ON THE STRUCTURE OF NOETHERIAN SYMBOLIC REES ALGEBRAS

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Let A be a Noetherian local ring and I an ideal of A. In this paper we use a slightly generalized notion of a symbolic power $I^{(n)}$ of I and consider $R = \bigoplus_{n\geq 0} I^{(n)}$. First we characterize the property of R to be Noetherian by an equimultiplicity condition of some symbolic power $I^{(k)}$. The main purpose of this note is to explore the problem when R, $R' = \bigoplus_{n\in Z} I^{(n)}$ and $G = \bigoplus_{n\geq 0} I^{(n)}/I^{(n+1)}$ are Cohen-Macaulay or Gorenstein algebras in the case that A is a normal domain and $\mathrm{ht}\, I = 1$.

1. Introduction

Let A be a Noetherian local ring and $\mathfrak{p}$ a prime ideal of A. The Noetherian property of the symbolic Rees algebra $R = \bigoplus_{n\geq 0} \mathfrak{p}^{(n)}$ was studied by many authors (e.g. [9], [10], [16], [18],[20], [21]). In this paper we use a slightly generalized notion of a symbolic power $I^{(n)}$ for any ideal I and consider $R = \bigoplus_{n\geq 0} I^{(n)}$. We are interested in the Cohen-Macaulay and Gorenstein property of R.

First we characterize the property of R to be Noetherian by an equimultiplicity condition of some symbolic power $I^{(k)}$. Under this aspect the problem was already studied in the case that $\dim(A/I) = 1$ ([10, Corollary], [16, Theorem 4]). Here we will show that for any ideal I of an unmixed local ring A the symbolic Rees algebra R is Noetherian if $I^{(k)}$ is equimultiple (i.e. the analytic spread of $I^{(k)}$ coincides with $\mathrm{ht}\, I^{(k)}$) for some k (see (3.3).). The converse is also true if $A/I^{(n)}$ is Cohen-Macaulay for large n (see (3.6).)

The main purpose of this note is to explore the problem when R, $R' = \bigoplus_{n\in Z} I^{(n)}$ and $G = \bigoplus_{n\geq 0} I^{(n)}/I^{(n+1)}$ are Cohen-Macaulay or Gorenstein algebras in the case that A is a normal domain and $\mathrm{ht}\, I = 1$. In

Theorem (4.1) we show that the following statements are equivalent:

R is Cohen-Macaulay,
R' is Cohen-Macaulay,
G is Cohen-Macaulay,

provided that the order of the class $[I]$ of I in the divisor class group $Cl(A)$ of A is finite. For the characterization of the Gorenstein property we describe in Theorem (4.5) the relations between the canonical classes $[K_A]$, $[K_R]$ and $[K_{R'}]$ of A, R and R' respectively as follows:

$$[K_R] = [I] + [K_A] ,$$
$$[K_{R'}] = [K_A] .$$

From the first equation we get that a Cohen-Macaulay ring R is Gorenstein if and only if $K_A \cong I^* := \operatorname{Hom}_A(I,A)$. The second equation implies that a Cohen-Macaulay ring R' or G is Gorenstein if and only if A is so.

In section 5 we give some examples. In particular we consider the following three conditions on A :

(i) A is quasi-unmixed,
(ii) A is reduced,
(iii) A is a Nagata ring.

These conditions imply that A is unmixed but not vice versa. Now our examples show that if we replace the condition "A is unmixed" by any two of the conditions (i), (ii), (iii), then Theorem (3.3) is no more true.

Throughout this paper we use the following notations:

(1) A is a Noetherian ring. If A is local we denote by $\mathfrak{m}$ the maximal ideal of A and by $\hat{A}$ the $\mathfrak{m}$-adic completion. We denote by $Q(A)$ the total quotientring of A .

(2) If I is a proper ideal of A we denote the ordinary Rees ring $\sum_{n\geq 0} I^n t^n \subset A[t]$, where t is an indeterminate, by $R(I)$, and the extended Rees algebra $\sum_{n\in Z} I^n t^n \subset A[t,t^{-1}]$ by $R'(I)$.

(3) For a given Z-graded ring R we denote the k-th Veronesean subring by $R^{(k)}$.

(4) We denote by $\ell(\mathfrak{a})$ the analytic spread of an ideal $\mathfrak{a}$ in a local ring A.

(5) If $\mathfrak{a}$ is an ideal of a Noetherian ring A we put $\overline{A^*}(\mathfrak{a}) = \bigcup_{n\gg 0} \mathrm{Ass}_A(A/\overline{\mathfrak{a}^n})$, where $\overline{\mathfrak{a}^n}$ means integral closure of $\mathfrak{a}^n$, and $A^*(\mathfrak{a}) = \bigcup_{n\gg 0} \mathrm{Ass}_A(A/\mathfrak{a}^n)$ (See [14]).

(6) For a Krull domain A we denote the divisor class group of A by $\mathrm{Cl}(A)$, and $[\mathfrak{a}]$ is the class in $\mathrm{Cl}(A)$ of an ideal $\mathfrak{a}$ in A. For a finitely generated A-module M of $\mathrm{rank}_A M = n$, we define $\det M = [(\overset{n}{\wedge} M)^{**}]$, where $(\ )^*$ means the A-dual.

(7) If a ring A has a canonical module we denote it by K_A.

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2. Preliminaries.

In this section we recall several results which play a key role in our investigation of symbolic Rees algebras. For some of them, in particular for (2.10) and (2.11), we give new proofs based on Itoh's paper [4].

Throughout this section I and J are ideals of A.

Definition (2.1). We say that a family $F = \{F_n\}_{n\in Z}$ of ideals of A is a filtration of A if F satisfies the following conditions:

(1) $F_n \supset F_{n+1}$ for any $n \in \mathbf{Z}$.

(2) $F_n F_m \subset F_{n+m}$ for any $n, m \in Z$.

(3) $F_0 = A$.

When this is the case we put

$$R(F) = \sum_{n \geqq 0} F_n t^n \subset A[t]$$

and

$$R'(F) = \sum_{n\in \mathbf{Z}} F_n t^n \subset A[t,t^{-1}] \,.$$

Then $R'(F) = R(F)[u]$, where $u = t^{-1}$.

Definition (2.2). A filtration $F = \{F_n\}_{n\in Z}$ is said to be an I-filtration (resp. I-stable) if $F_n \supset I^n$ for any $n \in \mathbf{Z}$ (resp. there is an integer r such that $F_n = I^{n-r}F_r$ for any $n \geq r$.). Obviously F is an I-filtration if and only if $F_1 \supset I$.

Lemma (2.3). Let $F = \{F_n\}_{n\in Z}$ be an I-filtration. Then the following conditions are equivalent:

(1) F is I-stable.

(2) $R'(F)$ is module-finite over $R'(I)$.

(3) $R(F)$ is module-finite over $R(I)$.

(4) There exists an integer r such that $F_n \subset I^{n-r}$ for any $n \geq r$.

Proof. (1) $\Rightarrow$ (4) and (3) $\Rightarrow$ (2) are trivial.

(4) $\Rightarrow$ (3): Since $\sum_{n\geq r} F_n t^n \subset R(I)t^r$, we know that $\sum_{n\geq r} F_n t^n$ is finitely generated over $R(I)$.

(2) $\Rightarrow$ (1): Let $R'(F)$ be generated by s homogeneous elements $c_1 t^{n_1}, \ldots, c_s t^{n_s}$ $(c_i \in F_{n_i}, n_i \in Z)$ over $R'(I)$. We put $r = \max\{n_1,\ldots,n_s\}$. Then if $n \geq r$, we have $F_n = \sum_{i=1}^{s} I^{n-n_i} c_i \subset$ $\subset I^{n-r}F_r \subset F_n$, hence $F_n = I^{n-r}F_r$.

Lemma (2.4). Let $F = \{F_n\}_{n\in Z}$ be any filtration. Then the following conditions are equivalent:

(1) $R(F)$ is a Noetherian ring.

(2) $[R(F)]^{(k)}$ is a Noetherian ring for any $k > 0$.

(3) $[R(F)]^{(k)}$ is a Noetherian ring for some $k > 0$.

Proof. (1) ⇒ (2): follows from [3, Chapter III, 1.3, Proposition 2].

(2) ⇒ (3): Trivial.

(3) ⇒ (1): Let $0 \leq i < k$ and put $L_i = \sum_{n \geq 0} F_{nk+i} t^{nk}$. Then $R(F) = \sum_{i=0}^{k-1} L_i t^i$ and L_i is Noetherian as an ideal of $[R(F)]^{(k)}$. Hence $R(F)$ is module-finite over $[R(F)]^{(k)}$.

Definition (2.5). For ideals I and J in A let $I : \langle J \rangle = \bigcup_{i \geq 1} [I \underset{A}{:} J^i]$. Then $F = \{I^n : \langle J \rangle\}_{n \in Z}$ is an I-filtration of A. And we write $R_J(I) := R(F)$ and $R'_J(I) := R'(F)$.

The Noetherian property of $R_J(I)$ and $R'_J(I)$ was studied by Schenzel in [19], [21]. In the rest of this section we will deal with these algebras from another point of view. For that we use the method of ideal transforms, and within this frame the results of Itoh ([4]) will be essential. So we follow his notations.

Definition (2.6). Let $T(I,A) = \{x \in Q(A) \mid I^n x \subset A \text{ for some } n > 0\}$. If A is a local ring with the maximal ideal $\mathfrak{m}$, then we denote $T(\mathfrak{m},A)$ by A^g.

Then we have the following two results of Itoh:

Proposition (2.7). ([4, (1.16)]) Let A be a residue ring of a local Cohen-Macaulay ring such that $\operatorname{depth} A \geq 1$. Then A^g is a finite A-module if and only if $\dim(A/\mathfrak{p}) \geq 2$ for every $\mathfrak{p} \in \operatorname{Ass}_A(A)$.

Proposition (2.8). ([4, (3.2)]) The following conditions are equivalent:

(1) $T(I,A)$ is a finite A-module.

(2) $\Delta = \operatorname{Ass}_A(Q(A)/A) \cap V(I)$ is a finite set and $(A_{\mathfrak{p}})^g$ is a finite $A_{\mathfrak{p}}$-module for every $\mathfrak{p} \in \Delta$.

The next lemma gives the link between $R'_J(I)$ and an ideal transform of $R'(I)$.

Lemma (2.9). Let $\mathfrak{a} = JR'(I) + uR'(I)$. Then we have:

$$R'_J(I) = T(\mathfrak{a}, R'(I)) .$$

Proof. Let $Q = Q(R'(I))$. Then $R'(I) \subset A[t,t^{-1}] \subset Q$. We take any $\varphi \in T(\mathfrak{a}, R'(I))$ i.e. $\mathfrak{a}^n \varphi \subset R'(I)$ for some $n > 0$. Then $u^n \varphi \subset R'(I) \subset A[t,t^{-1}]$, so $\varphi \in A[t,t^{-1}]$. Furthermore, since $J^n \varphi \subset R'(I)$, all coefficients of φ are in $I^m : J^n$ for some n , so we have $\varphi \in R'_J(I)$. Conversely, take any $\psi \in R'_J(I)$. For n large enough we get $u^n \psi \in A[t^{-1}] \subset R'(I)$ and $J^n \psi \subset R'(I)$. Therefore $\mathfrak{a}^\ell \psi \in R'(I)$ for some ℓ , hence $\psi \in T(\mathfrak{a}, R'(I))$.

Theorem (2.10). (c.f. [19, (6.4))]) Let A be a local ring with the maximal ideal $\mathfrak{m}$. Then the following conditions are equivalent.

(1) $R_{\mathfrak{m}}(I)$ is module-finite over $R(I)$.

(2) $\ell(I\hat{A} + \mathfrak{p}/\mathfrak{p}) < \dim(\hat{A}/\mathfrak{p})$ for any $\mathfrak{p} \in \mathrm{Ass}\,\hat{A}$.

Proof. First we note that, since $(I^n : \langle\mathfrak{m}\rangle)\hat{A} = I^n\hat{A} : \langle\mathfrak{m}\hat{A}\rangle$, we may assume that A is complete. Put $\mathfrak{a} = \mathfrak{m}R'(I) + uR'(I)$. Then by (2.3) and (2.9) condition (1) holds if and only if $T(\mathfrak{a}, R'(I))$ is module-finite over $R'(I)$. In this proof we put $R' = R'(I)$ and $Q' = Q(R')$.

(1) ⇒ (2): Assume that there is a prime ideal $\mathfrak{p} \in \mathrm{Ass}\,A$ such that $\ell(I + \mathfrak{p}/\mathfrak{p}) = \dim(A/\mathfrak{p})$. Then we know by [14, (4.1)] that $\mathfrak{m}/\mathfrak{p} \in \overline{A}^*(I + \mathfrak{p}/\mathfrak{p})$, hence we find by [14, (3.18)] a prime ideal $P \in \overline{A}^*(uS)$ such that $P \cap A/\mathfrak{p} = \mathfrak{m}/\mathfrak{p}$, where $S := R'(I + \mathfrak{p}/\mathfrak{p})$. Since $\dim S_P = \ell(uS_P) = 1$ by [14, (4.1)], we have $P \in \mathrm{Min}_S(S/uS)$.

Let $\psi : R' \longrightarrow S$ be the natural surjection and put $\mathfrak{p}^* = \ker \psi$, $P' = \psi^{-1}(P)$. Then $P' \in \mathrm{Ass}(R'/uR') \subset \mathrm{Ass}(Q'/R')$, and therefore $P' \in \mathrm{Ass}(Q'/R') \cap V(\mathfrak{a})$. This implies that $(R'(I)_{P'})^g$ is module-finite over $R'(I)_{P'}$ by (2.8). Since $\mathfrak{p}^* R'_{P'} \in \mathrm{Ass}(R'_{P'})$, we have $\dim(R'_{P'}/\mathfrak{p}^* R'_{P'}) \geq 2$ by (2.7). Hence we obtain $\dim S_P \geq 2$, which is a contradiction.

(2) ⇒ (1): Let $\Delta := \mathrm{Ass}_{R'}(Q'/R') \cap V(\mathfrak{a}) \subset \mathrm{Ass}_{R'}(R'/uR')$. Since Δ is a finite set, it is enough to show that $\dim(R'_{P'}/Q) \geq 2$ for any $Q \in \mathrm{Ass}\, R'_{P'}$ by (2.7) and (2.8).

Now we assume that there exists a prime ideal $Q \in \mathrm{Ass}(R'_{P'})$ such that $\dim(R'_{P'}/Q) \leq 1$. Then $Q = \mathfrak{p}^* R'_{P'}$ for some $\mathfrak{p} \in \mathrm{Ass}\, A$, where $\mathfrak{p}^*$ denotes the kernel of the natural surjection $\psi : R' \longrightarrow S = R'(I + \mathfrak{p}/\mathfrak{p})$. We put $P = \psi(P')$. Then $\dim S_P \leq 1$. Since $u \in P$, we have $P \in \mathrm{Min}(S/uS)$. This implies $P \in \overline{A}^*(uS)$. Hence we get $\mathfrak{m}/\mathfrak{p} = P \cap A/\mathfrak{p} \in \overline{A}^*(I + \mathfrak{p}/\mathfrak{p})$ and so $\ell(I + \mathfrak{p}/\mathfrak{p}) = \dim(A/\mathfrak{p})$. But this is a contradiction to condition (2).

<u>Theorem (2.11)</u>. (c.f.[19, (5.6)]) The following conditions are equivalent:

(1) $R_J(I)$ is module-finite over $R(I)$.

(2) $\ell(I\widehat{A_{\mathfrak{p}}} + Q/Q) < \dim(\widehat{A_{\mathfrak{p}}}/Q)$ for any $\mathfrak{p} \in V(I) \cap V(J)$ and any $Q \in \mathrm{Ass}\,\widehat{A_{\mathfrak{p}}}$.

<u>Proof</u>. (1) ⇒ (2): By (2.3) there exists an integer r such that $I^{n-r} \supset I^n : \langle J \rangle$ for any $n \geq r$. Let $\mathfrak{p} \in V(I) \cap V(J)$. Then $I^{n-r}A_{\mathfrak{p}} \supset I^n A_{\mathfrak{p}} : \langle J A_{\mathfrak{p}} \rangle \supset I^n A_{\mathfrak{p}} : \langle \mathfrak{p} A_{\mathfrak{p}} \rangle$. Therefore $R_{\mathfrak{p}A_{\mathfrak{p}}}(I A_{\mathfrak{p}})$ is module-finite over $R(IA_{\mathfrak{p}})$ by (2.3). So we get $\ell(I\widehat{A_{\mathfrak{p}}} + Q/Q) <$ $< \dim(\widehat{A_{\mathfrak{p}}}/Q)$ for any $Q \in \mathrm{Ass}\,\widehat{A_{\mathfrak{p}}}$ by (2.10).

(2) ⇒ (1): We put $R' = R'(I)$ and $\mathfrak{a} = JR' + uR'$. Then we have $T(\mathfrak{a}, R') = R'_J(I)$ by (2.9). Let $\Delta = \mathrm{Ass}_{R'}(Q(R')/R') \cap V(\mathfrak{a}) \subset$ $\subset \mathrm{Ass}_{R'}(R'/uR')$, which is a finite set. We take any $P \in \Delta$ and put $\mathfrak{p} = P \cap A$. Then $\mathfrak{p} \in V(I) \cap V(J)$, and from (2.10) and the assumption we conclude that $R'_{\mathfrak{p}A_{\mathfrak{p}}}(IA_{\mathfrak{p}})$ is module-finite over $R'(IA_{\mathfrak{p}})$. This implies that $T(PR'_P, R'_P)$ is module-finite over R'_P . Then $T(\mathfrak{a}, R')$ is module-finite over R' by (2.8).

3. Finite generation of symbolic Rees algebras.

Throughout this section I is a proper ideal of a Noetherian ring A and S denotes a multiplicative subset of A such that $I \cap S = \phi$.

Definition (3.1). We write $I^{(n)} = I^n A_S \cap A$ for each $n \in \mathbf{Z}$ and we put $R = R(\{I^{(n)}\}_{n \in \mathbf{Z}})$.

First we give a generalization of [20, Theorem (2.1)].

Theorem (3.2). Let A be a Noetherian ring and I an ideal of A . Then the following conditions are equivalent:

(1) $R = \bigoplus_{n \geq 0} I^{(n)} t^n$ is a Noetherian ring.

(2) There is a positive integer k such that $\ell(I^{(k)}\widehat{A_{\mathfrak{p}}} + Q/Q) < \dim \widehat{A_{\mathfrak{p}}}/Q$ for any $\mathfrak{p} \in V(I^{(k)})$ with $\mathfrak{p} \cap S \neq \phi$ and for any $Q \in \operatorname{Ass} \widehat{A_{\mathfrak{p}}}$.

(3) There is a positive integer k such that $[I^{(k)}]^n = I^{(kn)}$ for any $n \geq 1$.

(4) There is a positive integer k such that $[I^{(k)}]^n = I^{(kn)}$ for all $n >> 0$.

Proof. (1) ⇒ (3) comes from [3; Chapter III, 1.3 Lemma 2], and (3) ⇒ (4) is trivial.

(2) ⇒ (1): Put $\mathfrak{a} = I^{(k)}$. Then $\mathfrak{a} \cap S = \phi$ and $\mathfrak{a}^{(n)} = I^{(kn)}$ for any $n \geq 0$. Hence $R^{(k)} = R(\{\mathfrak{a}^{(n)}\}_{n \in \mathbf{Z}})$. Let $\mathcal{F} = A^*(\mathfrak{a})$, i.e. there is a positive integer r such that $\mathcal{F} = \operatorname{Ass}_A(A/\mathfrak{a}^n)$ for any $n \geq r$. Put $\mathcal{F}' = \{\mathfrak{p} \in \mathcal{F} \mid \mathfrak{p} \cap S \neq \phi\}$. We define an ideal J of A by

$$J = \begin{cases} \bigcap_{\mathfrak{p} \in \mathcal{F}'} \mathfrak{p} & \text{if } \mathcal{F}' \neq \phi \\ A & \text{if } \mathcal{F}' = \phi \end{cases} .$$

Then $\mathfrak{a}^n : \langle J \rangle \subset \mathfrak{a}^{(n)}$ for all $n \in \mathbf{Z}$ and $\mathfrak{a}^n : \langle J \rangle = \mathfrak{a}^{(n)}$ for $n \geq r$. Hence we get $R_J(\mathfrak{a}) \subset R^{(k)}$ and $[R_J(\mathfrak{a})]_n = [R^{(k)}]_n$ for any $r \geq r$.

Now the condition (2) of the theorem implies that $R_J(\mathfrak{a})$ is Noetherian by (2.11), so $R^{(k)}$ is also Noetherian. Hence R is Noetherian by (2.4).

(4) ⇒ (2): Put $\mathfrak{a} = I^{(k)}$. Let $\mathfrak{p} \in V(\mathfrak{a})$ and $\mathfrak{p} \cap S \neq \phi$. Then for any $n \gg 0$ we have $\mathfrak{a}^n = \mathfrak{a}^{(n)} \supset \mathfrak{a}^n : \langle\mathfrak{p}\rangle \supset \mathfrak{a}^n$ and so $\mathfrak{a}^n : \langle\mathfrak{p}\rangle = \mathfrak{a}^n$. Hence $R_{\mathfrak{p}}(\mathfrak{a})$ is Noetherian. Then we get (2) by (2.11).

As a corollary we obtain the following theorem.

Theorem (3.3). Let A be an unmixed local ring and I and ideal of A. If $\ell(I^{(k)}) = ht(I^{(k)})$ for some $k \geq 1$, then R is Noetherian.

Proof. Let $\mathfrak{p} \in V(I^{(k)})$ and $\mathfrak{p} \cap S \neq \phi$. We have to show the inequality (2) of (3.2). Note that for any $Q \in \operatorname{Ass}\widehat{A_{\mathfrak{p}}}$ we have

$$\ell(I^{(k)}\widehat{A_{\mathfrak{p}}} + Q/Q) \leq \ell(I^{(k)}) = ht(I^{(k)}) \leq \dim(A_{\mathfrak{p}}) = \dim(\widehat{A_{\mathfrak{p}}}/Q) .$$

Therefore it is enough to show that $ht(I^{(k)}) < \dim A_{\mathfrak{p}}$. If $ht\, I^{(k)} = \dim A_{\mathfrak{p}}$, then $\mathfrak{p} \in \operatorname{Min}(A/I^{(k)})$ and so $\mathfrak{p} \cap S = \phi$. This is a contradiction.

Proposition (3.4). Let A be a local ring. If A/I^n is Cohen-Macaulay for all $n \geq 1$, then $\ell(I) = \dim A - \dim(A/I)$. Hence, if A is quasi-unmixed, we have $\ell(I) = ht\, I$, i.e. I is equimultiple.

Proof. Put $s = \dim A/I$ and choose a subsystem of parameters $a_1, \ldots, a_s$ for A so that $a_1, \ldots, a_s$ form a system of parameters for A/I. Then as $a_1, \ldots, a_s$ is an A/I^n-sequence, we get $\mathfrak{q}I^n = \mathfrak{q} \cap I^n$ for all $n \geq 1$, where $\mathfrak{q} = (a_1, \ldots, a_s)A$. Therefore $G(I)/\mathfrak{q}G(I) \cong G(I + \mathfrak{q}/\mathfrak{q})$ and this implies $\ell(I) = \dim(A/\mathfrak{q}) = \dim A - s = \dim A - \dim(A/I)$.

Corollary (3.5). Let A be a local ring as in (3.4) and $F = \{F_n\}_{n \in \mathbf{Z}}$ any filtration of A. If $R(F)$ is Noetherian and A/F_n is Cohen-Macaulay for $n \gg 0$, then $\ell(F_k) = ht(F_k)$ for some k.

Proof. By assumption, there is a positive integer k such that $(F_k)^n = F_{kn}$ for all $n \geq 1$. Therefore, taking k large enough, we may assume that $A/(F_k)^n$ is Cohen-Macaulay for any $n \geq 1$. Then $\ell(F_k) = \mathrm{ht}(F_k)$ by (3.4).

From (3.3) and (3.5) we get the following theorem.

Theorem(3.6). Let A be an unmixed local ring and suppose that $A/I^{(n)}$ is Cohen-Macaulay for $n >> 0$. Then R is Noetherian if and only if $\ell(I^{(k)}) = \mathrm{ht}(I^{(k)})$ for some $k \geq 1$.

In the rest of this section we assume the following situation, labelled by (*):

(*) $(A,\mathfrak{m})$ is a Noetherian normal local domain of $\dim A = d > 0$ and $\mathrm{ht}\, I = 1$. When this is the case we choose in particular $S = A \setminus \bigcup_{\mathfrak{p} \in \mathcal{F}} \mathfrak{p}$, where $\mathcal{F} = \{\mathfrak{p} \in H_1(A) \mid I \subset \mathfrak{p}\}$, and $H_1(A)$ denotes the set of height one prime ideals of A. Then we define $I^{(n)}$ and R as in (3.1). Moreover we denote by R' and G the extended symbolic Rees algebra $R'(\{I^{(n)}\}_{n \in \mathbf{Z}})$ and the associated graded ring $\bigoplus_{n \geq 0} I^{(n)}/I^{(n+1)}$ respectively.

Remark. Note that $I^{(n)}$ is the divisorialization of I^n in the situation (*), since $I^{(n)} = (\bigcap_{\mathfrak{p} \in \mathcal{F}} I^n A_{\mathfrak{p}}) \cap A = \bigcap_{\mathfrak{p} \in H_1(A)} I^n A_{\mathfrak{p}}$.

Theorem (3.7). In the situation (*) we assume that A is an unmixed local ring of $\dim A \geq 2$ and that $A_{\mathfrak{p}}$ is factorial for all $\mathfrak{p} \in \mathrm{Spec}\, A \setminus \{\mathfrak{m}\}$. Then R is Noetherian if and only if $\ell(I^{(k)}) < \dim A$ for some k.

Proof. Let $\mathfrak{p} \in V(I) \setminus \{\mathfrak{m}\}$ and $\mathfrak{p} \cap S \neq \phi$. So $\mathrm{ht}\, \mathfrak{p} \geq 2$ and $A_{\mathfrak{p}}$ is factorial. Therefore $I^{(k)}A_{\mathfrak{p}}$ must be principal for any k since it is a divisorial ideal of $A_{\mathfrak{p}}$ by the fore-going remark. Hence we have for any $Q \in \mathrm{Ass}\, \widehat{A_{\mathfrak{p}}}$ and $k > 0$:

$$\ell(I^{(k)}\widehat{A_{\mathfrak{p}}} + Q/Q) \leq \ell(I^{(k)}\widehat{A_{\mathfrak{p}}}) = 1 < \dim \widehat{A_{\mathfrak{p}}} = \dim \widehat{A_{\mathfrak{p}}}/Q \quad ,$$

since $A_{\mathfrak{p}}$ is unmixed. Therefore we get the claim of the theorem by (3.2) and [14, (4.2)].

Now we characterize the property of R to be Noetherian by the order of the class $[I]$ of I in $Cl(A)$, which is denoted by $|[I]|$.

Theorem (3.8). In the situation (*), R is Noetherian if $|[I]| < \infty$. The converse holds if A is a quasi-unmixed local ring and $A/I^{(n)}$ is Cohen-Macaulay for $n >> 0$.

First we show the following lemma.

Lemma (3.9). Let k be a positive integer and assume that $I^{(k)} = aA$ with $a \in A$. Then the following assertions hold:

(1) at^k is a non-zero divisor on R and G.

(2) $R/at^kR = A \oplus I^{(1)} \oplus \ldots \oplus I^{(k-1)}$.

(3) $G/at^kG = A/I^{(1)} \oplus I^{(1)}/I^{(2)} \oplus \ldots \oplus I^{(k-1)}/I^{(k)}$.

Proof. For any $n \geq 0$ we write $n = ik + j$ with $i \geq 0$ and $0 \leq j < k$. Then we get $I^{(n)} = a^i I^{(j)}$. This proves (2) and (3). To show that at^k is a non-zero divisor on G, we assume that $x \in I^{(n)}$ $ax \in I^{(k+n+1)}$. Since $k + n + 1 = (i+1)k+j+1$, we have $I^{(k+n+1)} = a^{i+1}I^{(j+1)}$ and so $ax = a^{i+1}y$ for some $y \in I^{(j+1)}$. Then $x = a^i y \in I^{(n+1)}$. Therefore at^k is a non-zero divisor on G (and of course on R too).

Proof of (3.8). The first assertion follows immediately from (2) of (3.9). Conversely, let R be Noetherian and assume that A is quasi-unmixed and $A/I^{(n)}$ is Cohen-Macaulay for $n >> 0$. By (3.5) we have $\ell(I^{(k)}) = 1$ for some $k > 0$. Then $I^{(k)}$ must be principal, and so $|[I]| < k$.

Corollary (3.10). In the situation (*) we assume $\dim A = 2$. Then R is Noetherian if and only if $|[I]| < \infty$.

4. Cohen-Macaulay and Gorenstein properties.

Throughout this section A and I satisfy the condition (*) of section 3. Assuming $|[I]| < \infty$, we first investigate the Cohen-Macaulay property of the algebras R, R' and G defined in (*). Then we describe the relations between the canonical classes of A, R and R', which lead in particular to a characterization of the Gorenstein property of R, R' and G. Note that the divisor class groups of R and R' are available since these algebras are Krull domains (See (4.3).).

<u>Theorem (4.1)</u>. Suppose that $k = |[I]| < \infty$. Then the following statements are equivalent:

(1) R is a Cohen-Macaulay ring.

(2) R' is a Cohen-Macaulay ring.

(3) G is a Cohen-Macaulay ring.

(4) $I^{(n)}$ is a maximal Cohen-Macaulay module over A (i.e. a Cohen-Macaulay module over A with the same dimension as A) for $0 \leq n < k$.

(5) $I^{(n)}$ is a maximal Cohen-Macaulay module for any $n \in \mathbf{Z}$.

<u>Proof</u>. By assumption there is an element $a \in A$ such that $I^{(k)} = aA$.

(1)$\Leftrightarrow$(4): By (3.9) R is Cohen-Macaulay if and only if $T := R/at^kR = A \oplus I^{(1)} \oplus \ldots \oplus I^{(k-1)}$ is Cohen-Macaulay. Note that any system of parameters for A is a homogeneous system of parameters for T. Therefore conditions (1) and (4) are equivalent.

(4)$\Leftrightarrow$(5): We take any $n \in \mathbf{Z}$. Then $I^{(n)}$ is isomorphic to one of the ideals $A, I^{(1)}, \ldots, I^{(k-1)}$ as we have seen in the proof of (3.9).

(2)$\Leftrightarrow$(3): The element $u = t^{-1}$ is a homogeneous non-zero divisor on R' and we have $R'/uR' \cong G$. Hence R' is Cohen-Macaulay if and only if G is Cohen-Macaulay.

(5)$\Rightarrow$(3): We conclude from (5) that $\operatorname{depth}_A(I^{n)}/I^{(n+1)}) = d-1$ for any $n \in \mathbf{Z}$. Therefore $A/I^{(1)}$ is a Cohen-Macaulay ring and

$I^{(n)}/I^{(n+1)}$ ia a maximal Cohen-Macaulay module over $A/I^{(1)}$, i.e. $G/at^kG = A/I^{(1)} \oplus I^{(1)}/I^{(2)} \oplus \ldots \oplus I^{(k-1)}/I^{(k)}$ is a Cohen-Macaulay ring. Moreover G is a Cohen-Macaulay ring too since at^k is a non-zero divisor on G by (3.9).

(2) ⇒ (4): Condition (2) implies $R'_u = A[t,t^{-1}]$ is a Cohen-Macaulay ring, hence A must be Cohen-Macaulay and $\dim(A/I^{(1)}) = d-1$. Furthermore we know that $G/at^kG = A/I^{(1)} \oplus I^{(1)}/I^{(2)} \oplus \ldots \oplus I^{(k-1)}/I^{(k)}$ is a Cohen-Macaulay ring by (3.9), therefore $\operatorname{depth}_A(I^{(n)}/I^{(n+1)}) = d-1$ for $0 \leq n < k$. Then $I^{(1)}, I^{(2)}, \ldots, I^{(k-1)}$ are maximal Cohen-Macaulay A-modules by the depth-lemma applied to the exact sequence $0 \longrightarrow I^{(n+1)} \longrightarrow I^{(n)} \longrightarrow I^{(n)}/I^{(n+1)} \longrightarrow 0$, q.e.d.

If $\dim A = 2$ then $I^{(n)}$ is always a maximal Cohen-Macaulay A-module for all $n \in \mathbf{Z}$, since $I^{(n)}$ is the divisorialization of I^n. This yields the following Corollary.

<u>Corollary (4.2)</u>. If $\dim A = 2$ and $|[I]| < \infty$, then R, R' and G are Cohen-Macaulay rings.

To characterize the Gorenstein property of R, R' and G we first calculate the canonical classes of R and R'.

We start with a lemma.

<u>Lemma (4.3)</u>. Under the general assumptions of the section the following assertions are true:

(1) R and R' are Krull domains.

(2) If $P \in H_1(R)$ or $P \in H_1(R')$, then $\operatorname{ht}(P \cap A) \leq 1$.

(3) If $P \in H_1(R)$ and $P \cap A \neq 0$, then $(P \cap A)R_P = PR_P$.

(4) If $P \in H_1(R)$ and $Q \in H_1(R)$ with $P \cap A = Q \cap A \neq 0$, then $P = Q$.

<u>Proof</u>. (1) Since $I^{(n)} = (\bigcap_{\mathfrak{p} \in \mathcal{F}} I^n A_{\mathfrak{p}}) \cap A$, where $\mathcal{F} = \{\mathfrak{p} \in H_1(A) \mid I \subset \mathfrak{p}\}$, we have

$$(\#) \qquad R = \left(\bigcap_{\mathfrak{p}\in\mathcal{F}} R_{\mathfrak{p}}\right) \cap A[t] \quad \text{and} \quad R' = \left(\bigcap_{\mathfrak{p}\in\mathcal{F}} R'_{\mathfrak{p}}\right) \cap A[t,t^{-1}] \ .$$

For any $\mathfrak{p} \in H_1(A)$ there is an element $a \in A$ such that $IA_{\mathfrak{p}} = aA_{\mathfrak{p}}$, hence $R_{\mathfrak{p}} = A_{\mathfrak{p}}[at]$ and $R'_{\mathfrak{p}} = A_{\mathfrak{p}}[at,t^{-1}]$. These two rings are Krull domains. Therefore R and R' are Krull domains by (#) and [3, Chapter VII, 1.3, Example 3].

(2) We only consider the case where $P \in H_1(R')$. (The same proof works for $P \in H_1(R)$.) For every $\mathfrak{p} \in H_1(A)$ we put $w(\mathfrak{p}) = \{Q' \in H_1(R') \mid Q' \cap A \subset \mathfrak{p}\}$. Then by (#) we find the following defining family of discrete valuation rings for R' :

$$\left(\bigcup_{\mathfrak{p}\in\mathcal{F}} \left\{R'_{Q'} \;\middle|\; Q' \in w(\mathfrak{p})\right\}\right) \cup \left\{A[t,t^{-1}]_Q \;\middle|\; Q \in H_1(A[t,t^{-1}])\right\} \ .$$

Now, if $P \in H_1(R')$, we have the following two cases:

(i) $R'_P = R'_{Q'}$ for some $Q' \in w(\mathfrak{p})$ with $\mathfrak{p} \in \mathcal{F}$.

(ii) $R'_P = A[t,t^{-1}]_Q$ for some $Q \in H_1(A[t,t^{-1}])$.

For (i) we obtain $P \cap A = Q' \cap A \subset \mathfrak{p}$ and for (ii) we have $P \cap A = Q \cap A$. In both cases we get $\mathrm{ht}(P \cap A) \leq 1$.

(3) If $P \in H_1(R)$ and $P \cap A \neq 0$ we put $\mathfrak{p} = P \cap A$. Then $\mathfrak{p} \in H_1(A)$ by (2). Since $R_{\mathfrak{p}}$ is a polynomial ring over $A_{\mathfrak{p}}$, we know that $\mathfrak{p}R_{\mathfrak{p}} \in \mathrm{Spec}\, R_{\mathfrak{p}}$ and $0 \neq \mathfrak{p}R_{\mathfrak{p}} \subset PR_{\mathfrak{p}}$. Hence $\mathfrak{p}R_{\mathfrak{p}} = PR_{\mathfrak{p}}$ and so $\mathfrak{p}R_P = PR_P$.

(4) This follows from the proof of (3).

By (2) of (4.3) the inclusions $A \subset R$ and $A \subset R'$ satisfy the condition PDE of Samuel ([3, Chapter VII, 1.10]). Hence there are natural homomorphisms $i : \mathrm{Cl}(A) \longrightarrow \mathrm{Cl}(R)$ and $i' : \mathrm{Cl}(A) \longrightarrow \mathrm{Cl}(R')$. The next proposition describes these homomorphisms.

Proposition (4.4). (c.f.[22, Proposition 2.6]) With the assumptions as above the following statements hold:

(1) i is an isomorphism.

(2) i' is a splitting monomorphism.

Proof. (1) Let $T = A \setminus \{0\}$. Then $Cl(R_T) = 0$ since R_T is a polynomial ring over the quotient field of A . This implies $Cl(R) = \langle [P] \mid P \in H_1(R)\ ,\ P \cap A \neq 0 \rangle$, see [3, Chapter VII, 1.10]. On the other hand, if $P \in H_1(R)$ and $P \cap A \neq 0$, then $\mathfrak{p} := P \cap A \in H_1(A)$ and $i([\mathfrak{p}]) = [P]$ by (4.3) and [3, Chapter VII, 1.10, Prop. 14]. Hence i is surjective. To show that i is injective we assume $[\mathfrak{a}] \in \operatorname{Ker} i$, where $\mathfrak{a}$ is a divisorial ideal of A . Then there is a homogeneous element $f \in R$ such that $\mathfrak{a}R_P = fR_P$ for all $P \in H_1(R)$. Now take $\mathfrak{p} \in H_1(A)$. Since $R_{\mathfrak{p}}$ is flat over A , $\mathfrak{a}R_{\mathfrak{p}}$ is also divisorial. So we have $\mathfrak{a}R_{\mathfrak{p}} = fR_{\mathfrak{p}}$. Therefore $f \in A$ and $A_{\mathfrak{p}} = fA_{\mathfrak{p}}$. This implies $[\mathfrak{a}] = 0$.

(2) Let $j : Cl(R') \longrightarrow Cl(R'_u)$ be the homomorphism induced from the inclusion $R' \subset R'_u$, where $u = t^{-1}$. Since $R'_u = A[t,t^{-1}]$, we identify $Cl(R'_u)$ with $Cl(A)$, s. [3, Chapter VII, 1.10, Prop. 18]. Then the composition $j \circ i'$ is an identity map. Hence i' is a splitting monomorphism.

Now we describe the relations between canonical classes of A , R and R' .

Theorem (4.5). (c.f. [8, Theorem (c)]). Suppose that A is a homomorphic image of a regular local ring and R is Noetherian. Then the canonical modules K_A , K_R and $K_{R'}$ exist and we have the following equalities:

(1) $[K_R] = [I] + [K_A]$.

(2) $[K_{R'}] = [K_A]$.

Here we regard $[I]$ and $[K_A]$ as elements of $Cl(R)$ (resp. $Cl(R')$) via the group homomorphisms i (resp. i').

To prove (4.5) we recall the following fact, which is well-known.

Lemma (4.6). Let A be a discrete valuation ring and M a finitely generated free A-module with a free basis $h_1,\dots,h_n$. Suppose L is an A-submodule of M generated by $g_1,\dots,g_n$ such that M/L is a torsion A-module. If $g_j = \sum_{i=1}^{n} a_{ij}h_i\ (a_{ij} \in A)$ for $1 \le \ \le n$, then

$$\ell_A(M/L) = v(\det[a_{ij}]) \quad ,$$

where v denotes the normalized additive valuation of A.

We divide the proof of (4.5) into two steps:

Proof of (4.5) in case that $I^{(i)} = I^i$ for all $i \in \mathbf{Z}$: Let $I = (a_1,\dots,a_n)A$ with $a_i \neq 0$ for $1 \le i \le n$, and let $X_1,\dots,X_n,Y$ be indeterminates. We may assume $n \gtrsim 2$. We denote the kernels of the surjections

$$A[X_1,\dots,X_n] \longrightarrow R \text{ with } X_i \longmapsto a_it$$

and

$$A[X_1,\dots,X_n,Y] \longrightarrow R' \text{ with } X_i \longmapsto a_it,\ Y \longmapsto u=t^{-1}$$

by J and J' respectively.

Claim 1. The following equations hold:

(i) $\det(J/J^2) = -[K_R] + [K_AR]$.

(ii) $\det(J'/(J')^2) = -[K_{R'}] + [K_AR']$.

Proof. The equation (i) was already shown in the proof of [8, Theorem, (c)] . The same technique as in [8] works for (ii): Actually there is a regular local ring S together with an epimorphism $S \longrightarrow A$ by assumption. Let J_1 and J_2 be the kernels of the natural surjections $S[X_1,\dots,X_n,Y] \longrightarrow A[X_1,\dots,X_n,Y]$ and $S[X_1,\dots,X_n,Y] \longrightarrow R'$. Then by tensorizing with S the sequence $0 \longrightarrow J_1 \longrightarrow J_2 \longrightarrow J' \longrightarrow 0$

gives rise to a complex of R'-modules

$$0 \longrightarrow J_1/J_1^2 \otimes R' \longrightarrow J_2/J_2^2 \longrightarrow J'/(J')^2 \longrightarrow 0 \quad ,$$

which is split exact at prime ideals $P \in H_1(R')$. So we obtain (ii) by [8, Lemma p. 183].

<u>Claim 2</u>. The following equations hold:

(i) $\det(J/J^2) = -[IR]$.

(ii) $\det(J'/(J')^2) = 0$.

<u>Proof</u>. (i) We put $B = A[X_1,\ldots,X_n]$, $M = J/J^2$ and

$$g_i = a_nX_i - a_iX_n \in J \quad \text{for} \quad 1 \leq i \leq n-1 \quad .$$

Let L be the R-submodule of M generated by the classes of $g_1,\ldots,g_{n-1}$ in M . Since $JC = (g_1,\ldots,g_{n-1})C$, where $C = B \otimes_A Q(A)$, and since $g_1,\ldots,g_{n-1}$ form a regular sequence on C , we get $M \otimes_R Q(R) = L \otimes_R Q(R)$, and the rank of this vector space over $Q(R)$ is $n-1$. Hence we get:

$$\det M = - \sum_{P \in H_1(R)} \ell(T_P)[P] \quad ,$$

where $T = M/L$ (see [3, Chapter VII, §§ 4.5]). We want to show that

(*1) $\ell(T_P) = v_P(IR) + (n-2)v_P(a_n)$

for any $P \in H_1(R)$, where v_P is the valuation with respect to P . For that it is enough to consider the case where $IA_{\mathfrak{p}} = a_nA_{\mathfrak{p}}$ or $IA_{\mathfrak{p}} = a_{n-1}A_{\mathfrak{p}}$, where $\mathfrak{p} = P \cap A$. (Note $\operatorname{ht}\mathfrak{p} \leq 1$ by (2) of (4.3)).

If $IA_{\mathfrak{p}} = a_nA_{\mathfrak{p}}$, then there exists an element $\alpha_i \in A_{\mathfrak{p}}$ such that $a_i = \alpha_ia_n$ for all $1 \leq i \leq n-1$. We put

$$h_i = X_i - \alpha_iX_n \in JB_{\mathfrak{p}} \quad \text{for} \quad 1 \leq i \leq n-1 \quad .$$

Since $JB_{\mathfrak{p}} = (h_1,\ldots,h_{n-1})B_{\mathfrak{p}}$ and since $h_1,\ldots,h_{n-1}$ form a $B_{\mathfrak{p}}$-sequence, we know that M_P is R_P-free and the images of $h_1,\ldots,h_{n-1}$ in M_P

form a free basis over R_P . On the other hand we have $a_n h_i = g_i$, i.e.

$$(g_1, g_2, \dots, g_{n-1}) = (h_1, h_2, \dots, h_{n-1}) \begin{bmatrix} a_n & & & \\ & a_n & & \\ & & \ddots & \\ & & & a_n \end{bmatrix} .$$

Therefore $\ell(T_P) = (n-1)v_P(a_n)$ by (4.6), and this implies (*1) since $v_P(IR) = v_P(a_n)$.

Next we assume $IA_{\mathfrak{p}} = a_{n-1}A_{\mathfrak{p}}$. Then there exists an element $\alpha_i \in A_{\mathfrak{p}}$ such that $a_i = \alpha_i a_{n-1}$ for $1 \leq i \leq n$. In this case we put

$$h_i = X_i - \alpha_i X_n \in JB_{\mathfrak{p}} \quad \text{for} \quad 1 \leq i \leq n-2 \quad ,$$

$$h_{n-1} = X_n - \alpha_n X_{n-1} \in JB_{\mathfrak{p}} \quad .$$

Then we have

$$(g_1, \dots, g_{n-2}, g_{n-1}) = (h_1, \dots, h_{n-2}, h_{n-1}) \begin{bmatrix} a_n & & & \\ & \ddots & & \\ & & a_n & \\ -a_1 & \dots & -a_{n-2} & -a_{n-1} \end{bmatrix}$$

By the same argument as before we get $\ell(T_P) = v_P(a_{n-1}) + (n-2)v_P(a_n)$ and this implies (*1), since $v_P(a_{n-1}) = v_P(IR)$. This proves (i) of claim 2 since $[a_n^{n-2}R] = 0$.

(ii) Now we put $B = A[X_1, \dots, X_n, Y]$, $M = J'/(J')^2$ and

$$g_i = a_n X_i - a_i X_n \quad \text{for} \quad 1 \leq i \leq n-1 \quad ,$$

$$g_n = X_n Y - a_n \quad , \text{ both contained in } J' \quad .$$

Let L be the R'-submodule of M generated by the classes of $g_1, \dots, g_n$ in M . Then

$$\det M = - \sum_{P \in H_1(R')} \ell(T_P)[P] \quad ,$$

where $T = M/L$. We want to show

$$(*2) \qquad \ell(T_P) = (n-1)v_P(a_n)$$

for any $P \in H_1(R')$. For that it is enough to consider the case where $IA_{\mathfrak{p}} = a_n A_{\mathfrak{p}}$ and $IA_{\mathfrak{p}} = a_{n-1}A_{\mathfrak{p}}$, where $\mathfrak{p} = P \cap A$.

If $IA_{\mathfrak{p}} = a_n A_{\mathfrak{p}}$, there exists an element $\alpha_i \in A_{\mathfrak{p}}$ such that $a_i = \alpha_i a_n$ for $1 \leq i \leq n-1$. We put

$$h_i = X_i - \alpha_i X_n \qquad \text{for} \quad 1 \leq i \leq n-1 \quad ,$$

$$h_n = X_n Y - a_n \quad .$$

Since $J'B_{\mathfrak{p}} = (h_1, \ldots, h_n)B_{\mathfrak{p}}$ and since $h_1, \ldots, h_n$ form a $B_{\mathfrak{p}}$-sequence, we know that M_P is R'_P-free, and the images of $h_1, \ldots, h_n$ in M_P form a free basis over R'_P . On the other hand we have

$$(g_1, \ldots, g_{n-1}, g_n) = (h_1, \ldots, h_{n-1}, h_n) \begin{bmatrix} a_n & & & \\ & \ddots & & \\ & & a_n & \\ & & & 1 \end{bmatrix}$$

Therefore $\ell(T_P) = (n-1)v_P(a_n)$ by (4.6).

Next we assume $IA_{\mathfrak{p}} = a_{n-1}A_{\mathfrak{p}}$. Then there exists an element $\alpha_i \in A_{\mathfrak{p}}$ such that $a_i = \alpha_i a_{n-1}$ for $1 \leq i \leq n$. In this case we put

$$h_i = X_i - \alpha_i X_{n-1} \qquad \text{for} \quad 1 \leq i \leq n-2 \quad ,$$

$$h_{n-1} = X_n - \alpha_n X_{n-1} \quad ,$$

$$h_n = X_{n-1}Y - a_{n-1} \quad .$$

Then we have

$$(g_1, \ldots, g_{n-2}, g_{n-1}, g_n) = (h_1, \ldots, h_{n-2}, h_{n-1}, h_n) \begin{bmatrix} a_n & & & & \\ & \ddots & & & \\ & & a_n & & \\ -a_1 & \ldots & -a_{n-2} & -a_{n-1}Y & \\ 0 & \ldots & 0 & 0 & \alpha_n \end{bmatrix} .$$

By the same argument as before we get $\ell(T_p) = (n-2)v_p(a_n) + v_p(a_{n-1}\alpha_n)$ and this implies (*2), since $a_{n-1}\alpha_n = a_n$. This proves (ii) of claim 2.

Combining claim 1 and claim 2 we get (4.5) in the case where $I^{(i)} = I^i$ for all $i \in \mathbf{Z}$.

<u>Proof of (4.5) in the general case.</u> Since R is Noetherian, there exists a positive integer k such that $(I^{(k)})^n = I^{(kn)}$ for $n \geq 1$.

(1) We put $S = \sum_{n\geq 0} I^{(kn)}t^{kn}$. Then we are in the previous special case and therefore $[K_S] = [\mathfrak{a}S] + [K_A S]$, where $\mathfrak{a} = I^{(k)}$. Hence $K_S \cong K_S^{**} \cong (\mathfrak{a}K_A S)^{**}(\ell)$ for some $\ell \in \mathbf{Z}$, where $(\ )^*$ means the S-dual. It is easy to see that $\ell = -k$ by passing to $S_{\mathfrak{p}}$ for a prime ideal $\mathfrak{p} \in H_1(A)$. On the other hand, since R is module-finite over S , we have $K_R \cong \underline{\mathrm{Hom}}_S(R, K_S)$. Now we put $S_i = \sum_{n\geq 0} I^{(kn+i)}t^{kn}$ for $0 \leq i \leq k-1$. Then $R = \sum_{i=0}^{k-1} S_i t^i \cong \bigoplus_{i=0}^{k-1} S_i(-i)$ and so

$$K_R \cong \bigoplus_{i=0}^{k-1} \underline{\mathrm{Hom}}_S(S_i, (\mathfrak{a}K_A S)^{**})(i-k) \quad .$$

Therefore we may regard the canonical module of R as

$$K_R = \sum_{i=0}^{k-1} [[S \underset{F}{:} [S \underset{F}{:} \mathfrak{a}K_A S]] \underset{F}{:} S_i]t^{k-i} \quad ,$$

where $F = Q(R)$. For any $P \in H_1(R)$ the ideal $\mathfrak{p} = P \cap A$ has $\mathrm{ht}\,\mathfrak{p} \leq 1$. Therefore we find elements $a, b \in A$ such that $I_{\mathfrak{p}} = aA_{\mathfrak{p}}$, $(K_A)_{\mathfrak{p}} = bA_{\mathfrak{p}}$. Then we get $\mathfrak{a}K_A S_{\mathfrak{p}} = a^k bS_{\mathfrak{p}}$ and $(S_i)_{\mathfrak{p}} = a^i S_{\mathfrak{p}}$, and therefore $(K_R)_{\mathfrak{p}} = \sum_{i=0}^{k-1} a^{k-i}bS_{\mathfrak{p}}t^{k-i} = (IK_A Rt)_{\mathfrak{p}}$ and $(K_R)_P = (IK_A Rt)_P$.

This implies $[K_R] = [IR] + [K_A R]$.

(2) We use the same method as in (1): Let $S' = \sum_{n\in\mathbf{Z}} I^{(kn)}t^{kn}$. Then $[K_{S'}] = [K_A S']$ by the previous special case in the proof of (4.5). Hence $K_{S'} \cong (K_A S')^{**}$, $(\ )^*$ means the S'-dual. Now we put $S_i' = \sum_{n\in\mathbf{Z}} I^{(kn-i)}t^{kn}$ for $0 \leq i \leq k-1$. Then we get $R' = \sum_{i=0}^{k-1} S_i' t^{-i} \cong \bigoplus_{i=0}^{k-1} S_i'(i)$ and so

$$K_{R'} \cong \bigoplus_{i=0}^{k-1} \underline{\mathrm{Hom}}_S (S'_i, (K_A S')^{**})(-i) \quad .$$

Therefore we may regard the canonical module of R' as

$$K_{R'} = \sum_{i=0}^{k-1} [[S'_{F'} : [S' \underset{F'}{:} K_A S']] \underset{F'}{:} S'_i] \; t^i \;,$$

where $F' = Q(R')$. For any $P \in H_1(R')$ consider the ideal $\mathfrak{p} = P \cap A$. Similar to (1) we find elements $a, b \in A$ such that $K_A S'_{\mathfrak{p}} = bS'_{\mathfrak{p}}$ and $S'_i = a^{-i} S'_{\mathfrak{p}}$. Hence $(K_{R'})_{\mathfrak{p}} = \sum_{i=0}^{k-1} a^i b S'_{\mathfrak{p}} t^i = K_A R'_{\mathfrak{p}}$ and so $(K_{R'})_P = K_A R'_P$. This implies $[K_{R'}] = [K_A R']$, q.e.d. (4.5).

<u>Remark (4.7)</u>. If $k = |[I]| < \infty$, one can show the relations (1) and (2) of (4.5) without using the condition that A is a homomorphic image of a regular local ring, provided A has a canonical module. The reason is that in this situation $S = \sum_{n \geq 0} I^{(nk)} t^{nk}$ is a polynomial ring over A . Therefore we have immediately $K_S \cong K_A S(-k)$, and then the same method works as before.

Finally we come to the characterization of the Gorenstein property of R, R and G .

<u>Theorem (4.8)</u>. Let A be a homomorphic image of a regular local ring. Then the following assertions are equivalent:

(1) R is a Gorenstein ring.

(2) R is a Cohen-Macaulay ring and $I^* \cong K_A$, where $I^* = \mathrm{Hom}_A(I,A)$.

<u>Proof</u>. By (1) of (4.5) we have

$$\begin{aligned} [K_R] = 0 &\iff [I] + [K_A] = 0 \\ &\iff [K_A] = -[I] = [I^*] \\ &\iff K_A \cong I^* \quad . \end{aligned}$$

This proves the equivalence of (1) and (2) of (4.8).

Theorem (4.9). Let A be a homomorphic image of a rgular local ring. Then the following assertions are equivalent:

(1) R' is a Gorenstein ring.
(2) R' is a Cohen-Macaulay ring and A is a Gorenstein ring.
(3) G is a Gorenstein ring.
(4) G is a Cohen-Macaulay ring and A is a Gorenstein ring.

Proof. (1)$\Leftrightarrow$(2) follows from (2) of (4.5). (1)$\Leftrightarrow$(3) and (2)$\Leftrightarrow$(4) are trivial since $R'/uR' \cong G$.

Finally we collect some results under the condition $|[I]| < \infty$.

Proposition (4.10). Suppose $|[I]| < \infty$. Then the following statements are equivalent:

(1) R (resp. G) is a Gorenstein ring.
(2) $I^* \cong K_A$ (resp. A is a Gorenstein ring) and $I^{(i)}$ is a maximal Cohen-Macaulay A-module for all $i \geq 0$.

Proof. This follows from (4.1), (4.5) and (4.7).

Corollary (4.11). Suppose that $\dim A = 2$ and $|[I]| < \infty$. Then the following are true:

(1) R is a Gorenstein ring if and only if $I^* \cong K_A$.
(2) G is a Gorenstein ring if and only if A is a Gorenstein ring.

Proposition (4.12). Let A be a Gorenstein ring and $|[I]| < \infty$. Then R is Gorenstein if and only if $I^{(1)}$ is principal.

Proof. Let $k = |[I]|$. If R is Gorenstein, then $I^{(k-1)} \cong I^* \cong$ $\cong K_A \cong A$ and so $I^{(k-1)}$ if principal. Hence $k = 1$. Conversely if $I^{(1)}$ is principal, then R is a polynomial ring over A. Therefore R is Gorenstein.

Proposition (4.13). Let $k = |[I]| < \infty$. If A has a canonical module K_A and if $K_A \cong I$, then the following conditions are equivalent:

(1) R is a Gorenstein ring.

(2) $k \leq 2$ and $I^{(n)}$ is a maximal Cohen-Macaulay A-module for $n = 0,1$.

Proof. (1)⇒(2): We have $[K_A] = [I]$ by the assumption $K_A \cong I$. On the other hand we conclude from (1) and (4.10) that $[K_A] = -[I]$. Hence $[I] = -[I]$ and so $2[I] = 0$.

(2)⇒(1): Since $[I] = -[I] = [I^*]$, we have $K_A \cong I \cong I^*$. Hence R is Gorenstein by (4.10).

Proposition (4.14). Suppose that $|[I]| < \infty$. If R is a Cohen-Macaulay ring, then the following statements are equivalent:

(1) $R^{(n)}$ is a Gorenstein ring for some $n > 0$.

(2) The canonical module K_A of A exists and $[K_A] \in \langle [I] \rangle$.

Proof. (1)⇒(2): By (4.10) we have $[K_A] = -[I^{(n)}] = -n[I]$.
(2)⇒(1): There is an integer i such that $[K_A] = i[I]$. Let $k = |[I]|$ and take a positive integer n so that k divides $n+i$. Then $[I^{(n)}] + [K_A] = (n+i)[I] = 0$. Hence $R^{(n)}$ is Gorenstein by (4.5), (1).

Proposition (4.15). Suppose that $\dim A = 2$ and A has a canonical module K_A with $k = |[K_A]| < \infty$. If $I = K_A^{(k-1)}$, then R is a Gorenstein ring.

Proof. R is Cohen-Macaulay by (4.2). Moreover we have $[I] = (k-1)[K_A] = -[K_A]$. Hence R is Gorenstein by (4.10).

5. Examples and remarks.

Throughout this section A is a Noetherian local ring with the maximal ideal $\mathfrak{m}$.

First we consider the following three conditions on A :

(i) A is quasi-unmixed.

(ii) A is reduced.

(iii) A is a Nagata ring.

These condtions imply that A is unmixed. Therefore Theorem (3.3) is also true if we replace the assumption "A is unmixed" by the conditions (i), (ii), (iii). But in this situation we can give a simpler proof for the claim of (3.3) provided that the residue class field of A is infinite. (We use the same notations as in definition (3.1):

We choose a minimal reduction J of $\mathfrak{a} = I^{(k)}$ and put $T := \sum_{n\geq 0} J^n t^{kn}$. Since A is quasi-unmixed and J is an ideal of the principal class, we know by [17, Theorem 2.12] that $\mathrm{Ass}_A(A/\overline{J^n}) = \mathrm{Min}_A(A/\overline{J^n}) = \mathrm{Min}_A(A/\mathfrak{a})$ for $n > 0$, where $\overline{J^n}$ denotes the integral closure of J^n. This implies in particular that any element $s \in S$ is a non-zero divisor on $A/\overline{\mathfrak{a}^n}$. Therefore we have $\overline{\mathfrak{a}^n}A_S \cap A = \overline{\mathfrak{a}^n}$ and so $\mathfrak{a}^{(n)} \subset \overline{\mathfrak{a}^n}$. This shows

$$T \subset R^{(k)} \subset \overline{T} \subset A[t^k] \quad ,$$

where $\overline{T}$ is the integral closure of T in $A[t^k]$. If $\mathrm{ht}\,\mathfrak{a} > 0$, then $Q(T) = Q(A[t^k])$; therefore $R^{(k)}$ is module-finite over T since A is a reduced Nagata ring. If $\mathrm{ht}\,\mathfrak{a} = 0$, then $R^{(k)} = A$ since A is reduced. In both cases $R^{(k)}$ is Noetherian, hence R itself is Noetherian by (2.4).

The following three examples show that the claim of (3.3) under the assumptions (i),(ii), (iii) becomes false if any of these three conditions is omitted.

From now on we denote by $\mathfrak{p}^{(n)}$ the n-th symbolic power of a given prime ideal $\mathfrak{p}$ of A in the usual sense, i.e. $\mathfrak{p}^{(n)} = \mathfrak{p}^n A_{\mathfrak{p}} \cap A$.

Example (5.1). Let $(A, \mathfrak{m})$ be a two dimensional Nagata domain with the normalization $\overline{A}$ such that $\overline{A}/A \cong A/\mathfrak{m}$. Suppose that $\overline{A}$ has exactly two maximal ideals M and N with $\mathrm{ht}\,M = 1$, $\mathrm{ht}\,N = 2$ and $M \cap N = \mathfrak{m}$. Furthermore we assume that M is a principal ideal and N includes a prime element y. Such an example exists by [15, Appendix E 21]. Note that A is not quasi-unmixed by [15, 34.6]; otherwise

A has to satisfy the second chain-condition, hence $\overline{A}$ has to satisfy the first chain-condition. That would be a contradiction to the fact that M and N have different heights. We put $P = y\overline{A}$ and $\mathfrak{p} = P \cap A$. Then $\operatorname{ht}\mathfrak{p} = 1$ and the following assertions hold:

(1) $\mathfrak{p}^{(n)} = P^n M$ for $n \geq 1$.

(2) $\ell(\mathfrak{p}^{(n)}) = 1$ for $n \geq 1$.

(3) $R = \bigoplus_{n \geq 0} \mathfrak{p}^{(n)}$ is not a Noetherian ring.

Proof. (1) Since $A_{\mathfrak{p}} = (\overline{A})_{\mathfrak{p}} = \overline{A}_P$, we have $\mathfrak{p}^n A_{\mathfrak{p}} = P^n \overline{A}_P$ for $n \geq 0$. Therefore

$$\mathfrak{p}^{(n)} = \mathfrak{p}^n A_{\mathfrak{p}} \cap A = P^n \overline{A}_P \cap A = P^n \cap A = P^n \cap M = P^n M \quad .$$

(2) Let $M = x\overline{A}$. Then we get from (1) $(\mathfrak{p}^{(n)})^2 = a\mathfrak{p}^{(n)}$, where $a = xy^n$, and $a \in M \cap N = \mathfrak{m}$.

(3) Assume that R is Noetherian. Then there is a positive integer k such that $(\mathfrak{p}^{(k)})^2 = \mathfrak{p}^{(2k)}$. This implies $P^{2k}M^2 = P^{2k}M$ by (1), hence $M^2\overline{A}_M = M\overline{A}_M$, a contradiction.

Example (5.2). Let $S = k[[X,Y,Z]]$ be a formal power series ring over a field k and let $A = S/(X^2, X \cdot Y) =: k[[x,y,z]]$. Then A is a quasi-unmixed Nagata ring, but not reduced. We put $P = (x,z)A$. Then $P \in H_1(A)$ and the following assertions hold:

(1) $P^{(n)} = (x, z^n)A$ for $n \geq 1$

(2) $\ell(P^{(n)}) = 1$ for $n \geq 1$

(3) $R = \bigoplus_{n \geq 1} P^{(n)}$ is not a Noetherian ring.

Example (5.3). We use the noations of (5.2). Since $\operatorname{depth} A = 1$, there is a local domain $(R, \mathfrak{n})$ such that $\hat{R} \cong A$ by [11, Theorem 1]. Then R is quasi-unmixed and reduced, but not a Nagata ring (otherwise $A = \hat{R}$ would be reduced.). Let $\mathfrak{a} = (x,y)A + \mathfrak{m}^2$. Then $R/\mathfrak{a} \cap R \cong A/\mathfrak{a}$, since $\mathfrak{a}$ is $\mathfrak{n}$-primary. We get $\mathfrak{a} \cap R \subsetneqq \mathfrak{n}$, hence there is an element $f \in \mathfrak{n}$ such that $f \notin \mathfrak{a} \cap R$. Then $\mathfrak{m} = (x,y,f)A$.

Replacing z by f we may assume $z \in R$ from the beginning. Now we put $\mathfrak{p} = P \cap A$. Then $\operatorname{ht} \mathfrak{p} = 1$ and the following assertions hold:

(1) $P^{(n)} = \mathfrak{p}^{(n)}A$ for $n \geq 1$.

(2) $\ell(\mathfrak{p}^{(n)}) = 1$ for $n \geq 1$.

(3) $S := \bigoplus_{n \geq 0} \mathfrak{p}^{(n)}$ is not a Noetherian ring.

Proof. (1) Since $A/\mathfrak{p}^{(n)}A$ is Cohen-Macaulay, we have $\operatorname{Ass}_A(A/\mathfrak{p}^{(n)}A) = \operatorname{Min}_A(A/\mathfrak{p}^{(n)}A)$. On the other hand $V(\mathfrak{p}^{(n)}A) \subset V(z^nA) = \{P, \mathfrak{m}\}$. Hence $\operatorname{Ass}_A(A/\mathfrak{p}^{(n)}A) = \{P\}$ and so $\mathfrak{p}^{(n)}A$ is P-primary. Therefore we have $\mathfrak{p}^{(n)}A = P^{(n)}$, since $\mathfrak{p}^{(n)}A_P = P^nA_P = z^nA_P$.

(2) Since $(R/\mathfrak{n}) \otimes \mathcal{R}(\mathfrak{p}^{(n)}) \cong (A/\mathfrak{m}) \otimes_A (A \otimes_R \mathcal{R}(\mathfrak{p}^{(n)})) \cong (A/\mathfrak{m}) \otimes_A \mathcal{R}(P^{(n)})$ we get $\ell(\mathfrak{p}^{(n)}) = \ell(P^{(n)}) = 1$.

(3) Assume that S is Noetherian, then $S \otimes_R A \cong \mathcal{R} = \bigoplus_{n \geq 0} P^{(n)}$ is Noetherian. But this contradicts to (3) of (5.2).

The next example shows that the "only if" part of (3.6) is not true in general unless $A/I^{(n)}$ is Cohen-Macaulay for $n >> 0$.

Example (5.4). Let $A = k[[X,Y,Z,W]]/(XY - ZW) = k[[x,y,z,w]]$, where $k[[X,Y,Z,W]]$ is a formal power series ring over a field k. We put $\mathfrak{p} = (x,Z)A$. Then $\mathfrak{p} \in H_1(A)$ and the following assertions hold:

(1) $\mathfrak{p}^{(n)} = \mathfrak{p}^n$ for $n \geq 1$

(2) $\ell(\mathfrak{p}) = 2$

(3) $A/\mathfrak{p}^n$ is not a Cohen-Macaulay ring for $n \geq 2$.

Finally we construct a Gorenstein symbolic Rees algebra using the invariant theory.

Example (5.5). Let $S = \mathbb{C}[X,Y]$ be a polynomial ring and let $R = \mathbb{C}[X^3, X^2Y, XY^2, Y^3]$. We put $\mathfrak{p} = (X^3, X^2Y, XY^2)R$. Then the symbolic Rees algebra $\mathcal{R} = \bigoplus_{n \geq 1} \mathfrak{p}^{(n)}$ is a Gorenstein ring.

Proof. Let w be a primitive cubic root of unity and let G be the subgroup of $GL_2(\mathbb{C})$ generated by $\sigma = \begin{pmatrix} w & 0 \\ 0 & w \end{pmatrix}$. Then it is well-known, that $R = S^G$. We put $P = XS$. Therefore we obtain $p^{(n)} = P^n \cap R (= (P^n)^G)$ for $n \geq 1$, i.e.

$$R = R(P)^G \quad .$$

Since $R(P) = \mathbb{C}[X,Y,Xt]$ and σ acts on Xt as $\sigma(Xt) = wXt$, we get

$$R(P)^G = \mathbb{C}[X,Y,Xt]^{(3)} \quad ,$$

where Xt is taken with degree one. Hence R must be a Gorenstein ring by [23, Theorem 1].

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