STABILITY AND ALMOST PERIODICITY OF TRAJECTORIES OF PERIODIC PROCESSES

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Abstract. We prove that if the monodromy operator V of a linear periodic process $U(t,\tau)$ in a Banach space E is power-bounded, has countable peripheral spectrum, and if its peripheral point spectrum satisfies certain natural and simple duality condition (which always holds in reflexive spaces), then every positive trajectory $u(\tau) = U(0,\tau)x$, $\tau \geq 0$, $x \in E$, is asymptotically almost periodic. If, in particular, the peripheral point spectrum of V^* is empty, then every positive trajectory is asymptotically stable. We also obtain results on almost periodicity of complete bounded trajectories, and consider conditions under which nontrivial bounded complete trajectories exist

1. Introduction

We are mainly concerned with the asymptotic behaviour of solutions of the differential equation

$$u'(t) = A(t)u(t), \tag{1.1}$$

and also of the inhomogeneous equation

$$u'(t) = A(t)u(t) + f(t), \tag{1.2}$$

where $A(t)$ are closed linear operators in a Banach space E, which depend on t in a periodic manner, i.e. there exists a positive number ω such that $A(t+\omega) = A(t)$, $\forall t$. Asymptotic behaviour of solutions of Eqs.(1.1) and (1.2) is a subject of intensive research in recent years (see [Da], [Hl1-2], [Hr1-2], [He], [R-S], and references cited therein).

There are two related, but principally different, problems on the asymptotic behaviour of solutions of Eqs.(1.1)–(1.2). The first problem is concerned with solutions which are defined and satisfy (1.1) (resp., (1.2)) on the whole line $\mathbf{R}$, while the second problem is

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concerned with solutions on the positive half line $\mathbf{R}_+$. We shall investigate both types of behaviour, but we remark that the problem for the positive half line is as a rule more natural and difficult to resolve.

It is well known that if E is finite dimensional, then every solution on $\mathbf{R}_+$ of Eq.(1.1) approaches zero as $t \to \infty$ if and only if all the eigenvalues of the monodromy operator V associated with Eqs.(1.1)–(1.2) have absolute value less than 1 (see e.g. [Am], [Hl1]). If E is infinite dimensional and $r(V)$ – the spectral radius of V – is less than 1, then solutions on $\mathbf{R}_+$ of Eq.(1.1) also converge to 0 (see e.g. [D-K], [M-S]). But, to our knowledge, no general results on the asymptotic stability of (1.1) were known for the case when $r(V) = 1$. We prove, in section 3, that if the monodromy operator is power-bounded and its spectrum has countable intersection with the unit circle, and its peripheral point spectrum satisfies certain natural and simple duality condition (which holds automatically in reflexive spaces), then all solutions of Eq.(1.1) are asymptotically almost periodic (Theorem 3.2). In particular, if the peripheral point spectrum of V^* is empty, then all solutions on $\mathbf{R}_+$ of Eq.(1.1) converge to 0 as $t \to \infty$ (Corollary 3.3).

As concerning the question on the asymptotic behaviour of solutions on $\mathbf{R}$ of Eqs.(1.1) and (1.2), it is well known that if $E = \mathbf{R}^n$ and $A(t) \in C(\mathbf{R}, \mathcal{L}(\mathbf{R}^n, \mathbf{R}^n))$ is periodic, then every bounded solution $u(t), t \in \mathbf{R}$, is almost periodic (see e.g. [Hr1]). This fact, in general, is not true for periodic equations in infinite dimensional Banach spaces (see Remark 4.3). In section 4, we seek for a suitable extension of this result to infinite dimensional systems. More precisely, we show that if the intersection of the spectrum of the monodromy operator with the unit circle is countable and the function f is almost periodic, then every bounded uniformly continuous mild solution $u(t)$ of Eq.(1.2) is almost periodic, provided one of the following conditions holds: (i) the space X does not contain c_0 (the Banach space of numerical sequences converging to 0), or (i') the range of $u(t)$ is weakly relatively compact (Theorem 4.2). Furthermore, we show that if the intersection of the spectrum of the monodromy operator with the set $\overline{\{e^{i\mu} : \mu \in \Sigma(f)\}}$ is empty, where $\Sigma(f) = \{\lambda + 2n\pi/\omega : \lambda \in \mathrm{Sp}(f), n \in \mathbf{Z}\}$ and $\mathrm{Sp}(f)$ is the spectrum of f, then there exists a unique almost periodic solution $u(t)$ of Eq. (1.2) (Theorem 4.5).

Finally, in section 5, we present a general criterion under which there exist nontrivial bounded solutions on $\mathbf{R}$ of Eq.(1.1) (Theeorem 5.3).

Our approach is based on the notion of processes. There are known conditions when for every initial value x from a dense set in E, there exists a classical solution on $\mathbf{R}$ of Eqs.(1.1)–(1.2), with $u(0) = x$. Then the solutions generate in a natural way a process. Since processes arise when we consider not only equations of type (1.1)–(1.2), but also more general retarded functional differential equations (see e.g. [Hl1]), the results of the present paper can be applied to these classes of equations as well.

The main results of this paper are essentially based on the existing results and methods for linear operators and autonomous differential equations, which we have developed in our previous publications (see [L-V], [V-L], [V1], [V2]). Although this material is now known, it is not very well known so we think it is worthwhile to make it more available by giving precise formulations of the corresponding results, as we have done at relevant places in the paper.

2. De Leeuw-Glicksberg decomposition for periodic processes

Let E be a Banach space. A process on E is a two-parameter family of mappings $\mathcal{U} = \{U(t,\tau) : t \in \mathbf{R}, \tau \geq 0\}$ of E into itself, which satisfy the following properties:

(i) $U(t,0) = I$;

(ii) $U(t,\sigma+\tau) = U(t+\tau,\sigma)U(t,\tau)$, for all $t \in \mathbf{R}, \tau \geq 0, \sigma \geq 0$;

(iii) The function $u(t,\tau) = U(t,\tau)x$ is continuous for all $x \in X$.

A process $\mathcal{U}$ is said to be ω-periodic, if there is $\omega > 0$ such that $U(t+\omega,\sigma) = U(t,\sigma)$, $\forall t \in \mathbf{R}, \sigma \geq 0$. Without loss of generality, we assume that $\omega = 1$. A process $\mathcal{U}$ is said to be an *autonomous process* (or a *continuous dynamical system*) if $U(t,\sigma)$ is independent of t. In this case $T(\sigma) \equiv U(t,\sigma)$ is a continuous one-parameter semigroup: $T(0) = I$, $T(\sigma+\tau) = T(\sigma)T(\tau), \sigma \geq 0, \tau \geq 0$. A process $\mathcal{U}$ is called *linear* if $U(t,\tau)$ are bounded linear operators in E.

The *positive trajectory* of $\mathcal{U}$ through $x \in E$ is defined by $u(t) = U(0,t)x$, $t \geq 0$. A *complete trajectory* of $\mathcal{U}$ through $x \in E$ is by definition a continuous function $u : \mathbf{R} \to E$

such that $u(t+\tau) = U(t,\tau)u(t), \quad t \in \mathbf{R}, \tau \geq 0, \; u(0) = x$. While there always exists a positive trajectory through an arbitrary x in E, a complete trajectory through x may not exist, so that to say that there exists a complete trajectory under the process $\mathcal{U}$ through x may impose restrictions on x (see [Hl2], [V2]).

Let $\mathcal{U}$ be a linear 1-periodic process. Then the mapping $V \equiv U(0,1)$ is called the *monodromy operator*. It is easy to see that $V^n = U(0,n)$ (see e.g. [Hl1]). A complete orbit through x is, by definition, a two-sided sequence $\{x_n\}_{n \in \mathbf{Z}}$ in E such that $x_n = V^m x_{n-m}$, for all $n, m \in \mathbf{Z}, m \geq 0$.

LEMMA 2.1. *Let x be a vector in E. If there exists a complete orbit of the monodromy operator V through x, then there exists a complete trajectory $u(t)$ of the process $\mathcal{U}$ through x.*

PROOF: Indeed, we put $u(n) = x_n, \quad n \in \mathbf{Z}$, and define,for $0 \leq \tau \leq 1$,

$$u(-n+\tau) = U(-n,\tau)u(-n)$$

Then $u(t)$ is defined and continuous on $\mathbf{R}$, and represents a complete trajectory of U with $u(0) = x$. ∎

If, in Lemma 2.1, the complete orbit $\{x_n\}$ of V through x is bounded, then, as is easily seen, the corresponding complete trajectory of $\mathcal{U}$ is also bounded.

The following is well known as de Leeuw-Glicksberg's decomposition for operators whose orbits $\{T^n x : n = 0, 1, 2, \ldots\}$ are relatively compact for all $x \in E$. Let $\Gamma = \{\lambda \in \mathbf{C} : |\lambda| = 1\}$.

THEOREM 2.2. *Assume that T is a bounded linear operator in a Banach space E such that the orbits $\{T^n x : n = 0, 1, 2 \ldots\}$ are relatively compact for all $x \in E$. Then there are closed subspaces E_0, E_1 of E such that*

$$E = E_0 \oplus E_1, \tag{2.1}$$

where

$$E_0 = \{x \in E : \|T^n x\| \to 0 \text{ as } n \to \infty\},$$

and

$$E_1 = \overline{\operatorname{span}}\{x \in X : \exists\, \lambda \in \Gamma \text{ such that } Tx = \lambda x\}.$$

Let $\mathcal{G}$ be an infinite subgroup of the additive group $\mathbf{R}$. A function $f : \mathcal{G} \to E$ is said to be ***almost periodic***, if the family of translates $H_f \equiv \{f_t(s) \equiv f(t+s) : \; t, s \in \mathcal{G}\}$ is relatively compact in $C(\mathcal{G}, E)$ – the space of bounded continuous functions from $\mathcal{G}$ to E, with the topology of uniform convergence. Let $\mathcal{S} \subset \mathcal{G}$ be a subsemigroup such that $\mathcal{S} - \mathcal{S} = \mathcal{G}$. A function $f : \mathcal{S} \to E$ is said to be ***asymptotically almost periodic***, if there exist functions $g : \mathcal{G} \to E$ and $h : \mathcal{S} \to E$ such that g is almost periodic, $\|h(t)\| \to 0$ as $t \to \infty$, and $f = g|\mathcal{S} + h$. The function g (h) is called *almost periodic* (respectively, *stable*) part of f. Taking subgroups $\mathcal{G} = \mathbf{R}$ and $\mathcal{G} = \mathbf{Z}$ (with subsemigroups $\mathcal{S} = \mathbf{R}_+$ and $\mathcal{S} = \mathbf{Z}_+$, respectively), we get the definitions of almost periodic and asymptotically almost periodic functions and sequences (cf. [A-P], [F], [L-Z]).

The conclusion of Theorem 2.2 means that every sequence $\{T^n x\}_{n \geq 0}, x \in E$, is asymptotically almost periodic, its periodic part is $\{T^n x_1\}_{n \geq 0}$, and its stable part is $\{T^n x_0\}_{n \geq 0}$, where $x = x_0 + x_1$ is the decomposition of x according to (2.1). For more details the reader is referred to [D-G], [Ly].

Recently Haraux [Hr1-2] has shown that if $\mathcal{U}$ is a quasi-contractive, periodic process such that every its positive trajectory is relatively compact, then every positive trajectory is asymptotically almost periodic, and its almost periodic part is a complete trajectory. For linear periodic processes, the result of Haraux can be complemented in the following way, using the mentioned above de Leeuw-Glicksberg's decomposition.

THEOREM 2.3. *Assume that $\mathcal{U}$ is a linear 1-periodic process such that every its positive trajectory is relatively compact. Then there are closed subspaces E_0 and E_1 such that*

$$E = E_0 \oplus E_1,$$

where E_0 consists of $x \in E$ such that the positive trajectory through x converges to 0, *and E_1 consists of $x \in E$ such that there is a complete almost periodic trajectory through x.*

PROOF: From the conditions it follows that every orbit of the monodromy operator V is relatively compact. Therefore, by Theorem 2.2,

$$E = E_0 \oplus E_1,$$

where $E_0 = \{x \in E : \|V^n x\| \to 0\}$, on E_1 the operator V is invertible and the complete orbits $\{V^n x\}_{n \in \mathbf{Z}}$, $x \in E_1$, are relatively compact. Since, for each $x \in E_0$,

$$\|u(t)\| = \|U(t,0)x\| \le \sup_{0 \le t \le 1} \|U(t,0)\| \|V^n x\| \to 0, \quad \text{as} \quad t \to \infty$$

every positive trajectory starting at $x \in E_0$ converges to 0. On the other hand, if $x \in E_1$, then there exists a complete orbit with respect to V, which implies, by Lemma 2.1, that there exists a complete trajectory of $\mathcal{U}$ through x. It is easy to see that any such complete trajectory $u(t)$ has the property that $u(\mathbf{R}_-)$ is relatively compact. Therefore, by [Hr1, Theorem 8], $u(t)$ is almost periodic. ∎

3. A spectral criterion of asymptotic almost periodicity and stability

The following theorem [V-L] gives a spectral criterion under which a given bounded linear operator is such that all positive orbits $\{T^n x : n \ge 0\}$, $x \in E$, are relatively compact.

THEOREM 3.1. *Let T be a power-bounded operator in a Banach space E. Assume that:*

(i) $\sigma(T) \cap \Gamma$ is countable;

(ii) If there exists $\varphi \in E^*$ and $\lambda \in \Gamma$ such that $T^*\varphi = \lambda\varphi$, then there exists $x \in E$ such that $\varphi(x) \ne 0$ and $Tx = \lambda x$.

Then all positive orbits of T are relatively compact (so that T has the decomposition as in Theorem 2.2).

In particular, if under the conditions of Theorem 3.1, $P\sigma(T^*) \cap \Gamma = \emptyset$, then $\|T^n x\| \to 0$ as $n \to \infty$, for all $x \in E$ (here $P\sigma(V^*)$ is the point spectrum of T^*). This latter fact also has been obtained independently and by a completely different method in [A-B].

Using Theorem 3.1, Lemma 2.1, and the quoted result of Haraux, we obtain the following result.

THEOREM 3.2. *Assume that the monodromy operator V of a 1-periodic process $\mathcal{U}$ is power-bounded, i.e.* $\sup_{n\geq 0}\|V^n\| < \infty$, *and*

(i) $\sigma(V)\cap\Gamma$ *is countable;*

(ii) If there is a functional $\varphi \in X^*$ *and a complex number* $\lambda\in\Gamma$ *such that* $V^*\varphi = \lambda\varphi$, *then there is* $x\in X$ *such that* $\varphi(x)\neq 0$ *and* $Vx = \lambda x$.

Then every positive trajectory $u(t)$ of $\mathcal{U}$ is asymptotically almost periodic, and its periodic part is a complete trajectory of $\mathcal{U}$.

PROOF: The conditions imply that there exists de Leeuw-Glicksberg's decomposition for the monodromy operator V,

$$E = E_0 \oplus E_1. \tag{3.1}$$

We show that for every $x\in E$, the positive trajectory $u(t) = U(0,t)x$ is asymptotically almost periodic. According to De Leeuw-Glicksberg's decomposition of V, $x = x_0 + x_1$, where $x_0\in E_0$, $x_1\in E_1$. Thus, $u(t) = u_0(t) + u_1(t)$, where $u_0(t) = U(0,t)x_0$, $u_1(t) = U(0,t)x_1$. We have

$$\|u_0(t)\| = \|U(t,0)x_0\| \leq \sup_{0\leq t\leq 1}\|U(t,0)\|\|V^n x_0\| \to 0, \quad \text{as} \quad t\to\infty.$$

On the other hand, there exists a complete (in fact, almost periodic) orbit of the monodromy operator V, through x_1. By Lemma 2.1, there is a complete trajectory $v(t)$ of $\mathcal{U}$ through x_1 which of course must coincide with $u_1(t)$ for $t\geq 0$. By the same argument as in the previous part of the proof, one can show that the complete trajectory $v(t)$ has relatively compact range. Therefore, by [Hr1, Theorem 8], $v(t)$ is almost periodic. ∎

As a particular case of Theorem 3.2, we obtain the following condition for asymptotic stability of positive trajectories.

COROLLARY 3.3. *Assume that the monodromy operator V of a 1-periodic process $\mathcal{U}$ is power-bounded and*

(i) $\sigma(V) \cap \Gamma$ is countable;

(ii) $P\sigma(V^*) \cap \Gamma = \emptyset$.

Then every positive trajectory $u(t)$ of $\mathcal{U}$ satisfies $\|u(t)\| \to 0$ as $t \to \infty$.

PROOF: Under the conditions (i) and (ii) of the Corollary, the subspace E_1 in decomposition (3.1) is equal 0. The rest follows from the proof of Theorem 3.2. ∎

REMARK 3.4. Condition (ii) of Theorem 3.2 is necessary for the validity of the statement. Moreover, if the Banach space E is reflexive, then it is fulfilled automatically, and condition (ii) of Corollary 3.3 can be replaced by $P\sigma(V) \cap \Gamma = \emptyset$ (see [V-L]).

4. ALMOST PERIODIC SOLUTIONS ON $\mathbf{R}$.

In this section, we turn to Eqs.(1.1) and (1.2). Assume, that the operator-valued function $A(t)$ satisfies suitable conditions so that there exists, for every initial value $u(0) = x$ from a dense set in E, a classical solution on $\mathbf{R}_+$ of (1.2), with $u(0) = x$. Many such conditions are known (see e.g. [Pa], [He]). Then there exists a linear process $\mathcal{U} = \{U(t,\tau) : t \in \mathbf{R}, \tau \geq 0\}$ such that if $x(t)$ is a solution of Eq. (1.1), then $x(t+\tau) = U(t,\tau)x(t),\ t,\tau \geq 0$. In general, a function $x(t) = U(0,t)x$ is called a ***mild solution*** of Eq.(1.1). Thus, mild solutions correspond to positive trajectories. Assume that $A(t)$ is 1-periodic, i.e. $A(t+1) = A(t),\ \forall t \in \mathbf{R}$. Then the corresponding process is 1-periodic. Assume further that there exists a ***Floquet representation*** for the process $\mathcal{U}$, i.e. the monodromy operator V of the process $\mathcal{U}$ has a logarithm: there exists a bounded operator C such that $V = e^C$ (this is the case, for instance, if the spectrum of V does not surround the origin). The well known Floquet Theorem then states that there exists a continuous 1-periodic invertible operator-valued function $P(t)$ such that $P(0) = 0$ and

$$U(t,\tau)x = P(t+\tau)e^{\tau C}P^{-1}(t)x. \tag{4.1}$$

Formula (4.1) implies that if $u(t)$ is a solution of Eq.(1.2), then $v(s) = P^{-1}(s)u(s)$ is a solution of the equation

$$v'(s) = Cv(s) + g(s), \tag{4.2}$$

where $g(s) = P^{-1}(s)f(s)$ (see e.g. [D-K], [He], [M-S]).

We shall use the following result the proof of which can be found in [L-Z, p. 93].

THEOREM 4.1. *Assume that C is a generator of a C_0-semigroup in a Banach space E such that $\sigma(C) \cap i\mathbf{R}$ is countable, g is an almost periodic function, and $v(t)$ is a bounded uniformly continuous solution on $\mathbf{R}$ of Eq.(4.2). Then $v(t)$ is almost periodic provided one of the following conditions holds:*

(i) E does not contain c_0;

(ii) The set $\{u(t) : t \in \mathbf{R}\}$ is weakly relatively compact.

THEOREM 4.2. *Suppose that $\mathcal{U}$ is a 1-periodic process generated by Eq. (1.1) which has a Floquet representation, V is its monodromy operator, and f is an almost periodic function. Assume that $\sigma(V) \cap \Gamma$ is countable, and $u(t)$ is a bounded uniformly continuous solution of Eq.(1.2). Then $u(t)$ is almost periodic, provided one of the following conditions holds:*

(i) X does not contain c_0;

(ii) The set $\{u(t) : t \in \mathbf{R}\}$ is weakly relatively compact.

PROOF: By the Spectral Mapping Theorem (see e.g. [H-P], [Cle]), the set $\sigma(C) \cap i\mathbf{R}$ is countable. By the Floquet Theorem, if $u(t)$ is a solution of Eq.(1.2), then $v(s) = P^{-1}(s)u(s)$ is a solution of Eq.(4.2), and it is easy to see that all the properties of $u(t)$ are preserved for $v(t)$. By Theorem 4.1, $v(t)$ is almost periodic, which implies almost periodicity of $u(t)$. ∎

REMARK 4.3. It is not difficult to see that Theorem 4.2 does not hold without the condition of countability of the spectrum, even for autonomous processes. As an example consider a (bounded) self-adjoint operator A in a Hilbert space, with purely continuous spectrum (i.e. A has no eigenvalues). Then the autonomous process $U(t,\tau) \equiv e^{iA\tau}$ has the property that every its positive trajectory is bounded and can be extended to a bounded complete trajectory, but no complete trajectory is almost periodic (except the trivial one which is identically zero).

We recall that the *spectrum*, $\mathrm{Sp}(f)$, of the function $f \in L^\infty(\mathbf{R}, E)$ is defined as the complement in $\mathbf{R}$ of the set of points λ such that there is a neighborhood $\mathcal{U}$ of λ with the property that $f * \psi \equiv 0$ whenever $\psi \in L^1(\mathbf{R})$ and $\operatorname{supp}\hat{\psi} \subset \mathcal{U}$ (see [K], [L-Z]). It is well known that if f is almost periodic, then $\mathrm{Sp}(f)$ coincides with the closure of the set of Bohr's exponents of f, i.e.

$$\mathrm{Sp}(f) = \text{closure of } \{\lambda \in \mathbf{R} : a(\lambda; f) \equiv \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-i\lambda t} f(t) \neq 0\}.$$

Let $\Sigma(f) \equiv \overline{\{\lambda + 2n\pi : \lambda \in \mathrm{Sp}(f), n \in \mathbf{Z}\}}$. We shall need the following lemma.

LEMA 4.3. *Assume that $P(t)$ is a continuous 1-periodic operator-valued function, $f(t)$ is almost periodic and $g(t) = P(t)f(t)$. Then $g(t)$ is almost periodic and*

$$\mathrm{Sp}(g) \subset \Sigma(f).$$

PROOF: By the Approximation Theorem (see e.g. [L-Z]), there exist trigonometric polynomials $p_n(t)$ which converge to f uniformly on $\mathbf{R}$; moreover, the exponents of p_n can be chosen from the set of Bohr's exponents of f. The function $g_n(t) \equiv P(t)p_n(t)$ converge uniformly to g, and their spectra are contained in the set $\Sigma(f)$. Now the conclusion follows from lower semicontinuity of the spectrum $\mathrm{Sp}(g)$ (see [K]). ∎

PROPOSITION 4.4. *Assume that C is a bounded operator and g is an almost periodic function such that $\sigma(C) \cap i\mathrm{Sp}(g) = \emptyset$. Then there exists an almost periodic solution on $\mathbf{R}$ of Eq.(4.2). This solution is unique if we require $\mathrm{Sp}(u) \subset \mathrm{Sp}(g)$.*

This proposition is contained in [Pr]. Here we give a different, and shorter, proof.

PROOF: We put $\Lambda = \mathrm{Sp}(g)$, and consider the space $E(\Lambda) \equiv \{h \in BUC(\mathbf{R}, E) : \mathrm{Sp}(h) \subset \Lambda\}$. Since $E(\Lambda)$ is invariant with respect to translates, one can consider the C_0-group $S_\Lambda(t)h = h_t$, $t \in \mathbf{R}$, of the restrictions of translates to $E(\Lambda)$, and denote by D_Λ the corresponding generator. Since $\sigma(D_\Lambda) = i\Lambda$, from the condition of Proposition 4.4 it follows that there exists a bounded operator X from $E(\Lambda)$ to E such that $CX - XD_\Lambda = \delta_0$, where $\delta_0 : E(\Lambda) \to E$ is the Dirac operator defined by $\delta_0 h \equiv h(0)$ (see [V1]). We define a continuous linear operator $G : E(\Lambda) \to E(\Lambda)$ by $(Gh)(t) = Xh_t$. It is not hard to see that the operator G is correctly defined and, for every $f \in E(\Lambda)$, Gf represents a solution on $\mathbf{R}$ of the differetial equation $u'(t) = Cu(t) + f(t)$. In particular, Gg is a solution on $\mathbf{R}$ of Eq.(4.2). Since g is almost periodic, and G commutes with translates, it follows that Gg is also almost periodic. It is clear that $\mathrm{Sp}(Gg) \subset \mathrm{Sp}(g)$. If there is another solution $v(t), t \in \mathbf{R}$, of Eq.(4.2) with this property, then $w = u - v$ is a solution of the homogeneous equation $w'(t) = Cw(t),\ t \in \mathbf{R}$. Thus we have $i\mathrm{Sp}(w) \subset \sigma(C)$ (see [V2]). On the other hand, $i\mathrm{Sp}(w) \subset i\Lambda$. Thus, $\mathrm{Sp}(w) = \emptyset$, so $w \equiv 0$ (see, e.g. [K], or [V2]). ∎

THEOREM 4.5. *Assume that $\mathcal{U}$ is a 1-periodic process generated by Eq.(1.1), which has a Floquet representation, and f is an almost periodic function such that*

$$\sigma(V) \cap \overline{\{e^{it\mu} : \mu \in \Sigma(f)\}} = \emptyset.$$

Then there exits an almost periodic solution of Eq.(1.2). The solution is unique in the class of functions whose spectrum is contained in $\Sigma(f)$.

PROOF: The conditions of the theorem imply that there exists an almost periodic solution on $\mathbf{R}$ of Eq.(4.2), and hence there exists an almost periodic solution on $\mathbf{R}$ of Eq.(1.2). If u and v are two solutions of Eq.(1.2) and their spectra are contained in $\Sigma(f)$, then by the same reasoning as in the proof of Proposition 4.4 we can show that $\mathrm{Sp}(u - v) = \emptyset$, so that $u - v \equiv 0$. ∎

The existence of a Floquet representation is essential in the proof of Theorem 4.2. Without this assumption, one can show that the sequence $\{u(n)\}_{n\in\mathbf{Z}}$ is an almost periodic complete orbit of V. In general, a function $u(t)$ may not be almost periodic, even though its restriction to $\mathbf{Z}$, $\{u(n)\}_{n\in\mathbf{Z}}$, is an almost periodic sequence (cf. [F, p. 163]). But it is not known to the author whether the process $\mathcal{U}$ extends the almost periodic sequence $\{u(n)\}_{n\in\mathbf{Z}}$ to the function $u(t)$ almost periodically, i.e., whether Theorems 4.2 and 4.5 remain valid without the assumption on the existence of the Floquet representations ?

5. Existence of complete trajectories

Let $U(t,\tau)$ be a linear process which is uniformly bounded, i.e. $\sup\{\|U(t,\tau)\| : t \in \mathbf{R}, \tau \geq 0\} < \infty$ (such a process is called quasi-contractive in [Hr1]). Even for autonomous processes (i.e. continuous dynamical systems), it may happen that there is no complete trajectory of $\mathcal{U}$, except the trivial one through zero. In [V2], a spectral condition is given under which there exist a plenty of nontrivial bounded complete trajectories for a given bounded semigroup. Here we present an analogous condition under which there exist a plenty of nontrivial bounded complete trajectories for a periodic process.

We shall need the following result which is contained in [V2]. Recall that a power bounded operator is said to be *of class* $C_{0\cdot}$, if $\|T^n x\| \to 0$ as $n \to \infty$, for all $x \in E$.

THEOREM 5.1. *Let T be a power-bounded linear operator in a Banach space E. Assume that T is not in the class $C_{0\cdot}$, and one of the following condition holds:*

i) $\mathrm{ran}(T)$ *is dense in E, or*

ii) $\Gamma \not\subseteq \sigma(T)$.

Then there exist nontrivial bounded complete orbits for T^*.

COROLLARY 5.2. *Let E be reflexive, T be a power-bounded operator such that T^* is not in the class $C_{0\cdot}$, and one of the following conditions holds:*

i) $\ker(T) = \{0\}$, *or*

ii) $\Gamma \not\subseteq \sigma(T)$.

Then there exist nontrivial bounded complete orbits for T.

Using Corollary 5.2, and Lemma 2.1, we obtain the following result on the existence of nontrivial bounded complete trajectories for a periodic processes.

THEOREM 5.3. *Let E be reflexive, and $\mathcal{U}$ be a periodic linear process with the monodromy operator V. Assume that V satisfies the conditions in Corollary 5.2. Then there exist nontrivial bounded complete trajectories for $\mathcal{U}$.*

The proof in [V2] also gives a constructive method for obtaining a large family of bounded complete orbits for V, which together with the proof of Lemma 2.1 leads to a constructive method for obtaining a large family of bounded complete trajectories for the considered process. By a modification of the example in [V2], one can show that the reflexivity condition in Corollary 5.2, and hence in Theorem 5.3, is essential.

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