On a Notion of Resurgent Function of Several Variables

Boris Sternin
Victor Shatalov

MPG AG "Analysis"
Universität Potsdam
Fachbereich Mathematik
14415 Potsdam
Germany

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
53225 Bonn
Germany

Dept. of Compt. Math. and Cybern.
Moscow State University
Vorob'evy Gory
119899 Moscow
Russia

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Abstract

The paper presents the definition of a resurgent function of several independent variables. The main properties of the introduced notions are investigated.

Introduction

The aim of this paper is the definition of resurgent function of several independent variables. To introduce the appropriate definition of this notion, we shall first analyse the definition of the resurgent function of one independant variable introduced by Jean Ecalle [1, 2]. It is known that there are three representations of a resurgent function. They are:

a) The representation of a resurgent function as a *divergent power series.* More generally, one consideres the product of the exponent $e^{\omega x}$ and the divergent series. In the resurgent functions theory such a product is named an *elementary resurgent symbol.*

b) The representation of a resurgent function as a *tuple of analytic functions* in different sectors of the complex x-plane having the given asymptotical expansion of the type a).

c) The representation of a resurgent function as an *infinitely-continuable analytic function* in the dual space of the (dual) variable ξ.

Let us describe shortly the connection between these representations.

i) The correspondence between the representations b) and c), being the main one in the resurgent functions theory, is given by the isomorphism between the multiplicative algebra of

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functions defined in sectors of the complex x-plane and the convolutive algebra of infinitely continuable functions in the dual space. In the simplest case this isomorphism is given by the Borel-Laplace transformation.If so, the asymptotical expansion of a function at infinity corresponds to the asymptotical expansion of the image of this function on the dual space with respect to smoothness in a vicinity of singular points. In general, this isomorphism is given by some generalization of the Laplace transformation (see Jean Ecalle [1]).

ii) The correspondence between the representations a) and b) is given by the requirement that formal series (or, more generally, elementary resurgent symbols) of the representation a) are asymptotical expansions as $|x| \to \infty$ of the functions of the representation b). To establish this correspondance, one has to diminish the class of resurgent functions up to the class of so-called resurgent functions *with simple singularities*. This means that the image $\tilde{f}(\xi)$ of the function $f(x)$ has (in the dual space) singularities of the type

$$\tilde{f}(\xi) = \frac{a_0}{\xi - \omega} + \ln(\xi - \omega) \sum_{k=0}^{\infty} \frac{(\xi - \omega)^k}{k!} a_k$$

at every singular point $\xi = \omega$. The formal series corresponding to this representation is

$$\sum_{k=0}^{\infty} x^{-k} a_k. \tag{1}$$

Surely, in order to establish the correspondence between resurgent functions with simple singularities and elementary resurgent symbols one has to describe a class of formal series (1) for which there exists a resurgent function with this series as asymptotic expansion at infinity. This problem is solved in the framework of the resurgent functions theory.

iii) The correspondence between the representations a) and c) which is a consequence of i) and ii) is realized with the help of the formal Borel transformation which can be obtained by applying of the usual Borel transformation to each term of divergent series. The fact that the formal Borel transformation makes the series more rapidly convergent allows one to define a germ of analytic function, whose analytic continuation is the required element of the representation c).

To analyse the notion of a resurgent function in the case of *several variables* it is convenient to use the representation b) as a basic one. Thus, we start with the notion of an analytic function of exponential type defined in conical[1] subsets of the initial space $\mathbf{C}^n$ which are analogues of sectors in one-dimensional case.

To introduce the notion of the resurgent function of several variables, one have to perform the following program.

1. To introduce the correspondence between the initial functions and functions on the dual space (the analogue of the Borel-Laplace transformation) in such a way that it will

[1] that is, $\mathbf{R}_+$-invariant.

induce a correspondence between asymptotical expansions of initial functions at infinity and that of their images on the dual space by smoothness.

2. To investigate the form of asymptotic expansions of initial functions at infinity, that is, to introduce the appropriate notion of an elementary resurgent symbol. In doing so, one has to introduce also the appropriate notion of the resurgent functions with simple singularities.

As one can see from the analysis above, the problem of asymptotical expansions resurgent functions at infinity is one of the key problems of the theory. To investigate the asymptotical behavior of functions at infinity it is convenient to consider these functions on a compactification of the initial space $\mathbf{C}^n$. In doing so, it is natural to use the projective space $\mathbf{CP}^n$ (into which the initial space $\mathbf{C}^n$ is embedded as an affine chart) as the mentioned compactification, since the considered function $f(x)$ can have different asymptotic behaviour along different directions in $\mathbf{C}^n$.

Thus, the investigation of the asymptotic behaviour of the function $f(x)$ at infinity is reduced to the investigation of the behavior of this function in the vicinity of the manifold of the improper points of $\mathbf{CP}^n$. Usage of the affine coordinates of the projective space leads us to the problem of investigation of the asymptotical behavior of the function with respect to the transversal variable, all the rest (tangent) variables being parameters.

From this wievpoint it is natural that a resurgent function of several variables should be represented as a resurgent function of one (transversal) variable dependent on $n-1$ parameters in each affine chart of the projective space.

Below we present the *global* definition of a resurgent function. At the same time, we introduce also the *invariant* representation of the resurgent function of several variables as a function on the 'dual' space (the analogue of the representation c) above) and establish the correspondence between functions on initial and dual spaces.

We remark that the dual elements in the multidimensional theory are homogeneous functions of order -1 of $n+1$ variables.

Let us examine here an analogue of the representation of the type a) for resurgent functions of several variables. It can be shown that in multidimensional case the analogue of the expansion 1 is an asymptotic expansion of the form

$$e^{-S(x)}\sum_{k=0}^{\infty} A_k(x) \tag{2}$$

with respect to *arbitrary homogeneous* functions (not to power series) of variables $x = (x^1, \dots, x^n)$. Here $S(x)$ is a *first-order* homogeneous function and the function $A_k(x)$ are homogeneous functions of order $-k$ with respect to the standard action of the group $\mathbf{C}_*$:

$$\lambda(x^1 \dots x^n) = (\lambda x^1 \dots \lambda x^n).$$

However, as the amount of homogeneous functions of the first order regular in the whole $\mathbf{C}^n \backslash \{0\}$ not so big (such a function is not more that a linear function of the variable x), it is evident that the consideration of *ramifying* functions $f(x)$ is useful (and in applications even nesessary). We remark, however, that in vicinity of ramification points of $S(x)$ (such points are called *focal*, or *caustic*) it is nesessary to *uniformize* the representation (2), that is, to resolve singularities of $S(x)$. For uniformization we consider a *Lagrangian manifold* corresponding to the function $S(x)$. In doing so it is evident that both the notion of a resurgent function with simple singularities and the notion of the corresponding asymptotic expansion (the analogue of an elementary resurgent symbol in one-dimensional case) must be essentially modified in the multidimensional case. Below we describe in details such a modification based on the $\partial/\partial\zeta$- transformation, introduced by the authors (see [3], [4]).

Briefly about the contents of the paper.

The first section is aimed at the definition of the resurgent function of several variables. Here we introduce also the notion of the 'dual' representation of a resurgent function and investigate the correspondence between the main and the dual representations. The main properties of this correspondence (Theorem 1 on a homomorphism of algebras and Theorem 2 on commutation with differentiation operators) are also established.

The second section is aimed at the investigation of resurgent functions with simple singularities. Here we investigate asymptotical expansions of resurgent functions of several independent variables and introduce the needed modifications of the notion of resurgent function with simple singularities.

In the third section we show that *locally* a resurgent function of several variables can be represented as a resurgent function of one variable dependent on a certain number of parameters.

We remark that very important questions are out of the framework of this paper. Some of them are: the development of the calculus for working with the introduced notions (in one-dimensional case such calculus includes the alient (étranger) differentiation, the notion of resurgent equations, the notion of resurgent monomials and so on), as well as the applications of this theory to problems of mathematical physics such as the problem of investigation of the asymptotic behavior of solutions to differential equations at infinity, investigation of the field diagrams in electrodynamical problems and others. The authors are intended to discuss these questions in their further publications.

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1 Definition of a Resurgent Function

Let $\tilde{f}(\zeta, x)$ be a homogeneous function with respect to the variables $(\zeta, x) = (\zeta, x^1 \ldots x^n)$ in the space $\mathbf{C}^{n+1} \backslash \{0\}$ which is infinitely continuable as a function of ζ for given x in the following sense:

There exists such (ramifying) analytic function $S(x)$ that the function $\tilde{f}(\zeta, x)$ is analytically continuable along any path in the ζ-plane which does not come through points $\zeta = S(x)$.

Suppose that x belongs to sufficiently small simply connected conical domain $U \subset \mathbf{C}^n$ which does not contain singular points of the function $S(x)$. Then the function $S(x)$ is decomposed over U to single-valued branches

$$S_j(x), \quad j = 1, \ldots, N.$$

Definition 1 The function

$$f(x^1, \ldots, x^n) = \int_{\Gamma_j} e^{-\zeta} \tilde{f}(\zeta, x) d\zeta \stackrel{\text{def}}{=} \ell_j[\tilde{f}(\zeta, x)], \tag{3}$$

is called a *resurgent function* with the support in $S_j(x)$, the contour Γ_j is drawn on Figure 1 (and surrounds all singular points $S_k(x)$ $k \neq j$).

With the help of the standard continuation procedure of parameter-dependent integrals (see, for example F.Pham [5]) the integral (3) can be extended to all values of x (possibly, as a ramifying function).

If $Q = \{S_{j_1}(x), \ldots, Sj_k(x)\}$ is a subset of the set of values of ramifying function $S(x)$ then the sum of resurgent functions with supports in $S_{j_i}(x)$ over all $S_{j_i}(x) \in Q$ is called *a resurgent function* corresponding to the function $\tilde{f}(\zeta, x)$ *with support in* Q. We remark that, similar to the case of functions of one variable, the (mentioned above) procedure of continuation of a function with the support in a single point can lead to the resurgent function with an arbitrary support.

Several remarks on the introduced definition.

Remark 1 If the function $\tilde{f}(\zeta, x)$ is the function of the exponential type, that is it satisfies the inequaliity

$$\left| \tilde{f}(\zeta, x) \right| \leq C e^{c|\zeta|} \tag{4}$$

with some constants c and $C > 0$ independent of x, then the integral (3) converges for sufficiently large $|x|$ and, hence, determines a function in $U \backslash K_R$, $K_R = \{x : |x| < R\}$ being a unit ball in $\mathbf{C}^n$. If, on the opposite, the inequality (4) fails, one can make sense for the integral (3) (see [1, 6]) with the help of the procedure usual in the (one-dimensional) resurgent functions theory. In this case the function $f(x)$ is defined up to the terms decreasing exponentially with arbitrary constant c (see (4)).

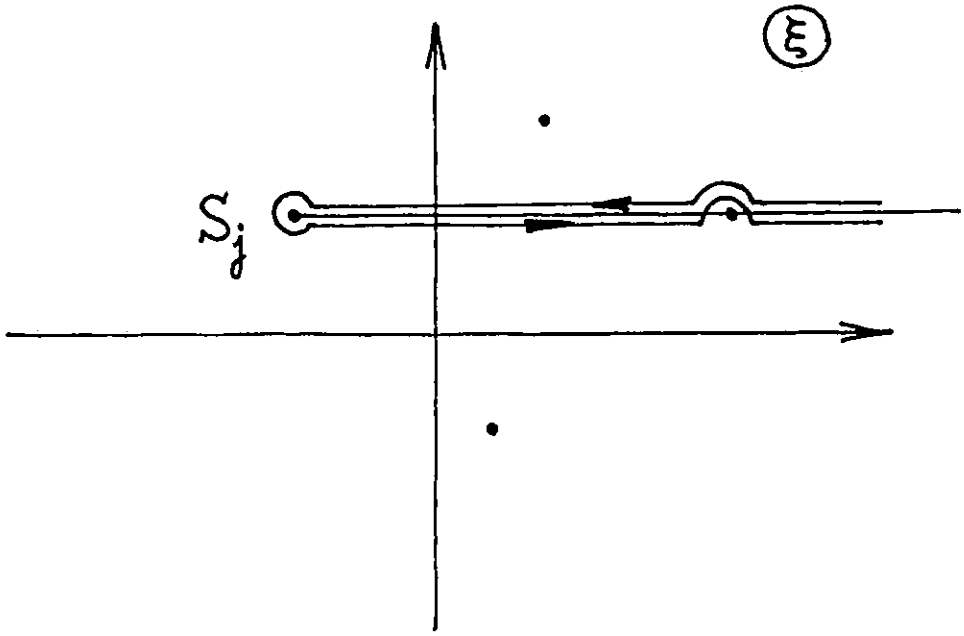

Figure 1

Remark 2 Evidently, the integral (3) vanishes if the function $\tilde{f}(\zeta, x)$ is regular near the point $\zeta = S_j(x)$. Hence, the function $\tilde{f}(\zeta, x)$ must be considered as *hyperfunction* (see, for example, [7]) that is, as an element of the factor of the space of analytic functions modulo holomorphic near $\zeta = S_j$ ones.

Let us present the equation for the function $\ell_j(\tilde{f})$ in the case when $\tilde{f}(\zeta, x)$ is a *regular* hyperfunction (this means that the inequality

$$\left|\tilde{f}(\zeta, x)\right| \leq C|\zeta - S_j(x)|^\alpha$$

for some $\alpha > -1$). In this case the function $\tilde{f}(\zeta, x)$ (as hyperfunction) is uniquely determined by its variation (see, for example, C.Tougeron [7]) over the point $\zeta = S_j(x)$. The operator ℓ_j in terms of the variation $\mathrm{var}\tilde{f}(\zeta, x)$ can be expressed as

$$\ell_j(\tilde{f}) = \int\limits_{\gamma_j} e^{-\zeta} \mathrm{var}\tilde{f}(\zeta, x) d\zeta, \tag{5}$$

the integration being performed over rays γ_j originated from $\zeta = S_j(x)$ and moving along the direction of the positive real axis. This ray must pass the singularity points of $\tilde{f}(\zeta, x)$ in the usual way. The representation (5) will be of use below.

Require in addition that the function $\tilde{f}(\zeta, x)$ have *simple singularities*[2] at points of the set $X = \{\zeta = S(x)\}$. This means that near these points

$$\tilde{f}(\zeta, x) = \frac{A_0(x)}{\zeta - S(x)} + \ln(\zeta - S(x)) \sum_{k=0}^{\infty} \frac{(\zeta - S(x))^k}{k!} A_k(x), \tag{6}$$

[2]This requirement makes sense only in the vicinity of points which are *not focal*. Focal points will be considered in the next section.

where $A_k(x)$ are homogeneous functions of order $-k$. Then the function (5) has the asymptotical expansion of the type (2) as $x \to \infty$.

Let us investigate now the connection between the representation of the resurgent function with the help of a homogeneous function $\tilde{f}(\zeta, x)$ and its representation with the help of the function $f(x)$ given by (3). To do this we introduce the following notions.

Denote by $\widetilde{\mathcal{R}}$ the set of infinitely continuable (hyper)functions $\tilde{f}(\zeta, x)$ homogeneous with respect to (ζ, x) of order -1. The *convolution* $\tilde{f} * \tilde{g}$ of two elements of $\widetilde{\mathcal{R}}$ is defined as a convolution of hyperfunctions of the variable ζ dependent on the parameters x. It is easy to check that the convolution of two elements of $\widetilde{\mathcal{R}}$ is a hyperfunction which is also homogeneous of order -1. Hence, the introduced operation induces the structure of algebra on $\widetilde{\mathcal{R}}$.

Let us calculate the convolution $\tilde{f} * \tilde{g}$ for the case of regular hyperfunctions $\tilde{f}$, $\tilde{g}$. In this case $\tilde{f}$ and $\tilde{g}$ can be represented by their variations $\mathrm{var}\tilde{f}$ and $\mathrm{var}\tilde{g}$ and the formula

$$\mathrm{var}(\tilde{f} * \tilde{g}) = \int_0^\zeta \mathrm{var}\tilde{f}(\eta, x)\mathrm{var}\tilde{g}(\zeta - \eta, x)d\eta \tag{7}$$

takes place.

Let us calculate the singularity set of the convolution $\tilde{f} * \tilde{g}$. For simplicity we shall carry out this calculation under the assumption that both $\tilde{f}$ and $\tilde{g}$ are regular hyperfunctions with singularity at the point 0. It is evident that the singularities of the integral (7) belong to the set of such points ζ, for which one of the following three conditions is valid:

a) One of singular points of $\tilde{f}(\eta, x)$ coincides with the upper limit of integration ζ.

b) One of the singular points of $\tilde{g}(\zeta - \eta, x)$ coincides with the lower limit of integration 0.

c) Some singular point of the function $\tilde{f}(\zeta, x)$ coincides with some singular point of the function $\tilde{g}(\zeta - \eta, x)$. In this case the integration contour can be 'pinched' between these singular points.

In the case a) we obtain $\zeta = S_j^{(f)}$ for some j, $\{S_j^{(f)}(x)\}$ being the singularity set of the function $\tilde{f}(x)$. Similarly, in the case b) we obtain $\zeta = S_k^{(g)}$ for some k. Finally, in the case c) we have

$$S_j^{(f)} = \zeta - S_k^{(g)}$$

for some j, k. In any case the singularities of the convolution $\tilde{f} * \tilde{g}$ can be represented in the form

$$\zeta = S_j^{(f)} + S_k^{(g)}$$

for some j, k (we recall that $\zeta = 0$ is a singular point both for $\tilde{f}$ and $\tilde{g}$). Thus, the following affirmation is valid.

Proposition 1 *The singularities of the function $\tilde{f} * \tilde{g}$ lye inside the set*

$$\{\zeta : \zeta = S_j^{(f)} + S_k^{(g)}\}$$

consisting of sums of singular values ζ for functions $\tilde{f}$ and $\tilde{g}$ consequently.

Remark 3 The proof of this Proposition in general case one can reduce to the proof above if one will use that the multiplication of the function $f(x)$ by the exponent $e^{\omega(x)}$ leads to the shift of the argument in the function $\tilde{f}(\zeta, x)$:

$$\ell_j(\tilde{f}(\zeta - \omega(x), x)) = e^{\omega(x)} \ell_j(\tilde{f}(\zeta, x)).$$

Denote by $\mathcal{R}$ the set of functions representable in the form (3). The following affirmation is valid.

Theorem 1 *The set $\mathcal{R}$ is an algebra with respect to the usual multiplication of functions. The operator ℓ_j defined by the formula (3) is a homomorphism of algebras*

$$\ell_j \; : \; \widetilde{\mathcal{R}} \to \mathcal{R}$$

Proof. It is evidently sufficient to show that

$$\ell_j(\tilde{f} * \tilde{g}) = \ell_j(\tilde{f}) \ell_j(\tilde{g}). \tag{8}$$

For simplicity we carry out the proof for the case of regular hyperfunctions. Due to formulas (5) and (7) we have

$$\ell_j(\tilde{f} * \tilde{g}) = \int_{\gamma_j} e^{-\zeta} \left\{ \int_0^{\zeta} \mathrm{var} \tilde{f}(\eta, x) \mathrm{var} \tilde{g}(\zeta - \eta, x) d\eta \right\} d\zeta.$$

Performing the variable change $\eta' = \zeta - \eta$ in the latter integral we obtain

$$\ell_j(\tilde{f} * \tilde{g}) = \int_{\gamma_j} e^{-\eta} \mathrm{var} \tilde{f}(\eta, x) d\eta \int_{\gamma_j} \mathrm{var} e^{-\eta'} \tilde{g}(\eta', x) d\eta' = \ell_j(\tilde{f}) \ell_j(\tilde{g})$$

similar to the procedure used in the theory of classical Laplace transformation, q.e.d.

To conclude this section, we examine the commutation formulas of differentiation operators $\partial / \partial x^i$ with the operator ℓ_j. To do this we note that the operator $\partial / \partial \zeta$ is invertible on the set of hyperfunctions homogeneous with respect to (ζ, x). Actually, the kernel of this operator consists of functions $\tilde{f}(\zeta, x)$ which are homogeneous with respect to the variables (ζ, x) and do not depend on ζ. However, the singularities of such functions cannot lay in the set $\zeta = S(x)$. Hence, such functions, being regular on the set X, determine zero element in the set of hyperfunctions. It is also evident that if $\tilde{f}(\zeta, x)$ is a homogeneous hyperfunction of order k, then $(\partial/\partial\zeta)^{-1}\tilde{f}(\zeta, x)$ is a hyperfunction homogeneous of order $k+1$.

For regular hyperfunctions having $\zeta = 0$ as a singular point, the action of the operator $(\partial/\partial\zeta)^{-1}$ is given by the formula

$$\mathrm{var}\{(\partial/\partial\zeta)^{-1}\tilde{f}\} = \int_0^\zeta \mathrm{var}\tilde{f}(\eta, x)d\eta.$$

The following affirmation is valid.

Theorem 2 *The following commutation formulas*

$$\frac{\partial}{\partial x^i}\ell_j(\tilde{f}(\zeta, x)) = \ell_j\left(\left(\frac{\partial}{\partial\zeta}\right)^{-1}\frac{\partial}{\partial x^i}\tilde{f}(\zeta, x)\right)$$

take place.

Proof. Due to the formula (3) we evidently have

$$\frac{\partial}{\partial x^i}\ell_j(\tilde{f}(\zeta, x)) = \int_{\Gamma_j} e^{-\zeta}\frac{\partial}{\partial x^i}\tilde{f}(\zeta, x)d\zeta. \tag{9}$$

As the function $\partial/\partial x^i\tilde{f}(\zeta, x)$ is a homogeneous function of order -2 (not of order -1), the right-hand part of formula (9) cannot be considered as the application of the operator ℓ_j to some function from $\widetilde{\mathcal{R}}$. However, representing the partial derivative $\partial/\partial x^i\tilde{f}$ in the form

$$\frac{\partial}{\partial x^i}\tilde{f}(\zeta, x) = \frac{\partial}{\partial\zeta}\left\{\left(\frac{\partial}{\partial\zeta}\right)^{-1}\frac{\partial}{\partial x^i}\tilde{f}(\zeta, x)\right\}$$

and integrating by parts in integrals (9) we obtain

$$\frac{\partial}{\partial x^i}\ell_j(\tilde{f}(\zeta, x)) = \int_{\Gamma_j} e^{-\zeta}\left\{\left(\frac{\partial}{\partial\zeta}\right)^{-1}\frac{\partial}{\partial x^i}\tilde{f}(\zeta, x)\right\}d\zeta. \tag{10}$$

The right-hand part of formula (10) can be considered as the application of the operator ℓ_j to the function

$$\left(\frac{\partial}{\partial\zeta}\right)^{-1}\frac{\partial}{\partial x^i}\tilde{f}(\zeta, x).$$

This completes the proof.

2 Resurgent Functions with Simple Singularities

In this section we shall consider only *regular* hyperfunctions. Note that the representation (5) of a resurgent function $f = \ell_j(\tilde{f}) \in \mathcal{R}$ by the corresponding regular hyperfunction $\tilde{f}(\zeta, x) \in \tilde{\mathcal{R}}$ is global. This means that this representation is valid everywhere including focal points. However, the globality fails when considering the asymptotical expansions of the integral (5) of the type (2). As we have already told above, this is connected with the fact that the notion of the function with simple singularities does not make sense in the vicinity of the focal points, here the function $S(x)$ itself has ramification. Thus (as it was already explained in the Introduction) to obtain the asymptotic expansion of a resurgent function near its focal point, one has to generalize the notion of the resurgent function with simple singularities.

The nesessity of such a generalization becomes obvious also when applying the resurgent functions theory to problems of construction of asymptotical expansions of solutions to differential equations for large values of independent variables. Indeed, the differential equation for a function f induces, due to Theorem 2 the $\partial/\partial\zeta$-differential equation [8, 9] for the corresponding homogeneous function $\tilde{f}$ in the dual space. However, it is known that asymptotical expansions (with respect to smoothness) of the type (6) are not in general invariant along solutions to differential equations. This means that if a solution of a differential equation has simple singularities near some point, this property in general will not be valid near some other point due to the differential equation.

To overcome this difficulty we use integral transformations (see [4, 10, 11]. The transformation appropriate for the considered situation is the so-called $\partial/\partial\zeta$-transformation introduced by the authors in [12].

We recall (see papers cited above) that the space $\mathcal{A}_q(X)$ is defined as a space of functions $f(\zeta, x)$ satisfying the inequality

$$|f(\zeta, x)| \leq C|\zeta - S(x)|^q,$$

in a vicinity of any regular point $x_0 \in X$. Here $\zeta = S(x)$ is an equation of the set X and $q > -1$.

If $f(\zeta, x) \in \mathcal{A}_q(X)$, then the $\partial/\partial\zeta$-transformation of this function is defined with the help of the integral[3]

$$F^{\partial/\partial\zeta}_{x\to p}[f(\zeta,x)] = \hat{f}(\zeta,p) = \left(\frac{i}{2\pi}\right)^{n/2}\left(\frac{\partial}{\partial\zeta}\right)^{n/2}\int\limits_{h(\zeta,p)} f(\zeta - xp, x)dx, \qquad (11)$$

where the half-integer powers of the operator $\partial/\partial\zeta$ are determined as it is described in [12] and the ramifying class $h(\zeta,p)$ is determined as follows.

Denote by $L_{(\zeta,p)}$ the plane in the space $\mathbf{C}^{n+1}$ with coordinates $(\tilde{\zeta},x)$ defined by the equation

$$\tilde{\zeta} = \zeta - xp.$$

Suppose that the plane $L_{(\zeta,p)}$ has quadratic contact with X at some point $(\tilde{\zeta}_0, x_0)$. Then for (ζ,p) being sufficiently close to (ζ_0,x_0) and such that $L_{(\zeta,p)}$ is not tangent to X, the intersection $L_{(\zeta,p)}\cap X$ is homeomorphic to the complex quadrics and $h(\zeta,p)$ is the vanishing class of this quadrics (see [5])

$$h(\zeta,p) \in H_n(L_{(\zeta,p)}, L_{(\zeta,p)}\cap X).$$

Later on, the class $h(\zeta,p)$ can be extended on other values of (ζ,p) with the help of the Thom theorem (see [13]). The inverse transformation to (11) has the form

$$F^{\partial/\partial\zeta}_{p\to x}[\hat{f}(\zeta,p)] = \left(\frac{i}{2\pi}\right)^{n/2}\left(\frac{\partial}{\partial\zeta}\right)^{n/2}\int\limits_{\tilde{h}(\zeta,x)} \hat{f}(\zeta + xp, p)dp,$$

the class $\tilde{h}(\zeta,x)$ is defined similar to the class $h(\zeta,p)$.

Let now $\tilde{f}(\zeta,x)$ be, as above, an infinitely continuable homogeneous function of order -1 with respect to (ζ,x) with singularities on the set X defined by the equation

$$X = \{\zeta = S(x)\},$$

the function $S(x)$ being a ramifying function of x. Let x_0 be a focal point, that is, a ramification point of the function $S(x)$. Require in addition that the homogeneous[4] Lagrangian manifold

$$\mathcal{L} = \left\{(x,p) : p = \frac{\partial S(x)}{\partial x}\right\}$$

[3]This formula admits the following interpretation:

$$F^{\partial/\partial\zeta}_{x\to p}[f(\zeta,x)] = \left(\frac{i}{2\pi}\right)^{n/2}\left(\frac{\partial}{\partial\zeta}\right)^{n/2}\int\limits_{h(\zeta,p)} e^{-xp\partial/\partial\zeta} f(\zeta,x)dx.$$

That is why this transformation was called $\partial/\partial\zeta$-transformation.

[4]with respect to the variables x.

determined by the function $S(x)$ could be continued up to a regular manifold over a neighbourhood of the point x_0. Then (see, for example, [14]) there exists such set of indices $I \subset \{1, 2, \ldots, n\}$ that the equations of the manifold $\mathcal{L}$ in the vicinity of the focal point will read

$$\mathcal{L} = \left\{(x,p) : p_I = \frac{\partial S_I(x^I, p_{\overline{I}})}{\partial x^I},\ x^{\overline{I}} = -\frac{\partial S_I(x^I, p_{\overline{I}})}{\partial p_{\overline{I}}}\right\}$$

for some *regular* function $S_I(x^I, p_{\overline{I}})$ which is homogeneous of order 1 with respect to the variables x^I. Here by $\overline{I}$ we denote the complement of the set I in $\{1, 2, \ldots, n\}$, $x^I = \{x^{i_1}, \ldots, x^{i_k}\}$ for $I = \{i_1, \ldots, i_k\}$ etc. By $|I|$ we denote the number of elements in the set I.

The main idea of the mentioned uniformization is the representation of the function $\tilde{f}(\zeta, x)$ in the form of inverse $\partial/\partial\zeta$-transformation of some other function $\tilde{f}_I(\zeta, x^I, p_{\overline{I}})$ which is homogeneous with respect to (ζ, x^I) (but of some other degree not equal to 1, this degree will be defined below) and having the simple singularities in the sense (6):

$$\tilde{f}_I(\zeta, x^I, p_{\overline{I}}) = \ln(\zeta - S_I(x^I, p_{\overline{I}})) \sum_{k=0}^{\infty} \frac{(\zeta - S_I(x^I, p_{\overline{I}}))^k}{k!} A_{kI}(x^I, p_{\overline{I}}). \tag{12}$$

(We recall that only regular hyperfunctions are considrerd). Thus, the function $\tilde{f}(\zeta, x)$ will be represented in the form

$$\begin{aligned}\tilde{f}(\zeta, x) &= F^{\partial/\partial\zeta}_{q_{\overline{I}} \to x^{\overline{I}}}[\tilde{f}(\zeta, x^I, p_{\overline{I}})] \\ &= \left(\frac{i}{2\pi}\right)^{n/2} \left(\frac{\partial}{\partial\zeta}\right)^{n/2} \int_{\tilde{h}(\zeta,x)} \tilde{f}_I(\zeta + x^{\overline{I}} p_{\overline{I}}, x^I, p_{\overline{I}}) dp_{\overline{I}}.\end{aligned} \tag{13}$$

The following affirmation shows what requirements have to be posed on degree of homogeneity of the function $\tilde{f}_I$ for $\tilde{f}$ to be a homogeneous function of degree -1.

Lemma 1 *If the function $\tilde{f}_I(\zeta, x^I, p_{\overline{I}})$ is a homogeneous function of variables (ζ, x^I) of degree $|\overline{I}|/2$ then the function $\tilde{f}(\zeta, x)$ given by (13) is a homogeneous function of (ζ, x) of degree -1.*

Proof. The only nontrivial point of the proof of the Lemma is the verification of the fact that the ramifyed class $\tilde{h}(\zeta, x)$ is invariant under the action of the group $\mathbf{C}_*$. This fact, however, is an easy consequence of homogeneity of the function $S_I(x^I, p_{\overline{I}})$ together with the definition of the ramifying class above if one takes into account that the equation of the intersection $\widetilde{L}_{(\zeta,x)} \cap \widetilde{X}$ reads

$$\zeta + x^{\overline{I}} p_{\overline{I}} = S_I(x^I, p_{\overline{I}}).$$

The latter equation shows that this intersection is invariant under the action of the group $\mathbf{C}_*$. Here

$$\widetilde{L}_{(\zeta,x)} = \{(\tilde{\zeta}, p_{\overline{I}}) : \tilde{\zeta} = \zeta + x^{\overline{I}} p_{\overline{I}}\}$$

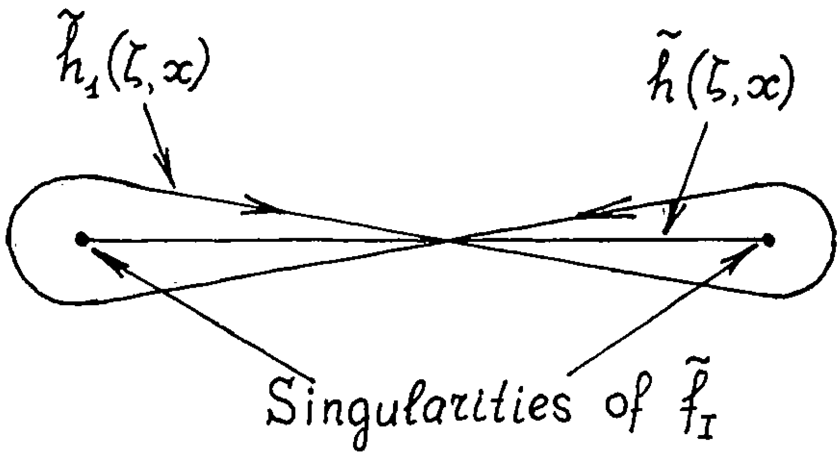

Figure 2

and $\widetilde{X} = \{\tilde{\zeta} = S_I(x^I, p_{\overline{I}})\}$ is the equation of the singularity set of the function $f_I(\tilde{\zeta}, x^I, p_{\overline{I}})$.

With the help of the technique described in the book [11] one can show that:

a) The singularity set of the integral (13) is given by the equation $\zeta = S(x)$.

b) If the function $f_I(\tilde{\zeta}, x^I, p_{\overline{I}})$ has the simple singularities (in the sense of the expansion (12)) then the function (13) has simple singularities *near each nonfocal point.*

Thus, it is natural to introduce the following generalization of the notion of the function with simple singularities.

Definition 2 A resurgent function $f(\zeta, x)$ is called *a resurgent function with simple singularities* if it can be locally represented in the form (13) the integrand $\tilde{f}_I(\zeta, x^I, p_{\overline{I}})$ having the asymptotic expansion (12).

Let us find out the form of asymptotic expansion of a resurgent function with simple singularities near its focal point. To do this, we note that if the function $\tilde{f}(\zeta, x)$ has the form (13) then its variation has the same form with class $\tilde{h}(\zeta, x)$ replaced by the certain class $\tilde{h}_1(\zeta, x)$. The latter can be constructed from the class $\tilde{h}(\zeta, x)$ as the sum of two its copies with opposite orientations lying on different sheets of the Riemannian surface of the function $\tilde{f}_I(\zeta + x^{\overline{I}} p_{\overline{I}}, x^I, p_{\overline{I}})$. The class $\tilde{h}_1(\zeta, x)$ can be considered as a homology class of the complement to the singularity set of the function $\tilde{f}_I$ in $\widetilde{L}_{(\zeta,x)}$ and surrounding these singularities in the fashion shown on Figure 2.

Using this fact together with formula (5) we obtain the following equation for the function $f(x)$:

$$f(x) = \int\limits_{\gamma_j} e^{-\zeta} \left\{ \left(\frac{i}{2\pi}\right)^{|\overline{I}|/2} \left(\frac{\partial}{\partial \zeta}\right)^{|\overline{I}|/2} \int\limits_{\tilde{h}(\zeta,x)} \tilde{f}_I(\zeta + x^{\overline{I}} p_{\overline{I}}, x^I, p_{\overline{I}}) dp_{\overline{I}} \right\} d\zeta.$$

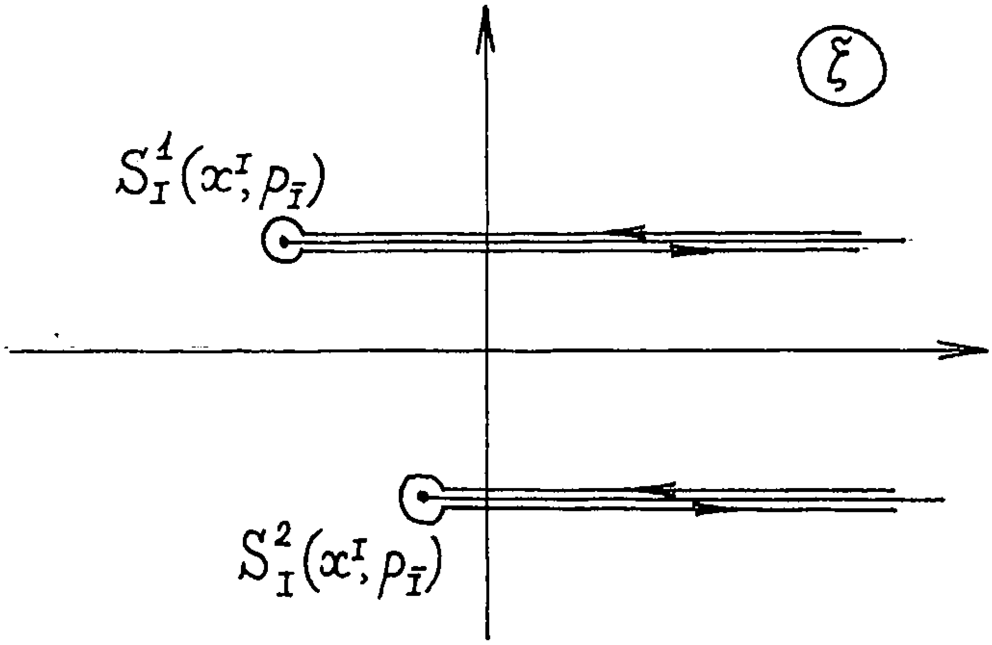

Figure 3

Taking into account that the integration contour $\tilde h(\zeta, x)$ of the inner integral degenerates to a single point ('vanishes') at the origin point of the ray γ_j we see that the latter integral can be rewritten in the form of integral

$$
\begin{aligned}
f(x) &= \left(\frac{1}{2\pi i}\right)^{|\bar I|/2} \int\limits_{H(x)} e^{-\zeta} \tilde f_I(\zeta + x^{\bar I} p_{\bar I}, x^I, p_{\bar I}) d\zeta \wedge dp_{\bar I} \\
&= \left(\frac{1}{2\pi i}\right)^{|\bar I|/2} \int\limits_{H(x)} e^{-\tilde\zeta + x^{\bar I} p_{\bar I}} \tilde f_I(\tilde\zeta, x^I, p^{\bar I}) d\tilde\zeta \wedge dp_{\bar I} \qquad (14)
\end{aligned}
$$

over the *absolute* (though noncompact) cycle $H(x)$ in the space with coordinates $(\tilde\zeta, p_{\bar I})$. It is evident that for $\left|(\tilde\zeta, p_{\bar I})\right| \to \infty$ the real part $Re\tilde\zeta$ increases on the (multidimensional) contour $H(x)$. Hence, deforming the contour $H(x)$ along the positive part of the real axis $\tilde\zeta$ for each fixed $p_{\bar I}$, we represent the integral (14) in the form

$$
f(x) = \left(\frac{1}{2\pi i}\right)^{|\bar I|/2} \int\limits_{\widetilde H(x)} e^{x^{\bar I} p_{\bar I}} \{\sum_j \int\limits_{\Gamma(S_I^j)} e^{-\tilde\zeta} \tilde f_I(\tilde\zeta, x^I, p_{\bar I}) d\tilde\zeta\} dp_{\bar I}, \qquad (15)
$$

$\widetilde H(x)$ being a projection of the deformed contour $H(x)$ and the contours $\Gamma(S_I^j) = \Gamma(S_I^j(x^I, p_{\bar I}))$ are drawn on Figure 3. Applying to the inner integral the usual asymptotic formula for one-dimensional resurgent functions we obtain as a result the asymptotical expansion for the function $f(x)$ near the focal point:

$$
f(x) = \sum_j \sum_{k=0}^{\infty} \left(\frac{1}{2\pi i}\right)^{|\bar I|/2} \int\limits_{\widetilde H(x)} e^{x^{\bar I} p_{\bar I} - S_I^j(x^I, p_{\bar I})} A_{kI}^j(x^I, p_{\bar I}) dp_{\bar I}. \qquad (16)
$$

It is easy to see that the functions $A^j_{kI}(x^I, p_{\overline{I}})$ are homogeneous with respect to the variables x^I of degree $|\overline{I}|/2 - k - 1$.

Of course, in the general situation a more explicit asymptotic expansion of the function $f(x)$ cannot be written down. However, for concrete Lagrangian manifolds one can obtain more explicit expansion obtaining different special functions. This procedure is out of the framework of this paper and we shall not consider it here.

3 Resurgent functions in the affine chart

In this section we show that in any affine chart the definition of a resurgent function introduced above can be reduced to the definition of a resurgent function of one variable dependent on parameters.

To begin with, let us introduce the needed notations.

Let $\mathbf{CP}^n$ is a projective compactification of $\mathbf{C}^n$, the space $\mathbf{C}^n$ being identified with the projective chart $U_0 = \{z^0 \neq 0\}$, that is

$$x^1 = z^1/z^0, \dots, x^n = z^n/z^0.$$

Here by $(z^0 : \dots : z^n)$ we denoted the homogeneous coordinates in $\mathbf{CP}^n$.

To be definite, let us consider the affine chart $U_1 = \{z^1 \neq 0\}$ of the projective space $\mathbf{CP}^n$. The coordinates in this chart are $y = (y^0, y^2, \dots, y^n) = (y^0, y')$ where

$$y^0 = z^0/z^1,\ y^2 = z^2/z^1, \dots, y^n = z^n/z^1.$$

The change of coordinates in the intersection of the charts U_0 and U_1 is

$$\begin{aligned} &x^1 = 1/y^0,\ x^2 = y^2/y^0, \dots, x^n = y^n/y^0; \\ &y^0 = 1/x^1,\ y^2 = x^2/x^1, \dots, y^n = x^n/x^1. \end{aligned} \tag{17}$$

As it was mentioned in the Introduction, the investigation of the asymptotic behavior of a function $f(x)$ as $|x| \to \infty$ in the chart U_1 is reduced to the investigation of the behavior of the corresponding function

$$f_1(y) = f(1/y^0, y^2/y^0, \dots, y^n/y^0)$$

as $y^0 \to 0$. The variables $(y^2, \dots, y^n)$ are then simply parameters.

Let us consider now a ramifying analytic function $f(x)$ defined in some cone Ω of the space $\mathbf{C}^n$. We denote $\Omega_R = \Omega \backslash K_R$ where $K_R = \{x : |x| < R\}$. Suppose that:

a) The function $f(x)$ is regular in Ω_R.

b) This function $f(x)$ is of an exponential type, that is, the inequality

$$|f(x)| \leq Ce^{c|x|}$$

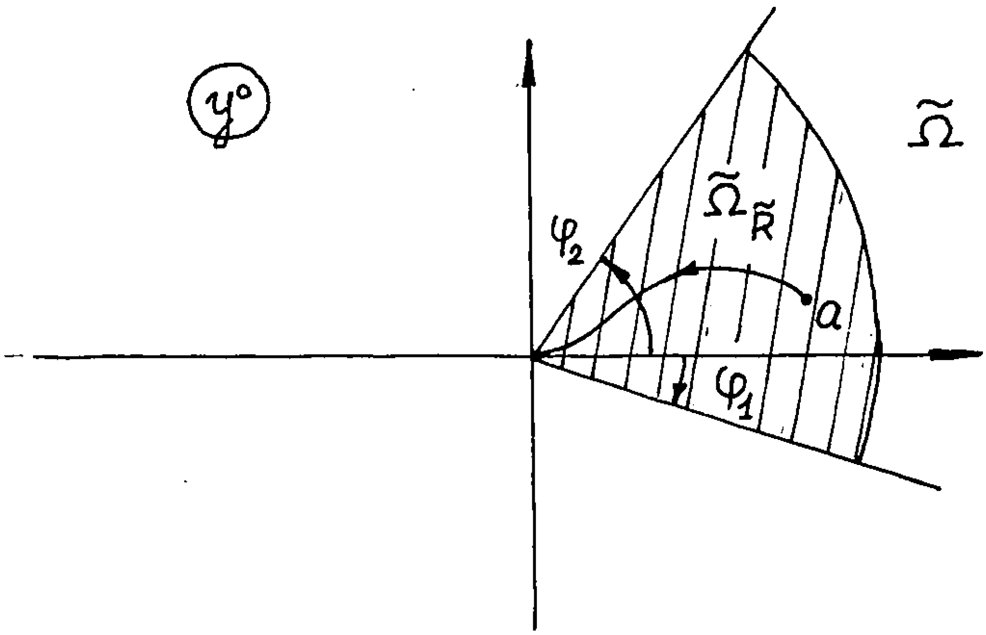

Figure 4

takes place for some c and $C > 0$.

Note that the action of the group $\mathbf{R}_+$ has the form

$$\lambda(x^1, \ldots, x^n) = \mu(y^0, y^2, \ldots, y^n) = (\mu y^0, y^2, \ldots, y^n)$$

in the chart U_1 where $\mu = \lambda^{-1}$. Hence, in U_1 the intersection of Ω with $y' = \text{const}$ is a cone in the plane $\mathbf{C}$ with the coordinate y^0. Without loss of generality one can assume that this cone is a sector

$$\varphi_1 < \arg y^0 < \varphi_2$$

(see Figure 4). Later on, in the set Ω_R one has

$$|x| = \frac{1}{|y^0|}\sqrt{1 + |y^2|^2 + \ldots + |y^n|^2} \geq R.$$

Thus, the image of the set Ω_R in the chart U_1 for fixed $(y^2, \ldots, y^n)$ is the intersection of the mentioned sector with the ball of the radius

$$\widetilde{R} = \frac{\sqrt{1 + |y^2|^2 + \ldots + |y^n|^2}}{R}.$$

This intersection we denote by $\widetilde{\Omega}_{\widetilde{R}}$ (see Figure 4).

The function $f(x^1, \ldots, x^n)$ has in the chart U_1 the coordinate representation of the form

$$f(x^1, \ldots, x^n) = f(y^0, y^2, \ldots, y^n) = f(1/y^0, y^2/y^0, \ldots, y^n/y^0)$$

and, hence, the inequality

$$f(y^0, y^2, \ldots, y^n) \leq Ce^{\frac{1}{|y^0|}\sqrt{1+|y^2|^2+\ldots+|y^n|^2}} \tag{18}$$

is valid if (that is always supposed) the variables $(y^2, \dots, y^n)$ lye in the compact set. Thus, the considerations in the chart U_1 lead to the problem of investigation of the behavior of the function $f_1(y)$ as $y^0 \to \infty$, $(y^2, \dots, y^n)$ being *parameters*. (When considering the problem locally, one can assume, without loss of generality, that $(y^2, \dots, y^n)$ lye in a sufficiently small neighbourhood of some 'central' point $(y_0^2, \dots, y_0^n)$). As it was already mentioned above, this is the problem of one-dimensional resurgent analysis with parameters.

Let us denote by $\tilde{f}_1(\xi, y^2, \dots, y^n)$ the Borel transform of the function $f_1(y)$ with respect to the variable y^0. This transform is given by the integral

$$\tilde{f}_1(\xi, y^2, \dots, y^n) = -\frac{1}{2\pi i} \int\limits_{\gamma_a} e^{-\xi/y^0} f(y^0, y^2, \dots, y^n) \frac{dy^0}{(y^0)^2}, \tag{19}$$

the contour γ_a being drawn on Figure 4 and a point a being chosen in an arbitrary way. The function $\tilde{f}_1(\xi, y')$, $y' = (y^2, \dots, y^n)$, is defined up to entire functions of y^0 dependent on y'. Thus, the expression (19) assigns to the function $f_1(y)$ not a function but the *class* of analytic functions modulo entire functions, that is, a hyperfunction (see [7]).

Let us describe the domain of $\tilde{f}_1$. Denoting $y^0 = re^{i\varphi}$, $\xi = \rho e^{i\theta}$ we have

$$\left|\exp(\xi/y^0)\right| = \left|\exp\left(\frac{\rho}{r}e^{i(\theta-\varphi)}\right)\right| = \exp\left(\frac{\rho}{r}\cos(\theta - \varphi)\right).$$

This relation together with estimate (18) gives the following inequality for the integrand of the integral (19):

$$\left|e^{\xi/y^0}\right| \le Ce^{\frac{\rho}{r}\cos(\theta-\varphi)+\frac{c_1}{r}} = Ce^{\frac{1}{r}(\rho\cos(\theta-\varphi)+c_1)}.$$

The latter inequality shows that the integral (19) converges for $\rho\cos(\theta - \varphi) + c_1 < 0$ (we suppose that the contour γ_a is a segment of the ray $\arg y^0 = \varphi$ at least near the point $y^0 = 0$). Thus, for $\cos(\theta - \varphi) < 0$, that is for $\pi/2 < \theta - \varphi < 3\pi/2$, the integral (19) converges if ρ is sufficiently large.

Now, changing the value of the angle φ between φ_1 and φ_2 we obtain that the considered integral is defined in the domain $S(\varepsilon, R_1)$ drawn on Figure 5 for some positive ε and R_1.

In the case when the magnitude $\varphi_2 - \varphi_1$ of the initial sector $\widetilde{\Omega}_{\widetilde{R}}$ is more than π, the Borel transformation (19) is invertible (as it is known from the Borel-Laplace transformation theory for hyperfunctions, see [7]). In this case the domain $S(\varepsilon, R_1)$ has the form of the Riemannian surface drawn on Figure 6 and the inverse transformation for (19) (the Laplace transformation) is given by

$$f_1(y) = \int\limits_{\Gamma_\theta} e^{-\xi/y^0} \tilde{f}_1(\xi, y') d\xi, \tag{20}$$

the contour of integration being also drawn on Figure 6. We omit the investigation of convergence of the integral (20) since it is well-known.

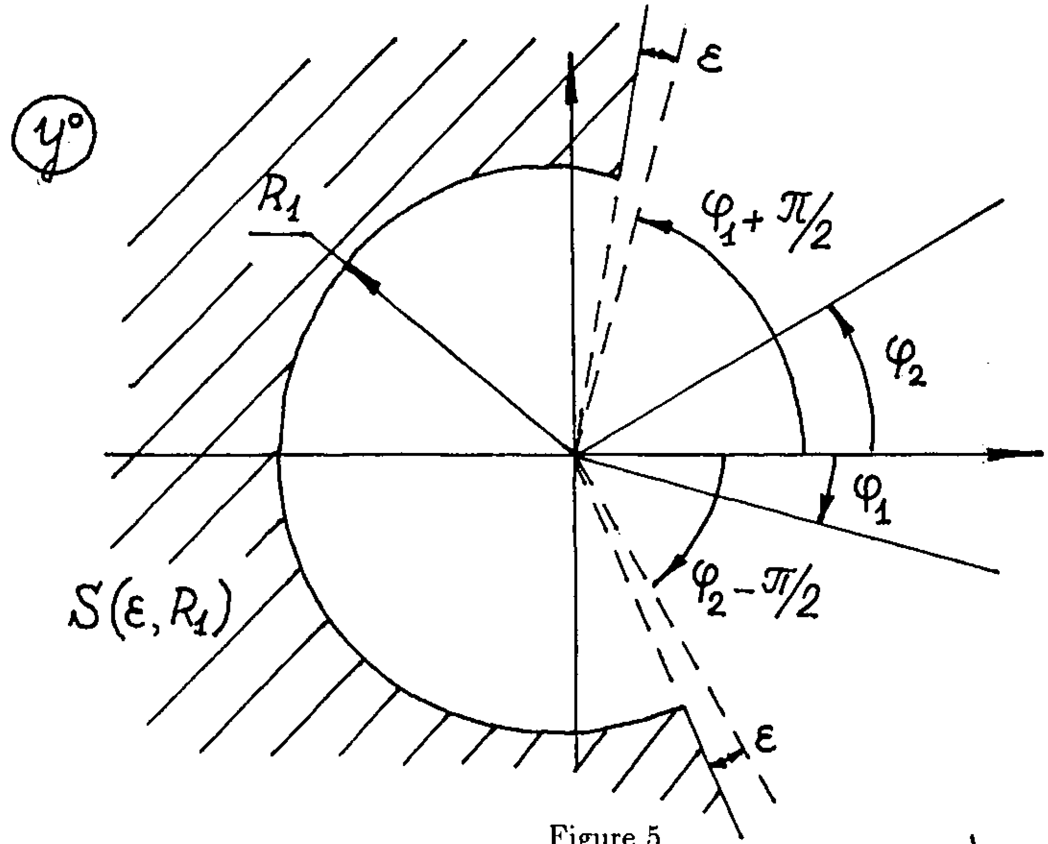

Figure 5

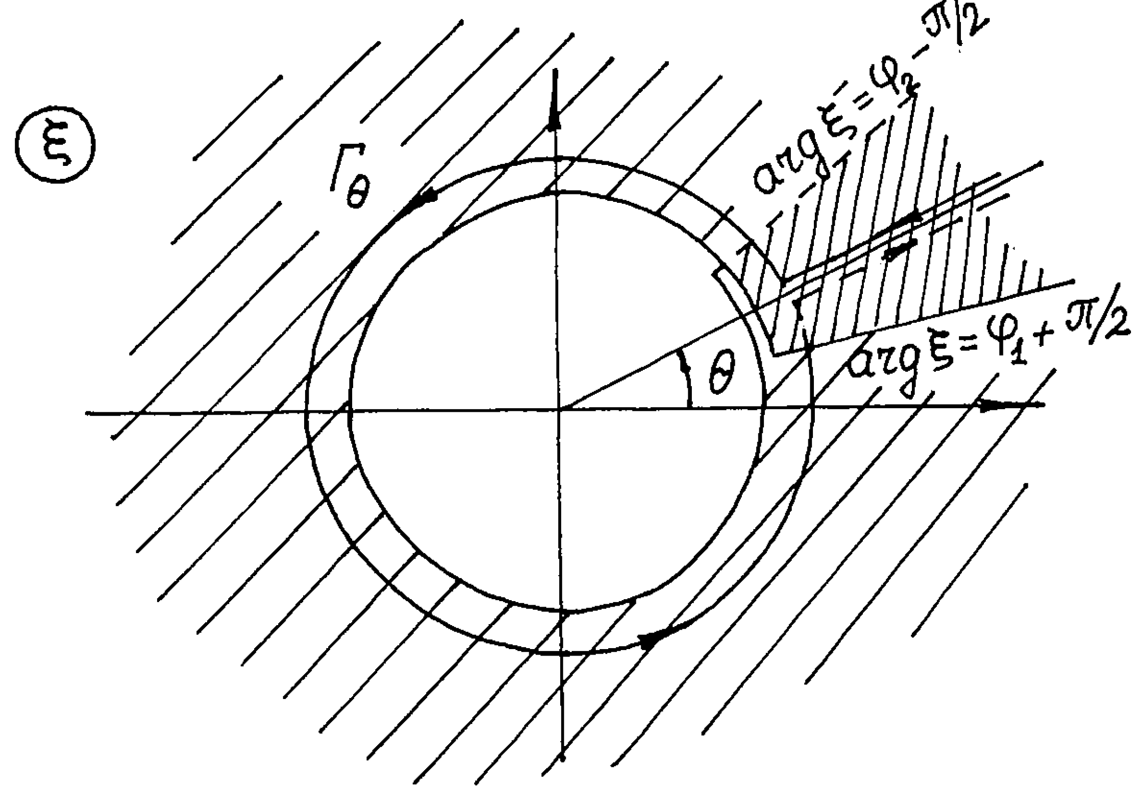

Figure 6

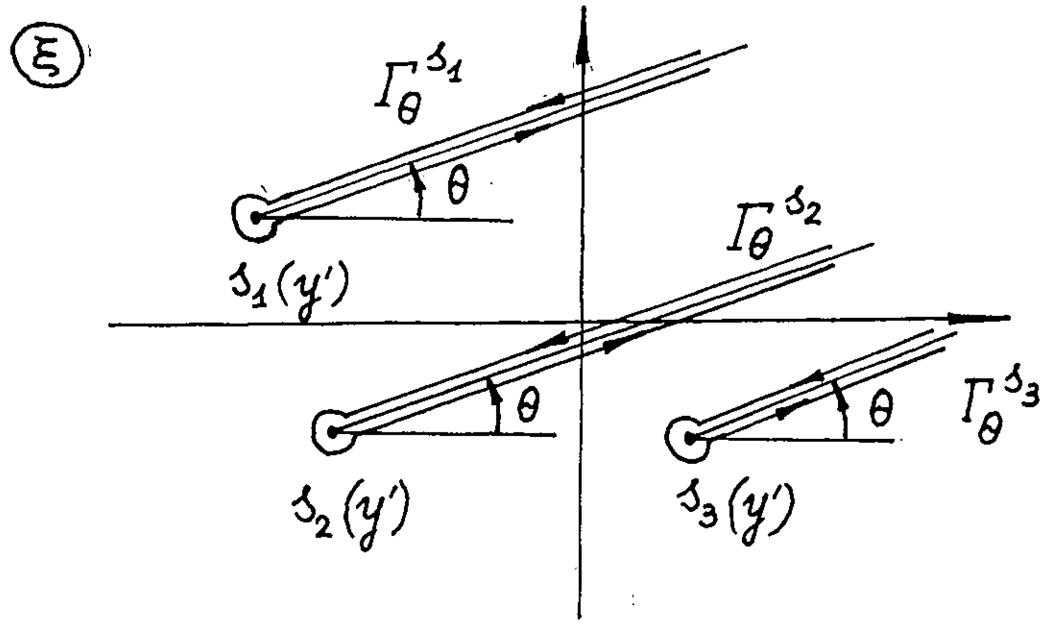

Figure 7

Suppose now that the Borel transform $\tilde{f}_1(\xi, y')$ is infinitely continuable for any fixed y', that is that this function can be continued up to a ramifying analytic function on the whole plane ξ with not more than a discrete set of singular points $\{\xi = s_i, i = 1, 2, \ldots\}$. Note, that under such assumptions $s_i = s_i(y')$ where $s_i(y')$ are (in general) ramifying analytic functions of y'. Then the integral (20) can be represented as a finite sum of the integrals of the form

$$f_1(y) = \sum_j \int_{\Gamma_\theta^{s_j}} e^{-\xi/y^0} \tilde{f}_1(\xi, y') d\xi, \tag{21}$$

the contours $\Gamma_\theta^{s_j}$ being drawn on Figure 7.

Thus, in the considered case the function $f_1(y)$ is *a resurgent function* of the variable y^0. Evidently, if two or more singular points s_j are lying on one and the same ray going in the direction θ, to represent the function $f_1(y)$ as a sum of integrals of the form (21) *over standard contours* $\Gamma_\theta^{s_j}$ one has to fix how these contours pass the singularity points. We do not describe here this well-known procedure.

However, in the case when the magnitude of the sector $\widetilde{\Omega}_{\widetilde{R}}$ is smaller than π, the Borel transformation cannot be used. In this case, as it is usual in the resurgent functions theory, the representation of the type (21) is already *a definition* of the resurgent function.

Let us now reduce the integral (21) to the form which coincides with the definition (3) of the resurgent function of several variables. To do this, we use the change of variables (17):

$$f(x) = \sum_j \int_{\Gamma_\theta^{s_j}} \tilde{f}_1(\xi, x'/x^1) d\xi. \tag{22}$$

Now the change of variables $\xi x^1 = \zeta$ reduces the right-hand part of the formula (22) to the sum of integrals of the type (3) with

$$\tilde{f}(\zeta, x) = \frac{1}{x^1} \tilde{f}_1 \left(\frac{\zeta}{x^1}, \frac{x'}{x^1} \right),$$

q.e.d.

With the help of the reduction of the resurgent function of several variables to the resurgent function of one variable dependent on parameters one can check that natural asymptotic expansions at infinity of a resurgent function of several variables have the form (2). To illustrate this, we suppose in addition that the function $\tilde{f}_1(\xi, y')$ has simple singularities. We recall that this means that these singularities are of the form

$$\tilde{f}_1(\xi, y') = \frac{a_0^j(y')}{\xi - s_j(y')} + \ln(\xi - s_j(y')) \sum_{k=0}^{\infty} \frac{(\xi - s_j(y'))^k}{k!} a_{k+1}^j(y'), \tag{23}$$

in a neighbourhood of each singular point $\xi = s_j(y')$, the series in the right-hand part of (23) being convergent in a vicinity of this point. It is well-known that each integral in the right-hand part of the representation (22) of the function $f_1(y)$ has an asymptotic expansion of the form

$$e^{-s_j(y')/y^0} \sum_{k=0}^{\infty} (y^0)^k a_k^j(y'). \tag{24}$$

With the help of the change of variables (17) the expression (24) becomes

$$e^{-S_j(x)} \sum_{k=0}^{\infty} A_k^j(x), \tag{25}$$

where

$$S_j(x) = \frac{1}{y^0} s_j \left(\frac{x^2}{x^1}, \ldots, \frac{x^n}{x^1} \right)$$

is a homogeneous function of the variables x of the first degree and the functions

$$A_k^j = (x^1)^{-k} a_k^j \left(\frac{x^2}{x^1}, \ldots, \frac{x^n}{x^1} \right)$$

are homogeneous functions of the degree $-k$. Thus, we see that in the multidimensional resurgent analysis expressions of the type (25) play the role of *elementary resurgent symbols*, that is, they are analogues of expressions $e^{-\omega/y} \sum a_k y^k$ where $\sum a_k y^k$ is a formal series of the Gevrey class G_1 (see [7]). Of course, the series included in (25) must also satisfy some inequalities of the same type.These estimates can be easily obtained using the evident relations between (24) and (25).

In conclusion we shall present an important remark. When considering the function $f_1(y^0, y')$ as a resurgent function of y^0 dependent on the parameters y', one can (for certain problems) exclude from consideration the so-called focal points. In doing so, the function $f_1(y^0, y')$ is considered *for all values of its main variable* y^0. However, if we had passed from the function $f_1(y^0, y')$ to the function $f(x)$, that is, from the terms of the form (24) to the terms of the form (25), to avoid the consideration of the focal points one must exclude from consideration some values of the *main* variables. In doing so, the asymptotic expansion of the function $f(x)$ remains uninvestigated near these points. That is why we have paid a lot of attention to investigation of focal points as well as to introduction of the appropriate notion of resurgent function with simple singularities.

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Max-Plank Arbeitsgruppe
" Analysis"
FB Mathematik, Universität Potsdam
Postfach 60 15 53, 14415 Potsdam
DEUTSCHLAND[5]
Fax: (331) 977 - 1173, (331) 977 - 1479

Dept. of Compt. Math. and Cybern.
Moscow State University
Vorob'evy Gory
119899 Moscow
RUSSIA
E-mail: sternin@cs.msu.su

[5] Until December 1,1993.