FANO THREEFOLDS V_{22}

ALEXANDER KUZNETSOV

1. Introduction

Let W be a fixed 4-dimensional vector space. In the paper [8] S. Mukai has announced Theorem according to which every Fano threefold V_{22} can be embedded into the Ellingsrud–Piene–Strømme compactification $\mathfrak{W}$ of the space of twisted rational cubic curves in the projective space $\mathbb{P}(W)$ as the zero locus of a certain global section of the vector bundle

$$\mathcal{K}^* \oplus \mathcal{K}^* \oplus \mathcal{K}^*, \tag{1}$$

where $\mathcal{K}$ is the universal 3-dimensional vector bundle over $\mathfrak{W}$ (see [3], [6]).

The natural isomorphism $\mathbf{H}^0(\mathfrak{W}, \mathcal{K}^*) \cong \mathbf{S}^2 W$ shows that the global sections of (1) are in one-one correspondence with the nets of quadrics in the dual projective space $\mathbb{P}(W^*)$ which can be classified via their Hesse curve and a theta-characteristic on this curve (see [12], [7]). The Hesse curve is the curve of the degenerate quadrics in the net. Since $\mathbf{dim}\, W = 4$ it is a plane quartic.

Now assume we are given a pair (C, θ), consisting of a plane quartic C and a theta-characteristic θ on C. Let $A_C : \mathbb{C}^3 \to \mathbf{S}^2 W$ be the corresponding net of quadrics, s_C the corresponding global section of (1) and $\mathfrak{W}_C$ the zero locus of s_C. We prove the following Theorem.

Main Theorem. *The space $\mathfrak{W}_C$ is a smooth Fano threefold V_{22}. Curve C is the curve, parameterizing lines in $\mathfrak{W}_C$. Intersection of lines in $\mathfrak{W}_C$ is managed by the theta-characteristic θ.*

The last statement means that lines $l_x, l_y \subset \mathfrak{W}_C$ corresponding to points $x, y \in C$ meet iff $|x+\theta-y| \neq \emptyset$.

The embedding $\mathfrak{W}_C \subset \mathfrak{W}$ of our Fano threefold into the space of twisted cubics in $\mathbb{P}(W)$ gives rise to the following construction. The space $\mathfrak{W}$ is equipped with a natural "naive universal curve". It is just the point-curve incidence subset in $\mathbb{P}(W) \times \mathfrak{W}$. In the paper [6] a certain nice desingularization $\widetilde{\mathcal{M}}$ of the "naive curve" was constructed. It turns out that both natural projections $\tilde{\pi}_0 : \widetilde{\mathcal{M}} \to \mathbb{P}(W)$ and $\tilde{\pi}_1 : \widetilde{\mathcal{M}} \to \mathfrak{W}$ are very simple. Both of them are consecutively projectivization of a vector bundle and a blow up (see [6, 4.7]).

We enlarge our picture by considering also the space $\widetilde{\mathcal{M}}_C \subset \widetilde{\mathcal{M}}$ that is the zero locus of the global section $\tilde{\pi}_1^* s_C$ of the vector bundle $\tilde{\pi}_1^*\mathcal{K}^* \oplus \tilde{\pi}_1^*\mathcal{K}^* \oplus \tilde{\pi}_1^*\mathcal{K}^*$ on $\widetilde{\mathcal{M}}$. The projections $\tilde{\pi}_i$ can be, of course, restricted on the space $\widetilde{\mathcal{M}}_C$. We prove that the situation with the maps remains the same, namely both maps $\tilde{\pi}_0 : \widetilde{\mathcal{M}}_C \to \mathbb{P}(W)$ and $\tilde{\pi}_1 : \widetilde{\mathcal{M}}_C \to \mathfrak{W}_C$ are consecutively projectivization of a vector bundle and a blow up (see Theorem 1). However, both vector bundles now are 2-dimensional. The first of them $\mathcal{E}_{|\mathfrak{W}_C}$ is just the restriction of the universal 2-dimensional bundle $\mathcal{E}$ over $\mathfrak{W}$ (see [3], [6]) to $\mathfrak{W}_C$. The second bundle $\mathcal{R}_C$, which lives over $\mathbb{P}(W)$ is a so-called mathematical instanton vector bundle with $c_2 = 3$. Relation of such bundles to the nets of quadrics was investigated in [7]. Now it turns out that the space $\widetilde{\mathcal{M}}_C$ and the projection $\tilde{\pi}_1$ can be described purely in terms of the rational curves in $\mathbb{P}(W)$ providing the most degenerate splitting of the bundle $\mathcal{R}_C$. E.g. the exceptional lifts of the 2-jumping lines of $\mathcal{R}_C$ span the surface $\Sigma^- \subset \mathbb{P}_{\mathbb{P}(W)}(\mathcal{R}_C)$ which is to be blown up to get $\widetilde{\mathcal{M}}_C$ (by the exceptional lift of the rational curve $X \subset \mathbb{P}(W)$ we mean the exceptional section of the ruled surface $\mathbb{P}_X(\mathcal{R}_{C|X}) \subset \mathbb{P}_{\mathbb{P}(W)}(\mathcal{R}_C)$). The

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exceptional lifts of 5-jumping twisted cubics are but the fibers of the map $\tilde{\pi}_1$. Thus the geometry of the space $\mathfrak{W}_C$ is found to be very strongly related to the geometry of the bundle $\mathcal{R}_C$.

We apply the above results to the following problems. First we prove that

$$(\mathcal{O}_{\mathfrak{W}_C}, \mathcal{K}^*_{|\mathfrak{W}_C}, \mathcal{E}^*_{|\mathfrak{W}_C}, \Lambda^2\mathcal{K}^*_{|\mathfrak{W}_C})$$

is a full and strong exceptional collection on $\mathfrak{W}_C$.

Second we give a precise description of rational curves of low degree (≤ 3) in $\mathfrak{W}_C$. In particular, we show that the linear system $|K_C|$ is naturally a compactification of the space $\mathbf{Rat}_2(\mathfrak{W}_C)$ of conics in $\mathfrak{W}_C$ and that the space $\mathbb{P}(W)$ is naturally a compactification of the space $\mathbf{Rat}_3(\mathfrak{W}_C)$ of rational cubic curves in $\mathfrak{W}_C$. The latter seems to be the most surprising because the space $\mathfrak{W}_C$ is by definition embedded into the compactification $\mathfrak{W}$ of the space of rational twisted cubics in $\mathbb{P}(W)$! It looks like the kind of duality the nature of which is quite mysterious.

Remark 1. In the paper [9] S. Mukai has introduced a construction of the Fano threefold V_{22} from a plane quartic F. This construction related to our as follows. The threefold $\mathbf{VSP}(6, F)$ constructed by Mukai coincides with the space $\mathfrak{W}_C$, when (C, θ) is the Scorza map (see [5]) of F.

The paper is organized as follows. In Section 2 we define the spaces $\mathfrak{W}_C \subset \mathfrak{W}$ and $\widetilde{\mathcal{M}}_C \subset \widetilde{\mathcal{M}}$ as the zero loci of the sections s_C and $\tilde{\pi}_1^* s_C$ respectively and give their description.

In Section 3 we deal with exceptional collections.

In Section 4 we give a description of rational curves in $\mathfrak{W}_C$.

This paper is a sequel to [6] and [7] so we will freely use notations and conventions thereof.

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2. The construction of the space $\mathfrak{W}_C$

2.1. Let W be a fixed 4-dimensional vector space, $\mathbb{P}(W)$ its projectivization, $\mathfrak{W}$ the Ellingsrud-Piene-Strømme compactification of the space of rational twisted cubic curves in $\mathbb{P}(W)$, $\widetilde{\mathcal{M}}$ the desingularization of the "naive universal curve" over $\mathfrak{W}$ (the "naive universal curve" is just the point-curve incidence subset in $\mathbb{P}(W) \times \mathfrak{W}$) which was constructed in [6]. Finally let $\tilde{\pi}_0 : \widetilde{\mathcal{M}} \to \mathbb{P}(W)$ and $\tilde{\pi}_1 : \widetilde{\mathcal{M}} \to \mathfrak{W}$ be the canonical projections.

Recall that projections $\tilde{\pi}_0$ and $\tilde{\pi}_1$ factor as follows

$$\begin{tikzcd}[column sep=small] & \widetilde{\mathcal{M}} \arrow[dl, "\hat{\pi}_0"'] \arrow[dr, "\hat{\pi}_1"] & \\ \mathbb{P}(\mathcal{R}) \arrow[d, "\pi_0"'] & & \mathbb{P}(\mathcal{E}) \arrow[d, "\pi_1"] \\ \mathbb{P}(W) & & \mathfrak{W} \end{tikzcd} \quad \overset{i}{\hookleftarrow} \quad \begin{tikzcd}[column sep=small] & \widetilde{M} = M^- \times_{M^0} M^+ \arrow[dl, "\tilde{\pi}_0"'] \arrow[dr, "\tilde{\pi}_1"] & \\ M^- = \mathbb{P}_{M^0}(S^-) \arrow[dr, "p^-"'] & & \mathbb{P}_{M^0}(S^+) = M^+ \arrow[dl, "p^+"] \\ & M^0 & \end{tikzcd}$$

Here $\mathcal{R}$ is a certain 11-dimensional exceptional vector bundle over $\mathbb{P}(W)$ (see [6], [7]) and $\mathcal{E}$ is the universal 2-dimensional vector bundle over $\mathfrak{W}$ (see [3], [6]). The maps π_0, π_1 are natural projections. The maps $\hat{\pi}_0$ and $\hat{\pi}_1$ are the blow ups of $\mathbb{P}_{\mathbb{P}(W)}(\mathcal{R})$ (resp. $\mathbb{P}_{\mathfrak{W}}(\mathcal{E})$) along the subvariety $M^- = \mathbb{P}_{M^0}(S^-) \subset \mathbb{P}_{\mathbb{P}(W)}(\mathcal{R})$ (resp. $M^+ = \mathbb{P}_{M^0}(S^+) \subset \mathbb{P}_{\mathfrak{W}}(\mathcal{E})$) with common exceptional divisor $\widetilde{M} = M^- \times_{M^0} M^+ \subset \widetilde{\mathcal{M}}$. Here $M^0 = \mathbb{P}(W^*) \times \mathbf{Gr}(2, W)$ and $S^{\pm}$ are certain 5-dimensional and 2-dimensional vector bundles over M^0.

Recall that over the space $\mathfrak{W}$ there is also a 3-dimensional universal vector bundle $\mathcal{K}$ (see [3], [6]) and we have the following exact sequence of coherent sheaves over $\widetilde{\mathcal{M}}$

$$0 \to \tilde{\pi}_0^* \mathcal{L}_{\mathcal{R}} \otimes \tilde{\pi}_0^* \mathcal{T}_{\mathbb{P}(W)}(-3) \to \tilde{\pi}_1^* \mathcal{K} \to i_*(p^* \Lambda^2 U \otimes \hat{\pi}_1^* \mathcal{L}^+). \tag{2}$$

Here $\mathcal{L}_{\mathcal{R}} \subset \pi_0^*\mathcal{R}$ is the relative tautological linear subbundle on $\mathbb{P}_{\mathbb{P}(W)}(\mathcal{R})$, $\mathcal{L}^{\pm} \subset p^{\pm *}S^{\pm}$ relative tautological linear subbundles on $M^{\pm} = \mathbb{P}_{M^0}(S^{\pm})$, $p^{\pm} : M^{\pm} \to M^0$ and $p : \widetilde{M} = M^- \times_{M^0} M^+ \to M^0$ natural projections, $U \subset W \otimes \mathcal{O}_{\mathbf{Gr}(2,W)}$ tautological 2-dimensional subbundle on $M^0 = \mathbb{P}(W^*) \times \mathbf{Gr}(2,W)$ (actually the pull back from the second factor) and $i : \widetilde{M} \to \widetilde{\mathcal{M}}$ is the embedding.

2.1.1. **Lemma.** *We have natural isomorphisms*

$$\mathbf{H}^0(\mathfrak{W},\mathcal{K}^*) \cong \mathbf{H}^0(\mathbb{P}_{\mathfrak{W}}(\mathcal{E}),\pi_1^*\mathcal{K}^*) \cong \mathbf{H}^0(\widetilde{\mathcal{M}},\tilde{\pi}_1^*\mathcal{K}^*) \cong \mathbf{H}^0(\widetilde{\mathcal{M}},\hat{\pi}_0^*\mathcal{L}_{\mathcal{R}}^* \otimes \tilde{\pi}_0^*\Omega_{\mathbb{P}(W)}(3)) \cong$$
$$\cong \mathbf{H}^0(\mathbb{P}_{\mathbb{P}(W)}(\mathcal{R}),\mathcal{L}_{\mathcal{R}}^* \otimes \pi_0^*\Omega_{\mathbb{P}(W)}(3)) \cong \mathbf{Hom}_{\mathbb{P}(W)}(\mathcal{R},\Omega_{\mathbb{P}(W)}(3)) = \mathbf{S}^2W.$$

Proof. The first, second, fourth and fifth isomorphisms follow from the projection formula applied to the maps π_1, $\hat{\pi}_1$, $\hat{\pi}_0$, π_0 respectively. In order to prove the third isomorphism consider the dual sequence of (2).

$$0 \to \tilde{\pi}_1^*\mathcal{K}^* \to \hat{\pi}_0^*\mathcal{L}_{\mathcal{R}}^* \otimes \tilde{\pi}_0^*\Omega_{\mathbb{P}(W)}(3) \to i_*(p^*\Lambda^2U^* \otimes \hat{\pi}_0^*\mathcal{L}^-) \to 0. \tag{3}$$

We applied here the fact that $\mathcal{N}_{\widetilde{M}/\widetilde{\mathcal{M}}} = \hat{\pi}_0^*\mathcal{L}^- \otimes \hat{\pi}_1^*\mathcal{L}^+$ (see [6, 4.6]). One can easily see that

$$\mathbf{H}^\bullet(\widetilde{\mathcal{M}},i_*(p^*\Lambda^2U^* \otimes \hat{\pi}_0^*\mathcal{L}^-)) = \mathbf{H}^\bullet(\widetilde{M},p^*\Lambda^2U^* \otimes \hat{\pi}_0^*\mathcal{L}^-) =$$
$$= \mathbf{H}^\bullet(M^-,p^{-*}\Lambda^2U^* \otimes \mathcal{L}^-) = \mathbf{H}^\bullet(M^0,\Lambda^2U \otimes p_*^-\mathcal{L}^-) = 0.$$

The third isomorphism follows. The last isomorphism can be proved by the direct calculation (alternatively, see [7, 3.2.4]). $\square$

2.1.2. The restriction of (3) on $\widetilde{M}$ gives rise to the pair of exact sequences.

$$0 \to p^*\Lambda^2U^* \otimes \hat{\pi}_1^*\mathcal{L}^{+*} \to \tilde{\pi}_1^*\mathcal{K}^*_{|\widetilde{M}} \to p^*(\eta^* \otimes W/U) \to 0; \tag{4}$$

$$0 \to p^*(\eta^* \otimes W/U) \to \left(\hat{\pi}_0^*\mathcal{L}_{\mathcal{R}}^* \otimes \tilde{\pi}_0^*\Omega_{\mathbb{P}(W)}(3)\right)_{|\widetilde{M}} \to p^*\Lambda^2U^* \otimes \hat{\pi}_0^*\mathcal{L}^- \to 0$$

(see [6, 5.5]). Here $\eta \subset W^* \otimes \mathcal{O}_{\mathbb{P}(W^*)}$ stands for the tautological linear subbundle on M^0 (actually, the pull back from the first factor.)

Lemma. *We have natural isomorphisms*

$$\mathbf{H}^0(\widetilde{M},p^*(\eta^* \otimes W/U)) \cong \mathbf{H}^0(M^0,\eta^* \otimes W/U) = W \otimes W; \qquad \mathbf{H}^\bullet(\widetilde{M},p^*\Lambda^2U^* \otimes \hat{\pi}_0^*\mathcal{L}^-) = 0.$$

Proof. Evident. $\square$

2.1.3. Fix a 3-dimensional vector space N. Let $(C,\theta) \in \mathbf{Qu} \setminus \mathbf{Qu}^L$ be a non-Lüroth rigid plane quartic $\alpha : C \hookrightarrow \mathbb{P}(N)$ (see [7, 2.1.1]) and $A_C : N \to \mathbf{S}^2W$ the corresponding net of quadrics in $\mathbb{P}(W^*)$, which accordingly to [7, 2.1.3] is a regular non-Lüroth net.

Taking into account the isomorphisms of Lemma 2.1.1 the net of quadrics A_C gives rise to the following data.

- $s_1 \in \mathbf{H}^0(\mathfrak{W}, N^* \otimes \mathcal{K}^*)$;
- $s_2 \in \mathbf{H}^0(\mathbb{P}_{\mathfrak{W}}(\mathcal{E}), N^* \otimes \pi_1^*\mathcal{K}^*)$;
- $s_3 \in \mathbf{H}^0(\widetilde{\mathcal{M}}, N^* \otimes \tilde{\pi}_1^*\mathcal{K}^*)$;
- $s_4 \in \mathbf{H}^0(\widetilde{\mathcal{M}}, N^* \otimes \hat{\pi}_0^*\mathcal{L}_{\mathcal{R}}^* \otimes \tilde{\pi}_0^*\Omega_{\mathbb{P}(W)}(3))$;
- $s_5 \in \mathbf{H}^0(\mathbb{P}_{\mathbb{P}(W)}(\mathcal{R}), N^* \otimes \mathcal{L}_{\mathcal{R}}^* \otimes \pi_0^*\Omega_{\mathbb{P}(W)}(3))$;
- $\mu_C \in \mathbf{Hom}(\mathcal{R}, N^* \otimes \Omega_{\mathbb{P}(W)}(3))$.

Let Z_k be the zero locus of s_k ($k = 1,\dots,5$) and $\mathcal{R}_C$ be the kernel of μ_C. Recall, that accordingly to the Theorem 1 of [7] $\mathcal{R}_C$ is an instanton 2-dimensional vector bundle over $\mathbb{P}(W)$.

Lemma. *We have*
1). $Z_2 = \pi_1^{-1}(Z_1) = \mathbb{P}_{Z_1}(\mathcal{E}_{|Z_1})$;
2). $Z_3 = \hat{\pi}_1^{-1}(Z_2)$;
3). $Z_4 = \hat{\pi}_0^{-1}(Z_5)$;
4). $Z_2 \setminus M^+ = Z_3 \setminus \widetilde{M} = Z_4 \setminus \widetilde{M} = Z_5 \setminus M^-$;
5). $Z_5 = \mathbb{P}_{\mathbb{P}(W)}(\mathcal{R}_C)$.

Proof. Evident. □

2.1.4. The Lemma 2.1.3, 4) shows that the spaces Z_2 and Z_5 coincide on the complements of M^+ and M^-. We should also describe $Z_2 \cap M^+$ and $Z_5 \cap M^-$. To this end consider the following additional data induced by A_C.
- $s'_0 \in \mathbf{H}^0(M^0, N^* \otimes \eta^* \otimes W/U)$;
- $s'_2 \in \mathbf{H}^0(M^+, N^* \otimes \pi_1^* \mathcal{K}^*_{|M^+})$;
- $s'_3 \in \mathbf{H}^0(\widetilde{M}, N^* \otimes \tilde{\pi}_1^* \mathcal{K}^*_{|\widetilde{M}})$;
- $s'_4 \in \mathbf{H}^0\left(\widetilde{M}, N^* \otimes (\hat{\pi}_0^* \mathcal{L}^*_{\mathcal{R}} \otimes \tilde{\pi}_0^* \Omega_{\mathbb{P}(W)}(3))_{|\widetilde{M}}\right)$;
- $s'_5 \in \mathbf{H}^0\left(M^-, N^* \otimes (\mathcal{L}^*_{\mathcal{R}} \otimes \pi_0^* \Omega_{\mathbb{P}(W)}(3))_{|M^-}\right)$.

Let Z'_k be the zero locus of s'_k $(k = 0, 2, 3, 4, 5)$. The sequence (4) shows that on the space ${p^+}^{-1}(Z'_0)$ the section s'_2 can be considered as section

$$s'_{2|{p^+}^{-1}(Z'_0)} = s'_1 \in \mathbf{H}^0({p^+}^{-1}(Z'_0), N^* \otimes {p^+}^* \Lambda^2 U^* \otimes {\mathcal{L}^+}^*).$$

Lemma. *We have*
1). $Z'_2 \subset {p^+}^{-1}(Z'_0)$ *is the zero locus of* s'_1;
2). $Z'_3 = \hat{\pi}_1^{-1}(Z'_2)$;
3). $Z'_4 = \hat{\pi}_0^{-1}(Z'_5)$;
4). $Z'_5 = {p^-}^{-1}(Z'_0)$;
5). $Z_2 \cap M^+ = Z'_2$, $Z_3 \cap \widetilde{M} = Z'_3$, $Z_4 \cap \widetilde{M} = Z'_4$, $Z_5 \cap M^- = Z'_5$.

Proof. Evident. □

2.1.5. Recall that in [7, 2.2.6] we have defined embeddings $\beta : C \to \mathbb{P}(W^*)$ and $\gamma : C \to \mathbf{Gr}(2, W)$ and proved that $(\beta \times \gamma)(C) \subset \mathbb{P}(W^*) \times \mathbf{Gr}(2, W)$ is the zero locus of the section $t_C \in \mathbf{H}^0(\mathbb{P}(W^*) \times \mathbf{Gr}(2, W), N^* \otimes \mathcal{O}_{\mathbb{P}(W^*)}(1) \otimes (W/U))$. This section evidently coincides with s'_0 (recall that $M^0 = \mathbb{P}(W^*) \times \mathbf{Gr}(2, W)$ and $\eta^* = \mathcal{O}_{\mathbb{P}(W^*)}(1)$), hence $Z'_0 = (\beta \times \gamma)(C) \subset \mathbb{P}(W^*) \times \mathbf{Gr}(2, W)$. Since

$$\mathbf{H}^0({p^+}^{-1}(\beta \times \gamma)(C), N^* \otimes {p^+}^* \Lambda^2 U^* \otimes {\mathcal{L}^+}^*) = \mathbf{H}^0(C, (\beta \times \gamma)^*(N^* \otimes \Lambda^2 U^* \otimes p^+_* {\mathcal{L}^+}^*)) =$$
$$= \mathbf{H}^0(C, N^* \otimes (\beta \times \gamma)^*(\Lambda^2 U^* \otimes {S^+}^*)) = \mathbf{Hom}_C((\beta \times \gamma)^* S^+, N^* \otimes \gamma^* \Lambda^2 U^*).$$

the section s'_1 induces a homomorphism $\nu_C : (\beta \times \gamma)^* S^+ \to N^* \otimes \gamma^* \Lambda^2 U^*$. The proof of the following Lemma will be postponed till 2.1.12.

Lemma. *The map ν_C is surjective.*

2.1.6. **Definition.** Let S_C^+ denote the kernel of the map ν_C and S_C^- denote $(\beta \times \gamma)^* S^-$. Both S_C^+ and S_C^- are 2-dimensional vector bundles over C. The Lemmas 2.1.4 and 2.1.5 imply.

Corollary. $Z'_2 = \mathbb{P}_C(S_C^+)$, $Z'_5 = \mathbb{P}_C(S_C^-)$, $Z'_3 = \mathbb{P}_C(S_C^-) \times_C \mathbb{P}_C(S_C^+) \subset \mathbb{P}_C(S_C^-) \times_C \mathbb{P}_C((\beta \times \gamma)^* S^+) = Z'_4$. *In particular,* $\mathbf{codim}_{Z_5} Z'_5 = 4 - 2 = 2$ *and* $\mathbf{codim}_{Z_2} Z'_2 = 4 - 2 = 2$.

Note that the Lemma 2.1.4, 5) implies that we have canonical embeddings $\mathbb{P}_C(S_C^+) \subset Z_2$, $\mathbb{P}_C(S_C^-) \subset Z_5$.

2.1.7. Now we can give the final description. To this end we introduce the following notation. Let $\mathfrak{W}_C = Z_1 \subset \mathfrak{W}$, $\widetilde{\mathcal{M}}_C = Z_3 \subset \widetilde{\mathcal{M}}$, $\widetilde{M}_C = Z_3' \subset \widetilde{M}$. Let us denote the restriction of the maps $\tilde{\pi}_i$, $\hat{\pi}_i$ on $\widetilde{\mathcal{M}}_C$ and of the maps π_0 on $\mathbb{P}_{\mathbb{P}(W)}(\mathcal{R}_C) \subset \mathbb{P}_{\mathbb{P}(W)}(\mathcal{R})$, π_1 on $\mathbb{P}_{\mathfrak{W}_C}(\mathcal{E}_{|\mathfrak{W}_C}) \subset \mathbb{P}_{\mathfrak{W}}(\mathcal{E})$ by the same letters.

Theorem 1. 1) *The projections $\tilde{\pi}_0 : \widetilde{\mathcal{M}}_C \to \mathbb{P}(W)$ and $\tilde{\pi}_1 : \widetilde{\mathcal{M}}_C \to \mathfrak{W}_C$ factor as follows:*

$$\begin{tikzcd}[column sep=small]
 & \widetilde{\mathcal{M}}_C \arrow[dl, "\hat{\pi}_0"'] \arrow[dr, "\hat{\pi}_1"] & & & & \widetilde{M}_C = \mathbb{P}_C(S_C^-)\times_C \mathbb{P}_C(S_C^+) \arrow[dl, "\hat{\pi}_0"'] \arrow[dr, "\hat{\pi}_1"] & \\
\mathbb{P}_{\mathbb{P}(W)}(\mathcal{R}_C) \arrow[d, "\pi_0"'] & & \mathbb{P}_{\mathfrak{W}_C}(\mathcal{E}_{|\mathfrak{W}_C}) \arrow[d, "\pi_1"] & \overset{i}{\hookleftarrow} & \mathbb{P}_C(S_C^-) \arrow[dr, "p^-"] & & \mathbb{P}_C(S_C^+) \arrow[dl, "p^+"'] \\
\mathbb{P}(W) & & \mathfrak{W}_C & & & C &
\end{tikzcd}$$

Here $\mathcal{R}_C$ is the mathematical instanton with $c_2 = 3$ corresponding to the curve C in the sense of [7, 3.1.2], *$\mathcal{E}$ is the universal* 2*-dimensional vector bundle over $\mathfrak{W}$ (see* [3], [6]*), $S_C^{\pm}$ are certain* 2*-dimensional vector bundles over C, defined in 2.1.6. The maps π_0 and π_1 are natural projections. The maps $\hat{\pi}_0$ and $\hat{\pi}_1$ are the blow ups along $\mathbb{P}_C(S_C^-)$ and $\mathbb{P}_C(S_C^+)$ respectively with common exceptional divisor $\widetilde{M}_C = \mathbb{P}_C(S_C^-) \times_C \mathbb{P}_C(S_C^+) \subset \widetilde{\mathcal{M}}_C$.*

2) *The birational transform $q = \hat{\pi}_1 \cdot \hat{\pi}_0^{-1} : \mathbb{P}_{\mathbb{P}(W)}(\mathcal{R}_C) \dashrightarrow \mathbb{P}_{\mathfrak{W}_C}(\mathcal{E}_{|\mathfrak{W}_C})$ is a flop at $\mathbb{P}_C(S_C^-)$.*

Proof. Since $\hat{\pi}_0 : \widetilde{\mathcal{M}} \to \mathbb{P}_{\mathbb{P}(W)}(\mathcal{R})$ is the blow up along M^- the irreducibility of $\hat{\pi}_0(\widetilde{\mathcal{M}}_C) = \mathbb{P}_{\mathbb{P}(W)}(\mathcal{R}_C)$ and of $\widetilde{M}_C = \widetilde{\mathcal{M}}_C \cap \widetilde{M} = \widetilde{\mathcal{M}}_C \cap \hat{\pi}_0^{-1}(M^-)$ implies that $\widetilde{\mathcal{M}}_C$ irreducible. Now the Lemma 2.1.3 implies that $\widetilde{\mathcal{M}}_C$ is a strict transform of $\mathbb{P}_{\mathbb{P}(W)}(\mathcal{R}_C)$ with respect to $\hat{\pi}_0$ and of $\mathbb{P}_{\mathfrak{W}_C}(\mathcal{E}_{|\mathfrak{W}_C})$ with respect to $\hat{\pi}_1$. Hence $\widetilde{\mathcal{M}}_C$ is the blow up of $\mathbb{P}_{\mathbb{P}(W)}(\mathcal{R}_C)$ along $\mathbb{P}_{\mathbb{P}(W)}(\mathcal{R}_C) \cap M^- = Z_5' = \mathbb{P}_C(S_C^-)$ and the blow up of $\mathbb{P}_{\mathfrak{W}_C}(\mathcal{E}_{|\mathfrak{W}_C})$ along $\mathbb{P}_{\mathfrak{W}_C}(\mathcal{E}_{|\mathfrak{W}_C}) \cap M^+ = Z_2' = \mathbb{P}_C(S_C^+)$. Now the rest of the first part of the Theorem follows from the same properties of $\widetilde{\mathcal{M}}$ (see 2.1). ☐

The proof of the second part of the Theorem will be given below.

2.1.8. Theorem 1 implies that $\mathbf{Pic}\, \widetilde{\mathcal{M}}_C = \mathbb{Z}^3$. We have a basis (h, r, d) in $\mathbf{Pic}\, \widetilde{\mathcal{M}}_C$ formed by the following elements: $h = c_1(\mathcal{O}_{\mathbb{P}(W)}(1))$ the positive generator of $\mathbf{Pic}\, \mathbb{P}(W) = \mathbb{Z}$, $r = c_1(\mathcal{L}^*_{\mathcal{R}\,|\mathbb{P}_{\mathbb{P}(W)}(\mathcal{R}_C)}) \in \mathbf{Pic}\, \mathbb{P}_{\mathbb{P}(W)}(\mathcal{R}_C)$, $d = [\widetilde{M}_C] \in \mathbf{Pic}\, \widetilde{\mathcal{M}}_C$ (we mean that all above Picard groups are embedded into $\mathbf{Pic}\, \widetilde{\mathcal{M}}_C$ via $\tilde{\pi}_i^*$, $\hat{\pi}_i^*$). We also have in $\mathbf{Pic}\, \widetilde{\mathcal{M}}_C$ elements $H = c_1(\mathcal{K}^*_{|\mathfrak{W}_C}) = c_1(\mathcal{E}^*_{|\mathfrak{W}_C}) \in \mathbf{Pic}\, \mathfrak{W}_C$ and $e = c_1(\mathcal{L}^*_{\mathcal{E}\,|\mathbb{P}_{\mathfrak{W}_C}(\mathcal{E}_{|\mathfrak{W}_C})}) \in \mathbf{Pic}_{\mathfrak{W}_C}(\mathcal{E}_{|\mathfrak{W}_C})$. All of them are restrictions of the analogous elements in $\mathbf{Pic}\, \widetilde{\mathcal{M}}$, hence satisfy the same relations

$$e = r + 2h, \quad H = 3r + 5h - d, \qquad r = 2H - 5e + 2d, \quad h = 3e - H - d \tag{5}$$

(compare with [6, (21)]). This implies that (H, e, d) is also a basis in $\mathbf{Pic}\, \widetilde{\mathcal{M}}_C$, hence H is a generator of $\mathbf{Pic}\, \mathfrak{W}_C = \mathbb{Z}H$.

The canonical classes of our spaces are expressed as follows:

$$\begin{array}{ll}
K_{\mathbb{P}(W)} = -4h, & K_{\mathfrak{W}_C} = -H, \\
K_{\mathbb{P}_{\mathbb{P}(W)}(\mathcal{R}_C)} = -4h - 2r, & K_{\mathbb{P}_{\mathfrak{W}_C}(\mathcal{E}_{|\mathfrak{W}_C})} = -2e, \\
K_{\widetilde{\mathcal{M}}_C} = -4h - 2r - d = -2e - d. &
\end{array}$$

2.1.9.

Proof of the Theorem 1, 2). Recall that spaces $\mathbb{P}_{\mathbb{P}(W)}(\mathcal{R})$ and $\mathbb{P}_{\mathfrak{W}}(\mathcal{E})$ are equipped by the maps p^- (resp. p^+) into a certain variety $\mathcal{M}_{2/3}$, such that p^- (resp p^+) is a contraction of $M^- = \mathbb{P}_{M^0}(S^-)$ (resp. $M^+ = \mathbb{P}_{M^0}(S^+)$) onto M^0 (see [6, Remark 12]). Therefore the composition of the natural embedding $\mathbb{P}_{\mathbb{P}(W)}(\mathcal{R}_C) \subset \mathbb{P}_{\mathbb{P}(W)}(\mathcal{R})$ (resp. $\mathbb{P}_{\mathfrak{W}_C}(\mathcal{E}_{|\mathfrak{W}_C}) \subset \mathbb{P}_{\mathfrak{W}}(\mathcal{E})$) and of the map p^- (resp. p^+) is a contraction of $\mathbb{P}_C(S_C^-)$ (resp. $\mathbb{P}_C(S_C^+)$) onto C. Hence we have the following commutative diagram

$$\begin{array}{ccc} \mathbb{P}_{\mathbb{P}(W)}(\mathcal{R}_C) & \xrightarrow{q} & \mathbb{P}_{\mathfrak{W}_C}(\mathcal{E}_{|\mathfrak{W}_C}) \\ {\scriptstyle p^-}\big\downarrow & & {\scriptstyle p^+}\big\downarrow \\ Y & = & Y \end{array}$$

where

$$\mathcal{M}_{2/3} \supset Y = p^-(\mathbb{P}_{\mathbb{P}(W)}(\mathcal{R}_C)) = \overline{p^-(\mathbb{P}_{\mathbb{P}(W)}(\mathcal{R}_C) \setminus \mathbb{P}_C(S_C^-))} =$$
$$= \overline{p^+(\mathbb{P}_{\mathfrak{W}_C}(\mathcal{E}_{|\mathfrak{W}_C}) \setminus \mathbb{P}_C(S_C^+))} = p^+(\mathbb{P}_{\mathfrak{W}_C}(\mathcal{E}_{|\mathfrak{W}_C})).$$

Applying [6, 5.10] and 2.1.8 we get

$$(K_{\mathbb{P}_{\mathbb{P}(W)}(\mathcal{R}_C)})_{|\mathbb{P}_C(S_C^-)} \in p^{-*}\,\mathbf{Pic}\,C, \quad (K_{\mathbb{P}_{\mathfrak{W}_C}(\mathcal{E}_{|\mathfrak{W}_C})})_{|\mathbb{P}_C(S_C^+)} \in p^{+*}\,\mathbf{Pic}\,C$$

and we see that q satisfies the definition of a flop. $\square$

Remark 2. It is easy to see that $\widetilde{\mathcal{M}}_C = \mathbb{P}_{\mathbb{P}(W)}(\mathcal{R}_C) \times_Y \mathbb{P}_{\mathfrak{W}_C}(\mathcal{E}_{|\mathfrak{W}_C})$. Indeed, we have

$$\widetilde{\mathcal{M}}_C = \widetilde{M}_C \sqcup \widetilde{\mathcal{M}}_C \setminus \widetilde{M}_C, \qquad \mathbb{P}_{\mathfrak{W}_C}(\mathcal{E}_{|\mathfrak{W}_C}) = \mathbb{P}_C(S_C^+) \sqcup \mathbb{P}_{\mathfrak{W}_C}(\mathcal{E}_{|\mathfrak{W}_C}) \setminus \mathbb{P}_C(S_C^+),$$
$$\mathbb{P}_{\mathbb{P}(W)}(\mathcal{R}_C) = \mathbb{P}_C(S_C^-) \sqcup \mathbb{P}_{\mathbb{P}(W)}(\mathcal{R}_C) \setminus \mathbb{P}_C(S_C^-), \qquad Y = C \sqcup Y \setminus C$$

and evidently

$$\widetilde{M}_C = \mathbb{P}_C(S_C^-) \times_C \mathbb{P}_C(S_C^+), \qquad \widetilde{\mathcal{M}}_C \setminus \widetilde{M}_C = Y \setminus C = \mathbb{P}_{\mathbb{P}(W)}(\mathcal{R}_C) \setminus \mathbb{P}_C(S_C^-) = \mathbb{P}_{\mathfrak{W}_C}(\mathcal{E}_{|\mathfrak{W}_C}) \setminus \mathbb{P}_C(S_C^+).$$

Remark 3. One can easily prove that

1). the space $Y \subset \mathbb{P}^7$ is a complete intersection of three quadrics;
2). the embedding $C \subset Y \subset \mathbb{P}^7$ is given by the complete linear system $|2K_C + \theta|$;
3). the curve C is the singular locus of Y;
4). the space $\widetilde{\mathcal{M}}_C$ is the blow-up of Y along C.

2.1.10. **Corollary.** *The space $\mathfrak{W}_C$ is a smooth threefold.*

Proof. First, note that $\mathbb{P}_{\mathfrak{W}_C}(\mathcal{E}_C)$ is a smooth 4-fold, because q is a flop and $\mathbb{P}_{\mathbb{P}(W)}(\mathcal{R}_C)$ is a smooth 4-fold. It is remained to recall that the projection $\pi_1 : \mathbb{P}_{\mathfrak{W}_C}(\mathcal{E}_C) \to \mathfrak{W}_C$ is a smooth $\mathbb{P}^1$-fibration. $\square$

2.1.11. Now we can prove the first part of the Main Theorem which has been formulated in the Introduction.

Proposition (S. Mukai, [8])**.** *The space $\mathfrak{W}_C$ is a smooth Fano threefold V_{22}.*

Proof. The Betti numbers of $\mathfrak{W}_C$ one can easily compute using the Theorem 1. The answer is

$$b_{2i}(\mathfrak{W}_C) = 1, \quad (i = 0,1,2,3) \qquad b_{2i+1}(\mathfrak{W}_C) = 0, \quad (i = 0,1,2).$$

Hence, from Iskovskih's classification Theorem [4] it follows that in order to prove the Proposition it suffices to check that the canonical class $K_{\mathfrak{W}_C}$ of $\mathfrak{W}_C$ generates the Picard group. But this has been done in 2.1.8. $\square$

Remark 4. If C is a double conic (the non-reduced case, see [7, 2.1.2]) then the space $\mathfrak{W}_C$ is the so-called "Japanese" threefold V_{22} which had been overlooked by Iskovskih and was found by S. Mukai and H. Umemura in [10].

Remark 5. Recalling that the space $\mathfrak{W}$ is the space of rational cubic curves in $\mathbb{P}(W)$ defined by a determinantal net of quadrics and the fact that the fiber of $\mathcal{K}$ over the point $P \in \mathfrak{W}$ is just the net $\mathcal{N}_P \subset \mathbf{S}^2 W^*$ corresponding to P imply that $\mathfrak{W}_C$ is nothing else then the subspace in $\mathfrak{W}$ of cubics P with the net of quadrics $\mathcal{N}_P$ orthogonal to the net $N \xrightarrow{A_C} \mathbf{S}^2 W$ (with respect to natural pairing $\mathbf{S}^2 W \otimes \mathbf{S}^2 W^* \to \mathbb{C}$) (compare with [8]).

2.1.12.

Proof of Lemma 2.1.5. Recall that

$$S^+ = \frac{W^*/\eta \otimes W/U}{\mathcal{O}} \cong \frac{W^*/\eta \otimes U^\perp}{\Lambda^2 U} \otimes \Lambda^2 U^*$$

and let

$$\hat{\nu}_C : (\beta \times \gamma)^*(W^*/\eta \otimes U^\perp) \to N^* \otimes \mathcal{O}_C, \qquad \hat{\nu}_C^* : N \otimes \mathcal{O}_C \to (\beta \times \gamma)^*(\eta^\perp \otimes W/U)$$

be the maps induced by ν_C. It suffices to prove that $\hat{\nu}_C^*$ is an embedding of vector bundles, that is for any $x \in C$ the induced map

$$N \to \beta(x)^\perp \otimes W/\gamma(x) = \mathbf{Hom}(W^*/\beta(x), W/\gamma(x))$$

is an embedding. But this map evidently coincides with the map

$$g_{\alpha(x)} : W^*/\beta(x) \to W/U, \qquad W^* \ni \psi \mapsto A_C(\alpha(x), \psi) \mod U.$$

But if for any $\psi \in W^*$ we have $g_{\alpha(x)}(\psi) = 0$ then it means that $\mathbf{dim}\,\mathbf{Im}\, A_C(\alpha(x)) \le 2$, that is $A_C(\alpha(x))$ is a quadric of corank 2 which contradicts the regularity of the net A_C. $\square$

3. The exceptional collection.

3.1. Recall that in [6, 4.1] on the space $\mathbb{P}_{\mathbb{P}(W)}(\mathcal{R})$ there was defined a 2-dimensional vector bundle E^- as a unique non-trivial extension of $\mathcal{O}(-r)$ by $\mathcal{O}(h)$

$$0 \to \mathcal{O}(h) \to E^- \to \mathcal{O}(-r) \to 0.$$

We will denote by the same letter its restriction on $\mathbb{P}_{\mathbb{P}(W)}(\mathcal{R}_C)$. Again, it is an unique extension of $\mathcal{O}(-r)$ by $\mathcal{O}(h)$.

Proposition. *We have*

$$\begin{aligned}
&q_* \mathcal{O}_{\mathbb{P}_{\mathbb{P}(W)}(\mathcal{R}_C)} = \mathcal{O}_{\mathbb{P}_{\mathfrak{W}_C}(\mathcal{E}_{|\mathfrak{W}_C})},\\
&q_* E^-(2r + 2h) = \pi_1^* \mathcal{E}^*_{|\mathfrak{W}_C},\\
&q_* \pi_0^* \Omega_{\mathbb{P}(W)}(r + 3h) = \mathcal{K}^*_{|\mathfrak{W}_C},\\
&q_* \pi_0^* \Omega^2_{\mathbb{P}(W)}(2r + 6h) = \Lambda^2 \mathcal{K}^*_{|\mathfrak{W}_C}
\end{aligned}$$

For any coherent sheaf $\mathcal{G}$ on $\mathbb{P}_{\mathbb{P}(W)}(\mathcal{R}_C)$ we have $q_*(\mathcal{G}(-r-2h)) = (q_*\mathcal{G})(-e)$.

Proof. The first is trivial. Restricting the second sequence of [6, 5.6] on $\widetilde{\mathcal{M}}_C$ we get an exact sequence

$$0 \to \hat{\pi}_0^* E^-(-r - 3h) \to \tilde{\pi}_1^* \mathcal{E}_{|\mathfrak{W}_C} \to i_*(\mathcal{L}^+ \otimes \dots) \to 0,$$

where $\dots$ stans for some sheaf which is the pull back from C. Twisting on $\mathcal{O}_{\widetilde{\mathcal{M}}_C}(3r+5h) = \mathcal{O}_{\widetilde{\mathcal{M}}_C}(H+d)$ (see (5)) and pushing forward along $\hat{\pi}_1$ we get

$$0 \to q_* E^-(2r + 2h) \to \hat{\pi}_{1*} \tilde{\pi}_1^* \mathcal{E}_{|\mathfrak{W}_C}(H + d) \to \hat{\pi}_{1*}\Big(i_*(\mathcal{L}^+ \otimes \dots) \otimes \mathcal{O}_{\widetilde{\mathcal{M}}_C}(3r + 5h)\Big) \to 0.$$

But by [6, 5.10] we have $\mathcal{O}_{\widetilde{\mathcal{M}}_C}(3r+5h)_{|\widetilde{M}_C} = \mathcal{L}^- \otimes \dots$, hence the fourth sheaf in the above sequence is $i_*\hat{\pi}_{1*}(\mathcal{L}^- \otimes \mathcal{L}^+ \otimes \dots) = 0$. On the other hand, pushing forward exact sequence

$$0 \to \tilde{\pi}_1^*\mathcal{E}_{|\mathfrak{W}_C}(H) \to \tilde{\pi}_1^*\mathcal{E}_{|\mathfrak{W}_C}(H+d) \to i_*i^*\tilde{\pi}_1^*\mathcal{E}_{|\mathfrak{W}_C}(H+d) \to 0,$$

and applying [6, 5.10] and evident formula

$$\mathcal{E}_{|\mathfrak{W}_C}(H) = \mathcal{E}_{|\mathfrak{W}_C} \otimes \mathcal{O}_{\widetilde{\mathcal{M}}_C}(c_1(\mathcal{E}^*_{|\mathfrak{W}_C})) = \mathcal{E}^*_{|\mathfrak{W}_C}$$

we get the second formula of the Proposition. The third and fourth formulas similarly follow from (3) and from the third sequence of [6, 5.6]. The fifth formula follows from (5) and from the projection formula. □

3.2. The following Theorem provides $\mathfrak{W}_C$ with an exceptional collection.

Theorem 2. *The following is full and strong exceptional collection on $\mathfrak{W}_C$.*

$$(\mathcal{O}_{\mathfrak{W}_C}, \mathcal{K}^*_{|\mathfrak{W}_C}, \mathcal{E}^*_{|\mathfrak{W}_C}, \Lambda^2\mathcal{K}^*_{|\mathfrak{W}_C}) \tag{6}$$

Proof. Consider the following collection on $\mathbb{P}_{\mathfrak{W}_C}(\mathcal{E}_{|\mathfrak{W}_C})$

$$(\mathcal{O}_{\mathfrak{W}_C}(-e), \mathcal{K}^*_{|\mathfrak{W}_C}(-e), \mathcal{E}^*_{|\mathfrak{W}_C}(-e), \Lambda^2\mathcal{K}^*_{|\mathfrak{W}_C}(-e), \mathcal{O}_{\mathfrak{W}_C}, \mathcal{K}^*_{|\mathfrak{W}_C}, \mathcal{E}^*_{|\mathfrak{W}_C}, \Lambda^2\mathcal{K}^*_{|\mathfrak{W}_C}) \tag{7}$$

(we omit π_1^* for short). According to [11] the collection (6) is full and exceptional iff the collection (7) is, so we will prove that (7) is full exceptional collection.

To this end we will represent it as the push forward along q of a certain exceptional collection on $\mathbb{P}_{\mathbb{P}(W)}(\mathcal{R}_C)$. Since the construction is long we will divide it on steps.

3.2.1. First we chose the following full exceptional collection on $\mathbb{P}(W)$

$$(\mathcal{O}_{\mathbb{P}(W)}(-h), \Omega^2_{\mathbb{P}(W)}(2h), \Omega_{\mathbb{P}(W)}(h), \mathcal{O}_{\mathbb{P}(W)})$$

3.2.2. Then we construct an exceptional collection on $\mathbb{P}_{\mathbb{P}(W)}(\mathcal{R}_C)$ as in [11]

$$(\mathcal{O}(-h), \Omega^2_{\mathbb{P}(W)}(2h), \Omega_{\mathbb{P}(W)}(h), \mathcal{O}, \mathcal{O}(r+h), \Omega^2_{\mathbb{P}(W)}(r+4h), \Omega_{\mathbb{P}(W)}(r+3h), \mathcal{O}(r+2h))$$

(we omit π_0^* for short). Its second part is the first twisted by $\mathcal{O}(r+2h)$ which can be considered as relative $\mathcal{O}(1)$. But this collection is not the required one. We yet have to make some mutations.

3.2.3. We translate $\mathcal{O}(-h)$ and $\Omega^2_{\mathbb{P}(W)}(2h)$ from left to right. Since $K_{\mathbb{P}(\mathcal{R}_C)} = 2r+4h$ (see 2.1.8) it follows from [1] that we'll get the following full exceptional collection

$$(\Omega_{\mathbb{P}(W)}(h), \mathcal{O}, \mathcal{O}(r+h), \Omega^2_{\mathbb{P}(W)}(r+4h), \Omega_{\mathbb{P}(W)}(r+3h), \mathcal{O}(r+2h), \mathcal{O}(2r+3h), \Omega^2_{\mathbb{P}(W)}(2r+6h))$$

3.2.4. We make the left mutation of $\mathcal{O}(f+h)$ through $\mathcal{O}$ and of $\mathcal{O}(2r+3h)$ through $\mathcal{O}(r+2h)$. Since

$$\mathbf{Hom}^i(\mathcal{O}, \mathcal{O}(r+h)) = \mathbf{H}^i(\mathcal{O}(r+h)) = \mathbf{H}^i(\mathbb{P}(W), \mathcal{R}^*_C(h)) = \mathbb{C}, \text{ for } i=1 \text{ and } 0, \text{ for others}$$

(see [7, 3.2.2]) the result of mutation is but the unique non-trivial extension of $\mathcal{O}$ by $\mathcal{O}(r+h)$ (resp. of $\mathcal{O}(r+2h)$ by $\mathcal{O}(2r+3h)$) that is $E^-(r)$ (resp. $E^-(2r+2h)$). Therefore the following collection is full and exceptional

$$(\Omega_{\mathbb{P}(W)}(h), E^-(r), \mathcal{O}, \Omega^2_{\mathbb{P}(W)}(r+4h), \Omega_{\mathbb{P}(W)}(r+3h), E^-(2r+2h), \mathcal{O}(r+2h), \Omega^2_{\mathbb{P}(W)}(2r+6h))$$

3.2.5. We make the left mutation of $\Omega^2_{\mathbb{P}(W)}(r+4h)$ through $\mathcal{O}$ and of $\Omega^2_{\mathbb{P}(W)}(2r+6h)$ through $\mathcal{O}(r+2h)$. Since

$$\begin{aligned}\mathbf{Hom}^\bullet(\mathcal{O}(r+2h),\Omega^2_{\mathbb{P}(W)}(2r+6h)) = \mathbf{Hom}^\bullet(\mathcal{O},\Omega^2_{\mathbb{P}(W)}(r+4h)) = \mathbf{H}^\bullet(\Omega^2_{\mathbb{P}(W)}(r+4h)) =\\ = \mathbf{H}^\bullet(\mathbb{P}(W),\mathcal{R}^*_C\otimes\Omega^2_{\mathbb{P}(W)}(4h)) = \mathbf{H}^\bullet(\mathbb{P}(W),\mathcal{R}^*_C\otimes\mathcal{T}_{\mathbb{P}(W)}) = \mathbf{Hom}^\bullet(\mathcal{R}_C,\mathcal{T}_{\mathbb{P}(W)}) = 0.\end{aligned}$$

(see [7, 3.1.6]) the mutation degenerates into the trivial transposition and the following collection is full and exceptional

$$(\Omega_{\mathbb{P}(W)}(h), E^-(r), \Omega^2_{\mathbb{P}(W)}(r+4h), \mathcal{O}, \Omega_{\mathbb{P}(W)}(r+3h), E^-(2r+2h), \Omega^2_{\mathbb{P}(W)}(2r+6h), \mathcal{O}(r+2h))$$

3.2.6. We translate $\mathcal{O}(r+2h)$ from right to left. Similarly to 3.2.3 we get the following full exceptional collection

$$(\mathcal{O}(-r-2h), \Omega_{\mathbb{P}(W)}(h), E^-(r), \Omega^2_{\mathbb{P}(W)}(r+4h), \mathcal{O}, \Omega_{\mathbb{P}(W)}(r+3h), E^-(2r+2h), \Omega^2_{\mathbb{P}(W)}(2r+6h))$$

3.2.7. Note, that the first part of the above collection coincides with the second part twisted by $\mathcal{O}(-r-2h)$. Therefore by the Proposition 3.1 its push forward along q is the collection (7). According to [2] the push forward along flop is an equivalence of categories hence the collection (7) (and therefore the collection (6)) is full and exceptional.

3.2.8. It is remained to prove that collection (6) is strong. But this evidently follows from [1]. $\square$

4. Rational curves on $\mathfrak{W}_C$

4.1. First we will describe a special class of rational curves in $\mathbb{P}_{\mathbb{P}(W)}(\mathcal{R}_C)$.

4.1.1. Let $M := \mathbf{H}^0(\mathbb{P}(W),\mathcal{R}_C(2))$.

Lemma. *We have* $\mathbf{H}^{>0}(\mathbb{P}(W),\mathcal{R}_C(2)) = 0$, $\mathbf{dim}\, M = 8$. *For any line* $L\subset\mathbb{P}(W)$ *the restriction map* $\mathbf{H}^0(\mathbb{P}(W),\mathcal{R}_C(2)) \to \mathbf{H}^0(L,\mathcal{R}_C(2)_{|L})$ *is surjective.*

Proof. Consider the following resolvent of the sheaf $\mathcal{O}_L$ on $\mathbb{P}(W)$

$$0\to L\otimes\mathcal{O}_{\mathbb{P}(W)}(-1)\to\mathcal{T}_{\mathbb{P}(W)}(-2)\to\mathcal{O}_{\mathbb{P}(W)}\to\mathcal{O}_L\to 0.$$

The spectral cohomology sequence of this resolvent tensored by $\mathcal{R}_C(2)$ with application of [7] ??? gives

$$0\to L\to\mathbf{H}^0(\mathbb{P}(W),\mathcal{R}_C(2))\to\mathbf{H}^0(L,\mathcal{R}_C(2)_{|L})\to 0,\qquad \mathbf{H}^{>0}(\mathbb{P}(W),\mathcal{R}_C(2)) = \mathbf{H}^{>0}(L,\mathcal{R}_C(2)_{|L})$$

and Lemma follows because according to [7, 3.3.2] $\mathcal{R}_C$ has no d-jumping lines with $d>2$. $\square$

4.1.2. **Definition.** For any rational curve $X\subset\mathbb{P}(W)$ (resp. $X\subset\mathfrak{W}_C$) let $\tilde{X}$ denote the exceptional section of the Hirzebruch surface $\pi_0^{-1}(X) = \mathbb{P}_X(\mathcal{R}_{C|X})$ (resp. $\pi_1^{-1}(X) = \mathbb{P}_X(\mathcal{E}_{|X})$).

Definition. Let $\Sigma^-\subset\mathbb{P}_{\mathbb{P}(W)}(\mathcal{R}_C)$ be the surface spanned by $\tilde{L}$ while L runs through the curve $\gamma(C)$ of 2-jumping lines of $\mathcal{R}_C$.

Note that $\mathbf{H}^0(\mathbb{P}_{\mathbb{P}(W)}(\mathcal{R}_C),\mathcal{O}(r+2h)) = \mathbf{H}^0(\mathbb{P}(W),\mathcal{R}_C(2)) = M$.

Lemma. *The complete linear system* $|r+2h|$ *on* $\mathbb{P}_{\mathbb{P}(W)}(\mathcal{R}_C)$ *is base-point free. The map*

$$\xi^-:\mathbb{P}_{\mathbb{P}(W)}(\mathcal{R}_C)\to\mathbb{P}(M^*)$$

given by this linear system is a contraction of $\Sigma^-\subset\mathbb{P}_{\mathbb{P}(W)}(\mathcal{R}_C)$ *on* C.

Proof. Since $\mathcal{R}_C$ has no d-jumping lines with $d > 2$ the restriction $\mathcal{R}_C(1)_{|L}$ is generated by its global sections for any non-2-jumping line L. Hense the restriction of $|r + 2h|$ on $\pi_0^{-1}(L)$ is ample if L is not 2-jumping.

On the other hand, if L is 2-jumping, then $\pi_0^{-1}(L) = \mathbb{P}_L(\mathcal{O}_L(-2) \oplus \mathcal{O}_L(2))$ hence the restriction $\xi^-_{|\pi_0^{-1}(L)}$ is a contraction of the exceptional section $\tilde{L}$. $\square$

4.1.3. Note that from the above Lemma it follows that for any 2-jumping lines $L_1 \neq L_2$ we have $\tilde{L}_1 \cap \tilde{L}_2 = \emptyset$, hence Σ^- is a smooth ruled surface over the curve C.

For any $x \in C$ let $l_x^- = {p^-}^{-1}(x) \subset \mathbb{P}_C(S_C^-) \subset \mathbb{P}_{\mathbb{P}(W)}(\mathcal{R}_C)$.

Lemma. *We have $\Sigma^- = \mathbb{P}_C(S_C^-) \subset \mathbb{P}_{\mathbb{P}(W)}(\mathcal{R}_C)$ as schemes over C.*

Proof. From [6, 5.10] it follows that the map ξ^- contracts $\mathbb{P}_C(S_C^-) \subset \mathbb{P}_{\mathbb{P}(W)}(\mathcal{R}_C)$ on C, hence it is remained to note that by [6, 5.4] the map

$$\gamma' : C \to \mathbf{Gr}(2, W) \qquad x \mapsto \pi_0(l_x^-)$$

coincides with γ. $\square$

Remark 6. Since $r + 2h = e$ the Remark 2 implies that these divisor classes are but pull-backs from Y along p^- and p^+ respectively. Therefore the map ξ^- factors as $\xi^- = \xi \cdot p^-$, where $\xi : Y \to \mathbb{P}(M^*) = \mathbb{P}^7$ is a closed embedding (compare with Remark 3).

4.1.4. **Corollary.** *a) The complete linear system $|e|$ on $\mathbb{P}_{\mathfrak{W}_C}(\mathcal{E}_{|\mathfrak{W}_C})$ is base-point free. The map ξ^+ given by $|e|$ factors as $\xi \cdot p^+$.*

b) The bundle $\mathcal{E}^*_{|\mathfrak{W}_C}$ on $\mathfrak{W}_C$ is generated by its global sections $\mathbf{H}^0(\mathfrak{W}_C, \mathcal{E}^*_{|\mathfrak{W}_C}) \cong M$.

4.2. **Lines in $\mathfrak{W}_C$.** For any $x \in C$ let $l_x^+ = {p^+}^{-1}(x) \subset \mathbb{P}_C(S_C^+) \subset \mathbb{P}_{\mathfrak{W}_C}(\mathcal{E}_{|\mathfrak{W}_C})$.

4.2.1. Recall that a curve $l \subset \mathfrak{W}_C$ is called line (resp. conic, cubic) if $K_{\mathfrak{W}_C} \cdot l = H \cdot l = 1$ (resp. 2, 3).

Now we will prove the second part of the Main Theorem.

Proposition. *For all $x \in C$ the curve $\pi_1(l_x^+)$ is a line in $\mathfrak{W}_C$. If $l \subset \mathfrak{W}_C$ is a line then $\tilde{l} = l_X^+$ for some $x \in C$.*

Proof. The first part immidiately follows from [6, 5.10].

Given a line $l \subset \mathfrak{W}_C$ we have $\mathcal{E}^*_{|l} = \mathcal{O}_l \oplus \mathcal{O}_l(1)$ (because $\mathcal{E}^*_{|\mathfrak{W}_C}$ is generated by its global sections), hence $e \cdot \tilde{l} = 0$, that is $\tilde{l}$ is contracted by p^+, hence $\tilde{l} = l_x^+$ for some $x \in C$. $\square$

4.3. **Intersection of lines.**

4.3.1. **Definition.** A rational cubic $X \subset \mathbb{P}(W)$ (possibly reducible) is called *superjumping* for $\mathcal{R}_C$ if we have $\mathbf{H}^1(X, \mathcal{R}_C(1)_{|X}) \neq 0$. The usual resolvent

$$0 \to \mathcal{O}_{\mathbb{P}(W)}(-3)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}(W)}(-2)^{\oplus 3} \to \mathcal{O}_{\mathbb{P}(W)} \to \mathcal{O}_X \to 0$$

tensored by $\mathcal{R}_C(1)$ and [7, 3.2.2] show that cubic X is superjumping iff the net of quadrics in $\mathbb{P}(W)$ intersecting on X is ortogonal (with respect to natural pairing $\mathbf{S}^2 W \otimes \mathbf{S}^2 W^* \to \mathbb{C}$) to the net $A_C : N \to \mathbf{S}^2 W$ defining instanton $\mathcal{R}_C$.

4.3.2. **Lemma.** *For any point $P \in \mathfrak{W}_C$ the curve $X = \tilde{\pi}_0(\tilde{\pi}_1^{-1}(P))$ is a superjumping cubic (possibly reducible).*

Proof. Follows from the Remark 5. $\square$

4.3.3. Assume that C is smooth.

Lemma. *If a superjumpig cubic $X \subset \mathbb{P}(W)$ is a union $X = L_1 \cup L \cup L_2$ of three lines such that $L_1 \cap L \neq \emptyset$ and $L_2 \cap L \neq \emptyset$ then $L_1 = \gamma(x_1)$, $L_2 = \gamma(x_2)$ for some $x_1, x_2 \in C$ and $(x_1, x_2) \in \mathbf{T}_\theta$.*

Proof. Let $\varphi_1 \in W^*$ (resp. φ_2) be an equation of the hyperplane H_1 (resp. H_2) containing $L_1 \cup L$ (resp. $L_2 \cup L$). Let $\psi_1 \in W^*$ (resp. ψ_2) be an equation of a hyperplane $H'_1 \neq H_1$ (resp. $H'_2 \neq H_2$) such that $L_1 = H_1 \cap H'_1$ (resp. $L_2 = H_2 \cap H'_2$). Then the net of quadrics intersecting on X is $\langle \varphi_1\psi_2, \varphi_1\varphi_2, \psi_1\varphi_2 \rangle \subset \mathbf{S}^2 W^*$. The orthogonality with A_C implies that $\varphi_1 = \beta(x_1)$, $\varphi_2 = \beta(x_2)$ for some $x_1, x_2 \in C$ and $\gamma(x_1) = L_2$, $\gamma(x_2) = L_1$. Since $L_1 \subset H_1$ it follows from [7, 3.3.3] that $(x_1, x_2) \in \mathbf{T}_\theta$. □

Remark 7. If C is a double conic (the non-reduced case, see [7, 2.1.2 and Remark 8]) then the same arguments show that no superjumping cubics for $\mathcal{R}_C$ are union of three lines.

4.3.4. Now we will prove the last part of the Main Theorem. Assume that C is smooth.

Proposition. *Let $x_1, x_2 \in C$. Lines $l_1 = \pi_1(l^+_{x_1})$ and $l_2 = \pi_1(l^+_{x_2})$ in $\mathfrak{W}_C$ meet iff $(x_1, x_2) \in \mathbf{T}_\theta$.*

Proof. Assume that $l_1 \cap l_2 = P \in \mathfrak{W}_C$. Then $\tilde{\pi}_0(\tilde{\pi}_1^{-1}(P)) = \pi_0(l^-_{x_1} \cup q^*\pi_1^{-1}(P) \cup l^-_{x_2})$ is a union of three lines and the Proposition follows from. □

Remark 8. If C is a double conic (i.e. $\mathfrak{W}_C$ is the Japanese threefold) then the Remark 7 implies that distinct lines in $\mathfrak{W}_C$ do not meet.

4.4. **Rational cubics in $\mathfrak{W}_C$.**

Proposition. *Let $P \in \mathbb{P}(W)$ be a point not lying on a 2-jumping line of $\mathcal{R}_C$ (i.e. $P \in \mathbb{P}(W) \setminus \pi_0(\Sigma^-)$). Then $\tilde{\pi}_1(\tilde{\pi}_0^{-1}(P))$ is an irreducible rational cubic curve in $\mathfrak{W}_C$. Any irreducible rational cubic curve in $\mathfrak{W}_C$ can be obtained thus.*

Proof. The first follows from (5) and 4.1.3.

Let X be an irreducible rational cubic in $\mathfrak{W}_C$. Then either $\mathcal{E}^*_{|X} = \mathcal{O}_X \oplus \mathcal{O}_X(3)$ or $\mathcal{E}^*_{|X} = \mathcal{O}_X(1) \oplus \mathcal{O}_X(2)$. The first is impossible because then $\tilde{X}$ were contracted by p^+ which is impossible by the Proposition 4.2.1. Therefore $e \cdot \tilde{X} = 1$. Since $\tilde{X} \not\subset \mathbb{P}_C(S^+_C)$ we have $d \cdot \tilde{X} \geq 0$, hence by (5) $h \cdot \tilde{X} \leq 0$. This means that $h \cdot \tilde{X} = 0$ and $d \cdot \tilde{X} = 0$, hence $\tilde{X} = \tilde{\pi}_0^{-1}(P)$ for some point $P \in \mathbb{P}(W)$ not lying on a 2-jumping line of $\mathcal{R}_C$. □

The Proposition shows that $\mathbb{P}(W)$ is a natural compactification of the space $\mathbf{Rat}_3(\mathfrak{W}_C)$ of irreducible rational twisted cubics in $\mathfrak{W}$.

4.5. **Conics in $\mathfrak{W}_C$.** Assume that C is smooth.

4.5.1. **Lemma.** *Let $X \subset \mathfrak{W}_C$ be an irreducible conic and $D_X = x_1 + \cdots + x_r$ is the divisor of points in C, corresponding to the lines in $\mathfrak{W}_C$ meeting X. Then $D_X \in |K_C|$.*

Proof. To prove the Lemma we degenerate X into the union $l_1 \cup l_2$ of intersecting lines. By the Proposition 4.3.4 we have $l_i = \pi_1(l^+_{y_i})$ for some $y_1, y_2 \in C$ such that $(y_1, y_2) \in \mathbf{T}_\theta$. This means that $\mathbf{T}_\theta(y_1) = y_2 + x_1 + x_2$, $\mathbf{T}_\theta(y_2) = y_1 + x_3 + x_4$. Therefore

$$D_X = x_1 + x_2 + x_3 + x_4 \sim \mathbf{T}_\theta(y_1) - y_2 + \mathbf{T}_\theta(y_2) - y_1 \sim (y_1 + \theta - y_2) + (y_2 + \theta - y_1) \sim 2\theta \sim K_C.$$

□

4.5.2. **Proposition.** *Let $P \in \mathbb{P}(W)$ be a point lying on exactly one 2-jumping line $L = \gamma(x)$ of $\mathcal{R}_C$ (i.e. P is a smooth point of the surface $\pi_0(\Sigma^-)$). Then $\tilde{\pi}_1(\tilde{\pi}_0^{-1}(P)) = X \cup \pi_1(l_x^+)$ is the union of an irreducible conic $X \subset \mathfrak{W}_C$ and of a line $\pi_1(l_x^+)$ meeting X. This defines a 4-fold covering $\zeta = \tilde{\pi}_1 \cdot \tilde{\pi}_0^{-1} : \pi_0(\Sigma^-) \setminus \mathbf{sing}(\pi_0(\Sigma^-)) \to \mathbf{Rat}_2(\mathfrak{W}_C)$ of the space $\mathbf{Rat}_2(\mathfrak{W}_C)$ of irreducible conics on $\mathfrak{W}_C$ by $\pi_0(\Sigma^-) \setminus \mathbf{sing}(\pi_0(\Sigma^-))$.*

Proof. The first is easy consequence of (5) and 4.1.3.

To prove the second we consider an arbitrary conic $X \subset \mathfrak{W}_C$. The same arguments as in proof of the Proposition 4.4 show that $\mathcal{E}^*_{|X} = \mathcal{O}_X(1) \oplus \mathcal{O}_X(1)$, hence $\pi_1^{-1}(X) \cong X \times \mathbb{P}^1$. On the other hand it is known that X meets exactly four lines (see 4.5.1) on $\mathfrak{W}_C$ hence the surface Σ^+ intersects $\pi_1^{-1}(X)$ in four points. Let $\tilde{X}_1$, $\tilde{X}_2$, $\tilde{X}_3$, $\tilde{X}_4$ be the horizontal sections of $\pi_1^{-1}(X)$ passing through one of these four points. Then the relations (5) show that $h \cdot \tilde{X}_i = 0$ and $d \cdot \tilde{X}_i = 1$, hence $\tilde{X}_i = \tilde{\pi}_0^{-1}(P_i)$ for some $P_i \in L_i \subset \mathbb{P}(W)$, where $L_i = l_{x_i}^-$ is a certain 2-jumping line of $\mathcal{R}_C$. □

4.5.3. Let $\tilde{\Sigma} \subset \mathbf{Gr}(2, N) \times C = \{(\nu, x) \mid \nu \ni \alpha(x)\}$ be the incidence subspace. We have a natural map

$$\sigma : \tilde{\Sigma} \to \pi_0(\Sigma^-) \qquad (\nu, x) \mapsto A_C(\nu, \beta(x)) \in \gamma(x) \subset \mathbb{P}(W).$$

It is easy to see that this map is a desingularization of $\pi_0(\Sigma^-)$.

Proposition. *Let $\mathbf{Rat}_2(\mathfrak{W}_C)$ be the space of conics on $\mathfrak{W}_C$. There is an embedding $\kappa : \mathbf{Rat}_2(\mathfrak{W}_C) \subset \mathbf{Gr}(2, N)$ such that the composition*

$$\tilde{\Sigma} \xrightarrow{\sigma} \pi_0(\Sigma^-) \xrightarrow{\zeta} \mathbf{Rat}_2(\mathfrak{W}_C) \xrightarrow{\kappa} \mathbf{Gr}(2, N) \tag{8}$$

is induced by the projection of $p_1 : \mathbf{Gr}(2, N) \times C \to \mathbf{Gr}(2, N)$ on the first factor.

Proof. The Lemma 4.5.1 implies that for any $X \in \mathbf{Rat}_2(\mathfrak{W}_C)$ we have $p_2(\zeta^{-1}(X)) \in |K_C|$, where $p_2 : \mathbf{Gr}(2, N) \times C \to C$ is the projection. Therefore there is defined a map $\kappa : \mathbf{Rat}_2(\mathfrak{W}_C) \to \mathbb{P}(N^*) = \mathbf{Gr}(2, N)$ such that $\alpha^{-1}(\kappa(X) \cap \alpha(C)) = p_2(\zeta^{-1}(X))$, hence the composition (8) is equal to p_1 (see definition of $\tilde{\Sigma}$). It is remained to show that κ is embedding. But this is evident because both ζ and p_1 are 4-fold coverings. □

The Propositions 4.5.2 and 4.5.3 show that $\mathbf{Gr}(2, N) = \mathbb{P}(N^*) = |K_C|$ is the natural compactification of the space $\mathbf{Rat}_2(\mathfrak{W}_C)$ of irreducible conics in $\mathfrak{W}_C$.

Remark 9. If C is a double conic (i.e. $\mathfrak{W}_C$ is the Japanese threefold) then slightly modified arguments of 4.5.2 and 4.5.3 show that in this case $\mathbb{P}(N^*)$ is also can be considered as the natural compactification of the space $\mathbf{Rat}_2(\mathfrak{W}_C)$.

Remark 10. One can apply the same method to description of the rational curves of higher degree. E.g. one can show that the space $\mathbf{Gr}(2, W)$ is the natural compactification of the space $\mathbf{Rat}_4(\mathfrak{W}_C)$ of irredusible rational curves of degree 4 in $\mathfrak{W}_C$.

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Independent Moscow University, 11 Bolshoj Vlasjevskij pereulok, Moscow 121002 Russia
E-mail address: sasha@ium.ips.ras.ru