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Intersection numbers of Heegner cycles

This is a report on work done to put into reality *Kudla's dream* which is to relate the central value of the derivative of certain Siegel-Eisenstein series of genus n with the intersection pairing of special cycles (Heegner cycles and their generalizations) on arithmetic models of Shimura varieties associated to orthogonal groups of signature $(n-1,2)$. In this abstract I will concentrate on my joint results with S.Kudla on the case $n=3$. In this case one is considering arithmetic Hirzebruch-Zagier cycles on arithmetic Hilbert-Blumenthal surfaces. Note that the choice of topic is very appropriate for a talk in Bonn, where this theme has a long tradition.

Let V be a quadratic space of signature $(2,2)$ over $\mathbf{Q}$. Then $B=C^+(V)$ is an indefinite quaternion algebra over a real quadratic field $\mathbf{k}$. Let

$$G=\{g\in B^\times ; gg^\iota\in\mathbf{Q}^\times\}\ \text{ as algebraic group over }\mathbf{Q}\quad .$$

Then G acts on $V\subset C^-(V)$ by conjugation. Let

$$\mathcal{D}=\text{space of oriented negative 2-planes in } V(\mathbf{R}).$$

The corresponding Shimura variety (a finite disjoint sum of classical Hilbert-Blumenthal surfaces)

$$Sh(G,\mathcal{D})=G(\mathbf{Q})\setminus[\mathcal{D}\times G(\mathbf{A}_f)/K]$$

is defined over $\mathbf{Q}$ as the moduli variety M parametrizing 4-tuples $(A,\lambda,\iota,\overline{\eta})$ consisting (roughly speaking) of an abelian variety of dimension 8 up to isogeny, a polarization of it, an action of $C(V)\otimes_{\mathbf{Q}}\mathbf{k}$ on it, and a level-K-structure. A *special endomorphism* of $(A,\lambda,\iota,\overline{\eta})$ is an element j of $\mathrm{End}(A)$ such that

$$\begin{aligned} j\circ\iota(c\otimes a)&=\iota(c\otimes a')\circ j\\ j^*&=j\qquad\text{(Rosati-involution)}\end{aligned}$$

Since $j^2=Q(j)\cdot\mathrm{id}_A$ for special j, the special endomorphisms of $(A,\lambda,\iota,\overline{\eta})$ form a quadratic space over $\mathbf{Q}$. For $n>0$ and $T\in\mathrm{Sym}_n(\mathbf{Q})_{\geq 0}$ and a K-invariant compact open subset $\omega\subset V(\mathbf{A}_f)^n$ we obtain the moduli space $Z(T,\omega)$ over M whose fibre over $(A,\lambda,\iota,\overline{\eta})$ parametrizes n-tuples $\mathbf{j}=(j_1,\dots,j_n)$ of special endomorphisms with $Q(\mathbf{j})=T$ and satisfying an integrality condition depending on ω. For $n=1$, $B=M_2(\mathbf{k})$ and suitable choices of $T\in\mathbf{Q}_{>0}$ and ω we obtain for $Z(T,\omega)$ the classical Hirzebruch-Zagier cycles T_N.

The scheme M can be extended from $\mathrm{Spec}\,\mathbf{Q}$ to $\mathrm{Spec}\,\mathbf{Z}[1/N]$ for sufficiently big N (which one can specify), so as to obtain a smooth quasi-projective scheme $\mathcal{M}$ of relative dimension 2 over $\mathrm{Spec}\,\mathbf{Z}[1/N]$. Similarly, for $T \in \mathrm{Sym}_n(\mathbf{Z}[1/N])_{\geq 0}$ and $\omega \subset V(\mathbf{A}_f)^n$ satisfying an integrality condition outside N we obtain an arithmetic cycle $\mathcal{Z}(T,\omega)$ on $\mathcal{M}$.

Theorem 1: *Let $T \in \mathrm{Sym}_3(\mathbf{Z}[1/N])_{\geq 0}$. If $\det(T) \neq 0$, then $\mathcal{Z}(T,\omega)$ has empty generic fibre. If furthermore $\mathcal{Z}(T,\omega) \neq \emptyset$, then it is concentrated in characteristic p for a unique p and in fact is contained in the supersingular locus of $\mathcal{M} \times_{\mathrm{Spec}\,\mathbf{Z}[1/N]} \mathrm{Spec}\,\mathbf{F}_p$.*

One might expect that the cycle $\mathcal{Z}(T,\omega)$ in the arithmetic 3-fold $\mathcal{M}$ should be a discrete set of points, for $T \in \mathrm{Sym}_3(\mathbf{Z}[1/N])_{>0}$. However, this is not true in general.

Theorem 2: *Let $T \in \mathrm{Sym}_3(\mathbf{Z}[1/N])_{>0}$ and let $\mathcal{Z}(T,\omega)(\overline{\mathbf{F}}_p) \neq \emptyset$. Then $\dim(\mathcal{Z}(T,\omega)) = 0$, iff p splits in $\mathbf{k}$ or p is inert in $\mathbf{k}$ and $T \neq 0 \pmod p$.*

If $T \equiv 0 \pmod p$ for inert p, then $\mathcal{Z}(T,\omega)$ is a union of certain of the projective lines that make up the supersingular locus in $\mathcal{M} \times_{\mathrm{Spec}\,\mathbf{Z}[1/N]} \mathrm{Spec}\,\mathbf{F}_p$, and these can be enumerated in terms of the Bruhat-Tits tree of $PGL_2(\mathbf{Q}_{p^2})$.

We form a generating series

$$(*) \qquad \sum\nolimits_T \deg(\mathcal{Z}(T,\omega))q^T \quad .$$

Here T runs over the matrices characterized in Theorem 2 and $\deg \mathcal{Z} = \log \#(\mathcal{O}_{\mathcal{Z}})$.

Theorem 3: *Consider the Siegel modular form of genus 3 and weight 2,*

$$\phi(\tau) = \det(v)^{-1} \cdot E'(h_\tau, 0, \Phi) \quad , \quad \tau = u + iv, \ h_\tau(i) = \tau \quad ,$$

where Φ depends on ω. Then the sum $(*)$ is part of the Fourier expansion of ϕ,

$$\phi(\tau) = \sum_{T \in \mathrm{Sym}_3(\mathbf{Z}[1/N])} a_T(v) q^T \quad , \quad q^T = e^{2\pi i\, tr(T\tau)} \quad .$$

The proof of the last theorem is by explicitly computing the coefficients of both series. For the coefficients of $(*)$ this is done by reducing it to a problem of deforming a homomorphism between two supersingular elliptic curves. This problem had been solved by Gross and Keating. For the Fourier coefficients of $\phi(\tau)$ one uses results of Kitaoka on local representation densities of quadratic forms.

The problem of intersecting arithmetic special cycles on $\mathcal{M}$ may be (partially) reduced to the previous three theorems by the following remark. Let $n_1, \ldots, n_r$ be positive integers with $n_1 + \ldots + n_r = 3$. For each $i = 1, \ldots, r$ let $T_i \in \mathrm{Sym}_{n_i}(\mathbf{Z}[1/N])_{>0}$ and $\omega_i \subset V(\mathbf{A}_f)^{n_i}$ as before, so that we obtain the special cycle $\mathcal{Z}(T_i, \omega_i)$ on $\mathcal{M}$. Then

$$\mathcal{Z}(T_1,\omega_1) \times_{\mathcal{M}} \ldots \times_{\mathcal{M}} \mathcal{Z}(T_r,\omega_r) = \coprod\nolimits_T \mathcal{Z}(T,\omega) \quad .$$

Here on the index runs over $T \in \mathrm{Sym}_3(\mathbf{Z}[1/N])_{\geq 0}$ with diagonal blocks $T_1, \ldots, T_r$.

For more details the reader can consult

S.Kudla, M.Rapoport: Arithmetic Hirzebruch-Zagier cycles. alg-geom 9904083.