Semi-infinite Weil complex and $N = 2$ superconformal algebra I.

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Contents.

0. Introduction.

Recently there was a significant progress in the understanding of infinite-dimensional graded Lie algebras. It turns out that the most appropriate generalization of the classical Lie algebra cohomology is the semi-infinite cohomology introduced by B.Feigin. He considered the semi-infinite wedge products of the elements of Lie algebra and defined a differential on such products. The cohomology of the corresponding chain complex (unbounded in both directions) is called the semi-infinite cohomology [F].

In [FF] B.Feigin and E.Frenkel constructed a semi-infinite analogue of the famous Weil complex based on the space

$$W^*(\mathfrak{g}) = S^*(\mathfrak{g}') \otimes \Lambda^*(\mathfrak{g}'),$$

where $\mathfrak{g}$ is a Lie algebra.

B.Feigin and E.Frenkel studied this complex in the case when $\mathfrak{g}$ is the Virasoro algebra, and computed the semi-infinite cohomology of $\mathfrak{g}$ with coefficients in its "adjoint semi-infinite symmetric powers" $H^{\frac{\infty}{2}+*}(\mathfrak{g}, S^{\frac{\infty}{2}+*}(\mathfrak{g}))$. The semi-infinite Weil complex of the Virasoro algebra has also been studied by physicists in relation to the string theory.

E. Getzler has shown that the semi-infinite Weil complex of the Virasoro algebra is a module over the $N = 2$ superconformal algebra [G].

In this work we consider the semi-infinite Weil complex $W^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}})$ of the loop algebra $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ of a complex finite-dimensional Lie algebra $\mathfrak{g}$.

In section 1 we review a construction of the semi-infinite Weil complex of a complex graded Lie algebra. In section 2 we show that the semi-infinite Weil complex of the loop algebra $\tilde{\mathfrak{g}}$ of a complex finite-dimensional Lie algebra $\mathfrak{g}$ is a module over the $N = 2$ superconformal algebra with central charge $3dim\mathfrak{g}$. We also obtain an action of the $N = 2$ superconformal algebra on the semi-infinite cohomology $H^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}, S^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}))$.

In section 3 we observe that given a complex graded vector space, one can repeat the construction of the semi-infinite Weil complex. In section 4 we describe an action of the $N = 2$ superconformal algebra on the semi-infinite Weil complex related to a standard module of type $\mathcal{F}_{\lambda,\mu}$ over the Witt algebra.

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1. Semi-infinite Weil complex.

The semi-infinite Weil complex was introduced by B. Feigin and E. Frenkel in [FF]. Recall the necessary definitions. Let $\mathfrak{g} = \oplus_{n\in\mathbb{Z}}\mathfrak{g}_n$ be a graded Lie algebra over $\mathbb{C}$, so that $dim\mathfrak{g}_n < \infty$. Let $\mathfrak{g}' = \oplus_{n\in\mathbb{Z}}\mathfrak{g}'_n$ be the restricted dual of $\mathfrak{g}$. Linear space $\mathfrak{g} \oplus \mathfrak{g}'$ carries non-degenerate exterior and symmetric forms: $(,)$ and $\{,\}$. Let $H(\mathfrak{g})$ and $C(\mathfrak{g})$ be the quotients of the tensor algebra $T^*(\mathfrak{g} \oplus \mathfrak{g}')$ by the ideals generated by the elements of the form $xy - yx - (x, y)$ and $xy + yx - \{x, y\}$, respectively, where $x, y \in \mathfrak{g} \oplus \mathfrak{g}'$. We fix $N \in \mathbb{Z}$. Let $\mathfrak{n}_+ = \oplus_{n>N}\mathfrak{g}_n$, $\mathfrak{n} = \oplus_{n\leq N}\mathfrak{g}_n$.

The symmetric algebra $S^*(\mathfrak{n}_+ \oplus \mathfrak{n}')$ is a subalgebra of $H(\mathfrak{g})$ and the exterior algebra $\Lambda^*(\mathfrak{n}_+ \oplus \mathfrak{n}')$ is a subalgebra of $C(\mathfrak{g})$. Let $S^{\frac{\infty}{2}+*}(\mathfrak{g})$, $\Lambda^{\frac{\infty}{2}+*}(\mathfrak{g})$ be the representations of $H(\mathfrak{g})$ and $C(\mathfrak{g})$ induced from the trivial representations $< 1_S >$ et $< 1_\Lambda >$ of
$S^*(\mathfrak{n}_+ \oplus \mathfrak{n}')$ and $\Lambda^*(\mathfrak{n}_+ \oplus \mathfrak{n}')$, respectively. Thus we obtain some semi-infinite analogues of symmetric and exterior power modules. Denote the actions of $H(\mathfrak{g})$ and $C(\mathfrak{g})$ on these modules by $\beta(x), \gamma(x')$ and $\tau(x), \varepsilon(x')$, respectively, for $x \in \mathfrak{g}$, $x' \in \mathfrak{g}'$. Notice that each element of $S^{\frac{\infty}{2}+*}(\mathfrak{g})$ and of $\Lambda^{\frac{\infty}{2}+*}(\mathfrak{g})$ is a finite linear combination of the monoms of the type $\gamma(x'_1)\ldots\gamma(x'_k)\beta(y_1)\ldots\beta(y_m)1_S$ and of the type $\varepsilon(x'_1)\ldots\varepsilon(x'_k)\tau(y_1)\ldots\tau(y_m)1_\Lambda$, respectively, where $x'_1, \ldots, x'_k \in \mathfrak{n}'_+$, $y_1, \ldots y_m \in \mathfrak{n}$.

Let $\{e_i\}_{i\in\mathbb{Z}}$ be a homogeneous basis of $\mathfrak{g}$. In other words, for any $i \in \mathbb{Z}$, $e_i \in \mathfrak{g}_n$ for some $n \in \mathbb{Z}$, and if $e_i \in \mathfrak{g}_n$, then $e_{i+1} \in \mathfrak{g}_n$ or $e_{i+1} \in \mathfrak{g}_{n+1}$. Let $\{e'_i\}_{i\in\mathbb{Z}}$ be the dual basis. Let $i_0 \in \mathbb{Z}$ be such that $e_{i_0} \in \mathfrak{g}_N$ and $e_{i_0+1} \in \mathfrak{g}_{N+1}$.

Notice that one can think of $\Lambda^{\frac{\infty}{2}+*}(\mathfrak{g})$ as the vector space spanned by the elements $w = e'_{i_1} \wedge e'_{i_2} \wedge \ldots$ such that there exists $K(w) \in \mathbb{Z}$ such that $i_{k+1} = i_k - 1$ for $k > K(w)$. Then $1_\Lambda = e'_{i_0} \wedge e'_{i_0-1} \wedge \ldots$ is a vacuum vector in this space. The actions of $\varepsilon(x'), \tau(x)$ are, respectively, the exterior multiplication and contraction in the space of semi-infinite exterior products.

Let ρ and π be the projective representations of $\mathfrak{g}$ in $\Lambda^{\frac{\infty}{2}+*}(\mathfrak{g})$ and $S^{\frac{\infty}{2}+*}(\mathfrak{g})$, respectively, defined as follows :

$$\rho(x) = \Sigma_{i\in\mathbb{Z}} : \tau([x,e_i])\varepsilon(e'_i) :, \tag{1.1}$$

$$\pi(x) = \Sigma_{i\in\mathbb{Z}} : \beta([x,e_i])\gamma(e'_i) :, \tag{1.2}$$

where $x \in \mathfrak{g}$, and where the double colons : : denote a normal ordering operation :

$$: \tau(e_j)\varepsilon(e'_i) := \left\{ \begin{array}{c} \tau(e_j)\varepsilon(e'_i) \text{ if } i \leq i_0 \\ -\varepsilon(e'_i)\tau(e_j) \text{ if } i > i_0 \end{array} \right\} \tag{1.3}$$

$$: \beta(e_j)\gamma(e'_i) := \left\{ \begin{array}{c} \beta(e_j)\gamma(e'_i) \text{ if } i \leq i_0 \\ \gamma(e'_i)\beta(e_j) \text{ if } i > i_0 \end{array} \right\} \tag{1.4}$$

Thus

$$\rho(x)1_\Lambda = \pi(x)1_S = 0 \text{ for } x \in \mathfrak{g}_0 \tag{1.5}$$

and

$$[\rho(x), \rho(y)] = \rho([x,y]) + c_\Lambda(x,y), \tag{1.6}$$

$$[\pi(x), \pi(y)] = \pi([x,y]) + c_S(x,y), \tag{1.7}$$

where $x, y \in \mathfrak{g}$ and c_Λ, c_S are 2-cocycles. Notice that $c_\Lambda = -c_S$. Let

$$W^{\frac{\infty}{2}+*}(\mathfrak{g}) = S^{\frac{\infty}{2}+*}(\mathfrak{g}) \otimes \Lambda^{\frac{\infty}{2}+*}(\mathfrak{g}). \tag{1.8}$$

Since the cocycles corresponding to the projective representations cancel, the representation $\theta(x) = \rho(x) + \pi(x)$ of $\mathfrak{g}$ in $W^{\frac{\infty}{2}+*}(\mathfrak{g})$ is well-defined. We obtain the semi-infinite Weil complex

$$\{W^{\frac{\infty}{2}+*}(\mathfrak{g}),\ d + \mathrm{h}\}, \tag{1.9}$$

where

$$d = \Sigma_{i<j} : \tau([e_i,e_j])\varepsilon(e'_j)\varepsilon(e'_i) : + \Sigma_{i,j} : \beta([e_j,e_i])\gamma(e'_i)\varepsilon(e'_j) : \tag{1.10}$$

$$\mathrm{h} = \Sigma_i \gamma(e'_i)\tau(e_i). \tag{1.11}$$

The differential d is the analogues of the classical differential for the Lie algebra cohomology and h is the Koszul differential. Notice that

$$d^2 = 0, \mathrm{h}^2 = 0, \{d, \mathrm{h}\} = 0, (d + \mathrm{h})^2 = 0. \tag{1.12}$$

Notice that if $\mathfrak{g}$ is a finite-dimensional Lie algebra, $\mathfrak{n}_+ = \mathfrak{g}$, $\mathfrak{n} = 0$, then (1.9) is the classical Weil complex.

Let $Deg\ \varepsilon(x') = Deg\ \gamma(x') = 1$, $Deg\ \tau(x) = Deg\ \beta(x) = -1$ for $x \in \mathfrak{g}, x' \in \mathfrak{g}'$. Then

$$\Lambda^{\frac{\infty}{2}+*}(\mathfrak{g}) = \oplus_{j\in\mathbb{Z}}\Lambda^{\frac{\infty}{2}+j}(\mathfrak{g}), S^{\frac{\infty}{2}+*}(\mathfrak{g}) = \oplus_{l\in\mathbb{Z}}S^{\frac{\infty}{2}+l}(\mathfrak{g}), \tag{1.13}$$

and we obtain a $\mathbb{Z}$-grading on $W^{\frac{\infty}{2}+*}(\mathfrak{g})$:

$$W^{\frac{\infty}{2}+i}(\mathfrak{g}) = \oplus_{2l+j=i}S^{\frac{\infty}{2}+l}(\mathfrak{g}) \otimes \Lambda^{\frac{\infty}{2}+j}(\mathfrak{g}). \tag{1.14}$$

As in the case of the classical Weil complex, on can construct two filtrations, for one of them $(F_1^p, l+m \geq p)$ the complex is acyclic, the second term of the spectral sequence associated to the other filtration $(F_2^p, 2l \geq p)$ is the semi-infinite cohomology of Lie algebra $\mathfrak{g}$ with coefficients in its "adjoint semi-infinite symmetric powers" $H^{\frac{\infty}{2}+*}(\mathfrak{g}, S^{\frac{\infty}{2}+*}(\mathfrak{g}))$. B. Feigin and E. Frenkel calculated this cohomology in the case when $\mathfrak{g}$ is the Virasoro algebra ([FF]). The next natural example is the loop algebra of a complex finite-dimensional Lie algebra.

2. An action of the $N = 2$ superconformal algebra on the semi-infinite Weil complex of a loop algebra.

In this section we fix $N = 0$. Thus $\mathfrak{n}_+ = \oplus_{n>0}\mathfrak{g}_n$, $\mathfrak{n} = \oplus_{n\leq 0}\mathfrak{g}_n$.

Recall that *the* $N = 2$ *superconformal algebra* is spanned by the Virasoro generators $\mathcal{L}_n$, Heisenberg generators H_n, two fermionic fields $G_r^{\pm}$, and a central element C, where $n \in \mathbb{Z}, r \in \mathbb{Z} + 1/2$, and where the nonvanishing commutation relations are as follows [FST]:

$$\begin{aligned}
&[\mathcal{L}_n, \mathcal{L}_m] = (n-m)\mathcal{L}_{n+m} + \frac{\mathsf{C}}{12}(n^3 - n)\delta_{n,-m},\\
&[\mathcal{L}_n, H_m] = -mH_{n+m},\\
&[\mathcal{L}_n, G_r^{\pm}] = (\frac{n}{2} - r)G_{n+r}^{\pm},\\
&[G_r^+, G_s^-] = 2\mathcal{L}_{r+s} + (r-s)H_{r+s} + \frac{\mathsf{C}}{3}(r^2 - \frac{1}{4})\delta_{r,-s},\\
&[H_n, H_m] = \frac{\mathsf{C}}{3}n\delta_{n,-m},\\
&[H_n, G_r^{\pm}] = \pm G_{n+r}^{\pm}.
\end{aligned} \tag{2.1}$$

Theorem 2.1. Let $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, where $\mathfrak{g}$ is a complex finite-dimensional Lie algebra. Then $W^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}})$ is a module over the $N = 2$ superconformal algebra with central charge

$3 dim\mathfrak{g}$.

Proof. We fix a basis in $\mathfrak{g}$. Let element u run through this basis. Define

$$\begin{aligned}
L_n &= -\Sigma_{u,m\in\mathbb{Z}}[m : \tau(u_{m+n})\varepsilon(u'_m) : +m : \beta(u_{m+n})\gamma(u'_m) :] \\
H_n &= -\Sigma_{u,m\in\mathbb{Z}} : \gamma(u'_{m+n})\beta(u_m) : \\
\mathfrak{h}_n &= \Sigma_{u,m\in\mathbb{Z}}\gamma(u'_{m+n})\tau(u_m), \\
\mathfrak{p}_n &= \Sigma_{u,m\in\mathbb{Z}} m\beta(u_{m-n})\varepsilon(u'_m).
\end{aligned} \tag{2.2}$$

Notice that $\mathfrak{h}_0$ is the Koszul differential and $\mathfrak{p}_0$ is an anologue of the homotopy operator [Fu]. Let

$$\mathfrak{L}_n = -L_{-n} + \frac{n+1}{2}H_n. \tag{2.3}$$

The nonvanishing commutation relations are:

$$\begin{aligned}
[\mathfrak{L}_n, \mathfrak{L}_k] &= (n-k)\mathfrak{L}_{n+k}, \\
[\mathfrak{L}_n, H_k] &= -kH_{n+k}, \\
[\mathfrak{L}_n, \mathfrak{h}_k] &= (1/2)(n-2k+1)\mathfrak{h}_{n+k}, \\
[\mathfrak{L}_n, \mathfrak{p}_k] &= (1/2)(n-2k-1)\mathfrak{p}_{n+k}, \\
[H_n, \mathfrak{h}_k] &= \mathfrak{h}_{n+k}, \\
[H_n, \mathfrak{p}_k] &= -\mathfrak{p}_{n+k}, \\
[\mathfrak{h}_n, \mathfrak{p}_k] &= \mathfrak{L}_{n+k} - (1/2)(k-n+1)H_{n+k}.
\end{aligned} \tag{2.4}$$

Set

$$G^+_{n-\frac{1}{2}} = \sqrt{2}\mathfrak{h}_n, G^-_{n+\frac{1}{2}} = \sqrt{2}\mathfrak{p}_n. \tag{2.5}$$

Then the central extension of the superalgebra spanned by $\mathfrak{L}_n, H_n, G^+_{n-\frac{1}{2}}, G^-_{n+\frac{1}{2}}$ is the $N=2$ superconformal algebra. To show that we obtain a representation of this algebra in $W^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}})$, one should check the commutation relations on the vacuum vector $\mathbb{1} = \mathbb{1}_S \otimes \mathbb{1}_\Lambda$. For example,

$$[H_n, H_{-n}]\mathbb{1} = (H_nH_{-n} - H_{-n}H_n)\mathbb{1}. \tag{2.6}$$

Let $n > 0$. Then $H_{-n}\mathbb{1} = 0$. Hence

$$\begin{aligned}
&[H_n, H_{-n}]\mathbb{1} = -H_{-n}(-\Sigma_u\Sigma^0_{m=1-n}\gamma(u'_{m+n})\beta(u_m))\mathbb{1} = \\
&(-\Sigma_u\Sigma^n_{m=1}\gamma(u'_{m-n})\beta(u_m))(\Sigma_u\Sigma^0_{m=1-n}\gamma(u'_{m+n})\beta(u_m))\mathbb{1} = \\
&-\Sigma_u\Sigma^0_{m=1-n}\gamma(u'_m)\beta(u_{m+n})\gamma(u'_{m+n})\beta(u_m)\mathbb{1} = -dim\mathfrak{g}(-n)\mathbb{1},
\end{aligned} \tag{2.7}$$

since $\gamma(u'_i)\beta(u_i)-\beta(u_i)\gamma(u'_i)=1$, and u runs through a basis of $\mathfrak{g}$. Notice that

$$[H_n,H_m]\mathbb{1}=0,\ \text{if } m\neq -n. \tag{2.8}$$

Hence

$$[H_n,H_m]\mathbb{1}=n dim\mathfrak{g}\delta_{n,-m}\mathbb{1}. \tag{2.9}$$

Thus the central charge is $3dim\mathfrak{g}$. The other commutation relations on the vacuum vector $\mathbb{1}$ are calculated in the same way.

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Corollary 2.1. The space $H^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}},S^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}))$ is a module over the $N=2$ superconformal algebra with central charge $3dim\mathfrak{g}$.

Proof. We will show that the $N=2$ superconformal algebra defined by (2.2) commutes with d. Recall that

$$d=d_1+d_2, \tag{2.10}$$

where

$$d_1=(1/2)\Sigma_{u,v,i,j}:\tau([u_i,v_j])\varepsilon(v'_j)\varepsilon(u'_i):, \tag{2.11}$$

$$d_2=\Sigma_{u,v,i,j}:\beta([u_i,v_j])\gamma(v'_j)\varepsilon(u'_i):, \tag{2.12}$$

u,v run through a fixed basis of $\mathfrak{g}$, and $i,j\in\mathbb{Z}$. Notice that

$$\begin{aligned}
&[L_n,\tau(u_m)]=-m\tau(u_{m+n}),\\
&[L_n,\varepsilon(u'_m)]=(m-n)\varepsilon(u'_{m-n}),\\
&[L_n,\beta(u_m)]=-m\beta(u_{m+n}),\\
&[L_n,\gamma(u'_m)]=(m-n)\gamma(u'_{m-n}),
\end{aligned} \tag{2.13}$$

Thus

$$\begin{aligned}
&[L_n,d_1]=(1/2)\Sigma_{u,v,i,j}:-(i+j)\tau([u,v]_{i+j+n})\varepsilon(v'_j)\varepsilon(u'_i):+\\
&:\tau([u_i,v_j])(j-n)\varepsilon(v'_{j-n})\varepsilon(u'_i):+:\tau([u_i,v_j])\varepsilon(v'_j)(i-n)\varepsilon(u'_{i-n}):=0.
\end{aligned} \tag{2.14}$$

Similarly,

$$\begin{aligned}
&[L_n,d_2]=\Sigma_{u,v,i,j}:-(i+j)\beta([u,v]_{i+j+n})\gamma(v'_j)\varepsilon(u'_i):+\\
&:\beta([u_i,v_j])(j-n)\gamma(v'_{j-n})\varepsilon(u'_i):+:\beta([u_i,v_j])\gamma(v'_j)(i-n)\varepsilon(u'_{i-n}):=0.
\end{aligned} \tag{2.15}$$

Notice that

$$[H_n, \gamma(u'_m)] = \gamma(u'_{m+n}), \qquad (2.16)$$
$$[H_n, \beta(u_m)] = -\beta(u_{m-n}).$$

Thus

$$[H_n, d_1] = 0, \qquad (2.17)$$

and

$$[H_n, d_2] = \Sigma_{u,v,i,j} : -\beta([u,v]_{i+j-n})\gamma(v'_j)\varepsilon(u'_i) : +\beta([u,v]_{i+j})\gamma(v'_{j+n})\varepsilon(u'_i) := 0. \qquad (2.18)$$

Next,

$$[\mathsf{h}_n, \varepsilon(u'_m)] = \gamma(u'_{m+n}), \qquad (2.19)$$
$$[\mathsf{h}_n, \beta(u_m)] = \tau(u_{m-n}).$$

Thus

$$[\mathsf{h}_n, d_1] = (1/2)\Sigma_{u,v,i,j} : -\tau([u,v]_{i+j})\gamma(v'_{j+n})\varepsilon(u'_i) : + : \tau([u,v]_{i+j})\varepsilon(v'_j)\gamma(u'_{i=n}) :=$$
$$-\Sigma_{u,v,i,j} : \tau([u,v]_{i+j})\gamma(v'_{j+n})\varepsilon(u'_i) :, \qquad (2.20)$$

$$[\mathsf{h}_n, d_2] = \Sigma_{u,v,i,j} : \tau([u,v]_{i+j-n})\gamma(v'_j)\varepsilon(u'_i) : + : \beta([u,v]_{i+j})\gamma(v'_j)\gamma(u'_{i+n}) :=$$
$$\Sigma_{u,v,i,j} : \tau([u,v]_{i+j})\gamma(v'_{j+n})\varepsilon(u'_i) :, \qquad (2.21)$$

since $\Sigma_{u,v,i,j} : \beta([u,v]_{i+j})\gamma(v'_j)\gamma(u'_{i+n}) := 0$. Hence

$$[\mathsf{h}_n, d_2] = -[\mathsf{h}_n, d_1]. \qquad (2.22)$$

Finally,

$$[\mathsf{p}_n, \tau(u_m)] = m\beta(u_{m-n}),$$
$$[\mathsf{p}_n, \gamma(u'_m)] = -(m+n)\varepsilon(u'_{m+n}). \qquad (2.23)$$

Thus

$$[\mathsf{p}_n, d_1] = (1/2)\Sigma_{u,v,i,j} : (i+j)\beta([u,v]_{i+j-n})\varepsilon(v'_j)\varepsilon(u'_i) :, \qquad (2.24)$$

$$[\mathfrak{p}_n, d_2] = \Sigma_{u,v,i,j} : -\beta([u,v]_{i+j})(j+n)\varepsilon(v'_{j+n})\varepsilon(u'_i) :=\tag{2.25}$$
$$\Sigma_{u,v,i,j} : -\beta([u,v]_{i+j-n}) j \varepsilon(v'_j)\varepsilon(u'_i) := -(1/2)\Sigma_{u,v,i,j} : (j+i)\beta([u,v]_{i+j-n})\varepsilon(v'_j)\varepsilon(u'_i) : .$$

Hence

$$[\mathfrak{p}_n, d_2] = -[\mathfrak{p}_n, d_1].\tag{2.26}$$

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3. Semi-infinite Weil complex of a graded vector space.

Let $V = \oplus_{n\in\mathbb{Z}} V_n$ be a graded vector space over $\mathbb{C}$, such that $dim V_n < \infty$. Let $V' = \oplus_{n\in\mathbb{Z}} V'_n$ be its restricted dual. We fix $N \in \mathbb{Z}$. Let

$$V_+ = \oplus_{n>N} V_n, V_- = \oplus_{n\leq N} V_n.\tag{3.1}$$

Using the space $T^*(V \oplus V')$ one can construct the infinite-dimensional Heisenberg and Clifford algebras $H(V)$ and $C(V)$, and define their representations induced from the trivial representations 1_S and 1_Λ of $S^*(V_+ \oplus V'_-)$ and $\Lambda^*(V_+ \oplus V'_-)$, respectively. We denote these representations by $S^{\frac{\infty}{2}+*}(V)$ and $\Lambda^{\frac{\infty}{2}+*}(V)$.

Let $\mathfrak{g} = \oplus_{n\in\mathbb{Z}} \mathfrak{g}_n$ be a graded Lie algebra over $\mathbb{C}$. Let ϕ be a representation of $\mathfrak{g}$ in V:

$$\phi(\mathfrak{g}_n)V_k \subset V_{k+n}.\tag{3.2}$$

Let $\{v_i\}_{i\in\mathbb{Z}}$ be a homogeneous basis in V, $\{v'_i\}_{i\in\mathbb{Z}}$ be the dual basis. Let i_0 be such that $e_{i_0} \in \mathfrak{g}_N$ and $e_{i_0+1} \in \mathfrak{g}_{N+1}$.

One can define the projective representations ρ and π of $\mathfrak{g}$ in $\Lambda^{\frac{\infty}{2}+*}(V)$ and $S^{\frac{\infty}{2}+*}(V)$, respectively:

$$\rho(x) = \Sigma_{i\in\mathbb{Z}} : \tau(\phi(x)v_i)\varepsilon(v'_i) :,\tag{3.3}$$
$$\pi(x) = \Sigma_{i\in\mathbb{Z}} : \beta(\phi(x)v_i)\gamma(v'_i) :,\tag{3.4}$$

where $x \in \mathfrak{g}$, and the normal ordering operation is defined by analogy to (1.3) and (1.4). Let

$$W^{\frac{\infty}{2}+*}(V) = S^{\frac{\infty}{2}+*}(V) \otimes \Lambda^{\frac{\infty}{2}+*}(V).\tag{3.5}$$

We obtain the representation $\theta(x) = \rho(x) + \pi(x)$ of $\mathfrak{g}$ in $W^{\frac{\infty}{2}+*}(V)$, since the corresponding cocycles cancel. We fix $N = 0$.

Theorem 3.1. Let $\tilde{V} = V \otimes \mathbb{C}[t, t^{-1}]$, where V is a complex finite-dimensional vector space. Then $W^{\frac{\infty}{2}+*}(\tilde{V})$ is a module over the $N=2$ superconformal algebra with central charge $3dimV$.

Proof. The elements of the $N=2$ superconformal algebra are defined by the quadratic expansions (2.2), where now u runs through a fixed basis of the vector space V.

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4. Semi-infinite Weil complex of the standard modules over the Witt algebra.

Let $L=\oplus_{i\in\mathbb{Z}}\mathbb{C}L_i$ be the Witt algebra:

$$[L_i,L_j]=(i-j)L_{i+j}. \tag{4.1}$$

Let $\lambda,\mu\in\mathbb{C}$. Let $\boldsymbol{\mathcal{F}}_{\lambda,\mu}=\oplus_{j\in\mathbb{Z}}\mathbb{C}v_j$ be the module over L defined as follows:

$$\phi(L_i)v_j=(-j+\mu-(i-1)\lambda)v_{i+j}. \tag{4.2}$$

We fix $i_0\in\mathbb{Z}$ and consider the corresponding polarization of $\boldsymbol{\mathcal{F}}_{\lambda,\mu}$:

$$\boldsymbol{\mathcal{F}}^+_{\lambda,\mu}=\oplus_{i>i_0}\mathbb{C}v_i,\boldsymbol{\mathcal{F}}^-_{\lambda,\mu}=\oplus_{i\leq i_0}\mathbb{C}v_i. \tag{4.3}$$

Then $W^{\frac{\infty}{2}+*}(\boldsymbol{\mathcal{F}}_{\lambda,\mu})$ is defined. Let $i_0=0$.

Theorem 4.1. The space $W^{\frac{\infty}{2}+*}(\boldsymbol{\mathcal{F}}_{\lambda,\mu})$ is a module over the $N=2$ superconformal algebra with central charge $3-6\lambda$.

Proof. Notice that the action of the element L_n on $W^{\frac{\infty}{2}+*}(\boldsymbol{\mathcal{F}}_{\lambda,\mu})$ is given by the quadratic expansion

$$\theta(L_n)=\Sigma_m(-m+n+\mu-n\lambda+\lambda)(:\tau(v_m)\varepsilon(v'_{m-n}):+:\beta(v_m)\gamma(v'_{m-n}):). \tag{4.4}$$

Let us define the action of the $N=2$ superconformal algebra on $W^{\frac{\infty}{2}+*}(\boldsymbol{\mathcal{F}}_{\lambda,\mu})$ using the following quadratic expansions:

$$\theta(H_n)=\lambda\Sigma_m:\tau(v_m)\varepsilon(v'_{m+n}):+(\lambda-1)\Sigma_m:\beta(v_m)\gamma(v'_{m+n}):+\mu\delta_{n,0}, \tag{4.5}$$

$$\theta(\mathsf{h}_n)=\Sigma_m\gamma(v'_{m+n})\tau(v_m), \tag{4.6}$$

$$\theta(\mathsf{p}_n)=\Sigma_m(m-\mu-(n+1)\lambda)\beta(v_{m-n})\varepsilon(v'_m), \tag{4.7}$$

$$\theta(\mathfrak{L}_n)=-\theta(L_{-n})+\frac{n+1}{2}\theta(H_n). \tag{4.8}$$

One can check that θ is a representation of the $N=2$ superconformal algebra, and calculate the central charge checking the commutation relations on the vacuum vector $\mathbb{1} = \mathbb{1}_S \otimes \mathbb{1}_\Lambda$. For example, let $n > 0$. Then

$$\begin{aligned}
&\theta([H_n, H_{-n}])\mathbb{1} = -\theta(H_{-n})\theta(H_n)\mathbb{1} = \\
&-\theta(H_{-n})(\lambda\Sigma^0_{m=1-n}\tau(v_m)\varepsilon(v'_{m+n}) + (\lambda-1)\Sigma^0_{m=1-n}\beta(v_m)\gamma(v'_{m+n}))\mathbb{1} = \\
&-\lambda^2\Sigma^0_{m=1-n}\tau(v_{m+n})\varepsilon(v'_m)\tau(v_m)\varepsilon(v'_{m+n})\mathbb{1} - \\
&(\lambda-1)^2\Sigma^0_{m=1-n}\beta(v_{m+n})\gamma(v'_m)\beta(v_m)\gamma(v'_{m+n})\mathbb{1} = \\
&(-\lambda^2 n - (\lambda-1)^2(-n))\mathbb{1} = n(1-2\lambda)\mathbb{1},
\end{aligned} \tag{4.9}$$

since $\varepsilon(v'_i)\tau(v_i) + \tau(v_i)\varepsilon(v'_i) = 1$, and $\gamma(v'_i)\beta(v_i) - \beta(v_i)\gamma(v'_i) = 1$. Hence,

$$\theta([H_n, H_m])\mathbb{1} = n(1-2\lambda)\delta_{n,-m}\mathbb{1}. \tag{4.10}$$

Thus the central charge is $3-6\lambda$. The other commutation relations on the vacuum vector $\mathbb{1}$ are calculated in the same way.

∎

Corollary 4.1. In the case when $\lambda = -1, \mu = 1$, the module $\boldsymbol{\mathcal{F}}_{\lambda,\mu}$ is the adjoint representation of L. The corresponding central charge is 9 (cf. [G]).

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