TWO GENERATORS FOR THE MAPPING CLASS GROUP OF A SURFACE

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Let F be an orientable surface of genus $g \geq 1$, either closed or with one boundary component. Let M be the mapping class group of F. Then M can be generated by two elements.

1. Introduction.

Let $F_{n,r}$ be a compact orientable surface of genus n with r boundary components. Let $M_{n,r}$ be the mapping class group of $F_{n,r}$, i.e. the group of isotopy classes of orientation preserving homeomorphisms of $F_{n,r}$ which leave the boundary pointwise fixed. We shall only consider the case of $r = 0$ or $r = 1$. Dehn proved in [D] that $M_{n,r}$ is generated by twists with respect to simple closed curves (see definition 1.) Lickorish proved in [L1] that twists with respect to curves $\alpha_1, \alpha_2, \ldots, \alpha_{2n}$, $\delta_2, \delta_3, \ldots, \delta_n$ (Fig. 1) suffice. He also proved that if we do not require the generators to be twists then four generators suffice. In fact it is easy to find three generators for $M_{n,r}$. We can find homeomorphisms S (defined later) and E, a "rotation", such that the conjugation by S acts transitively on Dehn twists with respect to $\alpha_1, \alpha_2, \ldots, \alpha_{2n}$ and the conjugation by E acts transitively on Dehn twists with respect to $\delta_1, \delta_2, \delta_3, \ldots, \delta_n$ ($\delta_1 = \alpha_1$.) Thus S, E, and the twist with respect to α_1 generate $M_{n,r}$. Humphries proved in [H] that $M_{n,r}$ can be generated by $2n+1$ twists with respect to curves $\alpha_1, \alpha_2, \ldots, \alpha_{2n}$, δ_2 and that this is the minimal number of twist generators for $M_{n,r}$.

In this paper we prove that $M_{n,r}$ can be generated by two elements.

Theorem. *Let F be a compact orientable surface of genus n, either closed or with one boundary component. Let M be the mapping class group of F. Then M can be generated by two elements.*

The result is known for $n \leq 2$. For $n = 0$ the group M is trivial. For $n = 1$ the group M is generated by twists with respect to curves α_1 and α_2. For $n = 2$ the group M is a quotient of the braid group B_6 (see [B]) and hence can be generated by two elements. So it suffices to prove the Theorem for $n \geq 3$.

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2. Definitions and notation.

We assume that $n \geq 3$ and that F is a surface of genus n represented on Fig. 1. The curve Δ on the right side of the picture is either the boundary of F or F is closed and Δ bounds a disk on F. F is oriented in such a way that the part of F facing us has the usual orientation of the plane. On an oriented surface we have the notion of left and right. If we move along a curve in some direction and u is the velocity vector and v is a normal vector then v is to the right of u if the pair (u, v) is negatively oriented.

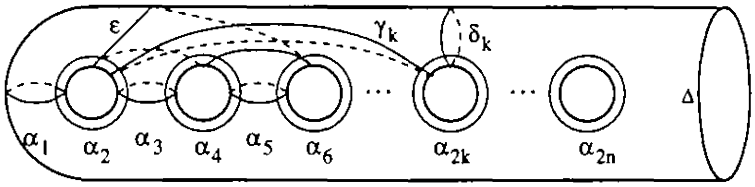

Figure 1

Definition 1. A (positive) Dehn twist with respect to a simple closed curve α (or along α) is an isotopy class of a homeomorphism h of F, supported in a tubular neighbourhood N of α, obtained as follows: we cut F along α, we rotate one side of the cut by 360 degrees to the right (clockwise) and then glue the surface back together damping the rotation to the identity at the boundary of N. If β is a curve on F which meets α transversely at one point and if β' is the image of β under the twist then β' is obtained by splitting $\alpha \cup \beta$ at the intersection point and reglueing it together into one simple closed curve, where β extends to the right into α (Fig. 2). If α and β meet at the far side of F, where the curves are denoted by broken lines, then β' turns "*left*" into α, which means *right* according to the orientation of the far side of F. For the inverse of a twist we have to interchange *left* and *right* .

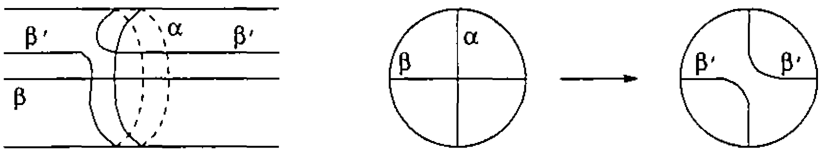

Figure 2

A Dehn twist with respect to α depends on the orientation of the surface F but not on the orientation of the curve α . The twist along α will be denoted T_α .

By a curve we always mean a simple closed curve on F. We shall say that curves are *equal* if they are isotopic on F.

We compose homeomorphisms from left to right and we write the symbol of a homeomorphism on the right side of the curve on which it acts. A conjugate of A by B is denoted $(A)B = B^{-1}AB$.

We let

$$S = T_{\alpha_1} T_{\alpha_2} \dots T_{\alpha_{2n}} \quad \text{and} \quad R = T_{\delta_{n-1}} T_{\delta_n}^{-1} .$$

We denote by G the subgroup of M generated by R and S .

3. Proof of the Theorem.

We assume as before that $n \geq 3$. In order to prove the Theorem it suffices to prove the following proposition

Proposition. *$G = M$, i.e. the group M is generated by R and S .*

Lemma 1. *If α is a curve, h is a homeomorphism and $\alpha' = (\alpha)h$ then $T_{\alpha'} = (T_\alpha)h$.*

Proof: Homeomorphism h^{-1} takes a neighbourhood N' of α' onto a neighbourhood N of α. Then T_α twists N along α and h takes N back to N' .

□

Lemma 2. *Let α and β be simple closed curves on F.*
(i) If α does not meet β then $T_\alpha T_\beta = T_\beta T_\alpha$.
(ii) If α and β meet transversely at one point then $T_\alpha T_\beta T_\alpha = T_\beta T_\alpha T_\beta$.

Proof: The first case is obvious. The second case is equivalent to $T_\alpha^{-1} T_\beta T_\alpha = T_\beta T_\alpha T_\beta^{-1}$. By Lemma 1 this is equivalent to the fact that the curves $(\beta)T_\alpha$ and $(\alpha)T_\beta^{-1}$ are equal, which is clear from definition 1 .

□

Lemma 3. $(T_{\alpha_i})S = T_{\alpha_{i-1}}$ *for* $i = 2, 3, \ldots, 2n$.

Proof: By Lemma 2 $(T_{\alpha_i})T_{\alpha_j} = T_{\alpha_i}$ for $|i - j| > 1$ and $(T_{\alpha_{i+1}})T_{\alpha_i} = (T_{\alpha_i})T_{\alpha_{i+1}}^{-1}$ for $i = 1, 2, \ldots, 2n-1$. Lemma 3 follows .

□

Lemma 4. $(\delta_n)S^m = \sigma_{2n-m}$ *for* $m = 0, 1, \ldots, 2n-1$ *(Fig. 3 .)*

Proof: For $m = 0$ we have $\delta_n = \sigma_{2n}$ (Fig. 3 .) Suppose $(\delta_n)S^m = \sigma_{2n-m}$ for some m . Let us apply S. For $j < 2n - m$ the curve σ_{2n-m} is disjoint from α_j so $(\sigma_{2n-m})T_{\alpha_j} = \sigma_{2n-m}$. Clearly $(\sigma_{2n-m})T_{\alpha_{2n-m}} = \sigma_{2n-m-1}$ by definition 1, and σ_{2n-m-1} is disjoint from α_j for $j > 2n - m$, thus $(\delta_n)S^{m+1} = \sigma_{2n-m-1}$. Lemma 4 follows by induction .

□

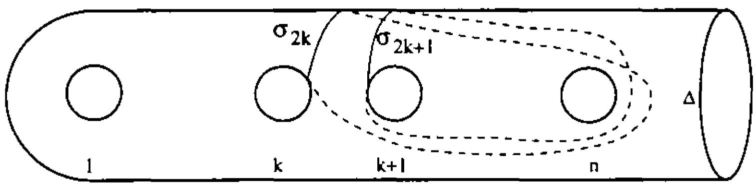

Figure 3

Lemma 5. $(\delta_{n-1})S^{2n-2} = \rho_2$ *and* $(\delta_{n-1})S^{2n-4} = \rho_4$ *(Fig. 4 .)*

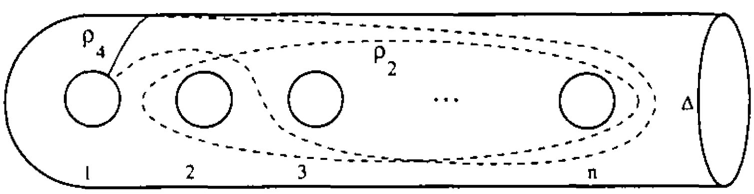

Figure 4

Proof: Let $\theta_{a,r} = (\delta_{n-1})S^aT_{\alpha_1}T_{\alpha_2}\dots T_{\alpha_r}$ for $a = 0,1,\dots,2n-3$ and $r = 1,2,\dots,2n$. $\theta_{a,2n} = (\delta_{n-1})S^{a+1}$.

For $k = 0,1\dots,n-2$ we define curves $\lambda_{k,1}$, $\lambda_{k,2}$, $\lambda_{k,3}$ (Fig. 5a) and curves $\mu_{k,1}$, $\mu_{k,2}$, $\mu_{k,3}$ (Fig. 5b for $k < n-2$ and Fig. 5c for $k = n-2$.) We prove by induction on k that

(a) $\theta_{2k,2n-2k-3+i} = \lambda_{k,i}$ for $i = 1,2,3$.

(b) $\theta_{2k+1,2n-2k-4+i} = \mu_{k,i}$ for $i = 1,2,3$.

(c) $(\delta_{n-1})S^{2k+2} = \mu_{k,3}$.

We start with $k = 0$ ($a = 0$) and apply the consecutive factors of S to δ_{n-1} . For $j < 2n-2$ curve α_j is disjoint from δ_{n-1} so $(\delta_{n-1})T_{\alpha_j} = \delta_{n-1}$. By definition 1 $(\delta_{n-1})T_{\alpha_{2n-2}} = \lambda_{0,1}$ (Fig. 5a) so $\theta_{0,2n-2} = \lambda_{0,1}$ as required. Suppose that for some $k \le n-2$ we have $\theta_{2k,2n-2k-2} = \lambda_{k,1}$. Clearly $(\lambda_{k,1})T_{\alpha_{2n-2k-1}} = \lambda_{k,2}$ and $(\lambda_{k,2})T_{\alpha_{2n-2k}} = \lambda_{k,3}$ so the equations (a) are true by the definition of $\theta_{a,r}$. For $j > 2n-2k$ curve α_j is disjoint from $\lambda_{k,3}$ so $(\delta_{n-1})S^{2k+1} = \lambda_{k,3}$. We apply again the consecutive factors of S. For $j < 2n-2k-3$ curve α_j is disjoint from $\lambda_{k,3}$. Now we apply $T_{\alpha_{2n-2k-3}}$ to $\lambda_{k,3}$. If $k < n-2$ we get the curve $\mu_{k,1}$ on Fig. 5b . If $k = n-2$ then $2n-2k-3 = 1$ so we apply T_{α_1} to $\lambda_{n-2,3}$. It is easy to see that we get the curve $\mu_{n-2,1}$ on Fig. 5c . Thus $\mu_{k,1} = (\delta_{n-1})S^{2k+1}T_{\alpha_1}\dots T_{\alpha_{2n-2k-3}} = \theta_{2k+1,2n-2k-3}$ as required. Clearly $(\mu_{k,1})T_{\alpha_{2n-2k-2}} = \mu_{k,2}$ and $(\mu_{k,2})T_{\alpha_{2n-2k-1}} = \mu_{k,3}$ so equations (b) are satisfied. For $j > 2n-2k-1$ curve α_j is disjoint from $\mu_{k,3}$ so $(\delta_{n-1})S^{2k+2} = \mu_{k,3}$ which proves (c) . If $k < n-2$ we go on and apply again the consecutive factors of S to $\mu_{k,3}$. For $j < 2n-2k-4$ curve α_j is disjoint from $\mu_{k,3}$. It is easy to see that when we apply $T_{\alpha_{2n-2k-4}}$ to $\mu_{k,3}$ we get $\lambda_{k+1,1}$. Thus $\theta_{2k+2,2n-2k-4} = \theta_{2(k+1),2n-2(k+1)-2} = \lambda_{k+1,1}$. This completes the induction step. From (c) we get

$(\delta_{n-1})S^{2n-4} = \mu_{n-3,3} = \rho_4$ (Fig. 5b and Fig. 4)

$(\delta_{n-1})S^{2n-2} = \mu_{n-2,3} = \rho_2$ (Fig. 5c and Fig. 4 .)

□

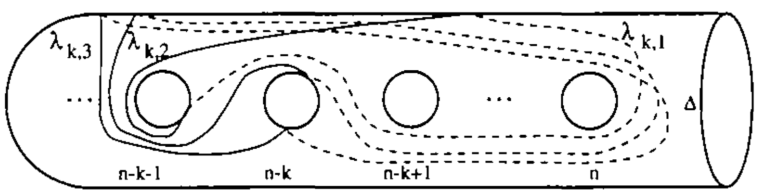

Figure 5a

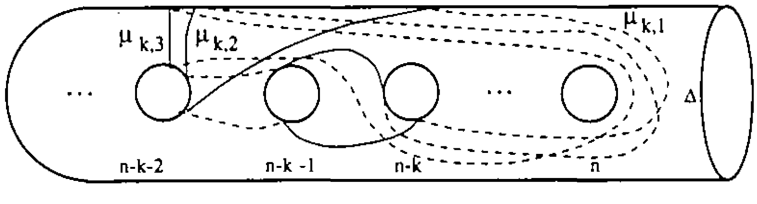

Figure 5b

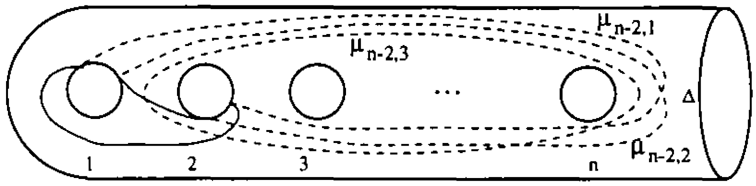

Figure 5c

Lemma 6. $(\gamma_i)S^2 = \delta_{i-1}$ *for* $i = 2, 3, \ldots, n$ *(Fig. 1 .)*

Proof: This is an easy exercise for a reader who followed the proof of Lemma 5 .

□

Let $U = (R)S^{2n-2}$. Then $U \in G$ and by Lemmas 4 and 5 $U = T_{\rho_2}T_{\sigma_2}^{-1}$.

Lemma 7. $(\delta_k)U = \gamma_k$ *for* $k = 2, 3, \ldots, n$ *(Fig. 1 .)*

Proof: Fig. 6a shows curves δ_k and ρ_2 . Clearly $(\delta_k)T_{\rho_2} = \nu_1$ (Fig. 6b .) Now in $(\nu_1)T_{\sigma_2}^{-1}$ (Fig. 6b) most of the picture "contracts" and we are left with γ_k .

□

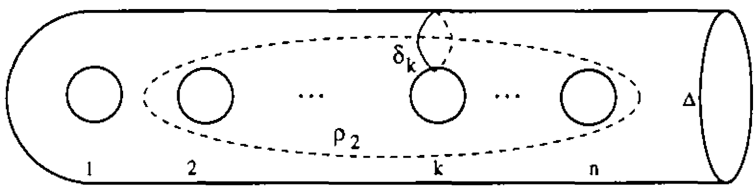

Figure 6a

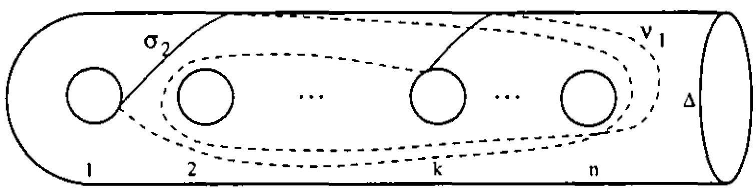

Figure 6b

Lemma 8. $T_{\alpha_1}T_{\delta_2}^{-1}$, $T_{\alpha_3}T_{\gamma_3}^{-1}$ *and* $T_{\delta_2}T_{\delta_3}^{-1}$ *belong to* G .

Proof: By Lemmas 6 and 7 we have $(\delta_k)US^2 = \delta_{k-1}$ for $k = 2, 3, \ldots, n$. Thus $(R)(US^2)^{n-3} = T_{\delta_2}T_{\delta_3}^{-1}$. Now by Lemma 7 $(R)(US^2)^{n-3}U = T_{\gamma_2}T_{\gamma_3}^{-1} = T_{\alpha_3}T_{\gamma_3}^{-1}$. Applying S^2 we get , by Lemma 6, $T_{\delta_1}T_{\delta_2}^{-1}$ which is equal to $T_{\alpha_1}T_{\delta_2}^{-1}$.

□

Lemma 9. $T_{\alpha_1}T_{\alpha_3}T_{\alpha_5}T_{\delta_3} = T_{\epsilon}T_{\delta_2}T_{\gamma_3}$ *(Fig. 1 .)*
Every factor on the left hand side commutes with every other factor in the relation.

Proof: When we cut F along the curves α_1, α_3, α_5 and δ_3 we get a disk with 3 holes (Fig. 7 .) The relation in Lemma 9 is the so called "lantern relation" . It was proven in [J] .

□

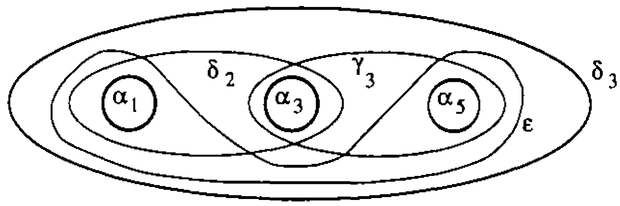

Figure 7

Let $W = T_{\alpha_3} T_{\gamma_3}^{-1} T_{\delta_3} T_{\delta_2}^{-1}$. By Lemma 8 $W \in G$. By Lemma 9
$W = T_\epsilon T_{\alpha_1}^{-1} T_{\alpha_5}^{-1}$.

Let $V = (R)S^{2n-4}$. Then $V \in G$ and by Lemmas 4 and 5 $V = T_{\rho_4} T_{\sigma_4}^{-1}$ (Fig. 3 and Fig. 4 .)

Lemma 10. $(\epsilon)V^{-1} = \delta_3$, $(\alpha_1)V^{-1} = \alpha_1$, $(\alpha_5)V^{-1} = \gamma_3$.

Proof: Clearly $(\epsilon)T_{\sigma_4} = \nu_2$ (Fig. 8a and Fig. 8b .) When we apply $T_{\rho_4}^{-1}$ to ν_2 (Fig. 8b) most of the picture "contracts" and we are left with δ_3 .

Clearly α_1 is disjoint from σ_4 and ρ_4 so $(\alpha_1)V = \alpha_1$.

Finally $(\alpha_5)T_{\sigma_4} = \nu_3$ (Fig. 8a and Fig. 8c .) When we apply $T_{\rho_4}^{-1}$ to ν_3 again most of the picture "contracts" and we are left with γ_3 .

□

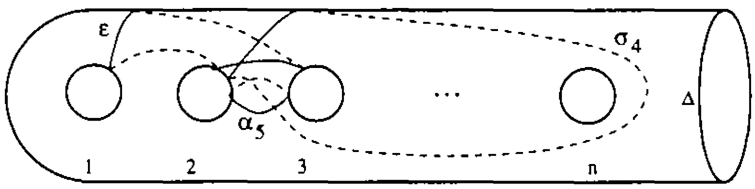

Figure 8a

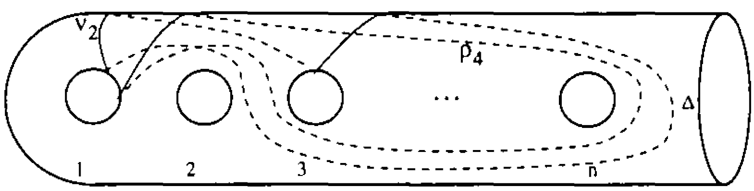

Figure 8b

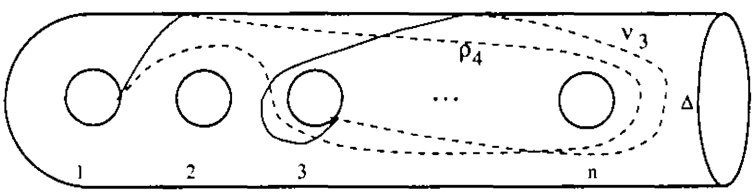

Figure 8c

We can now complete the proof of the Proposition . By Lemma 1 and Lemma 10 we have $(W)V^{-1} = T_{\delta_3} T_{\alpha_1}^{-1} T_{\gamma_3}^{-1} \in G$. Now by Lemma 8
$(T_{\alpha_1} T_{\delta_2}^{-1})(T_{\delta_2} T_{\delta_3}^{-1})(T_{\delta_3} T_{\alpha_1}^{-1} T_{\gamma_3}^{-1})(T_{\gamma_3} T_{\alpha_3}^{-1}) = T_{\alpha_3}^{-1} \in G$.

By Lemma 3 $T_{\alpha_i} \in G$ for each i . In particular $T_{\alpha_1} \in G$. Now $T_{\delta_2} \in G$ by Lemma 8 and $G = M$ by the result of Humphries in [H] . This concludes the proof of the Proposition and of the Theorem .

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