The first eigenvalue of the Laplacian for a positively curved homogeneous Riemannian manifold

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§0. Introduction. The purpose of this paper is to compute the first eigenvalue of the Laplacian for a certain positively curved homogeneous Riemannian manifold.

By a theorem of A.Lichnérowicz and M.Obata, if the Ricci curvature Ric_M of an n-dimensional compact Riemannian manifold M satisfies $Ric_M \geqq n-1$, then the first eigenvalue $\lambda_1(M)$ of the Laplacian of M is bigger than or equal to n, and the equality holds if and only if M is the standard sphere S^n of constant curvature one. Moreover, due to [L.Z.], [L.T], [C], [B.B.G], we know the following eigenvalue pinching theorem :

Theorem. Let M be a compact, n-dimensional Riemannian manifold whose sectional curvature $K_M \geqq 1$. Then there exist a constant $C(n) > 1$ depending only on n such that

$$C(n)n \geqq \lambda_1(M) \geqq n \Longrightarrow M \text{ is homeomorphic to } S^n.$$

On the other hand, we know the classification of compact homogeneous Riemannian manifolds with positive sectional curvature, due to [B], [Wℓ], [A.W], [B.B 1,2]. Therefore it would be interesting to know the first eigenvalues $\lambda_1(M)$ of these positively curved homogeneous manifolds. In this paper, we give a comparatively sharp estimate of $\lambda_1(M)$ of such manifolds and as its application we determine $\lambda_1(M)$ of 7-dimensional positively curved homogeneous Riemannian manifolds $SU(3)/T_{k,\ell}$ and the manifold $F_4/Spin(8)$ of flags in the Cayley plane (cf. Theorem 2.1). Moreover in the appendix, we give a complete

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list of $\lambda_1(M)$ of all compact simply connected irreducible Riemannian symmetric spaces, which was already given in [N1] for the classical cases. As its application, we obtain a complete list of stable, i.e., the identity map is stable as a harmonic map (cf. [Sm]), compact simply connected irreducible Riemannian symmetric spaces (cf. Theorem A.1).

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§1. Homogeneous Riemannian manifolds with positive curvature.

In this section, following [A.W], [W], [B.B 1,2], we prepare the results of classifying simply connected homogeneous Riemannian manifolds with positive curvature.

Let G be a compact connected Lie group, and H a closed subgroup. Let g be a Lie algebra of G, and h the subalgebra of g corresponding to H.

Definition 1.1 (cf. [A.W]). The pair (G,H) satisfies the condition (II) if there exists an $Ad(G)$-invariant inner product $(\cdot,\cdot)_0$ on g such that the orthogonal complement v of h in g has the orthogonal decomposition $v = v_1 \oplus v_2$ with the following properties :

(i) $[v_1,v_2] \subset v_2$, $[v_1,v_1] \subset h \oplus v_1$, $[v_2,v_2] \subset h \oplus v_1$, and

(ii) for $X=X_1+X_2, Y=Y_1+Y_2$ with $X_i,Y_i \in v_i, i=1,2$, $[X,Y]=0$ and $X \wedge Y \neq 0$ imply $[X_1,Y_1] \neq 0$.

Putting $k := h \oplus v_1$, k is a subalgebra of g and (g,k) is a symmetric pair of rank one. Furthermore, the connected Lie subgroup K of G corresponding to k is closed.

Definition 1.2 (cf. [A.W]). For $-1 < t < \infty$, we define an $Ad(H)$-invariant inner product $(\cdot,\cdot)_t$ on $v = v_1 \oplus v_2$ by

$$(X_1+X_2, Y_1+Y_2)_t = (1+t)(X_1,Y_1)_0 + (X_2,Y_2)_0 ,$$

for $X_i,Y_i \in v_i$, $i=1,2$, and let g_t be a G-invariant Riemannian metric on G/H induced from $(\cdot,\cdot)_t$.

Moreover let h be a G-invariant Riemannian metric on G/K induced from the inner product $(\cdot,\cdot)_0$ on v_2. Then the natural projection

π ; $G/H \longrightarrow G/K$ induces a Riemannian submersion π ; $(G/H, g_t) \longrightarrow (G/K, h)$ with totally geodesic fibers for all $-1 < t < \infty$ (cf. [B.B.B]).

Note that in case of $v_1 = \{0\}$, $(G/H, g_0) = (G/K, h)$ is a Riemannian symmetric space of rank one, and in case of $v_2 = \{0\}$, the condition (II implies the one such that the normally homogeneous Riemannian manifold $(G/H, g_0)$ has positive curvature.

Theorem 1.3 (cf. [A.W , Theorem 2.4], [H , Corollary 2.2]).

Let (G,H) be a pair with condition (II), and $v_1 \neq \{0\}$, and $v_2 \neq \{0\}$. Let g_t, $-1 < t < \infty$ be a G-invariant metric on G/H given in Definition 1.2. Then the Riemannian manifold $(G/H, g_t)$, $-1 < t < 0$, has positive curvature.

Theorem 1.4 (cf. [W] , [B.B 1,2], [B]). All the compact simply connected homogeneous spaces G/H which are not homeomorphi to S^n and have positively curved G-invariant Riemannian metrics exhaust the following table :

(I) in case of normally homogeneous spaces,

	G/H
1)	$SU(n+1)/S(U(n)\times U(1)) = P^n(\mathbb{C})$, $n \geqq 2$
2)	$Sp(n+1)/Sp(n)\times Sp(1) = P^n(H)$, $n \geqq 2$
3)	$F_4/Spin(9) = P^2(Cay)$
4)	$Sp(2)/SU(2)$

(II) in case of the condition (II) with $v_1 \neq \{0\}$ and $v_2 \neq \{$

	G/H	G/K
5)	$Sp(n)/Sp(n-1)\times T^1 \approx P^{2n-1}(\mathbb{C})$, $n \geqq 2$	$Sp(n)/Sp(n-1)\times Sp(1)$
6)	$SU(5)/Sp(2)\times T^1$	$SU(5)/S(U(4)\times U(1))$

7)	$SU(3)/T^2$	$SU(3)/S(U(2)\times U(1))$
8)	$SU(3)/T^1$	$SU(3)/S(U(2)\times U(1))$
9)	$U(3)/T^2 \approx SU(3)/T^1$	$SU(3)/S(U(2)\times U(1))$
10)	$SU(3)\times SU(2)/T^1\times\overline{SU(2)}$	$SU(3)/S(U(2)\times U(1))$
11)	$Sp(3)/SU(2)\times SU(2)\times SU(2)$	$Sp(3)/Sp(2)\times Sp(1)$
12)	$F_4/Spin(8)$	$F_4/Spin(9)$

Remark 1. Here we denote by T^k, $k=1,2$, k-dimensional tori. In case of 7), T^2 is a maximal torus in $SU(3)$. In cases of 8),9), the embeddings of T^1 and T^2 are given in [A.W], [B.B2] or §2. In case of 10), $T^1\times\overline{SU(2)} = \left\{(t\begin{pmatrix}1 & 0\\ 0 & X\end{pmatrix}, X);\ t\in T^1,\ X\in SU(2)\right\}$

$$\subset \left\{(t\begin{pmatrix}1 & 0\\ 0 & X\end{pmatrix}, Y);\ t\in T^1,\ X,Y\in SU(2)\right\} = T^1\times SU(2)\times SU(2$$

Here $T^1 = \left\{\begin{pmatrix}e^{-2i\eta} & & \\ & e^{i\eta} & \\ & & e^{i\eta}\end{pmatrix}\in SU(3);\ \eta\in\mathbb{R}\right\}$.

Remark 2. The examples 4),5) and 6) are due to [B], and the ones 7)$\sim$12) are due to [W], [A.W]. The inclusion $SU(2)\hookrightarrow Sp(2)$ in 4) is not canonical (cf. [B]). In the cases 5),6), the normally homogeneous Riemannian metric g_0 has positive curvature.

Remark 3. The pairs (G,H) with the condition (II) are classified in [B.B1,p.59]. All the simply connected homogeneous spaces G/H with the condition (II) which are not homeomorphic to S^n exhaust the above table in Theorem 1.4.

§2. The first eigenvalue of the Laplacian.

In this section, we show the following theorem :

<u>Theorem 2.1.</u> Let G/H be the homogeneous spaces as in Theorem 1.4. Let $(\cdot,\cdot)_0$ be the $Ad(G)$-invariant inner product on g given by

$$(X,Y)_0 = - B(X,Y), \quad X,Y \in g ,$$

for 1) $\sim$ 12), except 9), where B is the Killing form of g. For 9), we define $(\cdot,\cdot)_0$ by

$$(X,Y)_0 = -6 \, \mathrm{Trace}(XY), \quad X,Y \in u(3).$$

We define the inner product $(\cdot,\cdot)_t$, $-1 < t \leqq 0$, on the orthogonal complement v of h in g with respect to $(\cdot,\cdot)_0$ as in Definition 1.2 for 5) $\sim$ 12), and we consider only $(\cdot,\cdot)_0$ for the cases 1) $\sim$ 4). Then we can estimate the first eigenvalue $\lambda_1(g_t)$ of the G-invariant Riemannian metric g_t on G/H corresponding to $(\cdot,\cdot)_t$ as follows :

	G/H	$\lambda_1(g_t)$, $-1 < t \leqq 0$
1)	$SU(n+1)/S(U(n)\times U(1))$	$\lambda_1(g_0) = 1$
2)	$Sp(n+1)/Sp(n)\times Sp(1)$	$\lambda_1(g_0) = \frac{n+1}{n+2}$
3)	$F_4/Spin(9)$	$\lambda_1(g_0) = \frac{2}{3}$
4)	$Sp(2)/SU(2)$	$\frac{5}{12} \leqq \lambda_1(g_0)$
5)	$Sp(n)/Sp(n-1)\times T^1$	$\frac{2n+1}{4(n+1)} \leqq \lambda_1(g_t) \leqq \frac{n}{n+1}$
6)	$SU(5)/Sp(2)\times T^1$	$\frac{12}{25} \leqq \lambda_1(g_t) \leqq 1$
7)	$SU(3)/T^2$	$\frac{4}{9} \leqq \lambda_1(g_t) \leqq 1$
8)	$SU(3)/T^1$	$\lambda_1(g_t) = 1$

9)	$U(3)/T^2$	$\frac{1}{2} \leqq \lambda_1(g_t) \leqq 1$
10)	$SU(3)\times SU(2)/T^1\times\overline{SU(2)}$	$\frac{3}{8} \leqq \lambda_1(g_t) \leqq 1$
11)	$Sp(3)/SU(2)\times SU(2)\times SU(2)$	$\frac{7}{16} \leqq \lambda_1(g_t) \leqq \frac{3}{4}$
12)	$F_4/Spin(8)$	$\lambda_1(g_t) = \frac{2}{3}$

Remark. The cases 1)~ 3) are known in [C.W] .

For the proof of Theorem 2.1, we prepare Lemma 2.2.

Lemma 2.2. Under the assumptions of Theorem 2.1, the first eigenvalue $\lambda_1(g_t)$ of $(G/H, g_t)$, $-1<t\leqq 0$, can be estimated as

$$\lambda_1(g_o) \leqq \lambda_1(g_t) \leqq \lambda_1(G/K,h), \quad -1<t\leqq 0 ,$$

where $\lambda_1(G/K,h)$ is the first eigenvalue of $(G/K,h)$.

Proof. Since $\pi ; (G/H,g_t) \longrightarrow (G/K,h)$ is a Riemannian submersion with totally geodesic fibers, the (positive) Laplacian Δ_{g_t} , Δ_h of $(G/H,g_t)$, $(G/K,h)$ satisfy

$$\Delta_{g_t}(f\circ\pi) = (\Delta_h f)\circ\pi \quad , \quad f \in C^\infty(G/K)$$

(cf.[B.B.B ,p.188]). Then the spectrum $\mathrm{Spec}(\Delta_{g_t})$ includes the one $\mathrm{Spec}(\Delta_h)$, in particular, $\lambda_1(g_t) \leqq \lambda_1(G/K,h)$ for all t.

For the remaining inequality, we put $p = \dim(v_1)$ and $q = \dim(v_2)$. Let $\{X_i\}_{i=1}^p$, $\{Y_i\}_{i=1}^q$ be orthonormal bases of v_1 , v_2, respectively. Then since $\{X_i/\sqrt{t+1}\}_{i=1}^p$, $\{Y_i\}_{i=1}^q$ are orthonormal with respect to $(\cdot,\cdot)_t$, the Laplacian Δ_{g_t} of $(G/H,g_t)$ can be expressed (cf.[M.U , p.474,475]) as

$$\Delta_{g_t} = -\frac{1}{t+1}\hat{\lambda}(\sum_{i=1}^{p} X_i^2) - \hat{\lambda}(\sum_{i=1}^{q} Y_i^2),$$

in particular,

$$\Delta_{g_o} = -\hat{\lambda}(\sum_{i=1}^{p} X_i^2) - \hat{\lambda}(\sum_{i=1}^{q} Y_i^2),$$

where $\hat{\lambda}$ is the canonical isomorphism of the algebra of Ad(H)-invariant polynomials of $V = v_1 \oplus v_2$ into the space of G-invariant differential operators on G/H. Therefore we obtain

$$(2.1)\quad \Delta_{g_t} = \Delta_{g_o} + (1-\frac{1}{t+1})\hat{\lambda}(\sum_{i=1}^{p} X_i^2).$$

Here because of $-1<t\leqq 0$, the operator $P := (1-\frac{1}{t+1})\hat{\lambda}(\sum_{i=1}^{p} X_i^2) = \frac{t}{t+1}\lambda(\sum_{i=1}^{p} X_i^2)$ is non-negative, i.e., $\int_{G/H}(Pf)f\,dv_{g_t} \geqq 0$ for $f \in C^\infty(G/H)$, where dv_{g_t} is the volume element of $(G/H, g_t)$. Note that

$$(2.2)\qquad dv_{g_t} = (t+1)^{p/2} dv_{g_o}\ .$$

Therefore, together with (2.1), (2.2) and Mini-Max Principle (cf. [B.U, Proposition 2.1]), we obtain $\lambda_1(g_t) \geqq \lambda_1(g_o)$, $-1<t\leqq 0$. Q.E.D.

<u>Proof of Theorem 2.1</u>. The case 8) will be showed in Lemma 2.3. The upper estimate of $\lambda_1(g_t)$ can be obtained by the inequality $\lambda_1(g_t) \leqq \lambda_1(G/K,h)$ in Lemma 2.2 and Theorem 2.1, 1) ~ 3). In the lower estimate we use the inequalities

$$\lambda_1(G,g) \leqq \lambda_1(g_o) \leqq \lambda_1(g_t), \qquad -1<t\leqq 0.$$

Here $\lambda_1(G,g)$ is the first eigenvalue of (G,g) whose metric g is the bi-invariant one induced from the inner product $(\cdot,\cdot)_o$ on $\mathfrak{g}$. The computations of $\lambda_1(G,g)$ are accomplished in the appendix and note that $\lambda_1(U(n+1),g) = \frac{1}{2}$.

Case 8). A 1-dimensional torus $H = T^1$ in $G = SU(3)$ is conjugate in $SU(3)$ to

$$T_{k,\ell} = \left\{ \begin{pmatrix} e^{2\pi ik\theta} & & \\ & e^{2\pi i\ell\theta} & \\ & & e^{-2\pi i(k+\ell)\theta} \end{pmatrix} ; \theta \in \mathbb{R} \right\},$$

where k, ℓ are integers. We know by Lemma 3.1 and Theorem 3.2 in [A.W], that the pair $(SU(3), T^1)$ satisfies the condition (II) if and only if T^1 is conjugate in $SU(3)$ to $T_{k,\ell}$ with $k\ell(k+\ell) \neq 0$. Moreover, since $T_{k,\ell} = T_{mk,m\ell}$, $m \in \mathbb{Z}-(0)$, we can assume without loss of generality that $H = T^1 = T_{k,\ell}$ where $k\ell \neq 0$ and k,ℓ are relatively prime.

Let $K = S(U(2)\times U(1)) = \left\{ \begin{pmatrix} x & \\ & \det x^{-1} \end{pmatrix} ; x \in U(2) \right\}$, and g, k, h the corresponding Lie algebras of $G = SU(3)$, K, H, respectively. Let $(\cdot,\cdot)_o$ be the inner product on g defined by

$$(X,Y)_o = -B(X,Y) = -6\,\mathrm{Trace}(XY),\ X,Y \in g = su(3),$$

and we put $v_1 := h^\perp \cap k$, $v_2 := k^\perp$, and $v = h^\perp = v_1 \oplus v_2$, where $k^\perp$, $h^\perp$ are the orthogonal complements of k, h in g with respect to $(\cdot,\cdot)_o$, respectively. We define the Ad(H)-invariant inner product $(\cdot,\cdot)_t$, $-1 < t < \infty$, on $v = v_1 \oplus v_2$ as in Definition 1.2 and let g_t be the G-invariant Riemannian metric on $G/H = SU(3)/T_{k,\ell}$ induced from $(\cdot,\cdot)_t$. Then we have :

Lemma 2.3. Assume that $k\ell(k+\ell) \neq 0$. Then the first eigenvalue $\lambda_1(g_t)$ of $(SU(3)/T_{k,\ell}, g_t)$, $-1 < t \leqq 0$, is given by

$$\lambda_1(g_t) = 1 \quad \text{for every } -1 < t \leqq 0.$$

Proof. We know already that

$$\lambda_1(g_o) \leqq \lambda_1(g_t) \leqq \lambda_1(G/H,h) = 1.$$

So we have only to show $\lambda_1(g_o) = 1$. For this we use Theorem 1 in [U] which tells us that the eigenvalues of the Laplacian of $(SU(3)/T_{k,\ell}, g_o)$ are given by

$$f(n_1,n_2) := \frac{1}{9}(m_1{}^2+m_2{}^2-m_1m_2+3m_1),$$

where $m_1:=n_1+n_2$, $m_2:=n_2$, and n_1 and n_2 run over the set of non-negative integers satisfying $S^{k,\ell}_{n_1,n_2} \neq 0$. Here $S^{k,\ell}_{n_1,n_2}$ is the number of all the integer solutions (p',q,r) of the equations :

$$\begin{cases} kn_1 - n_2-(2k+\ell)p'+(-k+\ell)q+(k+2\ell)r = 0 \quad , \text{ and} \\ 0 \leqq p' \leqq n_1 \ , \ 0 \leqq q \leqq n_2 \ , \text{ and } \ 0 \leqq r \leqq p'+(n_2-q). \end{cases}$$

Then we can easily check

$$f(n_1,n_2) \geqq 1 = f(1,1),$$

except the cases $(n_1,n_2) = (1,0)$ or $(0,1)$. However $S^{k,\ell}_{1,0} = S^{k,\ell}_{0,1} = 0$ due to the assumption $k\ell(k+\ell) \neq 0$. Therefore we have the desired result. Q.E.D.

Appendix. The first eigenvalues of symmetric spaces.

The table of the first eigenvalues of the Laplacian of compact simply connected irreducible Riemannian symmetric spaces has been already given by [N1] for the classical cases. In this appendix, we give a complete list including the exceptional cases.

At first let G be a compact simply connected simple Lie group, $\mathfrak{g}$ its Lie algebra, and g the bi-invariant Riemannian metric on G induced from the inner product $(\cdot,\cdot)$ on $\mathfrak{g}$ given by

$$(A.1) \qquad (X,Y) = -B(X,Y), \quad X,Y \in \mathfrak{g} ,$$

where B is the Killing form of $\mathfrak{g}$. We denote by the same notation the inner product on $\mathfrak{g}^*$ canonically induced fron $(\cdot,\cdot)$ on $\mathfrak{g}$. Then it is known (cf. [Su]) that the spectrum of (G,g) can be given by the formula of Freudenthal as follows :

$$\begin{cases} \text{the eigenvalues ;} & (\lambda + 2\rho, \lambda), \\ \text{their multiplicities ;} & d_\lambda^{\,2} , \end{cases}$$

where 2ρ is the sum of all positive roots of the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$ relative to a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$, and d_λ is the dimension of the irreducible unitary representation of G with highest weight λ, and λ varies over the set $D(G)$ of all dominant weights in the dual $\mathfrak{t}^*$ of $\mathfrak{t}$.

By the last table in [Bo], we know $D(G)$, 2ρ, and the inner product $(\cdot,\cdot)$ in $\mathfrak{t}^*$, so we get the following table of the first eigenvalue of the Laplacian of (G,g) :

tape of G	$\lambda_1(G,g)$	
A_ℓ, $\ell \geqq 1$	$\frac{\ell(\ell+2)}{2(\ell+1)^2}$	X
B_ℓ, $\ell \geqq 2$	$\mathrm{Min}\left\{\frac{\ell}{2\ell-1}, \frac{\ell(2\ell+1)}{8(2\ell-1)}\right\}$	
	$= \begin{cases} \frac{5}{12}, & \ell=2 \\ \frac{21}{40}, & \ell=3 \\ \frac{\ell}{2\ell-1}, & \ell \geqq 4 \end{cases}$	X
C_ℓ, $\ell \geqq 2$	$\frac{2\ell+1}{4\ell+4}$	X
D_ℓ, $\ell \geqq 3$	$\mathrm{Min}\left\{\frac{2\ell-1}{4\ell-4}, \frac{\ell(2\ell-1)}{16(\ell-1)}\right\}$	
	$= \begin{cases} \frac{15}{32}, & \ell=3 \\ \frac{2\ell-1}{4\ell-4}, & \ell \geqq 4 \end{cases}$	X
E_6	$\frac{13}{18}$	
E_7	$\frac{57}{72}$	
E_8	1	
F_4	$\frac{2}{3}$	
G_2	$\frac{1}{2}$	

Table A.1. The first eigenvalue of the Laplacian of a compact simply connected simple Lie group.

In this table A.1, the symbol X means the unstability of (G,g).

Next, the spectrum of the Laplacian of an irreducible Riemannian symmetric space G/K of compact type is given as follows. Let G be a compact simply connected simple Lie group, K the corresponding closed subgroup of G. Let $\mathfrak{g}$, $\mathfrak{k}$ be the Lie algebras of G, K, respectively, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, the Cartan decomposition. We give the inner product $(\cdot,\cdot)$ on $\mathfrak{p}$ by the restriction of (A.1), and let h be the G-invariant Riemannian metric on G/K induced from $(\cdot,\cdot)$. Then it is known (cf. [Su]) that the spectrum of the Laplacian of $(G/K,h)$ is given by

the eigenvalues ; $(\lambda + 2\delta, \lambda)$,

their multiplicities ; d_λ .

Here λ varies over the set $D(G,K)$ of highest weights of all spherical representations of (G,K) ,which is determined by [Su] as follows.

Let $a \subset p$ be a maximal abelian subspace in p , and h , a maximal abelian subalgebra of g containing a . Let Π be a σ-fundamental root system, say $\Pi = \{\beta_1, \ldots, \beta_\ell\}$, $\ell = \dim(h)$, $\Pi_0 = \{\beta \in \Pi \,;\, \beta|_a \equiv 0\}$, and 2δ is the sum of all positive roots of $(g^{\mathbb{C}}, h)$ relative to Π . We denote by $\{\mu_1, \ldots, \mu_\ell\}$ the fundamental weights of g corresponding to Π , and put $q = \dim(a)$. Define

$$M_i \,,\ 1 \leqq i \leqq q \,, = \begin{cases} 2\mu_i & , \text{ if } \ p\beta_i = \beta_i \ \text{ and } \ (\beta_i, \Pi_0) = \{0\}, \\ \mu_i & , \text{ if } \ p\beta_i = \beta_i \ \text{ and } \ (\beta_i, \Pi_0) \neq \{0\}, \\ \mu_i + \mu_{i'} & , \text{ if } \ p\beta_i = \beta_{i'} \ \text{ and } \ \beta_i \neq \beta_{i'} \ , \end{cases}$$

where p is the Satake's involution. Then we have

$$D(G,K) = \left\{ \sum_{i=1}^{q} m_i M_i \,;\, m_i \geqq 0,\ m_i \in \mathbb{Z} \,,\ i=1,\ldots,q \right\}.$$

Then , since $(M_i + 2\delta, M_i) \gneqq 0$, we have

$$\lambda_1(G/K,h) = \operatorname{Min} \left\{ (M_i + 2\delta, M_i) \,;\, i=1,\ldots,q \right\}.$$

Let $\{\alpha_1, \ldots, \alpha_\ell\}$, $\{\varpi_1, \ldots, \varpi_\ell\}$, 2ρ be the fundamental root system, the corresponding fundamental weights, the sum of all positive roots, in the last table in [Bo], respectively. Then there exists an automorphism Φ of $g^{\mathbb{C}}$ preserving $(\cdot,\cdot)$ invariantly such that, for each $i=1,\ldots,\ell$, $\Phi(\alpha_i) = \beta_{i^*}$ for some $1 \leqq i^* \leqq \ell$. And then $\Phi(\varpi_i) = \mu_{i^*}$ and $\Phi(\rho) = \delta$. Since we know M_i by a Satake's diagram (cf. [Wr,p.30-32]), together with the last table in [Bo], we have a list of the first eigenvalue $\lambda_1(G/K,h)$ of simply connected irreducible Riemannian symmetric spaces $(G/K,h)$ of compact type :

tape of G/K	G/K	$\lambda_1(G/K,h)$
AI, $q \geqq 2$	$SU(q+1)/SO(q+1)$	$\frac{(q+3)q}{(q+1)^2}$
AII, $q \geqq 1$	$SU(2q+2)/Sp(q+1)$	$\frac{(2q+3)q}{2(q+1)^2}$ X
AIII, $\frac{\ell}{2} \geqq q \geqq 2$	$SU(\ell+1)/S(U(\ell+1-q)\times U(q))$	1
$q \geqq 2$	$SU(2q)/S(U(q)\times U(q))$	1
AIV, $\ell \geqq 1$	$SU(\ell+1)/S(U(\ell)\times U(1))$	1
BI, $\ell \geqq q \geqq 2$	$SO(2\ell+1)/SO(2\ell+1-q)\times SO(q)$	$\mathrm{Min}\left\{\frac{2\ell+1}{2\ell-1}, \frac{-q^2+(2\ell+1)q}{4\ell-2}\right\}$
		$= \begin{cases} 1, & q=2, \ell \geqq 2, \\ \frac{6}{5}, & q=3, \ell=3, \\ \frac{2\ell+1}{2\ell-1}, & \text{otherwise} \end{cases}$
BII, $\ell \geqq 2$	$SO(2\ell+1)/SO(2\ell)$	$\frac{\ell}{2\ell-1}$ X
CI, $q \geqq 3$	$Sp(q)/U(q)$	1
CII, $\frac{\ell-1}{2} \geqq q \geqq 1$	$Sp(\ell)/Sp(\ell-q)\times Sp(q)$	$\frac{\ell}{\ell+1}$ X
$q \geqq 2$	$Sp(2q)/Sp(q)\times Sp(q)$	$\frac{2q}{2q+1}$ X
DI, $\ell-2 \geqq q \geqq 2$	$SO(2\ell)/SO(2\ell-q)\times SO(q)$	$\mathrm{Min}\left\{\frac{\ell}{\ell-1}, \frac{-q^2+2\ell q}{4\ell-4}\right\}$
		$= \begin{cases} 1, & q=2, \\ \frac{\ell}{\ell-1}, & q \geqq 3, \end{cases}$

$q \geqq 2$	$SO(2q+2)/SO(q+2)\times SO(q)$	$\mathrm{Min}\left\{\frac{q+1}{q}, \frac{q+2}{4}\right\} = \begin{cases} 1, & q=2, \\ \frac{5}{4}, & q=3, \\ \frac{q+1}{q}, & q \geqq 4, \end{cases}$	
$q \geqq 2$	$SO(2q)/SO(q)\times SO(q)$	$\mathrm{Min}\left\{\frac{q}{q-1}, \frac{q^2}{4q-4}\right\} = \begin{cases} 1, & q=2, \\ \frac{9}{8}, & q=3, \\ \frac{q}{q-1}, & q \geqq 4, \end{cases}$	
DII, $\ell \geqq 2$	$SO(2\ell)/SO(2\ell-1)$	$\frac{2\ell-1}{4\ell-4}$	X
DIII, $q \geqq 2$	$SO(4q)/U(2q)$	1	
$q \geqq 2$	$SO(4q+2)/U(2q+1)$	1	
EI	$\widetilde{E_6/Sp(4)}$	$\frac{14}{9}$	
EII	$\widetilde{E_6/SU(2)\cdot SU(6)}$	$\frac{3}{2}$	
EIII	$\widetilde{E_6/Spin(10)\cdot SO(2)}$	1	
EIV	E_6/F_4	$\frac{13}{18}$	X
EV	$\widetilde{E_7/SU(8)}$	$\frac{5}{3}$	
EVI	$\widetilde{E_7/SO(12)\cdot SU(2)}$	$\frac{14}{9}$	
EVII	$\widetilde{E_7/E_6\cdot SO(2)}$	1	
EVIII	$E_8/SO(16)$	$\frac{31}{15}$	
EIX	$\widetilde{E_8/E_7\cdot SU(2)}$	$\frac{8}{5}$	
FI	$\widetilde{F_4/Sp(3)\cdot SU(2)}$	$\frac{4}{3}$	
FII	$F_4/Spin(9)$	$\frac{2}{3}$	X
G	$G_2/SU(2)\times SU(2)$	$\frac{7}{6}$	

Table A.2. The first eigenvalue of the Laplacian of a simply connected

Here in the Table A.2, $\widetilde{N}$ means the universal covering of N and X means also the unstability of $(G/K,h)$.

As these applications, we can state the stability or unstability of all compact simply connected irreducible Riemannian symmetric spaces. A compact Riemannian manifold (M,g) is <u>stable</u> (cf. [Sm]) if the identity map of (M,g) onto itself is stable as a harmonic map, that is, all the eigenvalues of the Jacobi operator coming from the second variation of a one parameter family of harmonic maps are non-negative. In case of an Einstein manifold (M,g), i.e., $Ric_g = cg$, where Ric_g is the Ricci tensor of (M,g), (M,g) is stable if and only if its first eigenvalue $\lambda_1(M,g)$ of the Laplacian on $C^\infty(M)$ satisfies $\lambda_1(M,g) \geqq 2c$ (cf. [Sm, Proposition 2.1]).

Since a compact simply connected Lie group (G,g) whose metric g is induced from the inner product (A.1) satisfies (cf. [K.N]) $Ric_g = \frac{1}{4}g$, we have :

$$(G,g) \text{ is stable if and only if } \lambda_1(G,g) \geqq \frac{1}{2}.$$

Moreover we know the Ricci tensor Ric_h of a simply connected irreducible Riemannian symmetric space $(G/K,h)$ of compact type satisfies $Ric_h = \frac{1}{2}h$, so we have :

$$(G/K,h) \text{ is stable if and only if } \lambda_1(G/K,h) \geqq 1.$$

Together with the Tables A.1, A.2, we obtain :

Theorem A.1. (1) Let G be a compact simply connected simple Lie group, g a bi-invariant Riemannian metric on G. Then (G,g) is unstable if and only if the type of G is one of the following : A_ℓ, $\ell \geqq 1$, B_2, C_ℓ, $\ell \geqq 2$ and D_3.

(2) Let $(G/K,h)$ be a simply connected irreducible

Riemannian symmetric space of compact type. Then $(G/K,h)$ is unstable if and only if the type of G/K if one of the following : AII, BII, CII, DII, EIV and FII.
That is, $SU(2q+2)/Sp(q+1)$, $q\geq 1$, the unit sphere S^n, $n\geq 3$, the quaternion Grassman manifolds $Sp(\ell)/Sp(\ell-q)\times Sp(q)$, $\ell-q\geq q\geq 1$, E_6/F_4 , and the Cayley projective space $F_4/Spin(9)$.

<u>Remark</u>. The classical stable or unstable irreducible Riemannian symmetric spaces have been known in [Sm, Proposition 2.13], and also see [N2]. However it should be noticed that the statement (3.1) in [N2] is false.

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