QUATERNIONIC GEOMETRY OF THE NILPOTENT VARIETY: AN EXAMPLE

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1. Introduction

If G is a compact simple Lie group, then Kirillov, Kostant & Souriau showed that any co-adjoint orbit in the complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$ is naturally a complex symplectic manifold (see for example [13]). In particular the real dimension of such an orbit is divisible by four and it is natural to ask whether it carries some type of quaternionic structure. A partial positive answer to this question was given by Kronheimer [16, 17] who showed that there is a *hyperKähler* structure on the co-adjoint orbits both of semi-simple and of nilpotent elements. Thus these orbits carry Ricci-flat metrics. Using Kronheimer's results, it was shown in [25] that the nilpotent orbits also admit an action of the multiplicative quaternions $\mathbb{H}^*$ and that the quotient by this action is a quaternionic Kähler manifold (so necessarily Einstein) of positive scalar curvature. In the case of the smallest nilpotent orbit one obtains a symmetric space of the form $G/(K \cdot Sp(1))$ and these are precisely the spaces studied by Wolf [26]. Since quaternionic Kähler manifolds have a twistor space there are techniques available for studying certain harmonic maps via complex geometry. The purpose of this paper is to investigate the geometry of the regular (largest) nilpotent orbit of $\mathfrak{sl}(3, \mathbb{C})$ and of a class of harmonic maps of a compact Riemann surface into $G_2/SO(4)$ in such a way as to illustrate some general theory to be described elsewhere, but which has been partially developed in [14, 23].

We start with a Morse theoretic description of the quaternionic Kähler manifold associated to the regular nilpotent orbit of $\mathfrak{sl}(3, \mathbb{C})$. The Grassmannian $\widetilde{\mathrm{Gr}}_3(\mathfrak{su}(3))$ of oriented three-planes in $\mathfrak{su}(3) \cong \mathbb{R}^8$ carries an $SU(3)$-invariant functional ψ defined by

$$\psi(V) = \psi(e_1, e_2, e_3) = -\langle e_1, [e_2, e_3]\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the metric determined by the Killing form of $\mathfrak{su}(3)$ and (e_1, e_2, e_3) is an oriented orthonormal basis for V. The gradient flow of ψ has two positive critical values; we define C_r to be the critical set corresponding to the least of these and let M_r be the associated unstable manifold, which by definition consists of the

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critical set C_r together with all points lying on trajectories $\gamma(t)$ of the gradient flow of ψ such that $\lim_{t\to-\infty}\gamma(t)$ lies in C_r. In §§2,3 we prove

Theorem 1.1. *The unstable manifold M_r is an eight-dimensional quaternionic Kähler manifold whose twistor space Z_r is the projectivisation $\mathbb{P}(\mathcal{O}_r)$ of the regular nilpotent orbit of $\mathfrak{sl}(3,\mathbb{C})$. The critical set C_r is the symmetric space $PSU(3)/SO(3)$ and M_r is isomorphic to the rank-three vector bundle associated to the Lie algebra of $SO(3)$.*

(Here $PSU(3)$ denotes $SU(3)$ modulo its centre $\mathbb{Z}_3$.)

In §4 we show that the quaternionic Kähler structure is locally symmetric by identifying a triple cover as an open subset of $G_2/SO(4)$. This renders some of the earlier arguments superfluous; however, it is the Morse theoretic approach which extends to arbitrary nilpotent orbits. Brylinski & Kostant [5] have recently classified which nilpotent orbits are finite covers of other nilpotent orbits; many orbits do not appear in this list and their associated quaternionic Kähler manifolds can not be locally symmetric (cf. [24]).

Both the proof of Theorem 1.1 and the discussion of the local symmetric structure, show that the complex contact structure on Z_r agrees with that induced by the complex symplectic structure on $\mathcal{O}_r$ defined by Kirillov, Kostant & Souriau. The complex contact structure on the twistor space is particularly important for the study of harmonic maps: if $f\colon \Sigma \to Z$ is a holomorphic map of a Riemann surface into a twistor space Z such that the pull-back of the contact form is zero, then the composition of f with the projection $Z \to M$ is a harmonic map $\Sigma \to M$ [6]. The complex contact structure of Z_r may be identified via the Springer resolution with that on an open set of $\mathbb{P}T'^*F_{12}(\mathbb{C}^3)$, the projectivised holomorphic cotangent bundle of the flag manifold of lines in planes in $\mathbb{C}^3$. Let L_1 and L_2 be the holomorphic tangent line bundles of the fibres of the projections p_1 and p_2 from $F_{12}(\mathbb{C}^3)$ to $\mathbb{C}\mathrm{P}(2)$ and $\mathbb{C}\mathrm{P}(2)^*$ respectively. The fact that M_r is locally isometric to $G_2/SO(4)$ leads to

Proposition 1.2. *There is a three-to-one correspondence between harmonic maps $f\colon \Sigma \to G_2/SO(4)$ (avoiding a $\mathbb{C}\mathrm{P}(2)$) which have a horizontal holomorphic lift to the twistor space of $G_2/SO(4)$ and triples (α,β,s), where*

1. *(α,β) is a pair of holomorphic maps $\Sigma \to \mathbb{C}\mathrm{P}(2)$ with the same ramification divisor and such that $\sum_i \alpha_i\beta_i = 0$,*
2. *if $\gamma = (\alpha,\beta)$ is the corresponding map into $F_{12}(\mathbb{C}^3)$, then $\gamma^*L_1 \cong \gamma^*L_2$,*
3. *s is a splitting of the inclusion $\gamma^*L_1 \hookrightarrow \gamma^*T'F_{12}(\mathbb{C}^3)/T'\Sigma$.*

This result may be regarded as generalisation of Loo's [19] description of harmonic maps $S^2 \to S^4$ in terms of pairs of holomorphic maps $S^2 \to \mathbb{C}\mathrm{P}(1)$.

It is well-known that $G_2/SO(4)$ can be viewed as the totally geodesic submanifold of $\widetilde{\mathrm{Gr}}_4(\mathbb{R}^7)$ consisting of co-associative four-planes [11]. It turns out that there is another relationship between the geometry of these two manifolds.

Proposition 1.3. *The quaternionic Kähler manifold $\widetilde{\mathrm{Gr}}_4(\mathbb{R}^7)$ admits an action of $U(1)$ such that the quaternionic Kähler quotient is the singular eight-dimensional space $M_r \cup \mathbb{C}P(2) = G_2/(SO(4) \times \mathbb{Z}_3)$.*

This result is proved in the final section of the paper where we also identify a specific flow line used in the earlier Morse theoretic arguments. During the latter calculation it becomes possible to obtain the hyperKähler potential of the regular orbit of $\mathfrak{sl}(3,\mathbb{C})$, this is a function which is simultaneously a Kähler potential for each of the Kähler structures. This also makes use of the hyperKähler quotient construction [12] which Kronheimer observed may be used to construct nilpotent orbits in $\mathfrak{sl}(n,\mathbb{C})$. Extensions to nilpotent orbits of other classical groups may be found in [15].

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2. The Nilpotent Variety and Morse Theory

We introduce the following notation. The set of all nilpotent matrices in $\mathfrak{sl}(3,\mathbb{C})$ will be denoted by $\mathcal{N} = \{ A \in \mathfrak{sl}(3,\mathbb{C}) : A^3 = 0 \}$ and is called the nilpotent variety of $\mathfrak{sl}(3,\mathbb{C})$. The group $SL(3,\mathbb{C})$ acts on $\mathcal{N}$ via the adjoint representation and has two non-trivial orbits: the smallest is the eight-dimensional orbit of highest root vectors $\mathcal{O}_h = \{ A \in \mathcal{N} : A^2 = 0, A \neq 0 \}$; the other is the regular orbit $\mathcal{O}_r$, which is 12-dimensional and is open and dense in $\mathcal{N}$. It is the quaternionic geometry associated to the regular orbit that is our main object of study.

Nilpotent elements are closely related to three-dimensional simple subalgebras. Let X be a nilpotent element in the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$ of a compact simple Lie group G. The Jacobson-Morosov Theorem (see [7]) states that there exist H and Y in $\mathfrak{g}^{\mathbb{C}}$ such that $[X,Y] = H$, $[H,X] = 2X$ and $[H,Y] = -2Y$, so the linear span of X, Y and H is a subalgebra of $\mathfrak{g}^{\mathbb{C}}$ isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. When X is a highest root vector, it is possible to take Y and H proportional to σX and $[X,\sigma X]$, respectively, where σ is the real structure on $\mathfrak{g}^{\mathbb{C}}$ with fixed point set $\mathfrak{g}$. This allows one to define an action of $\mathbb{H}^*$ on the set of highest root vectors by $(a+bj)\cdot X = a^2X - b^2\sigma X - ab[X,\sigma X]$ and the resulting quotient spaces are compact and symmetric [25, §6], [26]. When $G = Sp(n+1)$, this gives the quaternionic projective space $\mathbb{H}P(n) = Sp(n+1)/(Sp(n)\,Sp(1))$. Two other examples are $\mathbb{C}P(2) = SU(3)/S(U(1) \times U(2))$ and $G_2/SO(4)$ obtained by taking G to be $SU(3)$ or G_2 respectively.

Consider the gradient flow equations $\dot{V} = \nabla\psi(V)$ for the functional ψ on the Grassmannian $\widetilde{\mathrm{Gr}}_3(\mathfrak{su}(3))$ defined in the introduction. These may be written as

$$\begin{aligned}\dot{e}_1 &= -[e_2, e_3] - \psi e_1,\\ \dot{e}_2 &= -[e_3, e_1] - \psi e_2,\\ \dot{e}_3 &= -[e_1, e_2] - \psi e_3\end{aligned}$$

and the right-hand sides of these equations are just the components of the Lie brackets orthogonal to V. The critical points of ψ correspond to Lie algebra homomorphisms $\rho\colon \mathfrak{su}(2) \to \mathfrak{su}(3)$ and the critical sets are manifolds of the form $SU(3)/N_\rho$, where N_ρ is the normaliser of the subgroup with Lie algebra $\rho(\mathfrak{su}(2))$. The maximum value of ψ is $\sqrt{2}$ and the corresponding critical set C_h is the quaternionic Kähler manifold $\mathbb{C}P(2) = SU(3)/S(U(2) \times U(1))$. According to Dynkin [8] the only other positive critical value of ψ is $1/\sqrt{2}$ and its critical set C_r is $PSU(3)/SO(3)$. The corresponding subalgebra $\rho(\mathfrak{su}(2))$ may be taken to be the $\mathfrak{so}(3)$ consisting of those matrices in $\mathfrak{su}(3)$ with real entries. Changing the orientation of a three-plane V changes the sign of $\psi(V)$ so the remaining critical values are $-\sqrt{2}$ and $-1/\sqrt{2}$.

One motivation for the use of the gradient flow comes from the following lemma. Let $\mathcal{F}$ be the subset of $\widetilde{\mathrm{Gr}}_3(\mathfrak{su}(3))$ consisting of all oriented three-planes V such that every isotropic element of $V_{\mathbb{C}}$ is nilpotent.

Lemma 2.1. *The gradient flow of ψ preserves the variety $\mathcal{F}$ and the nilpotent orbits associated to elements of $\mathcal{F}$.*

Proof. Given an oriented, orthonormal basis e_1, e_2, e_3 for an element $V \in \mathcal{F}$, let

$$e_1' = (1 - t\psi)e_1 - t[e_2, e_3], \qquad \text{etc.},$$

be a path parameterised by $t \in \mathbb{R}$. Up to order t^2, e_1', e_2', e_3' are orthonormal, so we need to show that (up to first order) $e_1' + ie_2'$ is in the nilpotent orbit of $e_1 + ie_2$. However, the Jacobson-Morosov Theorem implies that there exists $H \in \mathfrak{sl}(3, \mathbb{C})$ such that $e_1 + ie_2 = \frac{1}{2}[H, e_1 + ie_2]$, and so

$$\begin{aligned}e_1' + ie_2' &= (1 - t\psi)(e_1 + ie_2) + t[e_1 + ie_2, ie_3]\\ &= e_1 + ie_2 + t[e_1 + ie_2, ie_3 + \tfrac{1}{2}\psi H],\end{aligned}$$

as required. □

The tangent space to the unstable manifold M_r at C_r is described by the Hessian of ψ. The following calculation was communicated to us by F. E. Burstall.

Proposition 2.2 (Burstall). *Let V be a point of the critical set C_r. Then, as an $SU(2)$-module,*

$$T_V\, \widetilde{\mathrm{Gr}}_3(\mathfrak{su}(3)) \cong S^6 + S^4 + S^2$$

and the Hessian of ψ is negative definite on the first summand, zero on the second and positive definite on the third. Thus $1/\sqrt{2}$ is a non-degenerate critical value of ψ in the sense of equivariant Morse theory and

$$T_V M_r \cong S^4 + S^2 \cong S^3 \otimes S^1.$$

Here $S^k = S^k\mathbb{C}^2$ is the $(k+1)$-dimensional irreducible representation of $SU(2)$. When k is even, this is the complexification of a real representation also denoted by S^k.

Proof. Identify $T_V\,\widetilde{\mathrm{Gr}}_3(\mathfrak{su}(3))$ with $V^* \otimes V^\perp \subset \Lambda^2\,\mathfrak{su}(3)$, so the Hessian of ψ is given by $d_V^2\psi(X,Y) = \psi(X.Y.e_1 \wedge e_2 \wedge e_3)$. Now the Killing form of $\mathfrak{su}(2)$ is $B_{\mathfrak{su}(2)}(e_i, e_j) = -2\delta_{ij}\psi(V)^2 = -\delta_{ij}$ and the Casimir operator of an $SU(2)$-representation p is

$$C_p = -\sum_{i=1}^{3} p(e_i)p(e_i),$$

which for the representation S^k is multiplication by $((k+1)^2 - 1)/8$. Calculating the negative of the Hessian we have

$$\begin{aligned}
-d_V^2\psi(X,Y) &= \langle XYe_1, [e_2, e_3]\rangle + \langle e_1, [XYe_2, e_3]\rangle + \langle e_1, [e_2, XYe_3]\rangle \\
&\quad + \langle Xe_1, [e_2, Ye_3]\rangle + \langle Ye_1, [e_2, Xe_3]\rangle + \langle Xe_1, [Ye_2, e_3]\rangle \\
&\quad + \langle Ye_1, [Xe_2, e_3]\rangle + \langle e_1, [Xe_2, Ye_3]\rangle + \langle e_1, [Ye_2, Xe_3]\rangle \\
&= -\langle Ye_1, X[e_2, e_3]\rangle - \langle Ye_2, X[e_3, e_1]\rangle - \langle Ye_3, X[e_1, e_2]\rangle \\
&\quad + \langle Ye_1, [e_2, Xe_3] - [e_3, Xe_2]\rangle + \langle Ye_2, [e_3, Xe_1] - [e_1, Xe_3]\rangle \\
&\quad + \langle Ye_3, [e_1, Xe_2] - [e_2, Xe_1]\rangle \\
&= \tfrac{1}{\sqrt{2}}\langle Ye_i, Xe_i\rangle + \langle Ye_i, A_X e_i\rangle,
\end{aligned}$$

where $A_X = \sqrt{2}\,\mathrm{ad}\, e_j \circ X \circ \mathrm{ad}\, e_j$. The Casimir element of $V^* \otimes V^\perp$ is given by

$$\begin{aligned}
C_{V^*\otimes V^\perp} X &= -(e_i(p(e_i)X) - (p(e_i)X)e_i) = -(e_ie_iX - 2e_iXe_i + Xe_ie_i) \\
&= C_{V^\perp}X + XC_V + \sqrt{2}A_X = 4X + \sqrt{2}A_X,
\end{aligned}$$

since $C_V = C_{S^2} = 1$ and $C_{V^\perp} = C_{S^4} = 3$. Thus $d_V^2\psi(X,Y) = \langle Ye_i, C_X e_i\rangle$, where $C_X = (3X - C_{V^*\otimes V^\perp}X)/\sqrt{2}$. Now $V^* \otimes V^\perp = S^2 \otimes S^4 = S^6 + S^4 + S^2$ and C has eigenvalues $-3/\sqrt{2}$, 0, and $\sqrt{2}$ on these summands. □

Thus the real normal bundle of the critical set $C_r = PSU(3)/SO(3)$ in the unstable manifold M_r is the vector bundle associated to the representation S^2. Since $SO(3)$ acts transitively on the unit sphere in $S^2 \cong \mathbb{R}^3$, this shows that $SU(3)$ acts transitively on the set of trajectories lying in $M_r \backslash C_r$. An example of such a trajectory is given (up to parameterisation) by $V(x,y) = (e_1, e_2, e_3)$,

$$e_1 + ie_2 = 2\begin{pmatrix} 0 & x^3 & 0 \\ 0 & 0 & y^3 \\ 0 & 0 & 0 \end{pmatrix}, \qquad e_3 = \sqrt{x^2+y^2}\begin{pmatrix} ix^2 & 0 & 0 \\ 0 & i(y^2-x^2) & 0 \\ 0 & 0 & -iy^2 \end{pmatrix},$$

where $2(x^6+y^6) = 1$ with $x \geqslant y \geqslant 0$. (One may check directly that this is a solution to the equations, but a method of obtaining such three-planes will be given in the last section.) Note that $V(x,y)$ lies in C_r when $x = y = 2^{-1/3}$ and is in C_h when $x = 2^{-1/6}$ and $y = 0$. A straightforward calculation shows that if $\alpha^2+\beta^2+\gamma^2 = 0$ then $X = \alpha e_1 + \beta e_2 + \gamma e_3$ has $X^3 = 0$. Thus X is nilpotent and lies either in $\mathcal{O}_r$ or in $\mathcal{O}_h$. However, if $X \in \mathcal{O}_h$, then by Lemma 2.1 the limit of X under the backwards flow must also be in $\mathcal{O}_h$. But such a limit lies in $V_{\mathbb{C}}$ for some $V \in C_r$ and such a $V_{\mathbb{C}}$ does not meet $\mathcal{O}_h$. Thus isotropic elements of $V_{\mathbb{C}}(x,y)$ lie in $\mathcal{O}_r$.

3. Twistor Space Structure

In [25] it was shown that any quaternionic Kähler manifold with positive scalar curvature has a principal $\mathbb{H}^*/\mathbb{Z}_2$-bundle $\mathcal{U}(M)$ whose total space carries a hyperKähler metric. Using the action of $\mathbb{C}^* \leqslant \mathbb{H}^*$ one obtains a quotient manifold Z which is a $\mathbb{C}P(1)$-bundle over M. The total space of Z is a complex manifold called the *twistor space* of M. For example if $M = \mathbb{H}P(n)$, then $\mathcal{U}(M) = (\mathbb{H}^{n+1} \setminus \{0\})/\mathbb{Z}_2$ and $Z = \mathbb{C}P(2n+1)$. The action of $j \in \mathbb{H}^*$ on $\mathcal{U}(M)$ descends to an anti-holomorphic involution σ on Z preserving the fibres. The $\mathbb{C}^*$-action on $\mathcal{U}(M)$ preserves one of the complex structures, say I. We may now construct a holomorphic two-form $\omega_J + i\omega_K$ which gives a complex symplectic structure on $\mathcal{U}(M)$. Contracting this two-form with the holomorphic vector field generating the $\mathbb{C}^*$-action gives a holomorphic bundle-valued one-form θ on Z. This has the property that $\theta \wedge (d\theta)^n$ is nowhere vanishing, in other words we have a *complex contact structure* on Z. Such information about Z may be used to recover the structure of M via the Inverse Twistor Construction (LeBrun [18], Pedersen & Poon [20]):

Let Z be a complex manifold of dimension $2n+1 \geqslant 5$ with a fixed-point-free anti-holomorphic involution σ. Then the set N of σ-invariant rational curves with normal bundle $2n\mathcal{O}(1)$ is naturally a quaternionic manifold of dimension $4n$.

If in addition Z admits a complex contact structure θ such that $\sigma^*\theta = \bar{\theta}$, then N is pseudo-quaternionic Kähler.

Here a quaternionic manifold should be regarded as the non-Riemannian analogue of a quaternionic Kähler structure (see [21] for a precise definition). The term pseudo-quaternionic Kähler indicates that the metric obtained need not be positive definite.

Let V be a point of the critical set C_r and let $\mathcal{C}$ denote the set of isotropic (null) elements of $V_{\mathbb{C}} = V \otimes \mathbb{C}$. Then V is $\rho(\mathfrak{su}(2))$ for some homomorphism ρ and $\mathcal{C}$ consists of the nilpotent elements of $\rho(\mathfrak{su}(2) \otimes \mathbb{C}) \subset \mathfrak{sl}(3,\mathbb{C})$. Direct calculation shows that

$SL(2,\mathbb{C})$ acts transitively on the set of nilpotent elements of $\mathfrak{sl}(2,\mathbb{C})$, so the elements of $\mathcal{C}$ are conjugate under the adjoint action of $SL(3,\mathbb{C})$, and that $\mathcal{C}$ is contained in the regular nilpotent orbit $\mathcal{O}_r$ (see [7]).

Lemma 3.1. *Let* $\mathbb{C}\mathrm{P}(1)$ *be the projectivisation* $\mathbb{P}(\mathcal{C})$. *Then the normal bundle* ν *of* $\mathbb{C}\mathrm{P}(1)$ *in* $\mathbb{P}(\mathcal{O}_r)$ *is*

$$\nu \cong 4\mathcal{O}(1).$$

Proof. The Lie algebra $\mathfrak{sl}(3,\mathbb{C})$ splits under the action of $\mathfrak{sl}(2,\mathbb{C})$ into a direct sum of $SU(2)$-modules as $\mathfrak{sl}(3,\mathbb{C}) = S^2 + S^4$. The tangent space to $\mathcal{O}_r$ at X is $T_X\mathcal{O}_r = (\operatorname{ad} X)\,\mathfrak{sl}(3,\mathbb{C})$, so the normal bundle at $x = [X]$ is $\nu_x \cong (\operatorname{ad} X)S^4$.

Since $V \cong \mathfrak{su}(2)$, there is an element $H \in V_{\mathbb{C}}$ such that $[H,X] = 2X$. On S^4, $\operatorname{ad} H$ has eigenvalues 4, 2, 0, -2 and -4. Restricting to the circle subgroup $\exp(itH)$, we have a further splitting of ν into $U(1)$-bundles which in this case are four line bundles with Chern classes 4, 2, 0 and -2 (respectively). In particular,

$$c_1(\nu) = 4.$$

Now $V_{\mathbb{C}}$ is a linear subspace of $\mathfrak{sl}(3,\mathbb{C})$, so $T\mathbb{P}(\mathfrak{sl}(3,\mathbb{C}))/T\mathbb{P}(V_{\mathbb{C}}) \cong 5(\mathcal{O}(1) \to \mathbb{C}\mathrm{P}(2))$. The map $\mathbb{C}\mathrm{P}(1) \to \mathbb{P}(V_{\mathbb{C}})$ has degree 2 and $\eta = T\mathbb{P}(\mathfrak{sl}(3,\mathbb{C}))/T\mathbb{P}(V_{\mathbb{C}}) \cong V_{\mathbb{C}}^{\perp} \otimes \mathcal{O}(2) = 5\mathcal{O}(2)$. The normal bundle ν is a rank four subbundle of η and so $\nu \cong \sum_{i\leqslant 2} a_i\mathcal{O}(i)$ with $\sum a_i = 4$. An $\mathcal{O}(2)$-summand gives a non-zero holomorphic section of $\nu(-2)$ and hence of $\eta(-2)$. This is a constant vector $v \in V_{\mathbb{C}}^{\perp}$ which lies in $T_x\mathbb{P}(\mathcal{O}_r)$ for all $x \in \mathbb{C}\mathrm{P}(1)$, that is v lies in

$$\bigcap_{[X]\in\mathbb{C}\mathrm{P}(1)} \operatorname{ad}(X)\,\mathfrak{sl}(3,\mathbb{C}). \tag{3.1}$$

However, since $\mathfrak{sl}(3,\mathbb{C}) = S^2 + S^4$ this intersection is $\{0\}$ so ν has no $\mathcal{O}(2)$-summand. As $c_1(\nu) = 4$, this implies that $\nu \cong 4\mathcal{O}(1)$. □

Proposition 3.2. *The unstable manifold* M_r *is an eight-dimensional quaternionic manifold with twistor space* $\mathbb{P}(\mathcal{O}_r)$.

Proof. Consider the flow line $V(x,y)$ defined at the end of §2 and let $\mathcal{C}(x,y)$ be the set of isotropic elements in $V(x,y)\otimes\mathbb{C}$. Then $\mathcal{C}(x,y)$ is contained in $\mathcal{O}_r$ and we need to determine the normal bundle ν of $\mathbb{C}\mathrm{P}(1) = \mathbb{P}(\mathcal{C}(x,y))$ in $\mathbb{P}(\mathcal{O}_r)$. As before we have $\operatorname{rank}\nu = 4$, $c_1(\nu) = 4$ (since the stabiliser of $V(x,y)$ under the action of $SU(3)$ is $U(1)$), ν is non-negative and to show $\nu \cong 4\mathcal{O}(1)$ it is sufficient to show that the intersection (3.1) is $\{0\}$.

Write $X(\zeta) = X(u+iv) = ue_1 + ve_2 + ie_3$. This is an element of $\mathcal{C}(x,y)$ if $|\zeta| = 1$. We have

$$\begin{pmatrix} \alpha & A & B \\ D & -\alpha+\beta & C \\ E & F & \beta \end{pmatrix} \in \operatorname{ad}(X(\zeta))\,\mathfrak{sl}(3,\mathbb{C})$$

which implies that

$$\bar{\zeta}A - \zeta D = \left(x^{-3}(2x^2 - y^2)\sqrt{x^2+y^2}\right)\alpha + y^3\gamma(\zeta),$$
$$\bar{\zeta}C - \zeta F = \left(y^{-3}(2y^2 - x^2)\sqrt{x^2+y^2}\right)\beta - x^3\gamma(\zeta),$$
$$\bar{\zeta}^2 B + \zeta^2 E = -x^{-3}y^3\alpha + x^3y^{-3}\beta + (x^2+y^2)^{3/2}\gamma(\zeta),$$

for some complex number $\gamma(\zeta)$. Eliminating $\gamma(\zeta)$ gives

$$(x^2+y^2)^{3/2}(\bar{\zeta}A - \zeta D) - y^3(\bar{\zeta}^2B + \zeta^2E) = x(2x^2+3y^2)\alpha - x^3\beta,$$
$$(x^2+y^2)^{3/2}(\bar{\zeta}C - \zeta F) - x^3(\bar{\zeta}^2B + \zeta^2E) = -y^3\alpha + y(3x^2+2y^2)\beta.$$

These equations hold for all $|\zeta| = 1$, so $A = B = C = D = E = F = 0$. The determinant of the matrix of coefficients of the right hand side is $6xy(x^2+y^2)^2$, so we also have $\alpha = \beta = 0$, as required.

The real structure on $\mathfrak{sl}(3,\mathbb{C})$ induces a real structure on the $\mathbb{C}P(1)$'s without fixed points and we may use the Inverse Twistor Construction to get a quaternionic structure on M_r.

We now wish to identify the twistor space Z_r of M_r with $\mathbb{P}(\mathcal{O}_r)$. Note that we already have a map $\varphi\colon Z_r \to \mathbb{P}(\mathcal{O}_r)$ since the twistor lines lie in $\mathbb{P}(\mathcal{O}_r)$. As $h^1(\nu) = 0 = h^1(\nu(-1))$, this map is a local diffeomorphism and so the image of φ is open (cf. [18]). Let (x_i) be a sequence of points in the image of φ converging to a point $x \in \mathbb{P}(\mathcal{O}_r)$. There exist oriented three planes $V_i \in M_r$ such that a pre-image of x_i lies in $V_i \otimes \mathbb{C}$. We also have a subsequence converging to a point V in $\overline{M_r} = M_r \cup C_h$ and $x \in \mathbb{P}V_{\mathbb{C}}$. However, if $V \in C_h$ then $x \in \mathbb{P}(\mathcal{O}_h)$ which is a contradiction. Therefore $V \in M_r$ and $x \in \varphi(Z_r)$. Thus φ is a covering map.

It remains to show that φ is injective. Choose $V = (e_1, e_2, e_3) \in C_r$ and let $X = e_1 + ie_2$. If $V' \in M_r$ is another three-plane with $X \in V'_{\mathbb{C}}$ then we can find $e'_3 \in \mathfrak{su}(3)$ such that (e_1, e_2, e'_3) is an oriented orthonormal basis for V'. Now,

$$\psi(V') = -\langle [e_1, e_2], e'_3\rangle = \langle \psi(V)e_3, e'_3\rangle \leqslant \psi(V).$$

However, $V' \in M_r$ and $V \in C_r$ implies $\psi(V') \geqslant \psi(V)$. Hence $\psi(V') = \psi(V)$ and $\langle e_3, e'_3\rangle = 1$. Since e_3 and e'_3 are unit vectors this gives $e_3 = e'_3$ and $V = V'$. Hence φ is a bijection. □

Proposition 3.3. *The unstable manifold M_r is quaternionic Kähler.*

Proof. The projectivised orbit $\mathbb{P}(\mathcal{O}_r)$ has a complex contact form θ defined at $[X] \in \mathbb{P}(\mathcal{O}_r)$ by $\theta([X, A]) = \langle X, A\rangle$. (This is obtained by contracting the holomorphic vector field generating the quotient $\mathcal{O}_r \to \mathbb{P}(\mathcal{O}_r)$ with the complex symplectic form on $\mathcal{O}_r$ defined by Kirillov, Kostant & Souriau, see [13].) To show M_r is pseudo-quaternionic Kähler it suffices to show that $\mathbb{C}P(1) = \mathbb{P}(C(x,y))$ is transverse to θ, in other words that $\theta|_{\mathbb{C}P(1)}$ is non-zero. At $X = e_1 + ie_2$ the tangent space to $\mathbb{C}P(1)$ is generated by e_3, thus a non-zero tangent vector at X is given by $[X, A]$, where

$$A = \begin{pmatrix} 0 & 0 & 0 \\ x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \end{pmatrix}.$$

Now $\theta([X, A]) = 2(x^2 + y^2)$ which is non-zero.

To complete the proof we need to show that the pseudo-quaternionic Kähler metric g on M_τ is positive definite. Note that by construction g is $SU(3)$-invariant, in particular at a point V of $C_\tau = PSU(3)/PSU(2)$ the metric g is $SU(2)$-invariant. Now $T_V C_\tau = S^2 + S^4$ so any $SU(2)$-invariant metric is either positive definite or has signature $(3,5)$. However, the fact that g is pseudo-quaternionic Kähler implies that g has signature $(4p, 4q)$ for some p and q. Thus the only possibility is that g is positive definite and M_τ is quaternionic Kähler. □

This completes the proof of Theorem 1.1. Note that we have also identified $\mathcal{O}_\tau \to \mathbb{P}(\mathcal{O}_\tau)$ as the $\mathbb{C}^*$-bundle associated to the contact line bundle on the twistor space of M_τ and this shows that $\mathcal{O}_\tau \to M_\tau$ is the bundle $\mathcal{U}(M_\tau)$.

4. Local Symmetric Space Structure and Harmonic Maps

It is well known that $\mathfrak{su}(3)$ is a subalgebra of $\mathfrak{g}_2$. This may be seen explicitly as follows: let $(0,1)$ be the highest root of $\mathfrak{g}_2^{\mathbb{C}}$ and suppose $(1,0)$ is a short root such that the long roots of $\mathfrak{g}_2$ are $\pm(0,1)$, $\pm(3,2)$, $\pm(3,1)$, and the short roots are $\pm(1,0)$, $\pm(1,1)$, $\pm(2,1)$; then $\mathfrak{sl}(3,\mathbb{C})$ is the subalgebra generated by the long roots. The Lie algebra $\mathfrak{g}_2^{\mathbb{C}}$ decomposes under the action of $SU(3)$ as

$$\mathfrak{g}_2^{\mathbb{C}} = \mathfrak{sl}(3,\mathbb{C}) \oplus \Lambda^{1,0}\mathbb{C}^3 \oplus \Lambda^{0,1}\mathbb{C}^3.$$

The space $G_2/SU(3)$ is 3-symmetric; the symmetry τ comes from the centre $\mathbb{Z}_3$ of $SU(3)$ and acts on this decomposition of $\mathfrak{g}_2^{\mathbb{C}}$ as $(1, e^{2\pi i/3}, e^{-2\pi i/3})$. The projection of the highest root orbit of $\mathfrak{g}_2^{\mathbb{C}}$ contains elements of both the nilpotent orbits of $\mathfrak{sl}(3,\mathbb{C})$ and so, counting dimensions, the image must be the whole nilpotent variety of $\mathfrak{sl}(3,\mathbb{C})$. To show that both orbits meet the image, we argue as follows. We have a subgroup $SO(4) = Sp(1)_+ Sp(1)_-$ of G_2 and Salamon [22] shows that the Lie algebra $\mathfrak{g}_2^{\mathbb{C}}$ decomposes under $SO(4)$ as

$$\mathfrak{g}_2^{\mathbb{C}} = S^2U_+ \oplus S^2U_- \oplus U_+S^3U_-,$$

where $U_\pm \cong \mathbb{C}^2$ are the two-dimensional non-trivial representations of $Sp(1)_\pm$. If we take $\mathfrak{sp}(1)_+$ to be the span of $E_{(0,1)}, H_{(0,1)}, E_{-(0,1)}$, where H_α, E_λ is a Cartan basis for $\mathfrak{g}_2^{\mathbb{C}}$, then $\mathfrak{sp}(1)_-$ is spanned by $E_{(2,1)}, H_{(2,1)}, E_{-(2,1)}$ and the remaining roots span $U_+S^3U_-$. Now, for definiteness, identify $E_{(0,1)}$ and $E_{(3,1)}$ with the matrices $\left(\begin{smallmatrix} 0&1&0\\0&0&0\\0&0&0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0&0&0\\0&0&1\\0&0&0 \end{smallmatrix}\right)$. This gives $[E_{(0,1)}, E_{(3,1)}] = E_{(3,2)} = \left(\begin{smallmatrix} 0&0&1\\0&0&0\\0&0&0 \end{smallmatrix}\right)$ and so $E_{(0,1)} + E_{(3,2)}$ is a highest root vector of $\mathfrak{su}(3)$ and hence of $\mathfrak{g}_2$. Consider the orbit of this element under $Sp(1)_-$. The first component lies in $\mathfrak{sp}(1)_+$ so it is $Sp(1)_-$-invariant. The other component lies in the $Sp(1)_-$-module S^3U_- spanned by $E_{(3,2)}, E_{(1,1)}, E_{(-1,0)}, E_{(-3,-1)}$. In particular, there is an element in the $Sp(1)_-$-orbit of $E_{(0,1)} + E_{(3,2)}$ which projects to $E_{(0,1)} + aE_{(3,2)} + bE_{(-3,-1)}$ with both a and b non-zero. Now this element lies in the regular orbit of $\mathfrak{sl}(3,\mathbb{C})$ and is the projection of a highest root element of $\mathfrak{g}_2^{\mathbb{C}}$, as required.

Quaternionically, we obtain the following. Let $1, i, j, k, e, ie, je, ke$ be a basis of the Cayley numbers $\mathbb{O}$, then the Wolf space $G_2/SO(4)$ is the space of quaternionic lines in $\mathbb{O}$. The action of $\tau \in \mathbb{Z}_3$ on $\mathbb{O}$ is via right multiplication by $e^{2\pi e/3} = (-1 + e\sqrt{3})/2$. The fixed point set of τ consists of those quaternionic lines which are spanned by 1, e and a complex line in $\mathbb{C}^3 = \langle i, j, k, ie, je, ke \rangle$, so this set is isomorphic to $\mathbb{C}P(2)$.

Proposition 4.1. *There is an open set in the highest root orbit of $\mathfrak{g}_2^{\mathbb{C}}$ which is a three-fold cover of the regular nilpotent orbit in $\mathfrak{sl}(3,\mathbb{C})$. The quaternionic Kähler manifold M_r associated to this $SL(3,\mathbb{C})$-orbit is $((G_2/SO(4)) \setminus \mathbb{C}P(2))/\mathbb{Z}_3$.* □

In the previous section we saw that M_r carries an $SU(3)$-invariant functional ψ. This functional defines an $SU(3)$-invariant functional on $G_2/SO(4)$ which may be described as follows. A quaternionic line in $\mathbb{O}$ necessarily contains 1 and so is determined by a three plane $\langle x, y, xy \rangle$ in $\operatorname{Im}\mathbb{O}$, the space spanned by $\mathbb{C}^3$ and e. (This is the usual identification of $G_2/SO(4)$ with space of associative three-planes in $\mathbb{R}^7$.) Any such three-plane meets $\mathbb{C}^3$ in at least a two-dimensional subspace which we may assume to be the span of x and y, with x and y orthogonal unit vectors. Since, away from the singular set, $(G_2/SO(4))/SU(3)$ is one-dimensional there is essentially only one $SU(3)$-invariant functional given by $(x, y, xy) \mapsto |\langle x, ey \rangle|$. This functional is 0 on $C_r = PSU(3)/SO(3)$ and 1 on $C_h = \mathbb{C}P(2)$. The gradient flow maps (x, y) to $(x\cos t - y\sin t, ex\sin t + ey\cos t)$.

Corollary 4.2. *Let V be a point of the critical set C_r and let $\mathcal{M}_V^-$ be the set of trajectories γ of ψ such that $\gamma(t) \in M_r \setminus C_r$ and $\lim_{t\to-\infty}\gamma(t) = V$. Then V determines a totally real three-dimensional subspace W of $\mathbb{R}^6 = \mathbb{C}^3$ and $\{\lim_{t\to\infty}\gamma(t) : \gamma \in \mathcal{M}_V^-\}$ is a two-sphere consisting of those complex lines which meet W non-trivially.* □

Twistor theory has found several applications in the study of harmonic maps; the ones that are most relevant here are the existence theorem of Bryant [4] for harmonic maps $\Sigma \to S^4$ and Loo's [19] description of the moduli space when $\Sigma = S^2$. These arise from maps to the twistor space $\mathbb{C}P(3)$ of S^4 which are horizontal with respect to the distribution defined by the complex contact structure and the results are obtained via suitable choices of complex contact manifolds birationally equivalent to (finite quotients of) $\mathbb{C}P(3)$. In our case the discussion above implies that the $\mathbb{Z}_3$-quotient of the twistor space of $G_2/SO(4)$ is birationally equivalent (as a complex contact manifold) to the twistor space obtained from the regular nilpotent orbit of $\mathfrak{sl}(3,\mathbb{C})$. We will now use the Springer resolution of the nilpotent variety of $\mathfrak{sl}(3,\mathbb{C})$ to obtain a contact-structure-preserving birational map to $\mathbb{P}T'^*F_{12}(\mathbb{C}^3)$, the projectivised holomorphic cotangent bundle of the flag manifold of lines in planes in $\mathbb{C}^3$.

Regarding $\mathfrak{sl}(3,\mathbb{C})$ as a matrix algebra in the usual way we have an $SL(3,\mathbb{C})$-equivariant map $\chi\colon \mathbb{P}(\mathcal{O}_r) \to \mathbb{P}(\mathcal{O}_h)$ defined by $\chi[X] = [X^2]$. If we identify $\mathbb{P}(\mathcal{O}_h)$ with the flag $F_{12}(\mathbb{C}^3)$ then $\chi[X] = (\ker X \subset \operatorname{Im} X \subset \mathbb{C}^3)$. An infinitesimal calculation shows that the fibres of χ are contact, that is if θ is the complex contact form on $\mathbb{P}(\mathcal{O}_r)$ then θ is zero on $T'_{[X]}\chi^{-1}\chi[X]$. This enables us to define a map $\tilde{\chi}\colon \mathbb{P}(\mathcal{O}_r) \to \mathbb{P}T'^*F_{12}(\mathbb{C}^3)$ by

$$\tilde{\chi}[X] = \chi_* \ker\theta_{[X]} = \left\{ [X^2, A] : \langle X, A\rangle = 0 \right\},$$

where $[\alpha] \in \mathbb{P}T'^*_x F_{12}(\mathbb{C}^3)$ is identified with $\ker\alpha_x$. The projectivised holomorphic cotangent bundle $p\colon \mathbb{P}T'^*F_{12}(\mathbb{C}^3) \to F_{12}(\mathbb{C}^3)$ has a natural complex contact structure for which the contact distribution at a point $[\alpha]$ is $(p_*)^{-1}\ker\alpha$.

Lemma 4.3. *The map $\tilde{\chi}$ maps the complex contact distribution on $\mathbb{P}(\mathcal{O}_r)$ into the complex contact distribution on $\mathbb{P}T'^*F_{12}(\mathbb{C}^3)$.*

Proof. This is follows from $p\tilde{\chi} = \chi$. □

Proposition 4.4. *The map $\tilde{\chi}$ is injective and its image consists of those two-planes in $T'F_{12}(\mathbb{C}^3)$ which are transverse to the fibres of the projections $p_1\colon F_{12}(\mathbb{C}^3) \to \mathbb{C}P(2)$ and $p_2\colon F_{12}(\mathbb{C}^3) \to \mathbb{C}P(2)^*$.*

Proof. Recall [2] that the Springer resolution of the nilpotent variety of $\mathfrak{sl}(3,\mathbb{C})$ is the map $\pi\colon T'^*F_{12}(\mathbb{C}^3) \to \mathfrak{sl}(3,\mathbb{C})$ defined as follows. Differentiation of the action of $SL(3,\mathbb{C})$ on $F_{12}(\mathbb{C}^3) = \mathbb{P}(\mathcal{O}_h)$ gives the map

$$\begin{aligned} \mathbb{P}(\mathcal{O}_h) \times \mathfrak{sl}(3,\mathbb{C}) &\longrightarrow T'\mathbb{P}(\mathcal{O}_h), \\ ([Y], A) &\longmapsto [Y, A]. \end{aligned}$$

To construct π, take the dual of this map, compose with projection to $\mathfrak{sl}(3,\mathbb{C})^*$ and then identify this with $\mathfrak{sl}(3,\mathbb{C})$ via the Killing form. Let $\mathbb{P}\pi$ be the map $\mathbb{P}T'^*F_{12}(\mathbb{C}^3) \to \mathbb{P}\mathfrak{sl}(3,\mathbb{C})$ induced by π. Then $\mathbb{P}\pi[\alpha] = \{ A : \alpha[Y, A] = 0 \}^\perp$ for $[\alpha] \in \mathbb{P}T'^*_{[Y]}\mathbb{P}(\mathcal{O}_h)$. Thus the above formula for $\tilde{\chi}$ shows that $\mathbb{P}\pi \circ \tilde{\chi} = \mathrm{Id}$ and $\tilde{\chi}$ is injective. The final statement now follows from an examination of the orbits of $SL(3,\mathbb{C})$ on $\mathbb{P}T'^*F_{12}(\mathbb{C}^3)$. □

We now turn to the proof of Proposition 1.2. Let $\mathcal{O}_2$ be the orbit of highest root vectors in $\mathfrak{g}_2^{\mathbb{C}}$. From the discussion above there is a triple cover $\phi\colon \mathbb{P}(\mathcal{O}_2) \to \mathbb{P}(\mathcal{O}_r \cup \mathcal{O}_h)$ branched over $\mathbb{P}(\mathcal{O}_h)$, which also gives a triple cover $\phi\colon G_2/SO(4) \to M_r \cup \mathbb{C}P(2)$ branched over $\mathbb{C}P(2)$. This map is $SL(3,\mathbb{C})$-equivariant and preserves the contact structures. By hypothesis the map $\gamma\colon \Sigma \to F_{12}(\mathbb{C}^3)$ has a contact lift to $\mathbb{P}T'^*F_{12}(\mathbb{C}^3)$. The contact distribution gives a rank two bundle η over Σ containing $T'\Sigma$ as a subbundle. Let $L = \eta/T'\Sigma$ and $\nu_\Sigma = \gamma^*T'F_{12}(\mathbb{C}^3)/T'\Sigma$. Transversality implies

$\eta \oplus \gamma^* L_1 \cong \gamma^* T' F_{12}(\mathbb{C}^3) \cong \eta \oplus \gamma^* L_2$, so $\gamma^* L_1 \cong \gamma^* L_2$, $L \oplus \gamma^* L_1 \cong \nu_\Sigma$ and the result follows.

Note that the condition that the inclusion be split may be rewritten as the vanishing of the corresponding cohomology class in $H^1(\gamma^* L_1 \otimes L^*)$. In principle one may determine this class directly in terms of the maps α and β, but it is still necessary to specify the splitting otherwise there need not be a unique choice for the lift of γ to $\mathbb{P}T''^* F_{12}(\mathbb{C}^3)$. For example, if $\nu_\Sigma \cong 2\gamma^* L_1$ then there is at least a complex one-dimensional family of such choices. Note also that a dimension count shows that the condition in Proposition 1.2 that $f(\Sigma)$ avoid $\mathbb{C}P(2)$ can always be satisfied by making a suitable choice of $SU(3)$ in G_2.

5. Quotient Constructions and the HyperKähler Potential

In [16, 17] Kronheimer describes semi-simple and nilpotent co-adjoint orbits as moduli spaces of instantons on non-compact four-manifolds and exhibits these orbits as infinite-dimensional hyperKähler quotients. Often finite-dimensional moduli spaces of instantons may also be constructed via a finite-dimensional hyperKähler quotient [1, 3]. We now explain how to obtain M_r from $\widetilde{\mathrm{Gr}}_4(\mathbb{R}^7)$ via the finite-dimensional quaternionic Kähler quotient construction of Galicki & Lawson [10] and prove Proposition 1.3.

In the previous section we noted that $G_2/SO(4)$ is the set of associative three-planes in $\mathbb{R}^7 = \operatorname{Im}\mathbb{O}$. If we identify $\widetilde{\mathrm{Gr}}_3(\mathbb{R}^7)$ with $\widetilde{\mathrm{Gr}}_4(\mathbb{R}^7)$ by sending V to the four-plane $W = V^\perp$, then the set of associative three-planes becomes the set of co-associative four-planes in $\operatorname{Im}\mathbb{O}$ [11].

Lemma 5.1. *Let W be an oriented four-plane in $\mathbb{R}^7$ and let $\{f_1, \dots, f_4\}$ be an oriented orthonormal basis of W. Then W is co-associative if and only if $\{f_1, \dots, f_4\}$ satisfies*

$$f_1 f_2 + f_3 f_4 = 0. \tag{5.1}$$

Proof. If W is co-associative we may use the action of G_2 to take $W^\perp$ to the three-plane spanned by i, j and k. It is then straightforward to verify the above identity. Conversely, if $f_1 f_2 + f_3 f_4 = 0$ then the co-associator $[f_1, f_2, f_3, f_4] = -8\,\mathrm{Alt}\langle f_2, f_3 f_4\rangle f_1$ vanishes and W is either co-associative or anti-co-associative [11, Theorem IV.1.18]. However, if W is anti-co-associative then one may show that $f_1 f_2 = f_3 f_4$ and hence $f_1 f_2 = 0$, which is a contradiction. □

The moment map for the action of a subgroup $G \leqslant SO(7)$ on $\widetilde{\mathrm{Gr}}_4(\mathbb{R}^7)$ is the map

$$\mu\colon \widetilde{\mathrm{Gr}}_4(\mathbb{R}^7) \to \mathfrak{g}^* \otimes \Lambda^2_+ \mathbf{W},$$

(where $\mathbf{W}$ is the tautological bundle) induced by composition of the inclusion $\Lambda^2_+ W \hookrightarrow \mathfrak{so}(7)$ with projection to $\mathfrak{g}$. (Note that it is more usual to consider $\Lambda^2_- W$, but these spaces differ only by a change in the orientation of W.)

Let x be a unit vector in $\operatorname{Im}\mathbb{O}$ and let ϕ be the G_2-invariant three-form on $\operatorname{Im}\mathbb{O}$ defined by $\phi(a,b,c) = \langle ab, c\rangle$. Then $\xi = x \lrcorner \phi$ is an element of $\Lambda^2\mathbb{R}^7 \cong \mathfrak{so}(7)$ and so defines a subgroup $U(1)$ of $SO(7)$. This subgroup $U(1)$ is the centre of the subgroup $U(3)$ preserving the complex structure defined by left multiplication by x on $x^\perp$. If $\{f_1,\dots,f_4\}$ is an oriented basis for a four-plane W, then this defines a basis for $\Lambda^2_+ W$ and the moment map for the $U(1)$-action is given by

$$\mu(W) = (\langle f_1\wedge f_2 + f_3\wedge f_4, \xi\rangle, \langle f_1\wedge f_3 + f_4\wedge f_2, \xi\rangle, \langle f_1\wedge f_4 + f_2\wedge f_3, \xi\rangle).$$

The quaternionic Kähler quotient construction now says that μ is invariant under the action of $U(1)$ and that if we restrict to an open set on which $U(1)$ acts freely, then $\mu^{-1}(0)/\,U(1)$ is a quaternionic Kähler manifold of dimension $\dim \widetilde{\mathrm{Gr}}_4(\mathbb{R}^7) - 4\dim U(1) = 8$. Note that we may rewrite the components of $\mu(W)$ as

$$\langle f_1\wedge f_2 + f_3\wedge f_4, \xi\rangle = \phi(f_1f_2 + f_3f_4, x), \qquad \text{etc.},$$

so equation (5.1) implies that the co-associative planes lie in $\mu^{-1}(0)$ and hence $U(1)\cdot(G_2/\,SO(4)) \subset \mu^{-1}(0)$.

Now $U(1)\cap G_2 \cong \mathbb{Z}_3$ and so to prove Proposition 1.3 it only remains to show that $\mu^{-1}(0) = U(1)\cdot(G_2/\,SO(4))$. Let W be a four-plane in $\mu^{-1}(0)$. Then $x^\perp \cap W$ is at least three-dimensional and we may choose an orthogonal set $\{u,v,w\}$ of vectors in $x^\perp\cap W$. If $\langle uv, x\rangle \neq 0$ replace v by $\langle uw, x\rangle v - \langle uv, x\rangle w$. Thus we may find an orthonormal basis $\{f_1,\dots,f_4\}$ of W such that x is orthogonal to f_1, f_2 and f_1f_2. Using the action of G_2 we may assume that $x = e$, $f_1 = i$ and $f_2 = j$ [11, Lemma IV.A.15]. By changing the choice of oriented basis of W we may also assume that $\langle f_4, e\rangle = 0$. Now $U(3)$ acts fixing e and contains a circle subgroup which also fixes i and j. Using this circle action we may take $\langle f_4, k\rangle = 0$ and choose the sign of $\langle f_4, ke\rangle$ independently of the sign of $\langle f_3, e\rangle$. Thus we may write

$$f_3 = \lambda k + a_0 e + a_1 ie + a_2 je + a_3 ke, \quad f_4 = b_1 ie + b_2 je + b_3 ke,$$

where λ, a_i, b_i are real and $a_0b_3 \leqslant 0$.

By Lemma 5.1 it is sufficient to show that $f_3f_4 = -f_1f_2 = -k$. The moment map equations imply

$$\lambda b_3 = 0, \quad b_2 = a_1 \quad \text{and} \quad b_1 = -a_2$$

and the fact that $\{f_3, f_4\}$ is orthonormal gives

$$a_3b_3 = 0, \quad \lambda^2 + a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1 \quad \text{and} \quad a_2^2 + a_1^2 + b_3^2 = 1.$$

The difference of the last two equations gives $b_3^2 = \lambda^2 + a_0^2 + a_3^2$. If $b_3 = 0$ then $\lambda^2 + a_0^2 + a_3^2 = 0$ so $\lambda = a_0 = a_3 = 0$ and $f_3f_4 = -k$, as required. If $b_3 \neq 0$ then $\lambda = 0 = a_3$ and $b_3^2 = a_0^2$. However we assumed that b_3 and a_0 had different signs so $b_3 = -a_0$ and one may now verify $f_3f_4 = -k$ completing the proof of Proposition 1.3.

Whenever one has a quaternionic Kähler quotient of M there is a corresponding hyperKähler quotient for $\mathcal{U}(M)$ [25]. Combining the above construction with the description [9] of $\widetilde{\mathrm{Gr}}_4(\mathbb{R}^7)$ as an $Sp(1)$-quaternionic Kähler quotient of $\mathbb{HP}(6)$ gives $\mathcal{U}(M_r)$ as a hyperKähler quotient of flat space. This is related to a construction of Kronheimer (private communication) for nilpotent orbits in $\mathfrak{sl}(n,\mathbb{C})$ as hyperKähler quotients (see below).

Since $\mathcal{O}_r \to M_r$ is precisely the bundle $\mathcal{U}(M_r)$ (see §3), not only is $\mathcal{O}_r$ a hyperKähler manifold, but its hyperKähler structure admits a *hyperKähler potential*: this is defined to be a function ϱ which is simultaneously a Kähler potential for each of the complex structures J of the hyperKähler structure on $\mathcal{O}_r$, that is each J satisfies $i\partial_J\bar{\partial}_J\varrho = \omega_J$. The hyperKähler metric then has the form $g = \nabla^2\varrho$. We calculate ϱ explicitly using the following construction of Kronheimer for the regular nilpotent orbit of $\mathfrak{sl}(3,\mathbb{C})$. Let $V_i = \mathbb{C}^i$, $i = 1,2,3$, let

$$M = \mathrm{Hom}(V_1,V_2) \oplus \mathrm{Hom}(V_2,V_1) \oplus \mathrm{Hom}(V_2,V_3) \oplus \mathrm{Hom}(V_3,V_2) \cong \mathbb{H}^8$$

and let $G = U(1) \times U(2)$, where $U(i)$ acts on $V_i = \mathbb{C}^i$ in the usual way. A point of M is a complex of linear maps

$$V_1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} V_2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} V_3$$

and moment maps for the action of G are given by

$$\begin{aligned}\mu^c &= (\beta_1\alpha_1, \beta_2\alpha_2 - \alpha_1\beta_1),\\ \mu^r &= (\beta_1\beta_1^* - \alpha_1^*\alpha_1, \alpha_1\alpha_1^* - \beta_1^*\beta_1 + \beta_2\beta_2^* - \alpha_2^*\alpha_2),\end{aligned}$$

where the hyperKähler moment map is $\mu = i\mu^r + j\mu^c \colon M \to \mathfrak{g}^* \otimes \mathrm{Im}\,\mathbb{H}$. Given such a complex define $X \in \mathrm{End}(\mathbb{C}^3)$ by $X = \alpha_2\beta_2$. If the complex lies in $\mu^{-1}(0)$ then we have

$$X^2 = \alpha_2\beta_2\alpha_2\beta_2 = \alpha_2\alpha_1\beta_1\beta_2$$

and $X^3 = 0$. Thus such elements correspond to the nilpotent variety of $\mathfrak{sl}(3,\mathbb{C})$ and we have $V_i \cong \mathrm{Im}\,X^{3-i}$ if each the α_i and β_i have maximal rank. Combining Kronheimer's construction with results in [25] gives

The open dense set of smooth points of $\mu^{-1}(0)/G$ is a hyperKähler manifold isomorphic to $\mathcal{O}_r$ and the quotient of this set by the right-action of $\mathbb{H}^*$ is the quaternionic Kähler manifold M_r.

Since the proof of this is little different from the general case we refer the interested reader to [15] where this result is extended to all classical groups.

To proceed we need the following lemma.

Lemma 5.2. *Under the action of $G \times SU(3)$, the α_i and the endomorphism X are equivalent to upper triangular matrices.* □

Since we are assuming that the α_i are injective, the complex moment map equations now imply

$$\alpha_1 = \begin{pmatrix} \alpha_{111} \\ 0 \end{pmatrix}, \qquad \alpha_2 = \begin{pmatrix} \alpha_{211} & \alpha_{212} \\ 0 & \alpha_{222} \\ 0 & 0 \end{pmatrix},$$

$$\beta_1 = \begin{pmatrix} 0 & \beta_{112} \end{pmatrix}, \qquad \beta_2 = \begin{pmatrix} 0 & \beta_{212} & \beta_{213} \\ 0 & 0 & \beta_{223} \end{pmatrix}.$$

The remaining moment map equations are now

$$\beta_{212}\alpha_{222} = \alpha_{111}\beta_{112}, \qquad \beta_{213}\overline{\beta_{223}} = \overline{\alpha_{211}}\alpha_{212}, \qquad |\alpha_{111}|^2 = |\beta_{112}|^2,$$

$$|\beta_{112}|^2 + |\alpha_{212}|^2 + |\alpha_{222}|^2 = |\beta_{223}|^2, \qquad |\alpha_{111}|^2 + |\beta_{212}|^2 + |\beta_{213}|^2 = |\alpha_{211}|^2.$$

Proposition 5.3. *At*

$$X = \alpha_2\beta_2 = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$$

in the regular nilpotent orbit of $\mathfrak{sl}(3,\mathbb{C})$, *the value of the hyperKähler potential is*

$$\varrho(X) = 2\sqrt{(|a|^{2/3} + |c|^{2/3})^3 + |b|^2}.$$

Proof. The hyperKähler potential on $\mu^{-1}(0)/G$ is the image of the restriction of the radial function r^2 on $M = \mathbb{H}^8$. So the moment map equations imply

$$\varrho(X) = \mathrm{Tr}((\alpha_1\alpha_1^* + \beta_1^*\beta_1) \oplus (\alpha_2\alpha_2^* + \beta_2^*\beta_2)) = 2(|\alpha_{211}|^2 + |\beta_{223}|^2).$$

Define $R = |\alpha_{211}|^2$, $S = |\beta_{223}|^2$ and $T = |\beta_{212}|^2$ so that $\varrho(X) = 2(R+S)$. Also, let $A = |a|^2$, $B = |b|^2$ and $C = |c|^2$.

For X to lie in the regular orbit of $\mathfrak{sl}(3,\mathbb{C})$ we must have that both a and c are non-zero, which implies that β_{212} and β_{223} are also non-zero. Thus we may write $\alpha_{211} = a/\beta_{212}$, $\alpha_{222} = c/\beta_{223}$, $\alpha_{212} = (b\beta_{212} - a\beta_{213})/(\beta_{212}\beta_{223})$ and $|\alpha_{111}|^4 = |\beta_{112}|^4 = |\alpha_{111}|^2|\beta_{112}|^2 = CT/S$. The second moment map equation now gives $\beta_{213} = \overline{a}b\beta_{212}/(ST+A)$ and hence by substituting into the previous equation we obtain $\alpha_{212} = bT\overline{\beta_{223}}/(ST+A) = bT\overline{\beta_{223}}/(R+S)$. The last two moment map equations give

$$|\alpha_{111}|^2 + \frac{A}{R} = R - \frac{BR}{(R+S)^2}, \tag{5.2}$$

$$|\beta_{112}|^2 + \frac{C}{S} = S - \frac{BS}{(R+S)^2}. \tag{5.3}$$

Thus, using $T = A/R$, we obtain

$$\frac{R}{S} = \frac{|\alpha_{111}|^2 + A/R}{|\beta_{112}|^2 + C/S} = \frac{\sqrt{\frac{AC}{RS}} + \frac{A}{R}}{\sqrt{\frac{AC}{RS}} + \frac{C}{S}} = \sqrt{\frac{AS}{CR}},$$

which implies

$$\left(\frac{R}{S}\right)^3 = \frac{A}{C}. \tag{5.4}$$

Subtracting (5.3) from (5.2) and multiplying through by R gives

$$\left(A - \frac{R}{S}C\right) = \left(1 - \frac{S}{R}\right)\left(R^2 - \frac{B}{\left(\frac{S}{R}+1\right)^2}\right).$$

Hence by (5.4) we have

$$R^2 = A^{2/3}\frac{(A^{1/3}+C^{1/3})^3+B}{(A^{1/3}+C^{1/3})^2}$$

and the result follows. □

The $\mathbb{H}^*$-action on $M \cong \mathbb{H}^8$ is given by right multiplication. This action descends to the hyperKähler quotient $\mu^{-1}(0)/G$ and the manifold $(\mu^{-1}(0)/G)/\mathbb{H}^*$ is precisely M_r (see [25]). For a three-plane $V \in M_r$, $V_{\mathbb{C}}$ is spanned by the isotropic elements. However, these isotropic elements are precisely one orbit of the $\mathbb{H}^*$-action and we obtain:

Proposition 5.4. *For $X = \alpha_2\beta_2 \in \mu^{-1}(0)$ the complexification of the corresponding three-plane in M_r is spanned by $\alpha_2\beta_2$, $\beta_2^*\alpha_2^*$ and $\beta_2^*\beta_2 - \alpha_2\alpha_2^*$. In particular, one obtains the three-planes $V(x,y)$ used in §2.*

Proof. It only remains to verify the last assertion. Let $X = \begin{pmatrix} 0 & x^3 & 0 \\ 0 & 0 & y^3 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}(e_1 + ie_2)$. Then e_3 is proportional to $\beta_2^*\beta_2 - \alpha_2\alpha_2^*$. By the proof of the previous proposition $\beta_2^*\beta_2 - \alpha_2\alpha_2^* = \operatorname{diag}(-R, R-S, S)$ and $R = x^2\sqrt{x^2+y^2}$, $S = y^2\sqrt{x^2+y^2}$. □

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