VECTOR BUNDLES AND THE BRAUER INDUCTION THEOREM

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Arunas Liulevicius[1]

The aim of this paper is to show that the language of vector bundles is useful in thinking about representations of groups. We give a simple and self-contained proof of the Brauer induction theorem which states that if G is a finite group, then each complex representation of G is a sum of virtual representations induced up from elementary subgroups of G.

The paper is organized as follows: §1 starts with the structure theorem on finite G-sets, introduces the notion of a family of orbits, and discusses topological induction; §2 defines the notion of a G-vector bundle over a G-set and introduces the pullback and transfer constructions on vector bundles - the key result here is the Mackey theorem which shows that the two operations commute in a pullback situation. Irreducible vector bundles are introduced, and Schur's Lemma allows the calculation of inner products. Topological induction and transfer combine to give us Frobenius induction of vector bundles, which is shown to satisfy Frobenius reciprocity. It is shown that if H is a finite group having an abelian normal subgroup with quotient a p-group, then each vector bundle over a finite H-set B is the transfer along an H-map $f: B' \longrightarrow B$ of a line bundle L over B'. In §3 the language of equivariant K-theory is introduced, as well as characters of representations, and Brauer's induction theorem is proved.

The language of K-theory has been introduced by M.F. Atiyah,

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F. Hirzebruch and G.B. Segal [1],[2],[3],[11]. The Burnside ring has been introduced in the study of induction theorems by A. Dress [8] and has been shown to be very useful in studying actions of compact Lie groups by T. tom Dieck [7]. Brauer's theorem was first proved in [4], and a new proof was presented by Brauer and Tate in [5]. Our proof is inspired by D.M. Goldschmidt, I.M. Isaacs and L. Solomon [10] (also see [6],[9]). Our treatment of characters is of course influenced by J.-P. Serre [12].

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1. Topological induction. A function $\beta: G\times B \longrightarrow B$ is called a left action of G on B if $\beta(g,\beta(h,b)) = \beta(gh,b)$, $\beta(e,b) = b$ for all g,h in G and all b in B, where e denotes the identity element of G. We normally write $\beta(g,b) = g.b = gb$. If $H \subset G$ is a subgroup, we define a left action of G on G/H by setting $g.g'H = gg'H$, and we call G/H with this action a standard orbit. If X and Y are G-sets, a function $f: X \longrightarrow Y$ is said to be a G-map if for all g in G we have $f(g.x) = g.f(x)$ for each x in x. We denote the set of all G-maps from X to Y by $\mathrm{Map}_G(X,Y)$. We will now determine the set $\mathrm{Map}_G(G/H,B)$. We let B^H be the set of all b in B such that $h.b = b$ for all h in H. We call B^H the fixed point set of H in B.

LEMMA 1. Evaluation at eH gives a one-to-one correspondence $\mathrm{Map}_G(G/H,B) \approx B^H$.

Proof. The result is immediate, since a G-map $f: G \longrightarrow B$ is determined by $b = f(e)$, and f induces a G-map $\underline{f}: G/H \longrightarrow B$ if and only if b is in B^H.

COROLLARY 2. $\mathrm{Map}_G(G/H,G/K) = \{gK \mid H \subset gKg^{-1}\}$.

A G-map $f: G/H \longrightarrow G/K$ determined by $f(eH) = aK$ with $i: H \subset aKa^{-1}$ factors

$$\begin{tikzcd}
G/H \arrow[r, "\underline{i}"] \arrow[rd, "f"'] & G/aKa^{-1} \arrow[d] \\
& G/K
\end{tikzcd}$$

with the vertical map induced by right multiplication by a in G, so is a G-equivalence. This explains why we will emphasize the maps $\underline{j}\colon G/H \longrightarrow G/J$ induced by inclusions of subgroups $j\colon H \longrightarrow J$.

COROLLARY 3. Two standard orbits G/H and G/K are G-equivalent if and only if H and K are conjugate subgroups of G.

We call a G-equivalence class of standard orbits an orbit type of G. This means that an orbit type of G corresponds to a conjugacy class of subgroups in G. Given a G-set B, we say that b and b' in B are in the same orbit if there exists a g in G such that $g.b = b'$. Notice that if $G_b = \{g \in G \mid g.b = b\}$ is the isotropy subgroup at b, then $.b\colon G \longrightarrow B$ induces a G-equivalence $.b\colon G/G_b \longrightarrow$ (orbit of b in B). Notice that $G_{hb} = hG_bh^{-1}$, so an orbit in B corresponds to a conjugacy class of subgroups in G, or equivalently to an orbit type of G. We let $fo(G)$ be the set of finite orbit types of G. If X is a finite G-set, then it determines a counting function

$$(X)\colon fo(G) \longrightarrow Z$$

defined by $(X)(\text{type } G/H) =$ number of orbits in X of type G/H. The function (X) is finitely non-zero and takes values in the set of natural numbers.

THEOREM 4. If X and Y are finite G-sets, then X is G-equivalent to Y if and only if $(X) = (Y)$.

We let $A(G)$ be the set of all finitely non-zero functions $fo(G) \longrightarrow Z$. We define addition in $A(G)$ by pointwise addition, or equivalently by setting $(X) + (Y) = (X \sqcup Y)$, the counting function of the disjoint sum of X with Y. We define multiplication by setting $(X) \times (Y) = (X \times Y)$, where we give $X \times Y$ the diagonal action of G: $g.(x,y) = (g.x,g.y)$. This defines a ring structure on $A(G)$ - it is called the Burnside ring of G.

If $f_1 \colon X_1 \longrightarrow X'$ and $f_2 \colon X_2 \longrightarrow X'$ are G-maps, then the pullback $X = \{(x_1,x_2) \in X_1 \times X_2 \mid f_1(x_1) = f_2(x_2)\}$ with maps $p_i(x_1,x_2) = x_i$, $i = 1,2$

$$\begin{tikzcd}
X \arrow[r, "p_2"] \arrow[d, "p_1"] & X_2 \arrow[d, "f_2"] \\
X_1 \arrow[r, "f_1"] & X'
\end{tikzcd}$$

has a unique G-structure making the maps p_i into G-maps, namely $g.(x_1,x_2) = (gx_1,gx_2)$. The case of the pullback of G-maps induced by inclusions of subgroups

$$\begin{tikzcd}
 & H_2 \arrow[d] \\
H_1 \arrow[r] & H'
\end{tikzcd}$$

that is

$$\begin{tikzcd} & G/H_2 \arrow[d] \\ G/H_1 \arrow[r] & G/H' \end{tikzcd}$$

is the archetypal case, as we have already seen. The reader should check that the diagram

$$\begin{tikzcd} G/(H_1 \cap H_2) \arrow[r] \arrow[d] & G/H_2 \arrow[d] \\ G/H_1 \arrow[r] & G/H' \end{tikzcd}$$

is a pullback diagram if and only if $H' = H_1H_2$. We shall examine the general case later once we introduce the notion of topological induction.

We now introduce the notion of a family of orbits. A set of orbit types $\mathcal{F}$ is called a family if whenever we have a G-map $G/H \longrightarrow G/K$ and type G/K is in $\mathcal{F}$ then type G/H is also in $\mathcal{F}$. Equivalently, a family is determined by a set of conjugacy classes of subgroups having the property that if K is in the family then every subgroup of K is in the family as well. If $\mathcal{F}$ is a family of orbit types, we say that a G-set X is a $\mathcal{F}$-set if all orbit types in X are in the family $\mathcal{F}$. Notice that if $f: X \longrightarrow Y$ is a G-map and Y is a $\mathcal{F}$-set, then X is also a $\mathcal{F}$-set. This means that the set of Z-linear combinations of (X) in the Burnside ring $A(G)$ of $\mathcal{F}$-sets X is an ideal $A_{\mathcal{F}}(G)$ of $A(G)$, since if X is a $\mathcal{F}$-set then $X \times Y$ is a

$\mathcal{F}$-set, since projection to X is a G-map. Suppose we are given a pullback diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_2 \arrow[d] \\ X_1 \arrow[r] & X' \end{tikzcd}$$

and X_1 is an $\mathcal{F}$-set, then X is a $\mathcal{F}$-set. Of course the special case with X = point was discussed just above.

Let us write G-Set for the category of G-sets and G-maps. If $f\colon G \longrightarrow G'$ is a homomorphism of groups, we have an obvious functor $f^*\colon G'\text{-Set} \longrightarrow G\text{-Set}$ called restriction along f: if X' is a G'-set, we define f^*X' to be the same set X' with the G-action $g.x' = f(g)x'$. We also define a functor $f_*\colon G\text{-Set} \longrightarrow G'\text{-Set}$ (called induction along f) by setting $f_*X = (G' \times X)/G$, where G acts on the right of $G' \times X$ by $(g',x).g = (g'f(g),g^{-1}x)$. If we let $[g',x]$ denote the orbit of (g',x) under this right action of G, then we define the left action of G' on f_*X by $g''.[g',x] = [g''g',x]$. There is an obvious G-map $i\colon X \longrightarrow f^*f_*X$ given by $i(x) = [e,x]$.

THEOREM 5. Composition with $i\colon X \longrightarrow f^*f_*X$ gives a natural one-to-one correspondence

$$\varphi\colon \mathrm{Map}_{G'}(f_*X,X') \longrightarrow \mathrm{Map}_G(X,f^*X').$$

<u>Proof</u>. Let us define a map $\psi\colon \mathrm{Map}_G(X,f^*X') \longrightarrow \mathrm{Map}_G(f_*X,X')$

as follows. Given a G-map $k: X \longrightarrow f^*X'$, consider the map $G' \times X \longrightarrow X'$ given by $(g',x) \to g'k(x)$. Since $g'f(g)k(x) = g'k(gx)$, this map induces a G'-map $\psi(k): f_*X \longrightarrow X'$. It is immediate that $\psi = \varphi^{-1}$.

A homomorphism of groups $f: G \longrightarrow G'$ induces a ring homomorphism $f^*: A(G') \longrightarrow A(G)$. If the image of f is of finite index in G', then f induces $f_*: A(G) \longrightarrow A(G')$. The map f_* is a homomorphism of the additive groups, but in general is not a homomorphism of rings. For example, it maps the unit element of $A(G)$ into the element $(G'/\text{Im } f)$. It does have one nice property for multiplication which we now study.

THEOREM 6. Let $f: G \longrightarrow G'$ be a homomorphism of groups, X a G-set, Y a G'-set, then $(f_*X) \times Y$ with the diagonal action is G-equivalent to $f_*(X \times f^*Y)$.

<u>Proof</u>. Define $K: f_*(X \times f^*Y) \longrightarrow (f_*X) \times Y$ by $K[g',x,y] = ([g',x],g'y)$, and $L: (f_*X) \times Y \longrightarrow f_*(X \times f^*Y)$ by $L([g',x],y) = [g',x,g'^{-1}y]$. It is immediate that $L = K^{-1}$.

COROLLARY 7. If $f: G \longrightarrow G'$ is a homomorphism of groups with $\text{Im } f$ of finite index in G', then the homomorphism of additive groups $f_*: A(G) \longrightarrow A(G')$ has the property: $f_*(u.f^*v) = f_*(u).v$.

COROLLARY 8. If $f: G \longrightarrow G'$ as above, then $f_*f^*v = (G'/\text{Im } f).v$.

If B is a G-set with action $\beta: G \times B \longrightarrow B$, then β

induces a map $\beta: i_*i^*B \longrightarrow B$, where $i: H \longrightarrow G$ is the inclusion of a subgroup H of G. Now Theorem 6 gives a G-equivalence $K: i_*i^*B \longrightarrow G/H \times B$, and under K the map β corresponds to projection into the second factor B.

If $i: H \longrightarrow G$ is the inclusion of a subgroup, then the following alternative notation is common, and we shall use it freely: $i_*X = G\times_H X$. Since i_*: H-Sets $\longrightarrow$ G-Sets is completely determined by what it does on orbits, we have the following description of i_*.

LEMMA 9. $i_*H/K = G/K$.

Proof. The inclusion $i: H/K \longrightarrow G/K$ defines a G-map $I: i_*H/K \longrightarrow G/K$ by $I[g,hK] = ghK$. The point $[e,eK]$ in i_*H/K defines a G-map $J: G/K \longrightarrow i_*H/K$. We have $IJ(gK) = I[g,eK] = gK$, and $JI[g,hK] = J\ ghK = [gh,eK] = [g,hK]$.

Here is another application of Theorem 6. If H and J are subgroups of G with $i: H \longrightarrow G$ the inclusion, then Theorem 6 says that i_*i^*G/J is G-equivalent to $G/H \times G/J$ with the diagonal action. This means that if we know the H-orbit structure of i^*G/J that is the double coset decomposition $G = \coprod_k Hg_kJ$ then

$$i^*G/J \approx \coprod_k H/(g_kHg_k^{-1} \cap J),$$

and according to Lemma 9

$$G/H \times G/J \approx \coprod_k G/(g_kHg_k^{-1} \cap J),$$

where of course $G/(g_k H g_k^{-1} \cap J)$ corresponds to the orbit of $(g_k H, eJ)$ in $G/H \times G/J$.

The notion of induction will help us to analyse the pullback P in the following diagram

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & G/H_2 \arrow[d, "\underline{i}_2"] \\
G/H_1 \arrow[r, "\underline{i}_1"] & G/H',
\end{tikzcd}$$

where $i_1 \colon H_1 \rightarrow H'$ and i_2 are inclusion maps. Define maps

$$f_k \colon H'/H_1 \times H'/H_2 \rightarrow G/H_k$$

for $k = 1,2$ by first projecting into H'/H_k and then including into G/H_k. Notice that the f_k define a map into P, since $\underline{i}_1\underline{f}_1 = \underline{i}_2\underline{f}_2$ is the constant map to the point H'/H'. This means that we have a G-map $F\colon G \times_{H'} (H'/H_1 \times H'/H_2) \rightarrow P$.

LEMMA 10. The map F is a G-equivalence.

Proof. The map F is given by $F[g, aH_1, bH_2] = (gaH_1, gbH_2)$. First we show that the map is onto P. Suppose we have $(gH_1, g'H_2)$ with $gH' = g'H'$, that is there is an h' in H' with $gh' = g'$. Then $F\,[g, eH_1, h'H_2] = (gH_1, g'H_2)$. Second, we show that F is one-to-one: say $F[g, aH_1, bH_2] = F[g', a'H_1, b'H_2]$, that is $ga = g'a'h_1$, $gb = g'b'h_2$ for some h_i in H_i. We want an x in H' with $g' = gx$, $aH_1 = xa'H_1$, $bH_2 = xb'H_2$. Now $g' = gah_1^{-1}a'^{-1}$,

$aH_1 = ah_1^{-1}H_1$, $gb = g'b'h_2 = gah_1^{-1}a'^{-1}b'h_2$, so $b = ah_1^{-1}a'^{-1}b'h_2$, thus setting $x = ah_1^{-1}a'^{-1}$ shows that $aH_1 = xa'H_1$, $bH_2 = xb'H_2$, so we have $[g, aH_1, bH_2] = [g', a'H_1, b'H_2]$, and F is one-to-one.

It is also useful to remark that if $f': G' \longrightarrow G''$ is a homomorphism, then $(f'f)^* = f^*f'^*$, $(f'f)_* = f'_*f_*$, $1^* = 1_* = 1$ for the identity map $1: G \longrightarrow G$. That is, restriction is a contravariant functor, induction is a covariant functor.

If $p': E' \longrightarrow G \times_H X$ is a G-map, we consider the pullback diagram

$$\begin{tikzcd}
E = i^! E' \arrow[r, "\tilde{i}"] \arrow[d, "p"] & E' \arrow[d, "p'"] \\
X \arrow[r, "i"] & i_* X
\end{tikzcd}$$

where $i^!E' = \{(x, e') \in X \times E' \mid ix = p'e'\}$, p and $\tilde{i}$ are the restrictions of the projection maps.

THEOREM 11. The map $\tilde{i}$ defines a G-equivalence $I: i_*E \relbar\relbar E'$ which makes the following diagram commute:

$$\begin{tikzcd}
i_*E \arrow[r, "I"] \arrow[d, "i_*p"] & E' \arrow[d, "p'"] \\
i_*X \arrow[r, "1"] & i_*X .
\end{tikzcd}$$

That is, G-maps over i_*X correspond to H-maps over X.

Proof. The map I is given by $I([g,x,e']) = ge'$. We wish to exhibit an inverse $J\colon E' \longrightarrow i_*E$. If $p'(e') = [g,x]$, we set $J(e') = [g,x,g^{-1}e']$. Notice that this is well-defined, for if h is in H, then $[g,x] = [gh,h^{-1}x]$, but then $[gh,h^{-1}x,h^{-1}g^{-1}e'] = [g,x,g^{-1}e']$. It is immediate that $J = I^{-1}$.

2. Vector bundles over G-sets and the transfer. A complex vector bundle p over a G-set B is a G-map $p: E(p) \longrightarrow B$ such that for each b in B the fiber $p_b = p^{-1}(b)$ has the structure of a complex vector space and for each g in G the map $g_*: p_b \longrightarrow p_{gb}$ is a C-linear map. The set $E(p)$ is called the total space of the bundle p. Notice that the fiber p_b over b is a complex representation of the isotropy group G_b. We shall now see that this completely determines p over each orbit.

COROLLARY 12. A G-vector bundle over G/H has the form $Gx_H M$, where M is the fiber over eH.

Proof. This is Theorem 11 applied to the special case X = point.

More generally:

COROLLARY 13. If $i: H \longrightarrow G$ is the inclusion of a subgroup, and $p': E(p') \longrightarrow i_* X$ is a G-vector bundle over $i_* X$ then there exists a unique H-vector bundle p over X such that $p' = i_* p$.

Proof. This is of course Theorem 11 over again. The uniqueness of p follows from the fact that p is the restriction of p' to the subset X under the inclusion $i: X \longrightarrow i_* X$.

Let us denote by $\mathrm{Vect}_G B$ the set of G-vector bundles over B. We have just shown the fact that if $i: H \longrightarrow G$ is the inclusion of a subgroup and X is an H-set, then topological induction

$$i_*: \mathrm{Vect}_H X \longrightarrow \mathrm{Vect}_G\, i_* X$$

is a one-to-one correspondence.

If $f\colon Y \longrightarrow X$ is a G-map, $p \in \mathrm{Vect}_G Y$, $q \in \mathrm{Vect}_G X$, we say that $F\colon p \longrightarrow q$ is a map of G vector bundles over f if we have a commutative diagram

$$\begin{tikzcd}
E(p) \arrow[r, "F"] \arrow[d, "p"] & E(q) \arrow[d, "q"] \\
Y \arrow[r, "f"] & X
\end{tikzcd}$$

F is a G-map, and for each y in Y the restriction $F_y\colon p_y \longrightarrow q_{f(y)}$ is a C-linear map.

If $f\colon Y \longrightarrow X$ is a G-map, $q \in \mathrm{Vect}_G X$, then the pullback $f^! q$ of q over f is a vector bundle over Y, $\bar{f}\colon f^! q \longrightarrow q$ is a map of vector bundles over f, moreover any map of G-vector bundles $F\colon p \longrightarrow q$ over f factors uniquely as a map $\bar{F}\colon p \longrightarrow f^! q$ over the identity map of Y followed by $\bar{f}\colon f^! q \longrightarrow q$.

$$\begin{tikzcd}
E(p) \arrow[r, "\bar{F}"] \arrow[d, "p"] & E(f^! q) \arrow[r, "\bar{f}"] \arrow[d, "f^! q"] & E(q) \arrow[d, "q"] \\
Y \arrow[r, "1"] & Y \arrow[r, "f"] & X
\end{tikzcd}$$

This takes care of maps into q over f, but what about maps from a bundle p over f? We shall construct a G-vector bundle $f_! p$ over X (called the transfer of p along f) which is defined up to isomorphism by the following property: there is a map of vector bundles $\underline{f}\colon p \longrightarrow f_! p$ such that if $f\colon p \longrightarrow q$ is a map of vector bundles over f, then there exists a unique map of vector bundles $\underline{F}\colon f_! p \longrightarrow q$ such that $F = \underline{F}\,\underline{f}$:

$$\begin{tikzcd}
E(p) \arrow[r, "\underline{f}"] \arrow[d, "p"] & E(f_! p) \arrow[r, "\underline{F}"] \arrow[d, "f_! p"] & E(q) \arrow[d, "q"] \\
Y \arrow[r, "f"] & X \arrow[r, "1"] & X
\end{tikzcd}$$

PROPOSITION 14. The transfer $f_! p$ exists.

Proof. We define $f_! p\colon E(f_! p) \longrightarrow X$ by setting

$$(f_! p)_x = \bigoplus_{f(y)=x} p_y$$

and the map $\underline{f}\colon p \longrightarrow f_! p$ to be the structure map into the direct sum $f_y\colon p_y \longrightarrow (f_! p)_x$. Given a map $F\colon p \longrightarrow q$ of G vector bundles over f, the definition of $\underline{F}_x\colon (f_! p)_x \longrightarrow q_x$ is forced on us, since on the summand p_y with $f(y) = x$ it must be $F_y\colon p_y \longrightarrow q_x$.

We now have two constructions: given $f\colon Y \longrightarrow X$ a G-map, we have the pullback over f:

$$f^!\colon \mathrm{Vect}_G X \longrightarrow \mathrm{Vect}_G Y$$

and the transfer over f:

$$f_!: \ \mathrm{Vect}_G Y \longrightarrow \mathrm{Vect}_G X.$$

Our first result is that transfers behave nicely with respect to topological induction.

PROPOSITION 15. Let $i: H \longrightarrow G$ be a subgroup, $f: Y \longrightarrow X$ a map of H-sets, then the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Vect}_H Y \arrow[r, dash, "i_*"] \arrow[d, "f_!"] & \mathrm{Vect}_G i_* Y \arrow[d, "(i_* f)_!"] \\
\mathrm{Vect}_H X \arrow[r, dash, "i_*"] & \mathrm{Vect}_G i_* X
\end{tikzcd} .$$

<u>Proof</u>. We have to show that $i_*(f_! p) = (i_* f)_!(i_* p)$ for each vector bundle $p \in \mathrm{Vect}_H Y$. The idea is simple - we check that the left hand side has the defining property of the right hand side. Given a commutative diagram

$$\begin{tikzcd}
i_* E(p) \arrow[rr, "F"] \arrow[dr] \arrow[dd, "i_* p"] & & E(q) \arrow[dd, "q"] \\
& i_* E(f_! p) \arrow[ur, dashed, "\underline{F}"'] \arrow[d, "i_*(f_! p)"] & \\
i_* Y \arrow[r, "i_* f"] & i_* X \arrow[r, "1"] & i_* X
\end{tikzcd} ,$$

with F a map of G-vector bundles, we have to show that there exists a unique G-vector bundle map $\underline{F}: i_*(f_! p) \longrightarrow q$ covering the identity of $i_* X$. We use Theorem 5 which says that i_* is left adjoint to restriction i^*: $\mathrm{Map}_G(i_*U,V) = \mathrm{Map}_H(U,i^*V)$. The above diagram becomes

$$\begin{tikzcd}
E(p) \arrow[rr, "F'"] \arrow[dr] \arrow[dd, "p"] & & i^*E(q) \arrow[dd, "i^*q"] \\
& E(f_!p) \arrow[ur, dashed, "\underline{F}'"] \arrow[d, "f_!p"] & \\
Y \arrow[r, "f"] & X \arrow[r, "1"] & X
\end{tikzcd}$$

where now we use the universal property of $f_!p$: there is a unique H-bundle map $\underline{F}'$: $f_!p \longrightarrow i^*q$ making this diagram commute. The adjoint of $\underline{F}'$ is our desired $\underline{F}$: $i_*(f_!p) \longrightarrow q$. The key result about transfers is that they work nicely in case we have a pullback diagram:

THEOREM 16. (Mackey). Given a pullback diagram of G-maps

$$\begin{tikzcd}
X \arrow[r, "f_2"] \arrow[d, "f_1"] & X_2 \arrow[d, "g_2"] \\
X_1 \arrow[r, "g_1"] & X'
\end{tikzcd}$$

then we have $g_1^! g_{2!} = f_{1!} f_2^!$.

<u>Proof</u>. Let p be a vector bundle over X_2, then we have

$$(g_1^! g_{2!} p)_{x_1} = (g_{2!} p)_{g_1(x_1)} =$$

$$= \bigoplus_{\substack{(x_1,x_2) \\ g_1(x_1)=g_2(x_2)}} p_{x_2}$$

$$= \bigoplus_{f_1(x)=x_1} (f_2^! p)_x$$

$$= (f_{1!} f_2^! p)_{x_1} \quad ,$$

so $g_1^! g_{2!} p = f_{1!} f_2^! p$ as claimed.

We will now study tensor products of vector bundles and will see how they behave under transfers. First, if $p \in \mathrm{Vect}_G B$, $p' \in \mathrm{Vect}_{G'} B'$, we define $p \times p' \in \mathrm{Vect}_{G \times G'} B \times B'$ by setting $(p \times p')_{(b,b')} = p_b \otimes p'_{b'}$. We define the internal tensor product of $p, p' \in \mathrm{Vect}_G B$ by setting $(p \otimes p')_b = p_b \otimes p_b$. That is, $p \otimes p'$ is the pullback of $p \times p'$ under the diagonal map $d: B \text{ —— } B \times B$, $d(b) = (b,b)$.

Now suppose $f': B' \longrightarrow B$ und $f'': B'' \longrightarrow B$ are G-maps, then we have a pullback diagram

$$\begin{tikzcd}
D \arrow[r, "d_2"] \arrow[d, "d_1"] & B' \times B'' \arrow[d, "f' \times f''"] \\
B \arrow[r, "d"] & B \times B
\end{tikzcd}$$

where $D = \{(b,b',b'') \in B \times B' \times B'' \mid f'(b') = b = f''(b'')\}$ and $d_1(b,b',b'') = b$, $d_2(b,b',b'') = (b',b'')$. Notice that $G_{(b,b',b'')} =$

$G_b \cap G_{b'} \cap G_{b''}$, so if we are given a family $\mathcal{F}$ and B' is a $\mathcal{F}$-set, then D is a $\mathcal{F}$-set as well.

COROLLARY 17. Suppose $\mathcal{F}$ is a family and B' is a $\mathcal{F}$-set, $f': B' \longrightarrow B$, $f'': B'' \longrightarrow B$ are G-maps, then

$$f'_! p' \otimes f''_! p'' = d_{1\,!}(d_2^!(p' \times p''))$$

is again the transfer of a bundle over a $\mathcal{F}$-set.

It is helpful to examine the special case of $f'' = 1: B \longrightarrow B$, the identity map of B.

COROLLARY 18. Suppose $p \in \mathrm{Vect}_G B$, $p' \in \mathrm{Vect}_G B'$, $f: B' \longrightarrow B$ a G-map. Then $f_!(p' \otimes f^! p) = (f_! p') \otimes p$.

<u>Proof</u>. Here $D = \{(b,b',b) \in B \times B' \times B \mid f(b') = b\}$, so we can identify it with B' by mapping (b,b',b) to b'. In the pullback diagram

$$\begin{tikzcd} B' \arrow[r, "(1\times f)d'"] \arrow[d, "f"] & B' \times B \arrow[d, "f \times 1"] \\ B \arrow[r, "d"] & B \times B \end{tikzcd}$$

we have

$$(f_! p') \otimes p = d^!(f \times 1)_!(p' \times p)$$
$$= f_! d'^!(1 \times f)^!(p' \times p)$$

$$= f_! d'^! (p' \times f^! p)$$

$$= f_! (p' \otimes f^! p),$$

as was to be shown.

Since we will be interested in finite dimensional vector bundles, we will assume that the base spaces B are finite sets (otherwise the transfer of a finite dimensional vector bundle need not be finite dimensional). From now on we will assume that the group G is finite, as well. These correspond to assuming compactness in the topological setting.

PROPOSITION 19. If $p \in \mathrm{Vect}_G B$ and G is finite, then there exists a G-invariant inner product on p.

Proof. Choose for each x in B a Hermitian inner product $(\, , \,)_x : p_x \times p_x \longrightarrow C$, and define for u,v in p_x

$$<u,v> = \sum_{y \in G} (y.u, y.v)_{yx} .$$

We notice that if g is in G, then $<gu,gv> = <u,v>$, so $< , >$ is an invariant inner product on p.

If $p,p' \in \mathrm{Vect}_G B$, then p' is called a subbundle of p if $p'_x \subset p_x$ for all x in B, and in this case we will write $p' \subset p$.

COROLLARY 20. If $p' \subset p \in \mathrm{Vect}_G B$, then there exists a $p'' \subset p$ such that $p = p' \oplus p''$.

Proof. Let $< , >$ be a G-invariant inner product on p, and set $p_x'' = \{u \in p_x | <u,v> = 0$ for all $v \in p_x'\}$. Then $p'' \subset p$ and $p = p' \oplus p''$.

A non-zero vector bundle p over B is said to be irreducible if the only subbundles of p are 0 and p itself.

LEMMA 21. (Schur's Lemma). If $f: p \longrightarrow q$ is a homomorphism of vector bundles, then

1) if p is irreducible, then either f is 0 or $\mathrm{Ker}\, f = 0$,

2) if q is irreducible, then either f is 0 or $\mathrm{Im}\, f = q$.

Schur's Lemma allows us to determine the irreducibles over G-sets for G an abelian group very easily.

COROLLARY 22. Let A be an abelian group, B an A-set, $\beta \in \mathrm{Irr}_A B$. Then there exists a homomorphism $L: D \longrightarrow C^x$ of a subgroup $j: D \longrightarrow A$ into the multiplicative group of nonzero complex numbers and an A-map $f: A/D \longrightarrow B$ such that $f_! j_* L = \beta$.

Proof. Let A/D be the unique orbit of B on which β is non-zero. Notice that according to Schur's Lemma each d in D acts on β as a scalar multiple $L(d)1$ of the unit. The function $L: D \longrightarrow C^x$ is a homomorphism of D into the multiplicative group of complex numbers. This means that $\beta = f_! j_* L$, as claimed, where $f: A/D \longrightarrow B$ is the inclusion of an orbit.

Line bundles over G-sets play such an important role that they deserve to have a special home. We let $\mathrm{Pic}_G B$ be the set of all complex line bundles p over B (that is, $p: E(p) \longrightarrow B$

has the property that $\dim_C p_x = 1$ for all x in B). Tensor product defines a group structure in $\mathrm{Pic}_G B$ - it is called the Picard group of the G-set B. If B is a point, then Pic_G^* is the set of all linear characters of G, that is homomorphisms $G \longrightarrow C^x$.

If $p,q \in \mathrm{Vect}_G B$, we let $\mathrm{Hom}_G(p,q)$ be the set of all bundle homomorphisms from p to q over the identity map of B. We define the Schur inner product

$$(\; , \;) : \mathrm{Vect}_G B \times \mathrm{Vect}_G B \longrightarrow Z$$

by setting $(p,q) = \dim_C \mathrm{Hom}_G(p,q)$.

COROLLARY 23. If $p,q \in \mathrm{Vect}_G B$ are irreducible, then

$$(p,q) = \begin{cases} 1 & \text{if } p \text{ is G-iso to } q \\ 0 & \text{if } p \text{ is not G-iso to } q. \end{cases}$$

<u>Proof</u>. If $f: p \longrightarrow q$ is an element of $\mathrm{Hom}_G(p,q)$, then according to Schur's Lemma is either 0 or an isomorphism. This means that $\mathrm{Hom}_G(p,p)$ is a division algebra over the complex numbers C. It is a finite dimensional algebra over C, since B is finite and each p_x is finite dimensional. Now a finite dimensional division algebra D over C with C in the center of D coincides with C, for given a d in D, left multiplication by d is a C-map, so there exists a c in C and a $v \neq 0$ in D with $dv = cv$, or $(d-c)v = 0$. Since v is nonzero, this

means that $d = c$.

Given an inclusion of subgroups $i\colon H \longrightarrow G$ and a G-set $\beta\colon G \times B \longrightarrow B$, we have the map $\beta\colon i^* i_* B \longrightarrow B$. We define $i_{\#}\colon \mathrm{Vect}_H i_* B \longrightarrow \mathrm{Vect}_G B$ by setting $i_{\#} = \beta_! i_*$ and call it Frobenius induction, since in the special case of $B = G/G =$ point we have

$$i_{\#}\colon \mathrm{Rep}\, H \longrightarrow \mathrm{Rep}\, G,$$

which to a representation V of H associates the representation $CG \otimes_{CH} V$, and this is known as Frobenius induction of representations. Another helpful way of looking at $i_{\#}V$ is to think of it as the G-module of all sections of the vector bundle $i_* V$.

We wish to show that Frobenius induction satisfies the famous Frobenius reciprocity condition:

THEOREM 24 (Frobenius reciprocity). Let $i\colon H \longrightarrow G$ be inclusion of subgroups, B a G-set,

$$i^*\colon \mathrm{Vect}_G B \longrightarrow \mathrm{Vect}_H i^* B$$

the restriction,

$$i_{\#}\colon \mathrm{Vect}_H i^* B \longrightarrow \mathrm{Vect}_G B$$

Frobenius induction. Given $p \in \mathrm{Vect}_H i^* B$, $q \in \mathrm{Vect}_G B$, then

$$\mathrm{Hom}_G(i_{\#}\, p, q) \approx \mathrm{Hom}_H(p, i^* q).$$

<u>Proof</u>. Consider the following diagram

$$\begin{tikzcd}
E(p) \arrow[r, "i"] \arrow[d, "p"] & E(i_* p) = i_* E(p) \arrow[r, "\underline{\beta}"] \arrow[d, "i_* p"] & E(i_{\#} p) \arrow[r, "F"] \arrow[d, "i_{\#} p"] & E(q) \arrow[d, "q"] \\
i^* B \arrow[r, "i"] & i_* i^* B \arrow[r, "\beta"] & B \arrow[r, "1"] & B .
\end{tikzcd}$$

Given $F: i_{\#}\, p \longrightarrow q$, composition with $\underline{\beta} i$ gives an H-map $p \rightarrow i^* q$. Conversely, given an H-map $k: p \longrightarrow i^* q$, it defines a G-map $K: i_* p \longrightarrow q$, hence by the universal property of the transfer, a map $\underline{K}: \beta_! i_* p \longrightarrow q$ such that $\underline{K\beta} i = k$. If $k = F\beta i$, then uniqueness gives $\underline{K} = F$, and the proof is complete.

It is helpful to introduce the <u>regular</u> <u>bundle</u> $r \in \mathrm{Vect}_G B$. The bundle is given by $E(r) = B \times CG$, where CG denotes the group ring of G over C with the G-action left multiplication, and $r: B \times CG \longrightarrow B$ is projection on the first factor.

LEMMA 25. For each q in $\mathrm{Vect}_G B$, $\mathrm{Hom}_G(r, q) \approx q$, the correspondence being given by evaluation at (x, e) where $e \in G$ is the unit element.

COROLLARY 26. The set $\mathrm{Irr}_G B$ of G-isomorphism classes of irreducible vector bundles over B is finite.

Let $\beta_1, \ldots, \beta_r$ be representatives for the elements of

$\mathrm{Irr}_G B$. Then given a $p \in \mathrm{Vect}_G B$, there exists an isomorphism

$$p \approx m_1\beta_1 + \ldots + m_r\beta_r ,$$

where the natural numbers $m_i = (p,\beta_i)$. Given an irreducible β, we define the β-isotypical part of p to be the subbundle of p spanned by the images of all $f \in \mathrm{Hom}_G(\beta,p)$ - we denote the β-isotypical part of p by $(\beta;p)$. We then have

$$p = (p;\beta_1) + \ldots + (p;\beta_r),$$

and this direct sum is canonical.

If $B \approx \coprod_i G/H_i$, then $\mathrm{Vect}_G B \approx \oplus_i \mathrm{Rep}\, H_i$, and under this correspondence an irreducible bundle β over B is determined by a sequence of representations r_i of H_i such that $r_i = 0$ for $i \neq i_0$ and r_{i_0} is an irreducible representation. Also $\mathrm{Pic}_G B \approx \oplus_i \mathrm{Pic}_G\, G/H_i \approx \oplus_i \mathrm{Lin}\, H_i$, where $\mathrm{Lin}\, H$ is the group of linear characters (= homomorphisms) $H \to C^x$ into the multiplicative group of nonzero complex numbers.

This is a natural place to examine what can be said in the situation of a normal subgroup of G - the results here are usually called Clifford theory.

Suppose $i: N \longrightarrow G$ is the inclusion of a normal subgroup, let $p \in \mathrm{Vect}_G B$ and $q \subset i^*p \in \mathrm{Vect}_G i^*B$. We notice that for each a in G we have $aq \in \mathrm{Vect}_N i^*B$, since $(aq)_{ax} \approx q_x$, and for each y in q_x and n in N we have $n.ay = a(a^{-1}na)y \in (aq)_{nax}$. It is also immediate that if q is irreducible, then so is aq. This means that if $q \in \mathrm{Irr}_N i^*B$, $q \subset i^*p$, then $a(q;i^*p) = (aq;i^*p)$, that is $a \in G$ maps the q-isotypical component of i^*p into the aq-isotypical component of i^*p.

THEOREM 27 (Clifford). Let $i: N \to G$ be the inclusion of a normal subgroup, $p \in \mathrm{Irr}_G B$, $q \in \mathrm{Irr}_N i^*B$, $\tilde{q} = (q;i^*p)$ the q-isotypical component of i^*p. Set $\tilde{N} = \{a \in G | a\tilde{q} = \tilde{q}\}$, $\tilde{i}: \tilde{N} \to G$ the inclusion. Then $\tilde{q}$ is a $\tilde{N}$-bundle over $\tilde{i}^*B$, and $p = \tilde{i}_*\tilde{q}$.

Proof. Since $a(q;i^*p) = (aq;i^*p)$ as we have already noticed, the sum $\bigoplus_{a \in G} a\tilde{q}$ is a nontrivial G-subbundle of p, hence is p itself, for p is irreducible. If we let S be a set of coset representatives in G for $G/\tilde{N}$, then $p = \bigoplus_{a \in S} a\tilde{q}$, which means that $p = \tilde{i}_*\tilde{q}$, as claimed.

COROLLARY 28 (Blichfeldt's Criterion). Let $i: A \to G$ be the inclusion of a normal abelian subgroup not in the center of G, and let $p \in \mathrm{Irr}_G B$ be an irreducible G vector bundle on which G acts faithfully ($g. = 1$ on $E(p)$ implies $g = e$ in G). Let $q \subset i^*p$, $q \in \mathrm{Irr}_A i^*B$, $\tilde{q} = (q;i^*p)$ the q-isotypical component of i^*p. Then $\tilde{A} = \{a \in G | a\tilde{q} = \tilde{q}\}$ is a proper subgroup of G.

Proof. If $\tilde{A} = G$, we first of all notice that G acts trivially on B, so we may as well assume that $B = *$, a point (by focusing on the orbit on which p is nonzero). Since A is abelian, we know that $\mathrm{Irr}_A * = \mathrm{Pic}_A * = \mathrm{Lin}\, A$, the group of linear characters, so $q = L: A \longrightarrow C^x$, and given an element g of G, the subbundle gq is the homomorphism $L(g(\ )g^{-1})$. This means that the hypothesis $\tilde{A} = G$ gives $L(gag^{-1}) = L(a)$, and $a \in A$ acts on $E(p)$ by multiplication via the scalar $L(a)$, so this means that A is in the center of G, a contradiction. This contradiction shows that $\tilde{A}$ must be a proper subgroup of G, as claimed.

Here is an important consequence, which shows that for a big class of finite groups irreducibles are transfers of line bundles.

COROLLARY 29. Let G be a finite group having an abelian normal subgroup with quotient group having order a power of a prime. If $p \in \mathrm{Vect}_G B$, then there exists a G-map $f: B' \longrightarrow B$ and a line bundle $L \in \mathrm{Pic}_G B'$ such that $p = f_! L$.

<u>Proof</u>. We first reduce to the case of p irreducible. Suppose $p' = f'_! L'$ and $p" = f"_! L"$ for G-maps $f': B' \longrightarrow B$, $f": B" \longrightarrow B$. We let $L = L' \sqcup L"$ the the obvious bundle over the disjoint sum $B' \sqcup B"$, and let $f: B' \sqcup B" \longrightarrow B$ be the map which restricts to f' and $f"$. Then $f_! L = p' \oplus p"$. This means that it remains to prove the result for $p \in \mathrm{Irr}_G B$.

We use induction on the order of G. If $|G| = 1$, p is 0, except at one point b of B, where $p_b = C$. We let $B' = b$ and $f: B' \longrightarrow B$ the inclusion map, then $p = f_! 1$.

Suppose the order of G is greater than 1, and the result is true for groups with order less than $|G|$. If G does not act effectively on $E(p)$, we let $K = \{g \in G | g.=1 \text{ on } E(p)\}$, $\pi: G \longrightarrow G/K$

the quotient map, then $B = \pi^* \underline{B}$ is a $\underline{G} = G/K$-set, $p = \pi^* \underline{p}$ with $\underline{p} \in \mathrm{Irr}_{\underline{G}} \underline{B}$. Now $|\underline{G}| < |G|$, so there exists a $\underline{G}$-map $\underline{f}: \underline{B}' \longrightarrow \underline{B}$ and a line bundle $\underline{L}$ over $\underline{B}'$ such that $\underline{p} = \underline{f}_! \underline{L}$. Applying π we obtain that $p = f_! L$. This means that we can assume that G acts effectively on $E(p)$. We now let A be a maximal abelian normal subgroup such that G/A has order a power of a prime. If $A = G$, the result is an immediate consequence of Schur's Lemma (this is Corollary 22), so we may assume that $|G/A| = s^t$, s a prime, $t > 0$. We claim: A is not in the center of G - if not, we choose an x in G such that xA is in the center of G/A and has order s, then the subgroup generated by x and A is abelian and normal, contradicting our hypothesis that A is maximal with this property. Now apply Blichfeldt's Criterion (Corollary 28) - there exists a proper subgroup $i: \tilde{A} \longrightarrow G$ and $\tilde{q} \in \mathrm{Vect}_{\tilde{A}} i^* B$ with $p = i_{\#} \tilde{q} = \beta_! i_* \tilde{q}$, where $\beta: G \times_{\tilde{A}} i^* B \longrightarrow B$ is induced by the action of G on B. Since p is irreducible, so is $\tilde{q}$. Now $|\tilde{A}| < |G|$ and $\tilde{A}$ satisfies the hypothesis of the Corollary, so by induction there exists an $\tilde{A}$-map $f': B' \longrightarrow i^* B$, $L' \in \mathrm{Pic}_{\tilde{A}} B'$ with $f_! L' = q$. According to Proposition 15 we have $i_*(f_! L) = (i_* f)_! (i_* L')$, and $i_* L'$ is a line bundle over $i_* B'$. Thus we have

$$
\begin{aligned}
p &= i_{\#} \tilde{q} \\
&= \beta_! (i_* \tilde{q}) \\
&= \beta_! (i_* f')_! (i_* L') \\
&= (\beta \circ i_* f')_! (i_* L'),
\end{aligned}
$$

which does the inductive step, so the corollary follows.

We apply this to a special situation. Suppose we have an extension

$$1 \to C \to H \to H/C \to 1$$

where H/C is of order s^t, s a prime. Such a group H is called hyperelementary. Let S be an s-Sylow subgroup of H, then $H = CS$, a semidirect product. Let $N = N_H(S) = DS$, where D is the centralizer of S in C (this because $n = cs$ in N and for x in S we have cxc^{-1} in S, so $cxc^{-1}x^{-1} \in C \cap S = \{e\}$). We let $i: N \to H$ be the inclusion.

COROLLARY 30. If 1_G denotes the trivial representation of G, then we have for $i: N \to H$ above

$$i_{\#}1_N = 1_H + j_{1\#}L_1 + \dots + j_{r\#}L_r ,$$

where $L_k \in \mathrm{Lin}\, H_k$ for proper subgroups $j_k: H_k \longrightarrow H$.

Caution: $r = 0$ is possible.

Proof. We apply Corollary 29 with B = point. We have $(i_{\#}1_N, 1_H) = (1_N, i^*1_H) = 1$, so 1_H appears exactly once in $i_{\#}1_N$. We have to exclude the possibility that $H_k = H$. Suppose that $L \in \mathrm{Lin}\, H$ and $1 \leq (i_{\#}1_N, L) = (1_N, i^*L)$, that is $N \subset \mathrm{Ker}\, L$. We claim: $L = 1_H$. Given $h \in$ in H, hSh^{-1} and S are both s-Sylow subgroups of $\mathrm{Ker}\, L$, so there exists an $x \in \mathrm{Ker}\, L$ such that $xhSh^{-1}x^{-1} = S$, that is $xh \in N \subset \mathrm{Ker}\, L$, so $h \in \mathrm{Ker}\, L$, and $L = 1_H$, as claimed.

3. Equivariant K-theory. So far we have only talked about vector bundles. We can add elements of $\mathrm{Vect}_G B$, but we also wish to be able to take differences of bundles. We let

$$h\colon \mathrm{Vect}_G B \longrightarrow K_G(B)$$

be the universal homomorphism with an abelian group as target, that is:

1) $h(p) = h(p')$ if p is isomorphic to p' over $1\colon B \longrightarrow B$.

2) if $p = p'+p''$, then $h(p) = h(p')+h(p'')$.

For example, we let $K_G(B)$ be the free abelian group on isomorphism classes of G-vector bundles over B (write $[p]$ for the generator corresponding to p), and divide by the subgroup generated by $[p] - [p']-[p'']$ for $p \approx p'+p''$.

Notice that the Schur inner product

$$(\ ,\ )\colon \mathrm{Vect}_G B \times \mathrm{Vect}_G B \longrightarrow Z$$

passes by universality to

$$(\ ,\ )\colon K_G(B) \times K_G(B) \longrightarrow Z.$$

LEMMA 31. $K_G(B)$ is a free abelian group on $p \in \mathrm{Irr}_G B$.

Proof. Mapping p to $[p]$ defines a homomorphism

$$F\colon Z(\mathrm{Irr}_G B) \longrightarrow K_G(B),$$

and the Schur inner product defines a homomorphism

$$S: K_G(B) \longrightarrow Z(\mathrm{Irr}_G B)$$

by setting $S(x)(p) = (x,p)$. We notice that $SF(p) = p$, that is the function which takes value 1 on p and 0 on other isomorphism classes of irreducibles. Conversely, to check that FS is the identity on $K_G(B)$, it is enough to check this on $[p] = F(p)$, but $FSF = F$, and we are done.

We notice that $K_G(B)$ is a ring with multiplication induced by tensor product of vector bundles, and $\mathrm{Pic}_G(B)$ is a subgroup of the group of units of the ring $K_G(B)$. In the case of $B = *$, a point, we have $K_G(*) = R(G)$, the complex representation ring of G. The subgroup $\mathrm{Pic}_G(*)$ is the group of linear characters of G. We will now identify $R(G)$ with the character ring of G. We define χ: Rep $G \longrightarrow \mathrm{Map}(G,C)$ by setting for a representation V of $G, \chi_V(g) = \mathrm{Trace}\ g.: V \longrightarrow V$. Since $\chi_{V+W} = \chi_V + \chi_W$, and $\chi_{V \otimes W} = \chi_V \chi_W$, χ induces $\chi: R(G) \longrightarrow \mathrm{Map}(G,C)$, a homomorphism of rings. We wish to show that this is an inclusion. Our strategy is to show that there is an inner product on $\mathrm{Map}(G,C)$ for which $(\chi_V, \chi_W) = (V,W)$. The first try works: let $f_1, f_2 \in \mathrm{Map}(G,C)$ and define

$$(f_1, f_2) = \frac{1}{|G|} \sum_{g \in G} f_1(g)\overline{f_2(g)} .$$

This means that we wish to show that if $V,W \in$ Rep G, then

$$(V,W) = \frac{1}{|G|}\sum_{g\in G}\chi_V(g)\overline{\chi_W(g)}$$

We do this in two steps. First, we notice that $E = \frac{1}{|G|}\sum_{g\in G} g$ is a projection operator onto the fixed point set: given a representation U of G, then $E.: U \longrightarrow U$ has the property that $EE = E$ and $\mathrm{Im}\, E. = U^G$, the fixed point set of U. This means in particular that $\mathrm{Trace}\, E. = \dim_C \mathrm{Im}\, E. = \dim_C U^G$. Given representations V and W, we let $U = \mathrm{Hom}_C(V,W)$ and define the action of g on $T: V \longrightarrow W$ by $g*T = g.Tg^{-1}$. This means that $U^G = \mathrm{Hom}_{CG}(V,W)$. We have: $\mathrm{Trace}_U g* = \mathrm{Trace}_W g.\ \mathrm{Trace}_V g^{-1}.$, so

$$\begin{aligned}
(V,W) &= \dim_C \mathrm{Hom}_{CG}(V,W) \\
&= \frac{1}{|G|}\sum_{g\in G}\chi_U(g) \\
&= \frac{1}{|G|}\sum_{g\in G}\chi_U(g^{-1}) \\
&= \frac{1}{|G|}\sum_{g\in G}\chi_V(g)\chi_W(g^{-1}) \\
&= (\chi_V,\chi_W)\ ,
\end{aligned}$$

the last because $\chi_W(g^{-1}) = \overline{\chi_W(g)}$, since $g.: W \longrightarrow W$ is diagonalizable (irreducible representations of cyclic groups over C are one dimensional). We have proved:

PROPOSITION 32. The character map $\chi: R(G) \longrightarrow \mathrm{Map}(G,C)$ preserves inner products: $(V,W) = (\chi_V,\chi_W)$.

We will be using this result to detect the unit element $1 \in R(G)$: an element $u \in R(G)$ is 1 if and only if $\chi_u(g) = 1$

for all $g \in G$.

We have an obvious construction C: G-Sets $\longrightarrow$ Rep G which to a G-set B associates the vector space CB with basis B. Since G acts on B, this defines a representation of G. We can describe this in another way: let $c: B \longrightarrow *$ be the collapsing map, 1_B the trivial bundle $B \times C \longrightarrow B$, then $CB = c_! 1_B$. The following result is immediate:

LEMMA 33. $\chi_{CB}(g) = |B^g|$ = number of elements in B fixed by g.

Given a family $\mathfrak{F} \in fo(G)$, we let $A_{\mathfrak{F}}(G) \subset A(G)$ be the ideal (see §1) consisting of all Z-linear combinations of $(G/H) \in \mathfrak{F}$. We will study under what conditions $C: A_{\mathfrak{F}}(G) \longrightarrow R(G)$ has $1 \in R(G)$ in its image. Caution: this need not mean that C is onto!

PROPOSITION 34. Given a family $\mathfrak{F}$, there exists a $u \in A_{\mathfrak{F}}(G)$ with $Cu = 1$ if and only if for each element g of G there exists a $v \in A_{\mathfrak{F}}(G)$ with $|v^g| = 1$.

<u>Proof</u>. If $Cu = 1$, then $|u^g| = 1$ for each g in G. Conversely, suppose the condition is satisfied. For each g in G choose a $v_g \in A\ (G)$ with $|v_g^g| = 1$, and contemplate the element

$$w = \prod_{g \in G} (1 - Cv_g).$$

Since

$$\chi_w(h) = \prod_{g \in G} (1 - |v^h{}_g|) = 0,$$

we have $w = 0$, hence $1 \in \text{Im } C: A_{\mathcal{F}}(G) \twoheadrightarrow R(G)$.

The way to show that the function $|\ {}^g| : A_{\mathcal{F}}(G) \twoheadrightarrow Z$ is onto for a given g in G is to exhibit for each prime p in Z an orbit type $u \in \mathcal{F}$ with $|u^g| \notin (p)$.

Here is a family $\mathcal{H}$ for which this works nicely. A subgroup H of G is said to be hyperelementary if H has a p-group quotient $\pi: H \twoheadrightarrow P$ with $\text{Ker } \pi$ a cyclic group of order prime to p. It is immediate that a subgroup or a conjugate of a hyperelementary group are again hyperelementary. We let $\mathcal{H}$ consist of orbit types (G/H) with H hyperelementary.

COROLLARY 35 (L. Solomon). There is an element $u \in A_{\mathcal{H}}(G)$ with $C(u) = 1$.

Proof. Given $g \in G$ and a prime p, we construct an $G/H \in \mathcal{H}$ such that $|(G/H)^g| \notin (p)$. The first thing to try is to split the cyclic group $\langle g \rangle$ into $\langle g \rangle = C \times D$, where D is the p-Sylow subgroup of $\langle g \rangle$. We let $N = N_G(C)$ be the normalizer of C in G, and contemplate the quotient map $\pi: N \twoheadrightarrow N/C$. Let P be a p-Sylow subgroup of N/C and let $H = \pi^{-1}(P)$. By its very definition we have an extension

$$1 \rightarrow C \rightarrow H \rightarrow P \rightarrow 1 \ ,$$

so H is hyperelementary. We now inspect $(G/H)^g$. If $gaH = aH$, this means that $a^{-1}Ca \subset a^{-1}\langle g\rangle a \subset H$. Since the elements of $a^{-1}Ca$ have order prime to p, we have $a^{-1}Ca \subset \operatorname{Ker} \pi = C$. This means that $a \in N$, so we have $|(G/H)^g| = |(N/H)^g|$. Now since C is normal in N, it acts trivially on N/H, so the action of $\langle g\rangle$ factors through the quotient $\langle g\rangle / C \approx D$ which is a p-group. Thus we have

$$\begin{aligned} |(G/H)^g| &= |(N/H)^g| \\ &= |(N/H)^D| \\ &= |(N/H)| \mod (p) \\ &\not\equiv \quad 0 \quad \mod (p), \end{aligned}$$

since H/C is a p-Sylow subgroup of N/C.

THEOREM 36. If G is a finite group, B G-set, then there exists a G-map $f\colon B' \longrightarrow B$ such that B' is an $\mathcal{H}$-set and $f_!\colon K_G(B') \longrightarrow K_G(B)$ is onto.

Proof. We know that there exists an element $u \in A_{\mathcal{H}}(G)$ such that $C(u) = 1 \in R(G)$. Now $u = (X) - (X')$, the difference of counting functions of two $\mathcal{H}$-sets X and X'. We thus have $1 = C(u) = CX - CX' = c_! 1_X - c'_! 1_{X'}$ where $c\colon X \longrightarrow *$, $c'\colon X' \longrightarrow *$ are the collapsing maps. Let $B' = B \times X \sqcup B \times X'$, $f = \pi_! \sqcup \pi'_!$; $B' \longrightarrow B$, the disjoint sum of projection maps. Let $w = 1_{B\times X} \sqcup^{-1}{}_{B\times X'} \in K_G(B')$. We claim: $1_B = f_!(w)$. This is a consequence of

$\pi_{1!}(1_{B\times X}) = 1_B . c_! 1_X$. Given $v \in K_G(B)$, we have $v = v.1_B = v.f_!(w) = f_!(f^! v).w)$, and we are done.

COROLLARY 37. If G is a finite group, B a finite set, $p \in \mathrm{Vect}_G B$, then there exist $\mathcal{H}$-sets B_1 and B_2, G-maps $f_i \colon B_i \longrightarrow B$, $L_i \in \mathrm{Pic}_G B_i$ such that

$$f_{1!}(L_1) + p = f_{2!}(L_2).$$

<u>Proof</u>. Let $f \colon B' \longrightarrow B$ be a G-map, B' an $\mathcal{H}$-set with $f_!(w) = p$ for some $w \in K_G(B')$. Write $w = q_1$, $q_1 \in \mathrm{Vect}_G B'$. According to Corollary 29 there exist G-maps $k_i \colon B_i \longrightarrow B'$ with $k_{i!}L_i = q_i$, hence $k_{1!}L_1 + w = k_{2!}L_2$. Letting $f_i = fk_i$ we have the corollary.

We can use Corollary 30 to show that we can get along with a smaller family of orbits.

Suppose we have inclusions

$$\begin{tikzcd}
K \arrow[r, "j"] \arrow[dr, "k"'] & H \arrow[d, "i"] \\
 & G
\end{tikzcd} ,$$

then j defines a map $\underline{j} \colon G/K \longrightarrow G/H$.

LEMMA 38. The following diagram commutes

$$\begin{tikzcd}
R(K) \arrow[r, "k_*"] \arrow[d, "j"] & K_G(G/K) \arrow[d, "\underline{j}_!"] \\
R(H) \arrow[r, "i_*"] & K_G(G/H)
\end{tikzcd} .$$

Proof. We recall that $j_*\colon \mathrm{Vect}_K(*) = \mathrm{Rep}(K) \approx \mathrm{Vect}_H(H/K)$ is a one to one correspondence. Under this correspondence our diagram becomes

$$\begin{tikzcd}
K_H(H/K) \arrow[r, "i_*"] \arrow[d, "c_!"] & K_G(G/K) \arrow[d, "\underline{j}_!"] \\
K_H(*) \arrow[r, "i_*"] & K_G(G/H) .
\end{tikzcd}$$

If we notice that $\underline{j} = i_* c$, the commutativity of this diagram is precisely Proposition 15.

We say that $H \subset G$ is elementary if $H = C \times S$, S an s-group for a prime s, C a cyclic group of order prime to s. Let $\mathcal{E}$ be the family of orbits (G/H), H elementary.

THEOREM 39 (Brauer). If G is a finite group, B a finite G-set, then there exists an $\mathcal{E}$-set B'' and a G-map $k\colon B'' \to B$ such that $k_!\colon K_G(B'') \to K_G(B)$ is onto.

Proof. We need only to show that $1_B = k_!(w)$ for some $w \in K_G(B'')$, for then the argument proceeds as in Theorem 36: given $v \in K_G(B)$, we have $v = f_!(f^! v).w)$.

We claim that the Corollary 30 implies that if B' is an $\mathcal{H}$-set, then there exists an $\mathcal{E}$-set B'' and a G-map f': $B'' \to B'$ such that $f'_!(x) = 1_B$, for some $x \in K_G(B'')$. We'll prove this in a moment - let's see how this will prove Theorem 39. Now suppose $f\colon B' \to B$ is a G-map such that $f_!$ is onto, and let $f_!(y) = 1_B$. We let $w = x.f^!(y)$, then setting $k = f f'$, we obtain $k_!(w) = 1_B$, and the theorem will be proved.

We still need to show that if H is a hyperelementary subgroup of G then there exists an $\mathcal{E}$-set X and a G-map $f: X \longrightarrow G/H$ such that $f_!$ is onto. We do this by induction on the order of H. If $|H| = 1$, then H is elementary and G is an $\mathcal{E}$-set. Let's assume that $|H| > 1$, H is hyperelementary but not elementary, and that the result is true for $(G/K) \in \mathcal{X}$ with $|K| < |H|$. Lemma 38 translates Corollary 30 to

$$\underline{i}_! 1_{G/N} = 1_{G/H} + \underline{j}_{1!} L_1 + \dots + \underline{j}_{r!} L_r ,$$

where $i: N \longrightarrow H$, $j_k: H_k \longrightarrow H$ are inclusions of proper subgroups and $L_k \in \mathrm{Pic}_G G/H_k$. By induction, there exist $\mathcal{E}$-sets $Y_0, \dots, Y_k$ and G-maps $f_0: Y_0 \longrightarrow G/N$, $f_k: Y_k \longrightarrow G/H_k$ such that $f_{k!}$ are onto. We let $u_i \in K_G(Y_i)$ be elements such that $f_{0!}(u_0) = 1_{G/N}$, $f_{k!}(u_k) = L_k$. Letting $X = Y_0 \sqcup \dots \sqcup Y_r$, we define $f: X \longrightarrow G/H$ by setti $f|Y_0 = \underline{i} f_0$, $f|Y_k = \underline{j}_k f_k$ for $k = 1, \dots, r$. We thus have

$$f_!(u_0, -u_1, \dots, -u_r) = 1_{G/H} ,$$

hence $f_!$ is onto, the inductive step works, and the claim (and hence also Theorem 39) is proved.

We immediately have:

COROLLARY 40. If G is a finite group, $p \in \mathrm{Vect}_G B$, then there exist $\mathcal{E}$-sets B_1, B_2, G-line bundles L_i over B_i and

G-maps $f_i : B_i \longrightarrow B$ such that

$$f_{1!}(L_1) + p = f_{2!}(L_2).$$

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