbitrary chosen orientation of $Y$ (in this sum, the summands with the first factor $h^i$ are of course 0 for $i > d-1$).

The variety $Y$ is a closed subvariety of $X_A$. The image of $\alpha$ under the push-forward $\mathrm{Ch}(\bar{X} \times \bar{Y}) \to \mathrm{Ch}(\bar{X} \times \bar{X}_A)$ with respect to the base change $X \times Y \hookrightarrow X \times X_A$ of the closed embedding $Y \hookrightarrow X_A$ is equal to

$$h^0 \times h^{D-rd} + a_1 h^1 \times h^{D-rd-1} + \cdots + a_{s-rd} h^{s-rd} \times h^{D-s},$$

where $D = \dim X_A = \deg A - 1$. It follows by Lemma 3.1 that $d - 1 \leq s - rd$ and $a_1 = \cdots = a_{d-1} = 1$. Choosing an appropriate orientation of $Y$, we may assume that $b_s = 0$.

It follows that

$$\alpha = h^0 \times l_{rd} + h^1 \times l_{rd+1} + \cdots + h^{d-1} \times l_{(r+1)d-1}.$$

Taking push-forward of $\alpha$ with respect to the projection $X \times Y \to Y$ we obtain $l_{(r+1)d-1} \in \mathrm{Ch}(\bar{Y})$. Therefore the cycle $l_{(r+1)d-1}$ is rational. □

**Theorem 3.3.** *Let $A$ be a central simple algebra over a field $F$. Let $\sigma$ be a quadratic pair on $A$. Then the Witt index of $\sigma$ over the function field of the Severi-Brauer variety of any central simple $F$-algebra Brauer-equivalent to $A$ is divisible by the Schur index of $A$.*

*Proof.* Let $Y$ be the variety of $\sigma$-isotropic ideals of reduced dimension 1 in $A$. Let $d = \operatorname{ind} A$ and let $r \geq 0$ be the largest integer with $\operatorname{ind} \sigma_{F(X)} \geq rd$. If $\operatorname{ind} \sigma_{F(X)} > rd$, then the cycle $l_{rd} \in \mathrm{Ch}(\bar{Y})$ is $F(X)$-rational. Therefore, by Proposition 3.2, the cycle $l_{(r+1)d-1}$ is $F$-rational (and in particular $F(X)$-rational) and it follows by §2.4 that $\operatorname{ind} \sigma_{F(X)} \geq (r+1)d$, a contradiction with the choice of $r$. Consequently, $\operatorname{ind} \sigma_{F(X)} = rd$. □

**Remark 3.4.** Proposition 3.2 is stronger than Theorem 3.3, because in the proof of Theorem 3.3 we have only used the fact that the cycle $l_{(r+1)d-1}$ is $F(X)$-rational while Proposition 3.2 states that this cycle is $F$-rational.

## 4. Motivic decomposition

Cycles constructed in the previous section produce a motivic decomposition which we now describe. Our motivic category is the category of graded correspondences $\mathrm{CR}(F, \mathbb{Z}/2\mathbb{Z})$ defined in [4, §63]. We write $\mathcal{M}(X)$ for the motive of a smooth projective variety $X$.

**Proposition 4.1.** *Let $A$ be a central simple $F$-algebra endowed with a quadratic pair $\sigma$. Let $Y$ be the variety of right $\sigma$-isotropic ideals in $A$ of reduced dimension 1. Let $d = \operatorname{ind} A$. Let $D$ be a central division algebra Brauer-equivalent to $A$. Let $X$ be the Severi-Brauer variety of $D$. Let $w$ be the Witt index of the quadratic pair $\sigma_{F(X)}$. Then the* Ch*-motive $\mathcal{M}(Y)$ of $Y$ has a direct summand isomorphic to*

$$S = \mathcal{M}(X) \oplus \mathcal{M}(X)(d) \oplus \cdots \oplus \mathcal{M}(X)(w-d)$$
$$\oplus\, \mathcal{M}(X)(m) \oplus \mathcal{M}(X)(m-d) \oplus \cdots \oplus \mathcal{M}(X)(m-w+d)$$

*where $m = \dim Y - \dim X$. If $\sigma_{F(X)}$ is hyperbolic (i.e., if $w = \deg A/2$), then $\mathcal{M}(Y) \simeq S$.*

*Proof.* We have seen in the proof of Proposition 3.2 that the cycles

$$\alpha_r = h^0 \times l_{rd} + h^1 \times l_{rd+1} + \cdots + h^{d-1} \times l_{(r+1)d-1} \in \mathrm{Ch}(\bar{X} \times \bar{Y})$$

with $r = 0, 1, \ldots, w/d - 1$ are rational. For the same $r$, taking pull-back with respect to the closed embedding $X \times Y \hookrightarrow X \times X_A$ of the rational cycle

$$h^0 \times h^{(r+1)d-1} + h^1 \times h^{(r+1)d-2} + \cdots + h^{d-1} \times h^{rd} \in \mathrm{Ch}(\bar{X} \times \bar{X}_A)$$

of Lemma 3.1, we get a rational cycle

$$\beta_r = h^0 \times h^{(r+1)d-1} + h^1 \times h^{(r+1)d-2} + \cdots + h^{d-1} \times h^{rd} \in \mathrm{Ch}(\bar{X} \times \bar{Y}).$$

Let $E/F$ be a splitting field of $Y$. Using the multiplication formula of §2.3, one checks that the morphism

$$(\alpha_0, \alpha_1, \ldots, \alpha_{w/d-1}, \beta_0, \beta_1, \ldots, \beta_{w/d-1}) : S_E \to \mathcal{M}(Y_E)$$

is right inverse to the morphism

$$(\beta_0^t, \beta_2^t, \ldots, \beta_{w/d-1}^t, \alpha_0^t, \alpha_2^t, \ldots, \alpha_{w/d-1}^t) : \mathcal{M}(Y_E) \to S_E$$

(where $^t$ stands for the transposition). Moreover, these are mutually inverse isomorphisms provided that $\sigma_{F(X)}$ is hyperbolic. Nilpotence theorem [4, Theorem 92.4 with Remark 92.3] finishes the proof. □

**Remark 4.2.** The summands of $S$ are indecomposable by [5]. The decomposition of $\mathcal{M}(Y)$ into a sum of indecomposable summands which we get in the case of hyperbolic $\sigma$ (or, at least, of hyperbolic $\sigma_{F(X)}$), is unique by [3].

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