# Around the section conjecture: motivic aspects

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Let $F$ be a number field and let $X \rightarrow \mathrm{Spec}(F)$ be a smooth and compact hyperbolic curve. After some choice of base point, we get an exact sequence

$$0 \rightarrow \hat{\pi}_1(\bar{X}) \rightarrow \hat{\pi}_1(X) \rightarrow \Gamma \rightarrow 0$$

where $\Gamma$ is the Galois group of $F$ and $\hat{\pi}_1$ refers to the profinite fundamental group. The section conjecture states that there is a one-to-one correspondence between conjugacy classes of splittings of this exact sequence and the geometric sections of the morphism $X \rightarrow \mathrm{Spec}(F)$. Since splittings will give rise to actions of the Galois group $\Gamma$ on the geometric fundamental group $\hat{\pi}_1(\bar{X})$, this conjecture can be compared to abelian conjectures that state that certain Galois representations must come from geometry, e.g., that of Fontaine and Mazur. In the letter to Faltings where Grothendieck states this conjecture as part of his vast anabelian program, he mentions the hope that the section conjecture will lead to a new proof of the Mordell conjecture. It should be remarked that even simple consequences of the section conjecture appear to be very difficult. For example, take a curve that evidently does not have points. Can one prove that there are no splittings either? Can one prove that the set of splittings is finite?

We described in our lecture a different methodology for deriving Diophantine consequences from a study of the fundamental group, at least when the field $F$ is the rationals. We allow $X$ to be non-compact but assume that it has a model over some ring of $S$-integers $R = \mathbb{Z}[1/S]$, for a finite set of primes $S$, with a smooth compactification in such a way that the compactification divisor is also smooth. The starting point is the method of Chabauty that uses the $\mathbb{Z}_p$-points for some prime $p \notin S$ and the log Albanese map

$$X(R) \rightarrow X(\mathbb{Z}_p) \rightarrow T_e J$$

where $J$ is the generalized Albanese variety of $X$. The main analytic tool we use is a lift of this map to classifying spaces of torsors for the De Rham fundamental group

$$\begin{array}{ccc}
\vdots & & \vdots \\
X(\mathbb{Z}_p) & \stackrel{j_3}{\rightarrow} & (\pi_1^{DR})_3/F^0 \\
|| & & \downarrow \\
X(\mathbb{Z}_p) & \stackrel{j_2}{\rightarrow} & (\pi_1^{DR})_2/F^0 \\
|| & & \downarrow \\
X(Z_p) & \stackrel{j_1}{\rightarrow} & (\pi_1^{DR})_1/F^0
\end{array}$$

Here, the notation is that $\pi_1^{DR}$ is the De Rham fundamental group of $X \otimes \mathbb{Q}_p$ that classifies unipotent bundles with flat connection and the subscript refers to quotients modulo the descending central series: $(\pi_1^{DR})_i = Z^{i+1} \backslash \pi_1^{DR}$ where $Z^1 = \pi_1^{DR}$, and $Z^{i+1} = [\pi_1^{DR}, Z^i]$. All these groups carry a Hodge filtration $F^i$ and the action of a Frobenius coming from comparison with the crystalline fundamental group of the special fiber. The horizontal arrows are given by the 'torsor of paths' map that chooses a basepoint $x$ and sends any other $y$ to the class of the torsor $\pi_1^{DR}(X; x, y)$ of De Rham paths from $x$ to $y$. The use of the global information is summarized by a diagram

$$\begin{array}{ccccc} X(R) & \rightarrow & X(\mathbb{Z}_p) & \rightarrow & (\pi_1^{DR})_n/F^0 \\ \downarrow & & \downarrow & \nearrow & \\ H^1_g(\Gamma_T, (\pi_1^{\text{ét}})_n) & \rightarrow & H^1_g(G_p, (\pi_1^{\text{ét}})_n) & & \end{array}$$

where the cohomology groups are certain subspaces non-abelian cohomology with coefficients in (quotients of) $\mathbb{Q}_p$-unipotent étale fundamental groups, defined by local Selmer conditions. So they should be viewed as non-abelian Selmer groups that carry naturally the structure of algebraic varieties. The vertical maps are again given by path spaces, this time acting on $\mathbb{Q}_p$-unipotent étale local systems, the lower horizontal map is simply localization, while the diagonal map is provided by a non-abelian version of Fontaine's Dieudonné functor. The group $\Gamma_T$, by the way, refers to the Galois group of the maximal extension of $\mathbb{Q}$ unramified outside $T$ for $T = S \cup \{p\}$, while $G_p$ is a decomposition group above $p$.

The hope then is to control the global points $X(R)$ by using its inclusion into the image of the map

$$H^1_g(\Gamma_T, (\pi_1^{\text{ét}})_n) \rightarrow (\pi_1^{DR})_n/F^0$$

obtained from the diagram.

We then make the following

**Conjecture:**

$$\dim H^1_g(\Gamma_T, (\pi_1^{\text{ét}})_n) < \dim(\pi_1^{DR})_n/F^0$$

for $n$ sufficiently large.

Finiteness theorems over $\mathbb{Q}$, that is, the theorems of Siegel and Faltings, follow from this conjecture in a manner entirely analogous to the method of Chabauty. Unfortunately, we can fully verify this conjecture only for genus zero curves. However, there is a conjecture of Bloch and Kato regarding the $p$-adic Chern class maps:

$$K^{(r)}_{2r-n-1}(V) \otimes \mathbb{Q}_p \rightarrow H^1_g(\mathbb{Q}, H^n(\bar{V}, \mathbb{Q}_p(r)))$$

for a smooth projective variety $V$ over $\mathbb{Q}$. The conjecture states that this map is an isomorphism. Within the remarkable circle of ideas surrounding the Bloch-Kato conjectures, this one amounts to a vast generalization of the finiteness of the Tate-Shafarevich group of elliptic curves. We are able to show:

*The Bloch-Kato conjecture implies the conjecture above.*

We remark that the 'sufficiently large $n$' of the conjecture can be computed precisely depending on the Mordell-Weil rank of the Jacobian of $X$, assuming the Bloch-Kato conjecture. (It can be computed in any case for genus zero curves.) The proof of this implication is quite simple using Poitou-Tate duality, the monodromy filtration on the Tate module of $J$ for the action of $G_l$, $l \neq p$, and the Hodge-Tate decomposition for the $G_p$-action.

One can work out other similar implications, but the overall picture appears to be that a sufficiently strong conjecture of the Birch and Swinnerton-Dyer type can even encompass non-linear theorems of the Faltings and Siegel type. That is to say, one can hope that the study of non-abelian Selmer groups will unify two rather different directions of investigation in Diophantine geometry.