SPACE OF HERMITIAN TRIPLES AS AN ANALOGY OF HERMITIAN CONNECTION SPACE

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Abstract. In this paper we continue to study spaces of hermitian triples. A generalized version of Kaehler — Einstein equation is derived as a natural equation on the space which is invariant under the $Diff X$- group action. One can construct a set of solutions to the generalized equation in a special situation using solutions of Seiberg — Witten system. At the end we illustrate our main thesis (claimed in the title of this paper), generalizing the Ashtekar - Isham quantization construction over the space of hermitian triples. For this one has to exploit special 2- dimensional knots instead of "classical" special 1- dimensional knots.

Introduction

This paper contains two examples which illustrate its title. But this one doesn't reflect an important feature belongs to the subject, growing around the spaces of hermitian triples. We are not so brave to claim directly in the title that in this new subject one considers the group of all diffeomorphisms of the based 4 - manifold as a gauge group. Thus the imagenary paralell, described in our two expamles, should contain all ingredients. So let us dilute the thesis claimed in the title adding details as follows. In a "classical" gauge theory one has the hermitian connection space on an appropriate complex G - bundle E over an appropriate based manifold X as the configuration space (where G as usual is $SU(n)$ or $U(n)$). On the configuration space, usualy denoted as $\mathcal{A}_h(E)$, one has the natural action of the corresponding gauge group $\mathcal{G}$ which consists of all maps

$$X \to G,$$

so every $g \in \mathcal{G}$ restricting to a point $x \in X$ gives an automorphism of the corresponding fiber E_x. And at the next step one considers the usual quotient

$$\mathcal{B}(E) = \mathcal{A}_h(E)/\mathcal{G}.$$

This quotient space should be called the phase space of the theory. The factorization is stipulated by the principle of relativity. It means that one can "observe" only gauge classes of connections and cann't distinguish two connections from the same class. This fixes the following condition for the equations to be meaningfull in the theory: each equation should be invariant under the gauge group action. This strong condition outlines a circle of physically meaningfull equations and as we now the number of such equations is small enough.

Now turn to a different subject. Let's consider the space of hermitian triples $\mathcal{M}_X$ over a based 4 - dimesnional manifold X. As well as in the previous "classical" case one has discrete parametrization of all components. While for $\mathcal{A}_h(E)$ we have the corresponding topological type of E in the present case we can characterize a component by the corresponding canonical class, so

$$\mathcal{M}^{K_i} = \{(g, J, \omega) \in \mathcal{M}_X \quad | \quad K_J = K_i \in H^2(X, \mathbb{Z})\}$$

(see [5]). So we take an appropriate component $\mathcal{M}^K$. On this component we have the natural action of the group $Diff X$ consists of all diffeomorphisms of X. Of course one could say that the component $\mathcal{M}^K$ is invariant only under the action of $Diff^0 X \subset Diff X$, containing such diffeomorphisms which generate identical transformation of the integer lattice $H^2(X, \mathbb{Z})$. But really the question of invariance can be reduced to the problem of gauge fixing in this case. We have here three levels of gauge fixing. The first one (which should be called "topological") corresponds to a fixed realization of the integer lattice $H^2(X, \mathbb{Z})$. If one didn't choose an appropriate realization then one cann't compaire two classes K_1 and K_2 over two different (but diffeomorphic!) 4 - manifolds X_1 and X_2. The second level gauge fixing corresponds to an appropriate choice of smooth atlas for any given smooth structure. Really by the definition a smooth structure on a based manifold X is a class of equivalent smooth atlases, and from this point of view any choice of smooth atlas corresponds to a gauge fixing. It remains to mention that for any atlas one has the group of automorphisms of this atlas. Just outlining the problem we say here that in this setup one gets the third (and the most definite) level of gauge fixing.

Thus one can see that the component $\mathcal{M}^K$ is invariant under the action of $Diff X$ so the natural question arises: to find over $\mathcal{M}^K$ "gauge invariant" equations. For the "phase space" $\mathcal{M}^K / Diff X$ every such equation (or an approrpiate system of equations) can be considered as a candidate to be a "physical low" in a "gauge theory". Really a number of such equations is well known, but as usual each such equation uses only one "variable" from three possibilities. For example, the condition for the second element J of hermitian triple (g, J, ω) to be integrable is a well defined "equation" on the quotient space $\mathcal{M}^K / Diff X$. But it doesn't require anything for the first and the third elements of the triple. In Subsection 1.2 we present a "gauge invariant" equation which uses all the variables. This equation is a generalization of the celebrated Kaehler — Einstein equation. The statement itself should be not so interesting if it were not connected with some other geometrical subjects. The point is that in some special situation we can construct the solutions from solutions of the Sieberg — Witten system of equations. The last one is a basic ingredient for a "classical" gauge theory which is (moreover) equivalent to the Donaldson theory. Thus one gets a realtionship between classical gauge theories and this imaginery "gauge" theory which isn't rigourisly defined or constructed yet.

Another aspect of the similarity manifested in the title is presented in Section 2. At the beginig of this Section we recall fundamental constructions of Ashtekar — Isham quantization of classical gauge theories. Over a 3 - dimensional manifold M one takes the set of framed knots γ_i each of them considered as a map from the phase space $\mathcal{B}_h(E)$ to G. Taking the trace in the fundamental representation one can regard these knots as real functions on the quotient space $\mathcal{B}_h(E)$. In the particular

$SU(2)$ - case one gets a C^* - algerba and its Gel'fand spectrum is defined as a compactification of $\mathcal{B}_h(E)$. Over this compactified space one considers a measure $d\mu$ which makes it possible to define the corresponding Hilbert space $L^2(\overline{\mathcal{B}}_h(E), d\mu)$. Moreover, we are interested in diffeomorphism invariant measures which give us some invariants of the framed knots ([1]).

The basic fact for all the constructions is that the space of connection is an affine space with the tangent space equals to the space of real 1 - forms, twisted by endomorphisms of E. The point is that one can integrate any 1 - form over any loop, representing a knot. Turning to the case of hermitian triple space one can use the fact that the tangent space to $\mathcal{M}^K$ is isomorphic to the space of real 2 - forms. It's the reason why 2 - dimensional knots are introduced to the picture. But the generalization for hermitian triple space is in some sense abelian since these new 2 - dimensional "Wilson loops" are weighted by elements of $U(1)$. In Subsection 2.4 we outline a pissible direct relationship between $SU(2)$ - connections over 3 - dimensional M and hermitian triples over some induced 4 - dimensional X. This simple and natural construction is based on so called "pumping up trick" (see, f.e., [4]) well known in spin network setup. This could give a way transporting the story to the framework of spin networks but we have not here niether place no time to develop this approach for the hermitian triple spaces, leaving this theme to the time, when such kind of generalizations will be meaningfull and usefull. Here in Subsection 2.3 we construct an ingredient, which corresponds to an invariant of 2 - dimensional knots. Let us recall (see [1]) that in prehistory every diff - invariant measure on the compactification $\overline{\mathcal{B}}_h(E)$ gives an invariant of 1 - dimensional knots. So the main part of the programme is dedicated to construct such a "faithfull" measure. In the case of hermitian triples one can construct such a measure almost immediately. This measure arises from real period map. Note, that the procedure which was quite appropriate in the "classical" case using cilindrical functions and so on (see [1]) looks as sufficiently abstract one applying to the case of hermitian triples. Nevertheless we have an example of diff - invariant measure.

We have to mention here what is the nearest perspective of the subject. Another usefull tool in the gauge theories comes from symplectic geometry. Symplectic geometry allows to move in quite different ways and to prove quite different things. Two examples show the width of its applications. The first one is the celebrated Taubes result which says that for any symplectic 4 - manifold the corresponding Seiberg — Witten invariant is nontrivial. This result is an example what one can derive for a symplectic object, applying a gauge theory. On the other hand we have a good example showing that an intrinsic for a gauge theory object can support some symplectic properties. We mean here the well known fact that the space of hermitian connections over a 4 - dimensional manifold is a symplectic space and (following S. Donaldson) one can see that the map "connection — curvature" can be regarded as a moment map whith respect to the gauge group action. So one could expect that an appropriate symplectization of hermitian triple space subject exists. Now we have some ideas which would lead to a universal construction of symplectic structures on i) the space of all Kaehler metrics from a fixed Kaehler class over a Kaehler surface and ii) the space of all almost complex structures compatible with a gvien symplectic form over a symplectic 4 - manifold. Leaving all the details outside of the present discussion we can say now only two words: this construction comes from consideration of the hermitian triple space.

Section 1. Solutions of generalized Kaehler — Einstien equation

1.1. Hermitian triple space: the canonical map.

First of all let us recall briefly what is the main object which we study during this paper. Let X be a smooth compact orientable riemannian 4 - manifold. Consider the space of all hermitian triples consists of all possible (g, J, ω) where g is a riemannian metric, J is a compatible almost complex structure and ω is the corresponding 2-form. All details of the definition together with all compatibility conditions can be found in [5], but we have to emphecise here that we don't require for the almost complex structures to be integrable and for 2 - forms which are often called below as "almost Kaehler forms" to be closed. The space of all such triples is called as hermitian triple space and is denoted as $\mathcal{M}_X$. One can divide the space into the union of disjoint components and it was proved in [5] that every such comoponent corresponds to a choice of canonical class $K \in \mathcal{K}_X \subset H^2(X, \mathbb{Z})$ so as

$$\mathcal{M}^K = \{(g, J, \omega) \in \mathcal{M}_X \quad | \quad K_J = K\}. \tag{1.1}$$

To the selfconnectedness of the present paper we recall here the definition of the canonical map τ. This is a natural map from $\mathcal{M}^K$ to $H^2(X, \mathbb{R})$, defined as follows. For each element $(g, J, \omega) \in \mathcal{M}^K$ consider the corresponding Hodge star operator $*_g$ and the orientation induced by J. Then by a famous Hodge theorem one can decompose any form into the direct sum of harmonic part, exact part and co- exact part. Let us decompose the third element in the triple due to the theorem

$$\omega = \omega_H + d\rho_1 + d^*\rho_2. \tag{1.2}$$

By another famous Hodge theorem the harmonic 2 - form ω_H represents a 2 - cohomology class over X with coefficients in $\mathbb{R}$ so

$$\tau(g, J, \omega) = [\omega_H] \in H^2(X, \mathbb{R}). \tag{1.3}$$

We have seen ([5]) that moreover $\tau(g, J, \omega)$ either has positive square or equals to zero. Here we'd like to add to the list of properties of τ from [5] the following

Proposition 1.1. *The canonical map τ is a contineous map with respect to three kinds of deformations.*

Recall that in [5] we've established that any pair of points

$$(g_1, J_1, \omega_1), (g_2, J_2, \omega_2) \in \mathcal{M}^{K_i}$$

can be joined by a "partially linear" path. It means that using three kinds of deformations:

(1). deformation of riemannian metric inside the conformal class with fixed almost complex structure;

(2). deformation of riemannian metric preserving the volume form with fixed almost complex strucutre;

(3). deformation of almost complex structure with fixed riemannian metric

— one can pass from the first point to the second. "Junction" Lemma ([5]) provides such a "partially linear" path.

Now to prove the statement we use the following schema: for each kind of the deformations listed above we can show that the canonical map τ is a contineous map.

First of all let us remark that the proof of such contineouty for the types 1 and 3 is quite obvious. In these two cases we have unchanged framework — given conformal class $*_g$ with respect to that one constructs the images of the canonical maps. For any pair (g_1, J_1, ω_1) and (g_2, J_2, ω_2) such that

either $g_1 = g_2$ and J_i are arbitrary almost complex strcuture

or $*_{g_1} = *_{g_2}$ with $J_1 = J_2$, so $\omega_1 = f \cdot \omega_2$ where f is a nonvanishing positive function.

we have the following ruling inequality

$$\int_X (\omega_1 - \omega_2) \wedge (\omega_1 - \omega_2) \geq \int_X ((\omega_1)_H - (\omega_2)_H) \wedge ((\omega_1)_H - (\omega_2)_H). \tag{1.4}$$

Really, the conformal class is fixed, so due to the harmonicity of the difference $((\omega_1)_H - (\omega_2)_H)$ we get the following identity

$$\int_X (\omega_1 - \omega_2)^{\Lambda 2} = \int_X ((\omega_1)_H - (\omega_2)_H)^{\Lambda 2} + \int_X (d^+(\rho_1 - \rho_2))^{\Lambda 2}, \tag{1.5}$$

where as usual

$$\omega_i = (\omega_i)_H + \tfrac{1}{2}(d\rho_i + d^*(*\rho_i)) = (\omega_i)_H + d^+\rho_i$$

(see the definition of τ above), and the second term in (1.5) is nonnegative and it implies the inequality (1.4). Further, since all the forms containing in (1.4) are self dual with respect to $*_{g_1}$ the left and the right sides of (1.4) represent positive nondegenerated distance functions on the space of 1st and 3d type deformations and on the space of self dual harmonic forms respectively. With respect to these distance functions the canonical map τ is a contineous map due to inequality (1.4).

The case which remains is much more delicate. Really for any deformation of the 2nd type a priori we haven't any preffered distance functions even on the 2-cohomology space which is the target space of τ. There we have just the intersection form Q but it doesn't play the role of an appropriate distance because it admits isotropical vectors. But let us start with the following remark, which endows us with a natural distance function on the space of deformations:

Lemma 1.1.1. *Let (g_1, J, ω_1) and (g_2, J, ω_2) are two points from $\mathcal{M}^{K_i}$ with the same almost complex structure and the same volume form*

$$\omega_1 \wedge \omega_1 \equiv \omega_2 \wedge \omega_2 = 2d\mu.$$

Then the following pointwise identity holds

$$|\omega_1 - \omega_2|^2_{g_1} \equiv |\omega_1 - \omega_2|^2_{g_2} \tag{1.6}$$

and the expressions vanishe iff $g_1 = g_2$ so the points coincide.

Really for every (1,1)- form $\alpha \in \Omega^{1,1}$ we have the following poinwise identiy

$$|\alpha|^2_{g_i} = 2(\frac{\alpha \wedge \omega_i}{\sqrt{2} d\mu})^2 - \frac{\alpha \wedge \alpha}{d\mu}. \tag{1.7}$$

Usual decomposition

$$\Omega^{1,1} = \mathbb{R} < \omega_i > \oplus \Omega^-_{g_i}$$

easily gives us formula (1.7). Substituting $\alpha = \omega_1 - \omega_2$ to (1.7) and comparing the expressions for g_1 and g_2 one gets the statement of Lemma 1.1.1.

Thus it's quite natural to define δ- neighbourhood of a point (g, J, ω) in the framework of the 2d type deformation as

$$O_\delta(g_0, J, \omega_0) = \{(g, J, \omega) \in \mathcal{M}^{K_i} \quad | \quad max_{x \in X} |\omega - \omega_0|^2_g < \delta\}. \tag{1.8}$$

To define an appropriate topology on the target space $H^+ \subset H^2(X, \mathbb{R})$ let us realize it as the subset of harmonic w.r.t. g_0 forms with positive squares. So we would like to use this basic riemannian metric g_0 to endow the space of deformations and the target space with appropriate topologies. We consider the space of deformations around (g_0, J, ω_0), defined above, with the L_0 - norm with respect to g_0 described in 1.8. As well we realize $H^2(X, \mathbb{R})$ as the space of harmonic with respect to g_0 2-forms $\mathcal{H}_{g_0}$ with natural distance function

$$\int_X |\alpha - \beta|^2_{g_0}, \tag{1.9}$$

where $\alpha, \beta \in \mathcal{H}^+_{g_0}$. We prove that for any positive $\epsilon \in \mathbb{R}_+$ there exists $\delta \in \mathbb{R}_+$ such that

$$max_{x \in X} |\omega_0 - \omega_1|^2_{g_0} < \delta \implies \int_X |(\omega_0)_{H_0} - (\omega'_1)_{H_0}|^2_{g_0} < \epsilon \tag{1.10}$$

where $(\omega'_1)_{H_0}$ is a harmonic with respect to g_0 2- form which represents the cohomological class $\tau(g_1, J, \omega_1)$. Let us recall that with respect to g_1 this class is represented by the corresponding harmonic component so

$$[(\omega_1)_{H_1}] = [(\omega'_1)_{H_0}] \in H^2(X, \mathbb{R}). \tag{1.11}$$

So our problem is as follows. Let $max_{x \in X} |\omega_0 - \omega_1|^2_{g_0} = M$. We would like to estimate the expression

$$\int_X |(\omega_0)_{H_0} - (\omega'_1)_{H_0}|^2_{g_0} = N \tag{1.12}$$

in terms of M. The first step is obvious: let us note that

$$N \leq \int_X |(\omega_0)_{H_0} - (\omega_1)_{H_0}|^2_{g_0} + \int_X |(\omega_1)_{H_0} - (\omega_1')_{H_0}|^2_{g_0}. \tag{1.13}$$

The first integral is easily bounded by $M \cdot VolX$. So our aim is to work with the second one.

Due to equality (1.11) we have

$$(\omega_1)_{H_1} = (\omega_1')_{H_0} + d\sigma, \tag{1.14}$$

where σ is a real 1-form. So the usual equality

$$\omega_1 = (\omega_1)_{H_1} + \tfrac{1}{2}(d\rho_1 + *_1 d\rho_1)$$

can be rearranged as follows

$$\omega_1 = (\omega_1')_{H_0} + d\sigma + \tfrac{1}{2}(d\rho_1 + *_1 d\rho_1). \tag{1.15}$$

But one can decompose the same 2- form with respect to g_0 getting

$$\omega_1 = (\omega_1)_{H_0} + d\rho_0 + d^{*_0}\rho_0'. \tag{1.16}$$

Since ω_1 is not a self dual form with respect to g_0 now ρ_0' doesn't depend on ρ_0. To compaire two decompositions (1.15) and (1.16) one has to express $*_1$ containing in (1.15) in terms of $*_0$. It's not hard to see that for any 2-form α we have the following equality:

Lemma 1.1.2. *For every 2- form α the following equality holds*

$$*_1\alpha - (\frac{\alpha \wedge \omega_1}{d\mu})\omega_1 = *_0\alpha - (\frac{\alpha \wedge \omega_0}{d\mu})\omega_0. \tag{1.17}$$

If we have two riemannian metric g_0, g_1 with the same volume form $d\mu$ and compatible with given almost complex structure J then one can compute $*\alpha$ as follows: first of all one divides α with respect to J

$$\alpha = (\alpha)^{1,1} + (\alpha)^{2,0} \tag{1.18}$$

(we denote as $(\alpha)^{2,0}$ the component belongs to $(\Omega^{2,0} \oplus \Omega^{0,2})_{\mathbb{R}}$ for breivity). Due to the well known facts the summands in (1.18) are pointwise orthogonal with respect to wedge product and *any* riemannian metric, compatible with J. The second summand is self dual everywhen. Separating self dual and anti self dual parts of the first summand with respect to g_i we get

$$\begin{aligned} ((\alpha)^{1,1})^+ &= \tfrac{1}{2} < \alpha, \omega_i >_i \cdot\omega_i \\ ((\alpha)^{1,1})^- &= (\alpha)^{1,1} - \tfrac{1}{2} < \alpha, \omega_i >_i \cdot\omega_i \end{aligned} \tag{1.19}$$

since ω_i has constant lenth $\sqrt{2}$ with respect to g_i. Thus we get

$$\begin{aligned} *_i\alpha =&(\alpha)^{2,0} + \tfrac{1}{2} < \alpha, \omega_i >_i \cdot \omega_i - ((\alpha)^{1,1} - \tfrac{1}{2} < \alpha, \omega_i >_i \cdot \omega_i) \\ &= (\alpha)^{2,0} - (\alpha)^{1,1} + < \alpha, \omega_i >_i \cdot \omega_i. \end{aligned} \tag{1.20}$$

Due to self duality of ω_i we can rearrange expression (1.20) forgetting about the metrics

$$*_i\alpha = (\alpha)^{2,0} - (\alpha)^{1,1} + \frac{\alpha \wedge \omega_i}{d\mu} \cdot \omega_i. \tag{1.21}$$

This gives us the lemma since the first and the second summands on the right side depend only on our fixed almost complex strucutre J.

This equality provides us with the relationship between the Hodge star operators and applying it to (1.15) we get

$$\omega_1 = (\omega_1')_{H_0} + d\sigma + \tfrac{1}{2}(d\rho_1 + *_0 d\rho_1) - \tfrac{1}{2}(\frac{d\rho_1 \wedge \omega_0}{d\mu})\omega_0 + \tfrac{1}{2}(\frac{d\rho_1 \wedge \omega_1}{d\mu})\omega_1. \tag{1.22}$$

But there is just unique decomposition of such type with respect to g_0 and thus we get the following identity

$$(\omega_1)_{H_0} - (\omega_1')_{H_0} = \{(\frac{d\rho_1 \wedge \omega_1}{2d\mu})\omega_1 - (\frac{d\rho_1 \wedge \omega_0}{2d\mu})\omega_0\}_{H_0} \tag{1.23}$$

where as usual simbol H_0 shows that we take the corresponding harmonic projection with respect to g_0. Now taking the corresponding integral we get from (1.23) that

$$\int_X |(\omega_1)_{H_0} - (\omega_1')_{H_0}|^2_{g_0} d\mu \leq \int_X |(\frac{d\rho_1 \wedge \omega_1}{2d\mu})\omega_1 - (\frac{d\rho_1 \wedge \omega_0}{2d\mu})\omega_0|^2_{g_0} d\mu.$$

Now we can bound the right hand side (which includes nothing more then the difference between two projections of the same element $d\rho_1$) by the term of the shape

$$O(||\omega_1 - \omega_0||^2_{g_0}) \cdot ||d\rho_1||^2_{g_0}. \tag{1.24}$$

To complete the proof we have two add the following two remarks. Firstly, due to L_0- closedness of ω_0 and ω_1 there are exist two real constant C_1 and C_2 such that for any real 2- form α one has

$$C_1|\alpha|^2_{g_0} \leq |\alpha|^2_{g_1} \leq C_2|\alpha|^2_{g_0}.$$

So we can correct bound (1.24) as

$$O(||\omega_1 - \omega_0||^2_{g_0}) \cdot \frac{1}{C_1}||d\rho_1||^2_{g_1} \tag{1.25}$$

and, secondly, from the basic decomposition (1.2) it is clear that the last norm is less or equal to $2VolX$. Combaining together all steps we get a constant C such that $N = C \cdot M$.

1.2. $Diff X$ **- invariant equations.**

Consider now a fixed component $\mathcal{M}^K$. One has the natural action of $Diff X$ on the component. Firstly we want to establish that the canonical map τ is invariant under the action. To do this let us take an element $\psi \in Diff X$ such that

$$\psi : X \to X_1 \tag{1.26}$$

where X_1 is another smooth realization. Then

$$\begin{aligned} \psi &: g \mapsto g_1 \\ \psi &: J \mapsto J_1 \\ \psi &: \omega \mapsto \omega_1 \end{aligned} \tag{1.27}$$

and the triple (g_1, J_1, ω_1) belongs to the component $\mathcal{M}^{K_1}$ over X_1 where $K_1 = \psi_* K$. From the gauge action point of view we cann't distinguish K and K_1 before some additional choice takes place so in our general situation without any additional specification we understand K and K_1 as the same class. More rigourisly the induced isomorphism

$$\psi_* H^2(X, \mathbb{Z}) \to H^2(X_1, \mathbb{Z})$$

identifies two integer lattices so K equals to K_1 (a tautology). Thus we can compare two images $\tau(g, J, \omega) \in H^2(X, \mathbb{R})$ and $\tau(g_1, J_1, \omega_1) \in H^2(X_1, \mathbb{R})$ and one gets

Proposition 1.2. *The canonical map* τ *is invariant under* $Diff X$ *- action.*

To prove it it's sufficient to mention that obviously

$$\psi_* \tau(g, J, \omega) = \tau(g_1, J_1, \omega_1) \tag{1.28}$$

and the last equality is equaivalent to the invariance.

Having this fact in mind we can define a $Diff X$ - invariant equation as follows. The point is that since $\tau(g, J, \omega)$ lies in $H^2(X, \mathbb{R})$ we can take any 2- cohomological class and consider

$$\tau(g, J, \omega) = h \in H^2(X, \mathbb{R}).$$

But this class h has to vary with respect to diffeomorphisms in the same manner as $\tau(g, J, \omega)$. So the simplest version is when h is the trivial one. Then one gets

$$\tau(g, J, \omega) = 0. \tag{1.29}$$

This $Diff X$ - invariant equation can be regarded as a generalization of the following known situation. Let for a solution (g, J, ω) the second element satisfies the integrability condition. Then the equation (1.29) reads as

$$\omega = d^+ \rho = (\partial \overline{\partial} \phi)^+,$$

where ϕ is a real smooth function. This leads to considerations very similar to the subject of subharmonic, plurisubharmonic and etc. functions.

The next possibility, arises from the invariance, is the following one

$$\tau(g, J, \omega) = c \cdot K \tag{1.30}$$

where c is a constant and $K = K_J$ is the canonical class corresponds to the component $\mathcal{M}^K$. This equation is $Diff X$ - invariant. The point is that the classes $\tau(g, J, \omega)$ and K vary in the same manner (it was discussed above). It remains to mention here the role of the constant c. We include it to the equation just for more familiarity of the constructed equation. Really one can forget about this constant because of the linearity of τ (see [5]) so equation (1.30) is equivalent to the following equation

$$\tau(g, J, \omega) = K. \tag{1.31}$$

We call it "generalized Kaehler — Einstein equation". The reason is an obvious one: if we additionally require for J to be integrable and for ω to be closed, then the equation (1.31) equiped with these two conditions coincides with the celebrated Kaehler — Einstein equation. In the framework of Kaehler geometry this equation is equivalent to constant scalar curvature condition for the corresponding Kaehler metric. It is the reason why the Kaehler — Einstein equation was introduced and studied and remains to be interesting and important. But now when we remove the conditions for J and ω such generalization looks like an unreasonable one. We intoduce this generalization having in mind the following remarkable relationship of the subject with the Seiberg — Witten theory. In the rest of Section 1 we will prove the following

Main Proposition. *Let K is a basic canonical class over X with positive self intersection $K^2 > 0$. It means that the Seiberg — Witten invariant of K is nontrivial. If K can be realised by self dual harmonic 2 - form with respect to a riemannian metric g then equation (1.31) has solutions.*

In other words we will construct a one - way correspondence between solutions of Seiberg — Witten system of equations and the "generalized Kaehler — Einstein" equation. Projecting the statement of Main Proposition to the subspace of Kaehler geometry we don't get any more then some usual and unsurprising results. Namely, if one has a Kaehler triple g, J, ω over X, being a complex surface, such that J is integrable, ω is closed then the Seiberg — Witten invariant of K is nontrivial automatically and the condition that the harmonic form, representing the canonical class K, is self dual means that K is proportional to $[\omega]$. But when we remove the Kaehler conditions lifting to the space of general almost complex geometry the statement of Main Proposition means that two equations (Kaehler — Einstein and Seiberg — Witten) vary in parallel ways. Even in symplectic setup this correspondence should present some additional properties .

This parallelism makes this generalization of Kaehler — Einstein equation defined by (1.31) such interesting for us. Returning to the question of $Diff X$ - invariant equations one could mention a number of such ones. But the number of well known such kind equations can be divided into two parts. The first one consists of the equations which have as the variable only riemannian metrics. The constant scalar curvature condition is an example of such type equation. The second one contains for example the integrability condition imposed on the second element in the triples. Of course there are exist other equations of "mixed" type but here we'd like to restrict our attention to the equation (1.31). In what follows we will establish how one can solve the equation using one - way correspondence mentioned above.

1.3. Hermitian triples and Seiberg — Witten equation.

Let X be as required above and g is a riemannian metric. For a $Spin^{\mathbb{C}}$- structure $c \in H^2(X, \mathbb{Z}), c \equiv w_2(X)(mod 2)$ one can consider the Seiberg — Witten system of equations:

$$\begin{aligned} &D_a(\phi) = 0 \\ &F_a^+ = -(\phi \otimes \bar{\phi})_0 \end{aligned} \tag{1.32}$$

where $\phi \in W^+$ is a spinor field of positive kirality and $a \in \mathcal{A}(det W^+)$ is an abelian hermitian connection.

If c is a canonical class

$$c = K \in \mathcal{K}_X \tag{1.33}$$

we know that the spinor bundle W^+ admits nonvanishing sections and moreover one can represent almost complex structures with canonical class K by such nonvanishing sections (see [5]).

We claim that any irreducible solution of the system (1.32) induces a set of hermitian triples from $\mathcal{M}^K$ with nontrivial images

$$\tau(g, J, \omega) \neq [0].$$

Namely one has

Proposition 1.3. *Let X is as required above, g is a generic riemannian metric s.t. there are no reducible solutions of (1.32) for $Spin^{\mathbb{C}}$- structure $c = K \in \mathcal{K}_X$. Then if there exists a solution (irreducible, of course) of the system then there exists a hermitian triple $(g_0, J_0, \omega_0) \in \mathcal{M}^K$ with nontrivial image.*

Proof. Consider first of all the case when the given spinor field ϕ, contains in the given irreducible solution (a, ϕ) is a nonvanishing section of W^+. Then we can construct a triple with nontrivial image almost immediately. Really for the riemannian metric g we have a real self dual nonvanishing 2- form

$$\omega_\phi = 2\pi\imath(\phi \otimes \bar{\phi})_0. \tag{1.34}$$

This form is not an almost kahler form for the given riemannian metric because it hasn't a priori length $\sqrt{2}$ at each point of X. But we can take another riemannian metric g_0 in the same conformal class $*_g$ such that the form ω_ϕ is an appropriate for g_0 to be almost kahler. Then we can reconstruct the second element in the corresponding hermitian triple getting

$$(g_0, J_0, \omega_0 = \omega_\phi) \in \mathcal{M}^K.$$

For the third element ω_0 we have the following representation

$$\omega_0 = \frac{\imath}{2\pi} F_a^+. \tag{1.35}$$

But $\frac{\imath}{2\pi}F_a$ is a real closed 2 - form, representing class K. Let us decompose this form with respect to the Hodge star operator $*_g$:

$$\frac{\imath}{2\pi}F_a = \alpha_K + d\rho_1, \tag{1.36}$$

where α_K is a harmonic 2 - form which represents class K. Now take the self dual part

$$\frac{\imath}{2\pi}F_a^+ = (\alpha_K + d\rho_1)^+ = \alpha_K^+ + \tfrac{1}{2}(d\rho_1 + d^*(*\rho_1))$$

where α_K^+ is self dual *harmonic* 2 - form. This form has to be nontrivial because there are no reducible solutions for the system (1.32) with respect to the given riemannian metric g. Thus we get that the Hodge decomposition for ω_0 has the form

$$\omega_0 = \alpha_K^+ + \tfrac{1}{2}d\rho_1 + \tfrac{1}{2}d^*(*\rho_1)$$

and the harmonic component has to be nontrivial. But the image of (g_0, J_0, ω_0) is by the definition equals to

$$\tau(g_0, J_0, \omega_0) = [\alpha_K^+] \neq [0] \subset H^2(X, \mathbb{R}) \tag{1.37}$$

so it has to be nontrivial to.

So it remains just one difficulty — the spinor field ϕ can vanish on a submanifold $Z_1 \cup ... \cup Z_p = Z \subset X$, where Z_i are irreducible smooth components. To avoid this difficulty let us remark first of all that for every i we have

$$dim Z_i < 3.$$

To see this let us exploit the first equation in the system (1.32). It reflects harmonicity of the spinor field. In other words, let Z_i has dimension 3. Being zeroset of a smooth section of W^+ it has to be homologically trivial. So one has a 4-dimensional submanifold $B \subset X$ such that

$$\partial B = Z_i.$$

Then we can restrict the first equation $D_a(\phi) = 0$ to this 4- dimensional domain B. The restriction is an elliptic differential equation of order 1 and applying the maximum principle one gets that ϕ has to be identically zero on B. But it's impossible.

Thus all the components Z_i have dimension at most 2 and we can arrange the following family of nonvanishing sections

$$\phi_\delta \in \Gamma(W^+)$$

where $\delta \in \mathbb{R}_+$ is a positive real parametr such that the difference $\phi - \phi_\delta$ has the norm

$$\int_X |\phi - \phi_\delta|^2 d\mu < \delta.$$

Remark, that the spinor bundle W^+ is topologically equivalent to $I \oplus K$, where I is the trivial complex line bundle and K is the line bundle with the first Chern class K. Let $e \in H_2(X, \mathbb{Z})$ is Poincare dual to the cohomological class K. Then we have two cases. The first one is when the 2- dimensional component of Z is homologious to e. Then one can choose two section ϕ_1 and ϕ_2 such that

ϕ_1 hasn't zeros (so trivializes summand I) and $\phi = \lambda\phi_1$, where ϕ_1 trivialize summand K and λ is a real valued smooth function.

Then one can take

$$\phi_\delta = \phi + \delta \cdot C_1 \phi_1,$$

where C_1 is a constant which can be easily computed.

If Z is homologically trivial we can repeat the procedure with a small modification.

Thus we get ϕ_δ which has no zeros and sufficiently near to ϕ. Therefore one can decompose the resulting self dual nonvanishing 2 - form $\omega_1 = 2\pi\imath(\phi_\delta \otimes \overline{\phi}_\delta)_0$ as the sum

$$\omega_1 = \omega_0 + \omega_\delta,$$

where ω_δ vanishes over all X exept small tubular δ - neibourhood of Z. Now we have a freedom to choose this deformation from ω_δ such that the harmonic part of ω_δ would be trivial. Then the image of ω_1 has to be the same as the image of ω_0 (note, that these images depend only on conformal class of given g so we can speak about these images without additional references to concrete riemannian metrics from the conformal class). And we can repeat the scaling procedure getting the triple (g_1, J_1, ω_1) with image in $[\alpha_K^+]$. As well one can remark that this construction could give a set of such triples with the same images.

Coming back to the statement of Main Propostion let us repeat that any solution of the Seiberg — Witten system defines a number of hermitian triples with images in $[\alpha_K^+]$ with respect to given metric. Then to prove our Main Proposition it is sufficient to remark that

(1) if the invariant of K is nontrivial then for any riemannian metric a solution exists;

(2) to solve the generalized equation (1.31) it remains to take such riemannian metric that the canonical basic class is represented by self dual harmonic 2- form. Possibility of the choice was included into the statement of Main Proposition so the proof is completed.

2. Hermitian triples and the Ashtekar quantization

2.1. The Ashtekar — Isham programme for the space of connection.

In the present subsection we follow [1], preserving the notations. The reader familiar with the subject can directly turn to subsection 2.2.

Let Σ be a real smooth connected orientable 3 - manifold. Consider a principal $SU(2)$ - bundle $P \to \Sigma$ which has to be trivial by the topological arguments. As usual denote the space of smooth $SU(2)$ - connections as $\mathcal{A}$ and the corresponding gauge group $\mathcal{G}$. In the focus of the interest we have the space of equivalence classes $\mathcal{A}/\mathcal{G}$. While the space $\mathcal{A}$ is an affine space under the vector space $\Omega^1(adP)$ the quotient space $\mathcal{A}/\mathcal{G}$ doesn't admit a linear structure having sufficiently complicated topological structure.

Further, let us fix a point $x_0 \in \Sigma$ and consider the space of based loops. It means that we consider all contineous piesewise analityc parametrized paths

$$p : [0; s_1] \cup ... \cup [s_{n-1}; 1] \to \Sigma$$

such that $p(0) = p(1) = x_0$. Such set admits a group structure: namely for any two based loops p_1 and p_2 one defines

$$p_2 \circ p_1 = \{ \begin{matrix} p_1(2s) for s \in [0; \frac{1}{2}] \\ p_2(2s-1) for s \in [\frac{1}{2}; 1] \end{matrix};$$

and

$$p^{-1}(s) = p(1-s).$$

The space of based loops is denoted as $\mathcal{L}_{x_0}$.

For every connection $A \in \mathcal{A}$ and loop $\alpha \in \mathcal{L}_{x_0}$ one can define the corresponding holonomy $H(\alpha, A) \in SU(2)$. Define using this an equivalence relation on $\mathcal{L}_{x_0}$ —

two loops $\alpha, \beta \in \mathcal{L}_{x_0}$ are holonomically equivalent iff $H(\alpha, A) = H(\beta, A)$ for any $A \in \mathcal{A}$.

We call the equivalence classes as hoops; the collection of all hoops is denoted as $\mathcal{HG}$; this set inherits naturally the group structure thus we get the hoop group.

Now we can introduce the Ashtekar - Isham C^* - algebra. To every hoop $\tilde{\alpha} \in \mathcal{HG}$ we associate a real valued function $T_{\tilde{\alpha}}$ on the quotient space $\mathcal{A}/\mathcal{G}$, bounded between -1 and 1, namely

$$T_{\tilde{\alpha}}([A]) = \frac{1}{2} Tr H(\alpha, A),$$

where $A \in \mathcal{A}$ represents the equivalence class $[A] \in \mathcal{A}/\mathcal{G}$ and $\alpha \in \mathcal{L}_{x_0}$ represents the class $\tilde{\alpha} \in \mathcal{HG}$. The trace is taken in the fundamental representation of $SU(2)$. According to $SU(2)$ trace identities we have

$$T_{\tilde{\alpha}} \cdot T_{\tilde{\beta}} = \frac{1}{2}(T_{\tilde{\alpha}\cdot\tilde{\beta}} + T_{\tilde{\alpha}\cdot\tilde{\beta}^{-1}}),$$

thus one can construct the corresponding complex algebra, considering finite linear combinations of $T_{\tilde{\alpha}_i}$ weighted with complex numbers. It has the structure of a C^* - algebra. This one, denoted as $\mathcal{HA}$ is a subalgebra of the C^* - algebra of all complex valued bounded contineous functions on $\mathcal{A}/\mathcal{G}$. Taking the completion $\overline{\mathcal{HA}}$ of $\mathcal{HA}$ under the supremum norm we get a C^* - algebra, which is the Ashtekar - Isham C^* - algebra.

Then one can apply the Gelfand representation theory, getting the following results:

1) our completion $\overline{\mathcal{HA}}$ is naturally isomorphic to the C^* - algebra of complex valued contineous functions on a compact Hausdorff space which is the Gel'fand spectrum $sp(\overline{\mathcal{HA}})$;

2) the quotient space $\mathcal{A}/\mathcal{G}$ is densely embedded in the spectrum, thus we can take into account that the spectrum is a natural compactification of the quotient space which would be denoted as $\overline{\mathcal{A}/\mathcal{G}}$;

3) every contineous cyclic representation of $\overline{\mathcal{HA}}$ by bounded operators on a Hilbert space has to have as the Hilbert space the space $L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu)$ where $d\mu$

is a regular measure and the corresponding operators on the Hilbert space are just functions on $\overline{\mathcal{A}/\mathcal{G}}$, represented by elements $\overline{\mathcal{HA}}$, acting on the space by pointwise multiplication;

4) each of these representations is uniquely determined by a contineous positive linear functional $< . >$ on the C^* - algebra.

The last functional gives the corresponding function $\Gamma(\alpha)$ on the loop space $\mathcal{L}_{x_0}$ via the following obvious relation

$$\Gamma(\alpha) =< T_{\tilde{\alpha}} >$$

which has to satisfy some special properties namely:

$$\sum_i a_i T_{\tilde{\alpha}_i} = 0 \implies \sum_i a_i \Gamma(\alpha_i) = 0$$

and

$$\sum_{i,j} \bar{a}_i a_j (\Gamma(\alpha_i \circ \alpha_j) + \Gamma(\alpha_i \circ \alpha_j^{-1})) \geq 0.$$

Thus we have an interesting relationship between regular measures on the completed space $\overline{\mathcal{A}/\mathcal{G}}$ and specila functions on $\mathcal{HG}$. Now we can impose the diffeomorphism group action which acts on the algebra $\overline{\mathcal{HA}}$, its spectrum $\overline{\mathcal{A}/\mathcal{G}}$ and the space of hoops $\mathcal{HG}$. So the relationship mentioned above gives us the following fact: the measure $d\mu$ is invariant under this action if and only if the corresponding function $\Gamma(\tilde{\alpha})$ is. Any diffeomorphism invariant function is an invariant of generalized knots therefore one gets a one - to - one correspondence between generalized knots invariants and regular diffeomorphism invariant measures on the space of generalized gauge equivalent connections.

2.2. Repeating the construction.

Now we want to apply the construction discribed above to the case of the space $\mathcal{M}^{K_0}$ instead of the quotient space $\mathcal{A}/\mathcal{G}$.

First of all let X be as usual a compact orientable smooth riemannian 4- manifold and let K_0 is an appropriate canonical class $K_0 \in H^2(X, \mathbb{Z})$ over X. Choosing the class one fixes the corresponding orientation as usual in our setup. The first question we are going to discuss is the question of an appropriate analogy of the loop space in the previous setup.

Let s be a smooth polymap consists of n smooth maps $s = (s_1, ..., s_n)$

$$s_i : [0; 1] \times [0; 1] \to X$$

such that $s_i(0; t) = s_i(1; t)$ and $s_i(t; 0) = s_i(1; t) \forall t$. It's clear that the image of s is n copies of smooth embedded 2- tori. One can define a group structure on the set of all such $s \in \mathcal{S}$ by usual rules. Every polymap $s \in \mathcal{S}$ has the following value on a hermitian triple (g, J, ω): let Σ be the total image of $s = (s_1, ..., s_n)$ with the internal orientation, induced by the parametrization then

$$H(s, (g, J, \omega)) = e^{\imath(sign Vol_g \Sigma + \int_\Sigma \omega)}$$

where $sign$ is the sign of the orientation ($+1$ if the internal orientation of Σ coincides with the given on X and -1 otherwise). Using this value we can define an equivalence on the space of polymaps: we say that two polymaps s_1 and s_2 are equivalent if for any hermitian triple $(g, J, \omega) \in \mathcal{M}^{K_0}$ one has

$$H(s_1, (g, J, \omega)) = H(s_2, (g, J, \omega)).$$

Factorization by the equivalence gives us the space of special 2 - dimensional loops which we denote as $\tilde{\mathcal{S}}$.

The next step is to consider for every element $\tilde{s} \in \tilde{\mathcal{S}}$ the corresponding function $T_{\tilde{s}}$ on the space of hermitian triples. This function has the same definition as ?? but we would like to distinguish these two definitions regarding the element $H(s, (g, J, \omega))$ as an element of S^1 and $T_{\tilde{s}}(g, J, \omega)$ as a complex number. Thus for every $\tilde{s} \in \tilde{\mathcal{S}}$ the corresponding function $T_{\tilde{s}}$ is a bounded contineous complex valued function. It's easy to construct a commutative algebra $\mathcal{HS}$ using the obvious property

$$T_{\tilde{s_1} \cdot \tilde{s_2}} = T_{\tilde{s_1}} \cdot T_{\tilde{s_2}}$$

thus

$$\mathcal{HS} = \{\sum_{i=1}^{n} a_i T_{\tilde{s_i}} \quad | \quad a_i \in \mathbb{C}\}.$$

Following the pattern above one could take the completion $\overline{\mathcal{HS}}$ using the supremum norm getting a subalgebra of the C^* - algebra. Taking the Gel'fand spectrum we come to the corresponding completion $\overline{\mathcal{M}^{K_0}} = sp(\overline{\mathcal{HS}})$ which we would like to consider as an analogy with the generalized quotient space $\overline{\mathcal{A}/\mathcal{G}}$. We can repeat all the construction above to state that there exists a one - to one correspondence between regular diffeomorphism invariant measures on the compactified space $\overline{\mathcal{M}^{K_0}}$ and some invariants of special two - dimesnional knots. We call such a knot "special" because it is not a knot in a rigouris sense. Really as in the case above we have configurations with self intersections and so on. But leaving all details and discussions about the subject of 2- dimesnional knots we want to point out here just one interesting thing which is included in the picture.

We can present now at least one diffeomorphism invariant measure on the hermitian triple space component $\mathcal{M}^K$. First of all let us construct a diffeomorphism invariant measure on the space of all riemannian metrics compatible with given smooth structure. For each metric g one has the corresponding maximal positive subspace $V_g \subset H^2(X, \mathbb{R})$. Positive means that the intersection form Q_X is positive definite being restricted on V_g. This subspace is nothing more then

$$V_g = [\mathcal{H}_g^+].$$

This gives us so called real period map

$$\Pi : \mathcal{H} \to Gr^+$$

where Gr^+ is the positive grassmanian of the vector space $H^2(X, \mathbb{R})$ endowed with the intersection form. It's a known fact ([3]) that this grassmanian is isomorphic to the corresponding Lobachevsky space. As well one has a natural riemannian metric on this Lobachevsky space which is invariant under the $Diff^+X$ - action. This riemannian metric defines the corresponding measure $d\mu$ on Gr^+ which is diffeomorphism invariant to. Then one takes

$$\Pi^* d\mu$$

as a diffeomorphism invariant measure on the space of all riemannian metrics $\mathcal{H}$ on X. And at the end one takes

$$d\tilde{\mu} = p^* \Pi^* d\mu$$

on $\mathcal{M}^K$ where p is the natural projection

$$p : \mathcal{M}^K \to \mathcal{H},$$

described in detials in [5]. This diffeomorphism invariant measure should correspond to some invariant of special 2 - dimensional knots.

Discussing the previous construction one could remark that this diffeomorphism invariant measure is not intrinsic to the hermitian triple case but is induced by riemannian metrics only. Now we want to construct a measure which would be more closer to the situation.

2.3. Diffeomorphism invariant riemannian metric.

For the purposes presented at the end of the previous subsecion we are going to introduce a natural structure on the component $\mathcal{M}^K$. Namely there exists a diffeomorphism invariant riemannian metric on the component. Moreover this riemannian metric admits an extension to the compatified space constructed above.

Lemma 2.3. *The tangent space to $\mathcal{M}^K$ in a point (g, J, ω) is isomorphic to the space of real 2 - form Ω^2_X.*

Really, let us separate (as usual in our practice during this paper) three types of deformations (or of variables if one wishes):

(1) the first type corresponds to the deformations of the almost complex structure with preserving reimannian metric;

(2) the second type corresponds to the defomration of the riemannian metric with fixed almost complex structure and fixed volume form;

and

(3) the third corresponds to the deformations of the riemannian metric with fixed almost complex structure and fixed conformal class.

We use the third elements of triples to determine these deformations: every such deformation affects on the 2- form part of the given hermitian triple (g_0, J_0, ω_0). On the other hand, using the induced increment $\delta\omega$ of the third element one can recover what the deformation could be. For the first type of the deformation we have

(1) $\delta\omega \in \Omega^+_{g_0}$ to be compatible with g_0 and $\delta\omega \wedge \omega_0 \equiv 0$ to preserve the volume form $d\mu_{g_0}$ which is $\frac{1}{2}\omega_0^{\wedge 2}$.

Due to the well known realtionship

$$\Omega^+_{g_0} = (\Omega^{2,0}_{J_0} \oplus \Omega^{0,2}_{J_0})_{\mathbb{R}} \oplus \mathbb{R} < \omega_0 >$$

it gives us that

(1) the first type variables correspond to the following condition

$$\delta\omega \in (\Omega^{2,0}_{J_0} \oplus \Omega^{0,2}_{J_0})_{\mathbb{R}}.$$

The second type of deformations is reflected by small 2 - forms which lie in the space $\Omega^{1,1}_{J_0}$ to be compatible with the given almost complex strucutre J_0 and which are pointwise orthogonal to ω_0 to preserve the volume form. Thus

(2) the second type variables correspond to the following condition

$$\delta\omega \in \Omega^-_{g_0}$$

due to the well known decomposition

$$\Omega^{1,1}_{J_0} = \Omega^-_{g_0} \oplus \mathbb{R} < \omega_0 > .$$

Finally the third type obviously gives just the following condition

(3) $\delta\omega \in \mathbb{R} < \omega_0 >$.

Collecting together the descriptions above we get the statement.

This description gives a hint why 2- dimensional knots came instead of 1- dimensional "classical knots". In the case of connection we have the space of 1- forms (bundle valued) as the space of variable deformations and all the construction based on the possibility of integration over 1- dimensional submanifolds. Now the space of deformations consists of 2- dimensional forms so to impose the integration one could introduce 2- dimensional analogy of "usual" knots.

Now with the description of $T\mathcal{M}^K$ in hand it's easy to define the riemannian metric. For two tangent vectors $v_1, v_2 \in T_{(g,J,\omega)}\mathcal{M}^K$, represented by two real forms ω_1, ω_2, the corresponding formula reads as

$$G(v_1, v_2) = \int_X < \omega_1, \omega_2 >_g d\mu.$$

Globalizing over X one gets the natural riemannian metric which we denote as G. For this one we have the following

Proposition 2.3. *Riemannian metric G is invariant under $Diff^+X$ - action. Moreover it is compatible with the compactification described in Subsection 2.2.*

The first part is obvious — the naturality of the definition ensures that the metric is invariant. The second part is more complicated but again the same agruments show that the extension is well defined. Such type compactification as usual uses distributions together with smooth 2 - forms and the point is that for sych type distributions the inner product defined above is still valid.

Thus one could try to exploit this riemannian metric to define the corresponding $Diff^+X$ - invariant measure on the compactified space.

2.4. Additional remarks on the example.

The construction recalled in Subsection 2.1 can be exploited in a different direction. We mean the following application: it's well known what are the rational cohomology of the quotient space $\mathcal{A}/\mathcal{G}$. The construction of μ - classes brilliantly described in [2] can be completed by a list of details in some special cases. One could derive, for example, an information about $H^*(\mathcal{A}/\mathcal{G}, \mathbb{Q})$ from the consideration above. Namely for every loop $\alpha \in \mathcal{L}_{x_0}$ one takes the corresponding map

$$H(\alpha, .) : \mathcal{A}/\mathcal{G} \to SU(2) \simeq S^3.$$

Then the cohomological class $H^*h \in H^*(\mathcal{A}/\mathcal{G}; \mathbb{Z})$ where $h \in H^3(S^3, \mathbb{Z})$ is defined and depends on the homological class of the loop α. This gives us a correspondence between $H_1(X, \mathbb{Z})$ and $H^3(\mathcal{A}/\mathcal{G}, \mathbb{Z})$. It's easy to see that the correspondence exactly the same as in the "classical" construction of μ - classes.

Now let us use the same idea in the situation around the space of hermitian triples. For every polymap s, representing a special 2 - dimensional loop Σ, one has

$$H(\Sigma, .) : \mathcal{M}^{K_0} \to S^1$$

(see Subsection 2.2). Then one takes $H^*h \in H^1(\mathcal{M}^{K_0}, \mathbb{Z})$ where $h \in H^1(S^1, \mathbb{Z})$ is the generator, and again it's clear that the pull back of the generator h depends only on the homological class of 2- dimensional loop Σ. This gives us a correspondence between $H_2(X, \mathbb{Z})$ and $H^1(\mathcal{M}^{K_0}, \mathbb{Z})$. It's naturally to call the induced classes as "generalized" μ - classes. Thus we have

$$\{\mu_{\Sigma_i}, i = 1, ..., b_2(X)\} \subset H^1(\mathcal{M}^{K_0}, \mathbb{Z}).$$

On the other hand we can repeat the main construction of μ - classes using some special universal $U(2)$ - bundle over the direct product $\mathcal{M}^{K_0} \times X$. Let us describe this bundle, which we will denote as $\mathcal{E}$. For a fixed point $x_0 \in X$ take the fiber of the tangent bundle $T_{x_0}X$. Tautologically over a point $(g, J, \omega) \times x_0$ we have the corresponding $U(2)$ - structure which gives us after globalization over $\mathcal{M}^{K_0}$ since the fiber $T_{x_0}X$ is fixed a $U(2)$ - bundle

$$\mathcal{E}_{x_0} \to \mathcal{M}^{K_0}.$$

Globalizing along X we get the universal bundle $\mathcal{E} \to \mathcal{M}^{K_0} \times X$. It has the first and the second Chern classes belong to $H^2(\mathcal{M}^{K_0} \times X, \mathbb{Z})$ and $H^4(\mathcal{M}^{K_0} \times X, \mathbb{Z})$ respectively which one could try to exploit to define a correspondence between any homological classes of based manifold and cohomological classes of the space of hermitian triples. But pure arithmetical arguments show that in this way we couldn't get the correspondence

$$H_2(X, \mathbb{Z}) \leftrightarrow H^1(\mathcal{M}^{K_0}, \mathbb{Z}).$$

It's an open question how one can relate these two known constructions extended to the case of the space of hermitian triples.

Another aspect we would like to mention here is the question of diffeomorphisms reversing the orientation. A reader has noted that in the previous Subsection we always use the group $Diff^+X$ instead of whole group $DiffX$, used in the Ashtekar — Isham — Lewandowski programme. It's more or less clear that all our constructions sustain the adding all diffeomorphisms not only preserving the orientation. Just one question could be advanced — what happens with the induced measure $d\tilde{\mu}$? But the answer is quite affirmative: if we take the "negative" grassmannian Gr^- (with the obvious definition to consist of all q - planes in $H^2(X, \mathbb{R})$ with negative definite restriction of the intersection form) then these two grassmanians Gr^+ and Gr^- are naturally isomorphic. They realize the same Lobachevsky space with the same riemannian metric. So if we reverse the orientation on X by some appropriate diffeomorphism then we get quite correct picture. Thus in all what above one can replace the subgroup $Diff^+X$ by whole group $DiffX$.

Ay the end the "pumping up trick" (see, f.e., [4]) gives an approach how one can construct a relatioship between two objects, the hermitian connection space and hermitian triples space. Let M be a 3 - dimensional smooth manifold, E is an appropriate $U(2)$ - bundle with hermitian structure h. Then one can consider a 4 - dimensional manifold X which corresponds to the object above. Namely let us take the determinant line bundle $detE$ and consider the associated bundle on unit circles. This gives the 4 - dimensional smooth manifold X. Topological type of E uniquelly defines the corresponding topological type of $\mathcal{M}^K$ (for example, if E is trivial then $K = 0$). Therefore we have the following diagramm

$$\begin{array}{ccc} \tilde{E} & \rightarrow & E \\ \downarrow & & \downarrow \\ X & \rightarrow & M \end{array}$$

On the manifold M we have the hermitian connection space $\mathcal{A}_h(E)$ and one the manifold X we have the correspondnig to E component $\mathcal{M}^k$. Locally we have $X = M \times S^1$. Consider any framed loop γ with a fixed point $p \in M$. For this loop one has the correspondnig torus $\tilde{\gamma} \in X$. This torus can be used to define a map

$$T_{\tilde{\gamma}} : \mathcal{M}^K \rightarrow U(1)$$

described above. At the same time the loop defines a map

$$T_\gamma : \mathcal{A}_h(E) \rightarrow U(2).$$

At the same time one could try to connect these two pictures over two related manifolds. For example if E is trivial then $X = M \times S^1$ and the euler charactericstic of X is trivial. For the corresponding component $\mathcal{M}^K$ one has the following associated map

$$\tilde{T}_{\tilde{\gamma}} : \mathcal{M}^K \rightarrow U(2),$$

based on considerations in the framework of hermitian geometry. If one takes a triple (g, J, ω) on X then the tangent bundle $TX \rightarrow X$ is reduced to the corresponding $U(2)$ - bundle over $\tilde{\gamma}$. Levi — Civita connection doesn't commute in general with the almost complex structure J but one could take a "projection" of the connection to the space of hermitian connections. Of course it requires some additional choices over M and X but it looks lake a way to construct a "Fourie transform": for this projected connection one takes the monodromy around the circle $p \times S^1$, getting an element of $U(2)$. Then a hermitian connection $a \in \mathcal{A}_h(E)$ would be "dual" to hermitian triple $(g, J, \omega) \in \mathcal{M}^K$ if for any path γ the results are compatible.

This idea (or a familiar one) can be exploited in a future to extend the imaginary relatioship between hermitian triple spaces and hermitian connection spaces which was illustrated in this paper.

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