Nonabelian Hodge Theory in Characteristic p

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1 Introduction

This talk is based on our joint work with Roman Bezrukavnikov and Arthur Ogus.

Let X be a smooth scheme over S, where S is a scheme over $spec\, \mathbb{F}_p$, $F_{X/S} : X \to X' = X \times_{F_S} S$ is the (relative) Frobenius morphism. For any quasi-coherent $\mathcal{O}_{X'}$-module E, the $\mathcal{O}_X$-module $F^*_{X/S}E$ has a canonical integrable connection. Moreover, the theory of Cartier descent says that the above gives an equivalence between the category of quasi-coherent $\mathcal{D}_{X/S}$-modules with zero p-curvature and the category of quasi-coherent $\mathcal{O}_{X'}$-modules.

The purpose of this work is to extend the above correspondence to a wider class of $\mathcal{D}_{X/S}$-modules.

A $\mathcal{D}_{X/S}$-module M is called nilpotent of level n if there exists a filtration $0 = M_{-1} \subset M_0 \subset \cdots \subset M_n = M$ by submodules such that M_i/M_{i-1} has zero p-curvature.

The main result says that, given a lifting $X \to S \;\; mod \;\, p^2$ and an integer $0 \leq n < p$, we get an equivalence between the category of nilpotent level n $\mathcal{D}_{X/S}$-modules and the category of $\mathcal{O}_{T^*_{X'/S}}$-modules on the cotangent bundle supported on the nth nilpotent neighborhood of the zero section.

2 The Cartier transform

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1 Azumaya property of the algebra of differential operators.

The material of this subsection is borrowed from [BMR].

Let X be a smooth scheme over S, where S is a scheme over $spec\, \mathbb{F}_p$. First, we observe that the ring of PD differential operators $\mathcal{D}_{X/S}$ has a large center $\mathcal{Z}_{X/S}$. Recall that, given a vector field $\theta \in T_{X/S}$, the differential operator θ^p acts on functions $\mathcal{O}_X$ as a derivation. We denote the corresponding vector field by $\theta^{[p]}$. Then one can check that the operator $c(\theta) = \theta^p - \theta^{[p]}$ is in $\mathcal{Z}_{X/S}$.

Denote by p the projection $p : T^*_{X'/S} \to X'$.

We define a homomorphism of sheaves of algebras

$$p_* \mathcal{O}_{T^*_{X'/S}} = S^{\cdot} T_{X'/S} \to F_{X/S\,*} \mathcal{Z}_{X/S} \tag{1}$$

sending $f \in \mathcal{O}_{X'} \subset p_* \mathcal{O}_{T^*_{X'/S}}$ to $F^*_{X/S} f \in F_{X/S\,*} \mathcal{O}_X \subset F_{X/S\,*}(\mathcal{D}_{X/S})$ and $\theta \in T_{X'/S}$ to $c(\theta)$.

Theorem 1 *The map (1) is an isomorphism.*

Via (1) we can view $F_{X/S\,*}\mathcal{D}_{X/S}$ as a sheaf of algebras over $T^*_{X'/S}$.

Recall that an Azumaya algebra over a scheme Y is a coherent sheaf of algebras $\mathcal{A}$ which is locally for the fppf topology isomorphic to the algebra of endomorphisms of a vector bundle. It follows that the coherent sheaf $\mathcal{A}$ is locally free. We denote by $rk\,\mathcal{A}$ its rank. A splitting of an Azumaya algebra is a vector bundle E over Y such that $\mathcal{A} \simeq End(E)$. The category $\mathcal{M}(\mathcal{A})$ of quasi-coherent sheaves of modules over a split Azumaya algebra is equivalent to the category $\mathcal{M}(\mathcal{O}_Y)$ of quasi-coherent sheaves $\mathcal{O}_Y$- modules. The functor assigning to a $End(E)$-module M the $\mathcal{O}_Y$-module $Hom_{End(E)}(E; M)$ gives the equivalence.

The following important observation is made in [BMR]

Theorem 2 $F_{X/S\,*}\mathcal{D}_{X/S}$ *is an Azumaya algebra over* $T^*_{X'/S}$ *of rank* $p^{2\,dim_S\,X}$

We shall give here a sketch of the proof reffering the reader to *loc. cit.* for the detailes. Consider the following diagram:

$$\begin{array}{ccc} T^*_{X'/S} \times_{X'} X & \xrightarrow{\delta} & T^*_{X'/S} \\ \downarrow & & \downarrow p \\ X & \xrightarrow{F_{X/S}} & X' \end{array}$$

We claim that the algebra $F_{X/S\,*}\mathcal{D}_{X/S}$ becomes split when pulled back via the map δ above. Indeed, $\mathcal{D}_{X/S}$ is a splitting module over $\delta^* F_{X/S\,*}\mathcal{D}_{X/S}$.

Sending a $\mathcal{D}_{X/S}$-module M to $F_{X/S\,*}M$ we obtain an equivalence of categories between the category of quasi-coherent $\mathcal{D}_{X/S}$-modules and the category of quasi-coherent modules over the Azumaya algebra $F_{X/S\,*}\mathcal{D}_{X/S}$.

Notice that the Azumaya algebra $F_{X/S\,*}\mathcal{D}_{X/S}$ is never split (unless $dim_S X = 0$). In effect, $\mathcal{D}_{X/S}$ has no zero divisors, whereas the matrix algebra $End(E)$ does have.

Let $i : X' \hookrightarrow T^*_{X'/S}$ be the zero section.

Lemma 3 *The Azumaya algebra* $i^* F_{X/S\,*}\mathcal{D}_{X/S}$ *splits canonically:*

$$i^* F_{X/S\,*}\mathcal{D}_{X/S} \simeq End_{\mathcal{O}_{X'}}(F_{X/S\,*}\mathcal{O}_X)$$

Indeed, the action of $\mathcal{D}_{X/S}$ on $\mathcal{O}_X$ gives the homomorphism $i^* F_{X/S\,*}\mathcal{D}_{X/S} \to End_{\mathcal{O}_{X'}}(F_{X/S\,*}\mathcal{O}_X)$. Since $rk(i^* F_{X/S\,*}\mathcal{D}_{X/S}) = rk(End_{\mathcal{O}_{X'}}(F_{X/S\,*}\mathcal{O}_X))$, it is an isomorphism.

As a corollary we get an important result of Cartier.

Corollary 4 *(**"Frobenius descent"**) The functor sending a* $\mathcal{D}_{X/S}$*-module* M *to the* $\mathcal{O}_{X'}$*-module* $F_{X/S\,*}\mathcal{H}om_{\mathcal{D}_{X/S}}(\mathcal{O}_X; M)$ *defines an equivalence of categories between the category of quasi-coherent* $\mathcal{D}_{X/S}$*-modules with zero* p*-curvature and the category of quasi-coherent* $\mathcal{O}_{X'}$*-modules.*

In effect, the category of $\mathcal{D}_{X/S}$-modules with zero p-curvature is equivalent to the category of $i^* F_{X/S\,*}\mathcal{D}_{X/S}$-modules.

2. Let $\tilde{S}$ be a flat scheme over $spec\,\mathbb{Z}_p/p^2$, $S = \tilde{S} \times spec\mathbb{F}_p$, $\tilde{F}_S : \tilde{S} \to \tilde{S}$ a lifting of the Frobenius $F_S : S \to S$ (i.e. a morphism whose restriction to S is F_S).

A lifting of $X \to S$ is a smooth scheme $\tilde{X} \to \tilde{S}$ over $\tilde{S}$ such that $X = \tilde{X} \times_{\tilde{S}} S$.

We are going show that given a lifting $\tilde{X} \to \tilde{S}$, the splitting of $F_{X/S\,*}\mathcal{D}_{X/S}$ over the zero section $i : X' \hookrightarrow T^*_{X'/S}$ extends to a splitting over a larger subscheme.

First, assume that, in addition to the lifting $\tilde{X} \to \tilde{S}$, we are given a lifting of the Frobenius: $\tilde{F}_{\tilde{X}/\tilde{S}} : \tilde{X} \to \tilde{X}'$.

The above lifting gives a map:

$$\mathcal{C}_{\tilde{F}_{\tilde{X}/\tilde{S}}} : T_{X/S} \to F^*_{X/S} T_{X'/S} \qquad (2)$$

Given a vector field $\theta \in \Gamma(T_{U/S})$ on a open subset $U \in X$ we can locally choose a lifting $\tilde{\theta}$ of θ on $\tilde{U}$ and put $\mathcal{C}_{\tilde{F}_{\tilde{X}/\tilde{S}}}(\theta) = \frac{d\tilde{F}_{\tilde{X}/\tilde{S}}(\tilde{\theta})}{p}|_{U'}$. It is easy to check that the right hand side of the above formula does not depend on the choice of lifting $\tilde{\theta}$, hence it gives a well defined map.

Next, we construct a functor:

$$\Phi_{\tilde{F}_{\tilde{X}/\tilde{S}}} : \mathcal{M}(\mathcal{O}_{T^*_{X'/S}}) \to \mathcal{M}(\mathcal{D}_{X/S}) \tag{3}$$

from the category of quasi-coherent $\mathcal{O}_{T^*_{X'/S}}$-modules to the category of quasi-coherent $\mathcal{D}_{X/S}$-modules.

Given a $\mathcal{O}_{T^*_{X'/S}}$-module E we define the $\mathcal{O}_X$-module $\Phi_{\tilde{F}_{\tilde{X}/\tilde{S}}}(E)$ to be $F^*_{X/S}(p_*E)$. The sheaf $\Phi_{\tilde{F}_{\tilde{X}/\tilde{S}}}(E)$ is a module over the commutative sheaf of algebras $F^*_{X/S}(S^{\cdot}T_{X'/S})$. We want to endow $F^*_{X/S}(p_*E)$ with an action of the Lie algebra of vector fields. Let $\theta \in \Gamma(T_{U/S})$ and $h = \sum_i f_i \otimes s_i$ be a section of $F^*_{X/S}(p_*E)$ over U, $f_i \in \mathcal{O}(U)$, $s_i \in p_*E(U')$. Put

$$\theta(h) = \sum_i (f_i(\mathcal{C}_{\tilde{F}_{\tilde{X}/\tilde{S}}}(\theta)(s_i)) + \theta(f_i) \otimes s_i) \tag{4}$$

It is easy to see that the formula (4) gives $F^*_{X/S}(p_*E)$ a $\mathcal{D}_{X/S}$-module structure.

We shall compute the action of the center $\mathcal{Z}_{X/S} \subset \mathcal{D}_{X/S}$ on $\Phi_{\tilde{F}_{\tilde{X}/\tilde{S}}}(E)$ more explicitly. Notice that $\mathcal{O}_{X'}$-module

$$F_{X/S\,*}\Phi_{\tilde{F}_{\tilde{X}/\tilde{S}}}(E) = F_{X/S\,*}\mathcal{O}_X \otimes_{\mathcal{O}_{X'}} p_*E$$

has a natural action of $S^{\cdot}T_{X'/S}$: the one which comes from the $S^{\cdot}T_{X'/S}$-module structure on p_*E. However, in general, it does not coincide with the action of $F_{X/S\,*}\mathcal{Z}_{X/S}$ identified with $S^{\cdot}T_{X'/S}$ via (1).

To describe the picture we shall introduce a (non-linear) X'- endomorphism of the cotangent space: $h_{\tilde{F}_{\tilde{X}/\tilde{S}}} : T^*_{X'/S} \to T^*_{X'/S}$. or, equivalently, an endomorphism of the sheaf of algebras:

$$h^*_{\tilde{F}_{\tilde{X}/\tilde{S}}} : S^{\cdot}T_{X'/S} \to S^{\cdot}T_{X'/S} \tag{5}$$

Given a vector field $v \in T_{X'/S}$ we define

$$h^*_{\tilde{F}_{\tilde{X}/\tilde{S}}}(v) = -v + (P_{\tilde{F}_{\tilde{X}/\tilde{S}}}(v))^p \in T_{X'/S} \oplus S^pT_{X'/S}$$

where $P_{\tilde{F}_{\tilde{X}/\tilde{S}}} : T_{X'/S} \to (F_{X/S}\pi)^*T_{X'/S}$ is the pullback of $\mathcal{C}_{\tilde{F}_{\tilde{X}/\tilde{S}}}$ with respect to $\pi : X' \to X$.

Notice that $h_{\tilde{F}_{\tilde{X}/\tilde{S}}}$ is identical on the zero section $X' \hookrightarrow T^*_{X'/S}$ and induces an automorphism of $X'_n \hookrightarrow T^*_{X'/S}$.

Lemma 5 *Let E be a quasi-coherent $\mathcal{O}_{T^*_{X'/S}}$-module. A central element $\theta \in F_{X/S\,*}\mathcal{Z}_{X/S}$ acts on $F_{X/S\,*}\Phi_{\tilde{F}_{\tilde{X}/\tilde{S}}}(E) = F_{X/S\,*}\mathcal{O}_X \otimes_{\mathcal{O}_{X'}} p_*E$ as $1 \otimes h^*_{\tilde{F}_{\tilde{X}/\tilde{S}}}(\theta)$. In the other words we have a canonical isomorphism of $\mathcal{O}_{T^*_{X'/S}}$-modules*

$$F_{X/S\,*}\Phi_{\tilde{F}_{\tilde{X}/\tilde{S}}}(E) \simeq h_{\tilde{F}_{\tilde{X}/\tilde{S}}\,*}(p^*F_{X/S\,*}\mathcal{O}_X \otimes_{\mathcal{O}_{T^*_{X'/S}}} E)$$

We omit the proof.

Let $\hat{i} : \hat{T}^*_{X'/S} \hookrightarrow T^*_{X'/S}$ be the formal completion of $T^*_{X'/S}$ along the zero section.

Theorem 6 *Let $\tilde{X} \to \tilde{S}$ be a lifting of X and $\tilde{F}_{\tilde{X}/\tilde{S}} : \tilde{X} \to \tilde{X}'$ a lifting of the Frobenius $F_{X/S}$. Then the Azumaya algebra $\hat{i}^*F_{X/S\,*}\mathcal{D}_{X/S}$ splits canonically.*

Proof:

Let $i_n : X'_n \hookrightarrow T^*_{X'/S}$ be the n-th infinitesimal neighborhood of X' in $T^*_{X'/S}$ (i.e. $\mathcal{O}_{X'_n} = i^{\cdot}\mathcal{O}_{T^*_{X'/S}}/i^{\cdot}J^{n+1}$, where J is the sheaf of ideals of $X' \hookrightarrow T^*_{X'/S}$).

We shall see that the $i_n^* F_{X/S\,*}\mathcal{D}_{X/S}$-module $i_n^* F_{X/S\,*}\Phi_{\tilde F_{\tilde X/\tilde S}}(i_{n\,*}\mathcal{O}_{X'_n})$ is a splitting of the Azumaya algebra $i_n^* F_{X/S\,*}\mathcal{D}_{X/S}$.

Indeed, according to the above lemma $i_n^* F_{X/S\,*}\Phi_{\tilde F_{\tilde X/\tilde S}}(i_{n\,*}\mathcal{O}_{X'_n})$ is a vector bundle over X'_n of rank $p^{dim_S X}$.

Therefore $i_n^* F_{X/S\,*}\mathcal{D}_{X/S}$-module structure on $i_n^* F_{X/S\,*}\Phi_{\tilde F_{\tilde X/\tilde S}}(i_{n\,*}\mathcal{O}_{X'_n})$ gives an isomorphism

$$i_n^* F_{X/S\,*}\mathcal{D}_{X/S} \to End_{\mathcal{O}_{X'_n}}(i_n^* F_{X/S\,*}\Phi_{\tilde F_{\tilde X/\tilde S}}(\mathcal{O}_{X'_n}))$$

It follows that

$$\lim_{\leftarrow} F_{X/S\,*}\Phi_{\tilde F_{\tilde X/\tilde S}}(i_{n\,*}\mathcal{O}_{X'_n})$$

is a splitting of $\hat i^* F_{X/S\,*}\mathcal{D}_{X/S}$.

Remarks:1) In fact, the Azumaya algebra $F_{X/S\,*}\mathcal{D}_{X/S}$ becomes split when pulled back via the map $h_{\tilde F_{\tilde X/\tilde S}} : T^*_{X'/S} \to T^*_{X'/S}$. Indeed, thanks to Lemma 5 $F_{X/S\,*}\Phi_{\tilde F_{\tilde X/\tilde S}}(\mathcal{O}_{T^*_{X'/S}})$ is a splitting of $h^*_{\tilde F_{\tilde X/\tilde S}} F_{X/S\,*}\mathcal{D}_{X/S}$.

2) The map $h_{\tilde F_{\tilde X/\tilde S}}$ is etale and surjective. Indeed, $h_{\tilde F_{\tilde X/\tilde S}}$ is a group endomorphism, where $T^*_{X'/S}$ is viewed as a group scheme over X'. Also, the differential $dh_{\tilde F_{\tilde X/\tilde S}}$ at any point at the zero section $X' \subset T^*_{X'/S}$ is equal to $-Id$. Hence, the image of $h_{\tilde F_{\tilde X/\tilde S}}$ is an open group subscheme and therefore it coincides with $T^*_{X'/S}$.

Next, we shall study the dependence of the functor $\Phi_{\tilde F_{X/S}}$ of the choice of lifting: $\tilde F_{X/S} : \tilde X \to \tilde X'$.

Notice that the set of such liftings is a torser over $\Gamma(F^*_{X/S} T_{X'/S})$.

Given a section $w \in \Gamma(F^*_{X/S} S^{\cdot} T_{X'/S})$ we define an automorphism of the algebra of differential operators:

$$e^w : \mathcal{D}_{X/S} \to \mathcal{D}_{X/S} \tag{6}$$

If w is written on an open subscheme $U \subset X$ in the form $w = \sum_i f_i \otimes v_i$, where $f_i \in \Gamma(\mathcal{O}_U)$, $v_i \in \Gamma(S^{\cdot} T_{U'/S})$ we put:

$$e^w(f) = f,\ for\ f \in \Gamma(\mathcal{O}_U),\ e^w(u) = u + \textstyle\sum_i u(f_i)\, v_i,\ for\ u \in \Gamma(T_{U\prime/S})$$

Here $u(f_i)$ denotes the Lie derivative and v_i are viewed as elements of the center of $\mathcal{D}_{U/S}$ via (1).

The other ingredient we will need is the endomorphism (5). Recall, that $h^*_{\tilde F_{\tilde X/\tilde S}}$ induces an automorphism of the completed symmetric algebra. We consider its pullback on X:

$$h^*_{\tilde F_{\tilde X/\tilde S}} : F^*_{X/S} \hat S^{\cdot} T_{X'/S} \to F^*_{X/S} \hat S^{\cdot} T_{X'/S}$$

Let $\hat{\mathcal{D}}_{X/S}$ denote the completed algebra of differential operators:

$$\hat{\mathcal{D}}_{X/S} = \mathcal{D}_{X/S} \otimes_{\mathcal{Z}_{X/S}} F^{\cdot}_{X/S} \hat S^{\cdot} T_{X'/S}$$

If M is a $\hat{\mathcal{D}}_{X/S}$-module and $\sigma : \mathcal{D}_{X/S} \to \mathcal{D}_{X/S}$ is an endomorphism, we shall denote by M^σ the twisted $\hat{\mathcal{D}}_{X/S}$-module.

Proposition 7 *Let E be a nilpotent quasi-coherent $\mathcal{O}_{\hat T^*_{X'/S}}$-module (i.e. E gets killed by some power of the ideal of the zero section) and $\tilde F_{\tilde X/\tilde S}, \overline F_{\tilde X/\tilde S} : \tilde X \to \tilde X'$ two liftings of the Frobenius, $\overline F_{\tilde X/\tilde S} - \tilde F_{\tilde X/\tilde S} = w \in \Gamma F^*_{X/S} T_{X'/S}$. Then*

$$\Phi_{\overline F_{\tilde X/\tilde S}}(E) = (\Phi_{\tilde F_{\tilde X/\tilde S}}(E))^\sigma$$

where $\sigma = e^{h^{*-1}_{\tilde F_{\tilde X/\tilde S}}(w)}$.

We shall not prove this assertion: it is an easy straightforward computation.

3. Splitting of the Azumaya algebra over the universal PD-envelope of the zero section.

Denote by $\hat{i}_{pd} : \hat{T}^*_{X'/S,pd} \to T^*_{X'/S}$ the universal PD-envelope of the zero section and by $\hat{\mathcal{D}}_{X/S,pd}$ the sheaf of algebras $\mathcal{D}_{X/S} \otimes_{\mathcal{Z}_{X/S}} F^{\cdot}_{X/S} p_* \mathcal{O}_{\hat{T}^*_{X'/S,pd}}$. A sheaf of $\mathcal{O}_{\hat{T}^*_{X'/S,pd}}$-modules is called nilpotent if it gets killed by some divided power of the ideal of the zero section. Similarly, we define the notion of nilpotent $\hat{\mathcal{D}}_{X/S,pd}$-modules

First, we shall see that the functor (3) extends to a functor:

$$\Phi_{\tilde{F}_{\tilde{X}/\tilde{S}}} : \mathcal{M}(\mathcal{O}_{\hat{T}^*_{X'/S,pd}}) \to \mathcal{M}(\hat{\mathcal{D}}_{X/S,pd})$$

from the category of nilpotent quasi-coherent $\mathcal{O}_{\hat{T}^*_{X'/S,pd}}$-modules to the category of nilpotent [1] quasi-coherent $\hat{\mathcal{D}}_{X/S,pd}$-modules.

Indeed, given a nilpotent $\mathcal{O}_{\hat{T}^*_{X'/S,pd}}$-module E, we already have a $\mathcal{D}_{X/S}$-module $\Phi_{\tilde{F}_{\tilde{X}/\tilde{S}}}(E)$ and it remains to show that the action of the center $\mathcal{Z}_{X/S}$ on $\Phi_{\tilde{F}_{\tilde{X}/\tilde{S}}}(E)$ naturally extends to an action of the divided power algebra $F^{\cdot}_{X/S} p_* \mathcal{O}_{\hat{T}^*_{X'/S,pd}}$ which commutes with $\mathcal{D}_{X/S}$-action.

Notice that we have an $F^{\cdot}_{X/S} p_* \mathcal{O}_{\hat{T}^*_{X'/S,pd}}$- action on $\Phi_{\tilde{F}_{\tilde{X}/\tilde{S}}}(E) = F^*_{X/S} p_* E$ since $p_* E$ is naturally a $p_* \mathcal{O}_{\hat{T}^*_{X'/S,pd}}$-module. Let us twist this action by the endomorphism

$$h^*_{\tilde{F}_{\tilde{X}/\tilde{S}}} : p_* \mathcal{O}_{\hat{T}^*_{X'/S,pd}} \to p_* \mathcal{O}_{\hat{T}^*_{X'/S,pd}}$$ [2]

Lemma (5) implies that the new action has all the required properties.

In fact, the map $h_{\tilde{F}_{\tilde{X}/\tilde{S}}} : \hat{T}^*_{X'/S,pd} \to \hat{T}^*_{X'/S,pd}$ is just multiplication by -1. It immediately follows from the definition.

Moreover, the above functors, for any two liftings of the Frobenius are canonically isomorphic. Indeed, for any $w \in \Gamma(F^*_{X/S} T_{X'/S}) = \mathcal{O}_X \otimes_{F^{\cdot}_{X/S} \mathcal{O}_{X'}} \mathcal{Z}_{X/S} \subset \hat{\mathcal{D}}_{X/S,pd}$ the automorphism

$$e^w : \hat{\mathcal{D}}_{X/S,pd} \to \hat{\mathcal{D}}_{X/S,pd}$$

is inner: it is the conjugation by $exp(w) = \sum_i \frac{w^i}{i!} \in \Gamma(\hat{\mathcal{D}}_{X/S,pd})$. Therefore, given any two liftings, $\tilde{F}_{\tilde{X}/\tilde{S}}$ and $\overline{F}_{\tilde{X}/\tilde{S}}$, with $\overline{F}_{\tilde{X}/\tilde{S}} - \tilde{F}_{\tilde{X}/\tilde{S}} = w$, and a quasi-coherent nilpotent $\mathcal{O}_{\hat{T}^*_{X'/S,pd}}$-module E, we have the isomorphism:

$$\Phi_{\overline{F}_{\tilde{X}/\tilde{S}}}(E) \xrightarrow{exp(-w)} (\Phi_{\tilde{F}_{\tilde{X}/\tilde{S}}}(E))$$

Since $exp(w + w') = exp(w) exp(w')$, the above isomorphisms satisfy the transitivity condition, for any three lifting.

It has a very important corollary: given just a lifting $\tilde{X} \to \tilde{S}$ of $X \to S$, we get a functor :

$$\Phi_{\tilde{X}/\tilde{S}} : \mathcal{M}(\mathcal{O}_{\hat{T}^*_{X'/S,pd}}) \to \mathcal{M}(\hat{\mathcal{D}}_{X/S,pd}) \tag{7}$$

In effect, we can choose an open covering $\tilde{X} = \cup_i \tilde{U}_i$ together with local liftings $\tilde{F}_{\tilde{U}_i/\tilde{S}}$ and then glue $\Phi_{X/S}$ out from $\Phi_{\tilde{F}_{\tilde{U}_i/\tilde{S}}}$.

Theorem 8 *i) A lifting $\tilde{X} \to \tilde{S}$ gives rise to a splitting of the Azumaya algebra $\hat{i}^*_{pd} F_{X/S\,*} \mathcal{D}_{X/S} = F_{X/S\,*} \hat{\mathcal{D}}_{X/S,pd}$*

ii) The functor $\Phi_{\tilde{X}/\tilde{S}}$ is the equivalence of categories.

[1] The nilpotency condition is technical and can be suppressed. But then one has use a different definition of quasi-coherent $\hat{\mathcal{D}}_{X/S,pd}$-modules: the problem is that $\hat{\mathcal{D}}_{X/S,pd}$ is not a quasi-coherent $\mathcal{O}_X$-module. For example, we can define the category of quasi-coherent $\hat{\mathcal{D}}_{X/S,pd}$-modules simply to be the category modules over the Azumaya algebra $F_{X/S\,*} \hat{\mathcal{D}}_{X/S,pd}$, which are quasi-coherent as $\hat{T}^*_{X'/S,pd}$-modules.

[2] Recall that the endomorphsm $h_{\tilde{F}_{\tilde{X}/\tilde{S}}} : T^*_{X'/S} \to T^*_{X'/S}$ is identical on the zero section. Hence, it extends canonically to the universal PD-envelope

Proof: i) In effect,

$$M_{pd} = \lim_{\leftarrow} F_{X/S\,*}\Phi_{\tilde{X}/\tilde{S}}(\mathcal{O}_{\hat{T}^*_{X'/S,pd}}/J^{[n]})$$

where $J^{[n]}$ is the divided power of the ideal of the zero section, is a splitting.

ii) It is easy to see that for any quasi-coherent nilpotent $\mathcal{O}_{\hat{T}^*_{X'/S,pd}}$-module E the $F_{X/S\,*}\hat{\mathcal{D}}_{X/S,pd}$-module $F_{X/S\,*}\Phi_{\tilde{X}/\tilde{S}}(E)$ is equal to $E \otimes_{\mathcal{O}_{\hat{T}^*_{X'/S,pd}}} M_{pd}$. Hence, by i) and the general property of Azumaya algebras $F_{X/S\,*}\Phi_{\tilde{X}/\tilde{S}}$ gives an equivalence of categories between the category of nilpotent $\mathcal{O}_{\hat{T}^*_{X'/S,pd}}$-modules and the one of nilpotent $F_{X/S\,*}\hat{\mathcal{D}}_{X/S,pd}$-modules.

We shall call the inverse functor:

$$C_{\tilde{X}/\tilde{S}} = \Phi^{-1}_{\tilde{X}/\tilde{S}} : \mathcal{M}(\hat{\mathcal{D}}_{X/S,pd}) \to \mathcal{M}(\mathcal{O}_{\hat{T}^*_{X'/S,pd}})$$

the Cartier transform.

4. Geometric construction of the splitting. The above we constructed a splitting $F_{X/S\,*}M_{pd}$ of the Azumaya algebra $F_{X/S\,*}\hat{\mathcal{D}}_{X/S,pd}$, first, locally, choosing a lifting of the Frobenius, and then glued the global splitting from those defined on small open subsets. Here we give an alternative geometric construction of M_{pd} " in one step".

We start with a smooth morphism $X \to S$ and a lifting $\tilde{S}$. We shall show that a lifting $\tilde{X}'$ gives rise to a splitting of $F_{X/S\,*}\hat{\mathcal{D}}_{X/S,pd}$.

Given a PD-thickining $(i : U \hookrightarrow T) \in Cris(X/S)$ there is a canonical morphism $f_{T/S} : T \to X'$ such that $f_{T/S}i = F_{U/S}$: the Frobenius $F_{T/S}$ factors uniquely through $U' \hookrightarrow T'$ and $f_{T/S}$ is the composition of the map to U' and the embedding $i' : U' \hookrightarrow X'$.

Theorem 9 *A lifting $\tilde{X}'$ gives rise to a crystal $\mathcal{L}_{\tilde{X}'/\tilde{S}}$ of $F^*_{X/S}T_{X'/S}$-torsers on $Cris(X/S)$ uniquely characterized by the following property:*

for any object $(U \hookrightarrow \tilde{T})$ of $Cris(X/\tilde{S})$ with $\tilde{T}$ flat over $\tilde{S}$ and $T = \tilde{T} \times_{\tilde{S}} S$ the $F^*_{X/S}T_{X'/S}(T)$ -set $\mathcal{L}_{\tilde{X}'/\tilde{S}}(T)$ is identified with the set of liftings $\tilde{f}_{T/S} : \tilde{T} \to \tilde{X}'$ of $f_{T/S}$.

The sheaf $\mathcal{L}_{\tilde{X}'/\tilde{S}}$ is represented by an affine bundle over X equipped with an integrable connection; the associated vector bundle is $F^*_{X/S}T_{X'/S}$ together with the p-flat connection on it. We shall use the same name $q : \mathcal{L}_{\tilde{X}'/\tilde{S}} \to X$ for the affine bundle.

Let $M_{pd} = \mathcal{H}om_{\mathcal{O}_X}(q_*\mathcal{O}_{\mathcal{L}_{\tilde{X}'/\tilde{S}}}; \mathcal{O}_X)$. By construction, M_{pd} has a $\mathcal{D}_{X/S}$-module structure. On the other hand the $F^*_{X/S}T_{X'/S}$-action on $\mathcal{L}_{\tilde{X}'/\tilde{S}}$ gives the embedding

$$F^*_{X/S}p_*\mathcal{O}_{\hat{T}^*_{X'/S,pd}} = \hat{S}^{\cdot}_{pd}(F^*_{X/S}T_{X'/S}) \hookrightarrow \hat{\mathcal{D}}^{\infty}_{\mathcal{L}_{X'/S}/X}$$

whose image consists of all $F^*_{X/S}T_{X'/S}$-invariant hyper-differential operators. Hence, the algebra the algebra $F^*_{X/S}p_*\mathcal{O}_{\hat{T}^*_{X'/S,pd}}$ acts naturally on M_{pd}. In particular, we have an action of $F^*_{X/S}p_*\mathcal{O}_{T^*_{X'/S}}$

Proposition 10 *The above action, twisted by the automorphism $-id : T^*_{X'/S} \to T^*_{X'/S}$, coincides with the one that comes from the $\mathcal{D}_{X/S}$-module structure and the canonical map $F^*_{X/S}p_*\mathcal{O}_{T^*_{X'/S}} \hookrightarrow \mathcal{D}_{X/S}$.*

Corollary 11 *The sheaf M_{pd} has a canonical $\hat{\mathcal{D}}_{X/S,pd}$-module structure. Moreover, $F_{X/S\,*}M_{pd}$ is a splitting of $F_{X/S\,*}\hat{\mathcal{D}}_{X/S,pd}$.*

3 The de Rham cohomology

1. Let $\tilde{X} \to \tilde{S}$ be, as in the previous section.

Given an integer $n \geq 0$ we denote by $HIG^n(X'/S)$ the category of quasi-coherent $\mathcal{O}_{X'_n}$-modules and by $MIC^n(X/S)$ the category of quasi-coherent modules over $\mathcal{D}_{X/S,n} = \mathcal{D}_{X/S} \otimes_{\mathcal{Z}_{X/S}} F^*_{X/S}p_*\mathcal{O}_{X'_n}$.

Theorem (8) has the following important corollary

Corollary 12 *For any $p > n \geq 0$, we have a canonical (depending on the lifting) equivalence of categories:*

$$C_{\tilde{X}/\tilde{S}} : MIC^n(X/S) \simeq HIG^n(X'/S)$$

Indeed, for any such n, there is a natural morphism $X'_n \to \hat{T}^*_{X'/S,pd} \to T^*_{X'/S}$, thus from Theorem (8) we get a splitting of the Azumaya algebra on X'_n. It defines the equivalence.

The purpose of this section is to prove the following result:

Theorem 13 *Let M be an object of $MIC^n(X/S)$, for $p > n \geq 0$. Put $m = p - n - 1$. There is a canonical isomorphism in the derived category of quasi-coherent $\mathcal{O}_{X'}$- modules:*

$$\gamma : \tau_{\leq m} R^{\cdot}\mathcal{H}om_{\mathcal{O}_{T^*_{X'/S}}}(\mathcal{O}_{X'}; C_{\tilde{X}/\tilde{S}}(M)) \simeq \tau_{\leq m} F_{X/S\,*} R^{\cdot}\mathcal{H}om_{\mathcal{D}_{X/S}}(\mathcal{O}_X; M)$$

In particular, if $n + dim_{\,S}X < p$ we have an isomorphism

$$R^{\cdot}\mathcal{H}om_{\mathcal{O}_{T^*_{X'/S}}}(\mathcal{O}_{X'}; C_{\tilde{X}/\tilde{S}}(M)) \simeq F_{X/S\,*} R^{\cdot}\mathcal{H}om_{\mathcal{D}_{X/S}}(\mathcal{O}_X; M)$$

A proof is given in Subsection 4. In the next two subsections we establish some auxiliary results.

2. Let Y be a scheme and $p : L \to Y$ a vector bundle on Y. We denote by $i_n : Y_n \hookrightarrow L$ the n-th infinitesimal neighborhood of of the zero section $i_0 : Y \hookrightarrow L$ (i.e. $\mathcal{O}_{Y_n} = i_0^{\cdot}(\mathcal{O}_L/J^{n+1})$). For $n \leq m$, we have the natural immersion $i_{n,m} : Y_n \hookrightarrow Y_m$.

There is a canonical morphism:

$$\alpha_{n,m} : R^{\cdot} i_n^*(i_{0\,*}\mathcal{O}_Y) \to R^{\cdot} i_{n,m}^*(i_{0\,*}\mathcal{O}_Y)$$

The purpose of this section is to show that, for $m \geq n + rk\, L$, there exists a canonical morphism $\beta_{n,m}$ in the opposite direction such that $\beta_{n,m}\alpha_{n,m} = Id$. In the other words, $R^{\cdot} i_n^*(i_{0\,*}\mathcal{O}_Y)$ is a direct summand of $R^{\cdot} i_{n,m}^*(i_{0\,*}\mathcal{O}_Y)$. More completely, for any $n \leq m$, we construct the morphism

$$\beta_{n,m} : \tau_{\geq n-m} R^{\cdot} i_{n,m}^*(i_{0\,*}\mathcal{O}_Y) \to \tau_{\geq n-m} R^{\cdot} i_n^*(i_{0\,*}\mathcal{O}_Y)$$

such that $\beta_{n,m}(\tau_{\geq n-m}\alpha_{n,m}) = Id$

The construction makes use of the $\mathbb{G}_m$-action on L and the fact that both the objects $R^{\cdot} i_n^*(i_{0\,*}\mathcal{O}_Y)$, $R^{\cdot} i_{n,m}^*(i_{0\,*}\mathcal{O}_Y)$ and the morphism $\alpha_{n,m}$ live in the derived category of $\mathbb{G}_m$-equivariant coherent sheaves on Y_n.

Notice the pushforward of a complex of $\mathbb{G}_m$-equivariant sheaves E to Y is endowed with a grading:

$$R^{\cdot} p_* E = \bigoplus_{i \in \mathbb{Z}} (R^{\cdot} p_* E)_i$$

For example, $(p_*\mathcal{O}_{Y_n})_i = S^{-i}L^*$, for $-n \leq i \leq 0$, and 0 otherwise. E is said to have weights greater (resp. less) than k if $(R^{\cdot} p_* E)_i = 0$, for all $i \leq k$ (resp. $i \geq k$). We have the natural truncation functor $w_{>k}$ (resp. $w_{<k}$) from the category of derived category of $\mathbb{G}_m$-equivariant sheaves to the subcategory of complexes whose weight are greater (resp. less) than k. For any object E,there is a natural morphism $E \to w_{>k}E$ (resp. $w_{<k}E \to E$).

Lemma 14 *i) The sheaf $R^j i_n^*(i_{0\,*}\mathcal{O}_Y)$ is pure of weight $j - n$, for any $j < 0$, and it has weight 0 if $j = 0$.*

ii) The morphism $\alpha_{n,m}$ induces an isomorphism $w_{>-m-1} R^{\cdot} i_n^*(i_{0\,*}\mathcal{O}_Y) \simeq w_{>-m-1} R^{\cdot} i_{n,m}^*(i_{0\,*}\mathcal{O}_Y)$.

Proof: i)Since the statement is local on Y, we may assume L to be the trivial vector bundle. For $n = 0$, it can be checked directly with help of the Koszul resolution. In general, we may proceed by induction on n using the exact sequence

$$S^n L^* \otimes R^j i_0^*(i_{0\,*}\mathcal{O}_Y) \to p_* R^j i_n^*(i_{0\,*}\mathcal{O}_Y) \to p_* R^j i_{n-1}^*(i_{0\,*}\mathcal{O}_Y)$$

ii)We have to show that $cone(\alpha_{n,m})$ has weights less than $-m$. By i) the cone C of the canonical morphism $R^{\cdot}i_m^*(i_{0*}\mathcal{O}_Y) \to i_{0*}\mathcal{O}_Y$ has weights less than $-m$. On the other hand, we have $cone(\alpha_{n,m}) = R^{\cdot}i_{n,m}^*C$. Thus the statement follows from the fact that $R^{\cdot}i_{n,m}^*$ does not increase weights.

As a corollary of i) we obtain that the complex $\tau_{\geq k}R^{\cdot}i_n^*(i_{0*}\mathcal{O}_Y)$ has weights greater than $k-n-1$. Using the lemma we can define $\beta_{n,m}$ to be the composition of the following arrows

$$\tau_{\geq n-m}R^{\cdot}i_{n,m}^*(i_{0*}\mathcal{O}_Y) \to w_{>-m-1}\tau_{\geq n-m}R^{\cdot}i_{n,m}^*(i_{0*}\mathcal{O}_Y) \simeq w_{>-m-1}\tau_{\geq n-m}R^{\cdot}i_n^*(i_{0*}\mathcal{O}_Y)$$
$$\simeq \tau_{\geq n-m}R^{\cdot}i_n^*(i_{0*}\mathcal{O}_Y)$$

3. Given a $\mathcal{D}_{X/S}$-module M (resp. $\mathcal{O}_{T^*_{X'/S}}$-module E), we put $\hat{M} = \hat{\mathcal{D}}_{X/S} \otimes_{\mathcal{D}_{X/S}} M$ (resp. $\hat{E} = \mathcal{O}_{\hat{T}^*_{X'/S}} \otimes_{\mathcal{O}_{T^*_{X'/S}}} E$).

Lemma 15 *i) For any $\mathcal{D}_{X/S}$-module M, there are canonical isomorphisms*

$$R^{\cdot}\mathcal{H}om_{\mathcal{D}_{X/S}}(\mathcal{O}_X; M) = R^{\cdot}\mathcal{H}om_{\hat{\mathcal{D}}_{X/S}}(\mathcal{O}_X; \hat{M})$$

ii) For any $\mathcal{O}_{T^*_{X'/S}}$-module E, we have

$$R^{\cdot}\mathcal{H}om_{\mathcal{O}_{T^*_{X'/S}}}(\mathcal{O}_{X'}; M) = R^{\cdot}\mathcal{H}om_{\hat{\mathcal{O}}_{T^*_{X'/S}}}(\mathcal{O}_{X'}; \hat{E})$$

Proof:i) Assigning to a $\mathcal{D}_{X/S}$-module M the $\hat{\mathcal{D}}_{X/S}$-module $\hat{M}$ we get an exact functor from the category of $\mathcal{D}_{X/S}$-modules to the category of $\hat{\mathcal{D}}_{X/S}$-modules. In turn, it gives a morphism: $R^{\cdot}\mathcal{H}om_{\mathcal{D}_{X/S}}(\mathcal{O}_X; M) \to R^{\cdot}\mathcal{H}om_{\hat{\mathcal{D}}_{X/S}}(\mathcal{O}_X; \hat{M})$. To complete the proof it suffices to check that, for any M, $\mathcal{H}om_{\mathcal{D}_{X/S}}(\mathcal{O}_X; M) = \mathcal{H}om_{\hat{\mathcal{D}}_{X/S}}(\mathcal{O}_X; \hat{M})$ which is fairly obvious. A similar argument gives ii).

4. Proof of the Theorem. We define the morphism γ to be the composition of the following morphisms:

$$\tau_{\leq m}R^{\cdot}\mathcal{H}om_{\mathcal{O}_{T^*_{X'/S}}}(\mathcal{O}_{X'}; C_{\tilde{X}/\tilde{S}}(M)) \simeq \tau_{\leq m}R^{\cdot}\mathcal{H}om_{\mathcal{O}_{X'_n}}(R^{\cdot}i_n^*\mathcal{O}_{X'}; C_{\tilde{X}/\tilde{S}}(M)) \simeq$$
$$\tau_{\leq m}R^{\cdot}\mathcal{H}om_{\mathcal{O}_{X'_n}}(\tau_{\geq -m}R^{\cdot}i_n^*\mathcal{O}_{X'}; C_{\tilde{X}/\tilde{S}}(M)) \overset{\beta_{n,p-1}}{\to} \tau_{\leq m}R^{\cdot}\mathcal{H}om_{\mathcal{O}_{X'_n}}(\tau_{\geq -m}R^{\cdot}i_{n,p-1}^*\mathcal{O}_{X'}; C_{\tilde{X}/\tilde{S}}(M)) \simeq$$
$$\tau_{\leq m}R^{\cdot}\mathcal{H}om_{\mathcal{O}_{X'_{p-1}}}(\mathcal{O}_{X'}; C_{\tilde{X}/\tilde{S}}(M)) \simeq \tau_{\leq m}F_{X/S*}R^{\cdot}\mathcal{H}om_{\mathcal{D}_{X/S,p-1}}(\mathcal{O}_X; M) \to$$
$$\tau_{\leq m}F_{X/S*}R^{\cdot}\mathcal{H}om_{\mathcal{D}_{X/S}}(\mathcal{O}_X; M)$$

We shall prove that γ is a quasi-isomorphism. Since the statement is local we may assume that there exists a Frobenius lifting: $\tilde{F}_{X/S} : \tilde{X} \to \tilde{X}'$. According to Theorem (6) the lifting $\tilde{F}_{X/S}$ gives a splitting of the Azumaya algebra $F_{X/S*}\hat{\mathcal{D}}_{X/S}$ over the formal completion $\hat{T}^*_{X'/S}$. The above combined with the lemma from the previous subsection implies that for any nilpotent $\mathcal{D}_{X/S}$-module there is an isomorphism

$$R^{\cdot}\mathcal{H}om_{\mathcal{O}_{T^*_{X'/S}}}(\mathcal{O}_{X'}; \Phi^{-1}_{\tilde{F}_{\tilde{X}/\tilde{S}}}(M)) \simeq F_{X/S*}R^{\cdot}\mathcal{H}om_{\mathcal{D}_{X/S}}(\mathcal{O}_X; M)$$

Consider the following commutative diagram

$$\begin{array}{ccc}
\tau_{\leq m}R^{\cdot}\mathcal{H}om_{\mathcal{O}_{T^*_{X'/S}}}(\mathcal{O}_{X'}; C_{\tilde{X}/\tilde{S}}(M)) \to & \tau_{\leq m}F_{X/S*}R^{\cdot}\mathcal{H}om_{\mathcal{D}_{X/S,p-1}}(\mathcal{O}_X; M) \to \\
\downarrow & \downarrow \\
\tau_{\leq m}R^{\cdot}\mathcal{H}om_{\mathcal{O}_{T^*_{X'/S}}}(\mathcal{O}_{X'}; C_{\tilde{X}/\tilde{S}}(M)) \to & \tau_{\leq m}R^{\cdot}\mathcal{H}om_{\mathcal{O}_{X'_{p-1}}}(\mathcal{O}_{X'}; C_{\tilde{X}/\tilde{S}}(M)) \to
\end{array}$$

$$\rightarrow \tau_{\leq m} F_{X/S\,*} R^{\cdot}\mathcal{H}om_{\mathcal{D}_{X/S}}(\mathcal{O}_X; M)$$

$$\downarrow$$

$$\rightarrow \tau_{\leq m} R^{\cdot}\mathcal{H}om_{\mathcal{O}_{T^*_{X'/S}}}(\mathcal{O}_{X'}; C_{\tilde{X}/\tilde{S}}(M))$$

All the vertical arrows in the diagram are isomorphisms while the composition of the morphisms in the second row is the identity morphism. It completes the proof.

4 Functoriality of the Cartier transform

1. Definition of the functors $R^{\cdot} f_{DR*}$,$R^{\cdot} f_{Hig*}$, f^*

Let $f : X \to Y$ be a morphism of smooth schemes over S

$$df : T_{X/S} \to f^* T_{Y/S}$$

be the differential of f. To fix notation we recall the definition of the direct and inverse of a $\mathcal{D}$-module.

First, let M be a left $\mathcal{D}_{Y/S}$-module. Then there exists a unique $\mathcal{D}_{X/S}$- module structure on $f^*M = f^{\cdot} M \otimes_{f^{\cdot}\mathcal{O}_Y} \mathcal{O}_X$ with the property that:

$$v(f^*(m)) = \sum_i f^*(w_i(m)) \otimes h_i$$

where $m \in M(U)$, $U \subset Y$, $v \in T_{f^{-1}(U)/S}$ and $w_i \in T_{U/S}$, $h_i \in \mathcal{O}_{f^{-1}(U)}$ such that $df(v) = \sum_i f^*(w_i) \otimes h_i$.

The definition of the pushforward is more involved. First, recall that, for a left $\mathcal{D}_{Y/S}$-module M, $M \otimes_{\mathcal{O}_Y} \omega_{Y/S}$ has a structure of a right $\mathcal{D}_{Y/S}$-module. Here $\omega_{Y/S}$ is the canonical sheaf on Y. The action is given by the formula

$$w(m \otimes u) =: -m \otimes Lie_w(u) - w(m) \otimes u$$

where $w \in T_{U/S}$, $m \in M(U)$, $u \in \omega_{U/S}$. The inverse functor from the category of right $\mathcal{D}$-modules to the one of left $\mathcal{D}$-modules sends M to $M \otimes_{\mathcal{O}_Y} \omega^{-1}_{Y/S}$.

In particular, we have the right $\mathcal{D}_{X/S}$-module structure on $f^*(\mathcal{D}_{Y/S} \otimes \omega^{-1}_{Y/S}) \otimes \omega_{X/S}$. In addition, the latter is endowed with a left action of $f^{\cdot}(\mathcal{D}_{Y/S}$. Now, given an object $N \in D^b(\mathcal{D}_{X/S} - modules)$ of the bounded derived category of $\mathcal{D}_{X/S}$-modules, we define it pushforward to be

$$R^{\cdot} f_{DR*} N := R^{\cdot} f_{\cdot}(f^*(\mathcal{D}_{Y/S} \otimes \omega^{-1}_{Y/S}) \otimes \omega_{X/S} \otimes_{\mathcal{D}_{X/S}} N)[dim\, Y - dim\, X]$$

In the case, when f is smooth, $R^{\cdot} f_{DR*} N$ can be computed by the relative De Rham complex:

$$R^{\cdot} f_{DR*} N = R^{\cdot} f_{\cdot}(N \otimes_{\mathcal{O}_X} \Omega^{\cdot}_{X/Y})$$

The $\mathcal{D}_{Y/S}$-module structure on the left hand side corresponds to the "Gauss-Manin connection" on the right one.

Lemma 16 *Assume that f is smooth. Then the functor $R^{\cdot} f^*$ is left adjoint to $R^{\cdot} f_{DR*}$*

Next, we define the functors $R^{\cdot} f^*$, $R^{\cdot} f_{Hig*}$ for the category of Higgs modules i.e. $\mathcal{O}$-modules on the cotangent bundle.

Given a morphism $f : X \to Y$ we have the following diagram

$$\begin{array}{ccc} f^* T^*_{Y/S} & \xrightarrow{i} & T^*_{X/S} \\ \downarrow{\scriptstyle p} & & \\ T^*_{Y/S} & & \end{array}$$

Denote by $D^b(\mathcal{O}_{T^*_{X/S}} - modules)$ the bounded derived category of Higgs modules. Given complexes E in $D^b(\mathcal{O}_{T^*_{Y/S}} - modules)$ and G in $D^b(\mathcal{O}_{T^*_{X/S}} - modules)$ we define:

$$R^{\cdot} f^* E = R^{\cdot} i_* R^{\cdot} p^* E$$

$$R^{\cdot} f_{Hig*} G = p_Y^* \omega_{Y/S}^{-1} \otimes_{\mathcal{O}_{T^*_{Y/S}}} R^{\cdot} p_* R^{\cdot} i^* (G \otimes_{\mathcal{O}_{T^*_{X/S}}} p_X^* \omega_{Y/S})[dim\, Y - dim\, X]$$

Here p_X (resp. p_Y) denotes the projection $T^*_{X/S} \to X$ (resp. $T^*_{Y/S} \to Y$). The properties of $R^{\cdot} f^*$, $R^{\cdot} f_{Hig*}$ for Higgs modules are completely parallel to those for $\mathcal{D}$-modules:

Lemma 17 *Assume that f is smooth. Then*
i) $R^{\cdot} f_{Hig*} G = R^{\cdot} f_{\cdot}(N \otimes_{\mathcal{O}_X} \Omega^{\cdot}_{X/Y})$
ii)The functor $R^{\cdot} f^*$ is left adjoint to $R^{\cdot} f_{Hig*}$

2. Compatibility with the Cartier transform. Let $\tilde{f} : \tilde{X} \to \tilde{Y}$ be a morphism of smooth schemes over $\tilde{S}$. Denote by $\tilde{f}' : \tilde{X}' \to \tilde{Y}'$ the corresponding morphism of the Frobenius twists.

Theorem 18 *For any Higgs module E in $HIG^{p-1}(Y'/S)$, $R^i f'^* E$ are in $HIG^{p-1}(X'/S)$ and there is a natural isomorphism $C^{-1}(R^i f'^* E) \simeq R^i f^* C^{-1}(E)$*

To formulate the second main result, we assume, in addition, that a lifting $\tilde{F}_{\tilde{Y}/\tilde{S}} : \tilde{Y} \to \tilde{Y}'$ is given.

Theorem 19 *Let M be an object of $MIC^n(X/S)$, for $p > n \geq 0$. Put $m = p - n - 1$. There is a canonical isomorphism in the derived category of $\mathcal{O}_{T^*_{Y'/S}}$-modules:*

$$\tau_{\leq m} \Phi^{-1}_{\tilde{F}_{\tilde{Y}/\tilde{S}}} R^{\cdot} f_{DR\,*} M \simeq \tau_{\leq m} R^{\cdot} f_{Hig\,*} C_{\tilde{X}/\tilde{S}} M$$

5 Some examples and applications

1. Motivic $\mathcal{D}$-modules.

In [F] Faltings introduced certain category $\mathcal{MF}^{\nabla}_{0;k}(X)$, an analog of the category of variations of Hodge structure in characteristic p. Using the notion of Cartier transform his definition can be reformulated as follows.

Definition 20 *An object of the category $\mathcal{MF}^{\nabla}_{0;k}(X)$, $(0 \leq k < p)$ consists of the following data:*

1) A coherent sheaf M on X with a integrable connection $\nabla : M \to M \otimes \Omega^1_{X/S}$

2) A filtration $0 = F^{k+1} \subset F^1 \subset \cdots \subset F^0 = M$ by $\mathcal{O}_X$-submodules satisfying Griffiths-transversality: $\nabla(F^i) \subset F^{i-1} \otimes \Omega^1_{X/S}$

The above gives the $\mathcal{O}_X$-linear maps $F^i/F^{i+1} \to F^{i-1}/F^i \otimes \Omega^1_{X/S}$. That makes $\bigoplus_i F^i/F^{i+1}$ a $p_* \mathcal{O}_{T^*_{X/S}}$-module.

3)An isomorphism of $p_* \mathcal{O}_{T^*_{X'/S}}$-modules:

$$\phi : p_* C_{\tilde{X}/\tilde{S}}(M) \simeq \pi_*(\bigoplus_i F^i/F^{i+1})$$

It is proven in [F] that the coherent sheaf M which admits the above structure is a vector bundle and that the category $\mathcal{MF}^{\nabla}_{0;k}(X)$ is abelian.

Let $\tilde{X} \xrightarrow{\tilde{f}} \tilde{Y}$ be smooth proper morphism of smooth schemes over $\tilde{S}$ and M an object of $\mathcal{MF}^{\nabla}_{0;k}(X)$. Recall that the $\mathcal{D}_{Y/S}$-module $R^i f_{DR*} M$ is equipped with the filtration $0 = G^{k+1+i} \subset G^1 \subset \cdots \subset G^0 = R^i f_{DR*} M$ satisfying Griffiths-transversality.

Theorem 21 *Assume that $k + i < p$. Then there is a canonical isomorphism*

$$\phi : p_* C_{\tilde{Y}/\tilde{S}}(R^i f_{DR*} M) \simeq \pi_*(\bigoplus_j G^j/G^{j+1})$$

i.e. $(R^i f_{DR*} M; G^{\cdot})$ is an object of $\mathcal{MF}^{\nabla}_{0;k+i}(Y)$

A proof can be found in [F]. It also follows from Theorem (19) combined with the following lemma.

Lemma 22 *Let $f : X \to Y$ be smooth proper morphism, $(M; F^{\cdot})$ a coherent sheaf together with a integrable connection and a filtration satisfying Griffiths-transversality. Assume that the Hodge spectral sequence associated to $(M; F^{\cdot})$ degenerates. Then there is a canonical isomorphism of $p_*\mathcal{O}_{T^*_{Y/S}}$-modules:*

$$\bigoplus_j G^j/G^{j+1} \simeq R^i f_{Hig\,*}(\bigoplus_j F^j/F^{j+1})$$

2. The theorem of Barannikov and Kontsevich

Recall the following striking result of Barannikov and Kontsevich.

Theorem 23 *Let $X_{\mathbb{C}}$ be a quasi-projective smooth scheme over $\mathbb{C}$. Suppose $f \in \mathcal{O}_{X_{\mathbb{C}}}(X_{\mathbb{C}})$ defines a proper morphism to $\mathbb{A}^1_{\mathbb{C}}$. Then the hypercohomologies of the complexes*

$$\mathcal{O}_{X_{\mathbb{C}}} \xrightarrow{d+df} \Omega^1_{X_{\mathbb{C}}} \xrightarrow{d+df} \cdots,$$

$$\mathcal{O}_{X_{\mathbb{C}}} \xrightarrow{df} \Omega^1_{X_{\mathbb{C}}} \xrightarrow{df} \cdots$$

have the same finite dimension in every degree.

We make use the theory of Cartier transform to prove the following more precise result.

Let K be a finite extension of $\mathbb{Q}$, O_K the ring of integers in K, and $S \subset spec\, O_K$ an open subscheme.

Theorem 24 *Let $\mathcal{X}$ be a smooth scheme over S and $f \in \mathcal{O}_{\mathcal{X}}(\mathcal{X})$ a function. For a point $s \in S$, denote by $\mathcal{X}_s$ the fiber $\mathcal{X}_s = \mathcal{X} \times_S spec\, k(s)$. Then, for almost all closed points $s \in S$, the complexes*

$$F_*(\mathcal{O}_{\mathcal{X}_s} \xrightarrow{d+df} \Omega^1_{\mathcal{X}_s} \xrightarrow{d+df} \cdots),$$

$$\mathcal{O}_{\mathcal{X}'_s} \xrightarrow{df} \Omega^1_{\mathcal{X}'_s} \xrightarrow{df} \cdots$$

are quasi-isomorphic.

In particular, it gives an alternative proof of the Barannikov-Kontsevich theorem.

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