SIMPLE COMPACT QUANTUM GROUPS I

SHUZHOU WANG

Abstract. The notion of simple compact quantum groups is introduced and some basic properties are established. As non-trivial (i.e. noncommutative and noncocommutative) examples, the following compact quantum groups are shown to be simple: The universal quantum groups $B_u(Q)$ for $Q \in GL(n, \mathbb{C})$ satisfying $Q\bar{Q} = \pm I_n$, $n \geq 2$; The quantum automorphism groups $A_{aut}(B, \tau)$ of finite dimensional C^*-algebras B endowed with the canonical trace τ when $\dim(B) \geq 4$, including the quantum permutation groups $A_{aut}(X_n)$ on n points ($n \geq 4$); The standard deformations K_q of simple compact Lie groups K and their twists K_q^u, as well as Rieffel's deformation K_J.

1. Introduction

The theory of quantum groups saw spectacular breakthroughs in the 1980's when on the one hand Drinfeld and Jimbo discovered the quantized universal enveloping algebras of semisimple Lie algebras [14, 18] based on the work of the Faddeev school on the quantum inverse scattering method, on the other hand Woronowicz independently discovered quantum deformations of compact Lie groups and formulated the axioms for compact quantum groups [51, 52, 53]. Moreover, further work of Rosso, Soibelman and Vaksman, Levendorskii [29, 31, 34, 33, 21] showed that the Drinfeld-Jimbo quantum groups admit "compact real forms" K_q and twists K_q^u that fit into the framework of Woronowicz, most notable of which is the work of Soibelman [33], in which a unified theory of deformations of simple compact Lie groups is developed based on his earlier joint work with Vaksman [34].

Starting in his Ph.D. thesis [39], the author of the present article took a different direction from the above by viewing quantum groups as intrinsic objects and found in a series of papers (including [38] in collaboration with Van Daele) several classes of compact quantum groups that can not be obtained as deformations of Lie groups. The most important of these are the universal compact quantum groups of Kac type $A_u(n)$ and their self-conjugate counterpart $A_o(n)$ [40], the more general universal compact quantum groups $A_u(Q)$ and their self-conjugate counterpart $B_u(Q)$ [38, 44], where $Q \in GL(n, \mathbb{C})$, and the quantum automorphism groups $A_{aut}(B, tr)$ of finite dimensional C^*-algebras B endowed with a tracial functional tr, including the quantum permutation groups $A_{aut}(X_n)$ on the space X_n of n points. Further studies of these quantum groups reveal remarkable properties. We mention a few. (1) According to deep work of Banica [2, 3, 4], the representation rings (also called

---

1991 *Mathematics Subject Classification.* Primary 46L65, 46L87, 46L55, 46L60, 46L89, 58B32; Secondary 16W30, 17B37, 20G42, 81R50, 81R60.

Key words and phrases. Simple Quantum Groups, Noncommutative Geometry, Quantizations, Deformations, Woronowicz C^*-algebras, Hopf Algebras.

Research supported in part by the National Science Foundation grant DMS-0096136.

the fusion rings) of the quantum groups $B_u(Q)$ and $A_{aut}(B,\tau)$ are isomorphic to those of appropriate compact Lie groups when $Q\bar{Q}$ is a scalar, $\dim(B) \geq 4$ (τ being the canonical trace on B), and the representation ring of $A_u(Q)$ is almost a free product of two copies of $\mathbb{Z}$; (2) The compact quantum groups $A_u(Q)$ admit ergodic actions on both finite and infinite injective von Neumann factors and the quantum groups $A_o(n)$ admit minimal (i.e. prime) actions on the Murray-von Neumann factor R [46]; (3) The special $A_u(Q)$'s for positive Q and $B_u(Q)$'s for Q satisfying the property $Q\bar{Q} = \pm 1$ have tractable isomorphic classifications and the general $A_u(Q)$'s and $B_u(Q)$'s for arbitrary Q can be decomposed explicitly as free products of the former special ones [48]; (4) The quantum permutation groups $A_{aut}(X_n)$ admit interesting quantum subgroups that appear in connection with other areas of mathematics, such as the quantum automorphism groups of finite graphs and the free wreath products discovered by Bichon [7, 8], as well as certain quantum symmetry groups in the theory of subfactors studied by Banica [6].

The purpose of this article is to initiate a study of simple compact quantum groups and show that the compact quantum groups mentioned in the last two paragraphs are simple in generic cases. An outline of its contents is as follows. From §2 to §4, we develop several basic notions related to simple compact quantum groups. In §2, we prove several equivalent conditions for a quantum subgroup of a compact quantum group to be normal, a notion introduced in [39, 40]. For more on the notion of normal quantum subgroups, we refer the reader to [49]. In §3, we develop some basic properties of normal quantum subgroups. In §4 the notion of simple compact quantum groups is introduced. A compact matrix quantum is called *simple* if it is connected and has no non-trivial connected normal quantum subgroups and no non-trivial representations of dimension one. We also introduce in this section quantum groups with property F and those with property FD, as well as almost classical compact quantum groups. In connection with noncommutative geometry, compact quantum groups with property F and almost classical compact quantum groups are two classes of quantum groups closest to compact Lie groups, though they need not be quantum deformations of the latter. All of the simple compact quantum groups studied in this paper belong to *both* classes. After preparatory work in the first few sections, the main examples of this paper are described in the last two sections, §5 and §6. Recall [4] that *the canonical trace* τ on a finite dimensional C^*-algebra B is the restriction of the unique tracial state on the algebra $L(B)$ of operators on B. In §5, we prove the main results on $B_u(Q)$ and $A_{aut}(B,\tau)$ (cf. Theorem 5.1 and Theorem 5.5):

Theorem 1.1. *Let $Q \in GL(n,\mathbb{C})$ be such that $Q\bar{Q} = \pm I_n$ and $n \geq 2$. Then $B_u(Q)$ is a simple compact quantum group with property F.*

Theorem 1.2. *Let B be a finite dimensional C^*-algebra with $\dim(B) \geq 4$ and τ its canonical trace. Then $A_{aut}(B,\tau)$ is an absolutely simple compact quantum group with property F.*

Besides the correspondence between Hopf $*$-ideals and Woronowicz Hopf C^*-ideals which we develop in §5, the proofs of these two results rely heavily on the fundamental work of Banica on the structure of fusion rings of these quantum groups [2, 4]. The last section §6 is devoted to the standard deformations K_q of simple compact Lie groups, their twists K_q^u [33, 21, 22], and Rieffel's quantum groups K_J

[28], where $q \in \mathbb{R}\backslash\{0\}$, $u \in \wedge^2(i\mathfrak{t})$ and J is an appropriate skew-symmetric transformation on the direct sum $\mathfrak{t} \oplus \mathfrak{t}$ of Cartan subalgebra $\mathfrak{t}$ of the Lie algebra of K (cf. Theorem 6.1, Theorem 6.6 and Theorem 6.4):

Theorem 1.3. *Let K be a connected and simply connected simple compact Lie group. Then both K_q and its twists K_q^u are simple compact quantum groups with property F.*

Theorem 1.4. *Let K be a simple compact Lie group with a toral subgroup T of rank at least two. Then K_J is a simple compact quantum group with property F.*

The proofs of Theorem 1.3 and Theorem 1.4 make use of the work of Lusztig and Rosso [23, 30] on representations of quantized universal enveloping algebras, the work of Soibelman and Levendorskii [33, 21, 22] on quantum function algebras of K_q and K_q^u, and the work of Rieffel [28] and the author [43] on strict deformations of Lie groups and quantum groups.

Classification of simple compact quantum groups and their irreducible representations up to isomorphism are two of the main goals in the study of compact quantum groups. Namely, one would like to develop a theory of simple compact quantum groups that parallels the Killing-Cartan theory and the Cartan-Weyl theory for simple compact Lie groups. To accomplish the first goal, one must first construct all simple compact quantum groups. Though we have given several infinite classes of examples of these in this article, it should be pointed out that the construction of simple compact quantum groups is far from being complete. In fact it is fair to say that we are only at the beginning stage for this task at the moment. One indication of this is that all the simple compact quantum groups known so far are almost classical in the sense that their representation rings are isomorphic to those of ordinary compact groups and are in particular commutative. The first candidates of simple compact quantum groups that are not almost classical would be $A_u(Q)$, where $Q \in GL(n, \mathbb{C})$ are positive, $n \geq 2$. The representation ring of $A_u(Q)$ is "very" noncommutative, being close to the free product of two copies of the ring of integers, according to Banica [3]. To construct the other ("the remaining") simple compact quantum groups, we believe the most profitable approach is to study quantum automorphism groups of appropriate quantum spaces, such as those in our papers [45, 46, 47] and the papers of Banica and Bichon [6, 7, 8]. In retrospect, both simple Lie groups and finite simple groups are automorphism groups, so it is natural to take a similar approach for the theory of simple quantum groups.

Convention and Notation. We assume that all Woronowicz C^*-algebras (also called Woronowicz Hopf C^*-algebras) considered in this paper to be full unless otherwise explicitly stated, since morphisms between quantum groups are meaningful only for full Woronowicz C^*-algebras cf. [40, 42]. For a compact quantum group G, A_G denotes the underlying Woronowicz C^*-algebra and $\mathcal{A}_G$ the associated dense Hopf $*$-algebra of quantum representative functions on G. Sometimes we also call A_G a compact quantum group, referring to G. See [40, 52] for more on other unexplained definitions and notations used in this paper.

2. The Notion of Normal Quantum Subgroups

Before making the notion of simple quantum groups precise, we study further the notion of normal quantum subgroups (of compact quantum groups) introduced in [39, 40]. Let (N, π) be a quantum subgroup of a compact quantum group G with surjections $\pi : A_G \longrightarrow A_N$ and $\hat{\pi} : \mathcal{A}_G \longrightarrow \mathcal{A}_N$. The quantum group (N, π) should be more precisely called a closed quantum subgroup, but we will omit the word *closed* in this paper, since we do not consider non-closed quantum subgroups. Define

$$A_{G/N} = \{a \in A_G | (id \otimes \pi)\Delta(a) = a \otimes 1_N\},$$
$$A_{N\backslash G} = \{a \in A_G | (\pi \otimes id)\Delta(a) = 1_N \otimes a\},$$

where Δ is the coproduct on A_G, 1_N is the unit of the algebra A_N. We omit the subscript N in 1_N when no confusion arises. Similarly, we define

$$\mathcal{A}_{G/N} = \mathcal{A}_G \cap A_{G/N}, \quad \text{and} \quad \mathcal{A}_{N\backslash G} = \mathcal{A}_G \cap A_{N\backslash G}.$$

Note that G/N, $N\backslash G$ shall be denoted more precisely by $G/(N, \pi)$, $(N, \pi)\backslash G$ respectively, if there is a possible confusion. Let h_N be the Haar measure on N. Let

$$E_{G/N} = (id \otimes h_N \pi)\Delta, \qquad E_{N\backslash G} = (h_N \pi \otimes id)\Delta.$$

Then $E_{G/N}$ and $E_{N\backslash G}$ are projections of norm one (completely positive and completely bounded conditional expectations) from A_G onto $A_{N\backslash G}$ and $A_{G/N}$ respectively (cf. [25] as well as Proposition 2.3 and Section 6 of [46]), and

$$\mathcal{A}_{G/N} = E_{G/N}(\mathcal{A}_G), \quad \text{and} \quad \mathcal{A}_{N\backslash G} = E_{N\backslash G}(\mathcal{A}_G).$$

From this, we see that the *-subalgebras $\mathcal{A}_{N\backslash G}$ and $\mathcal{A}_{G/N}$ are dense in $A_{G/N}$ and $A_{N\backslash G}$ respectively.

Assume N is a closed subgroup of an ordinary compact group G. Let π be the restriction morphism from $A_G := C(G)$ to $A_N := C(N)$. Let $C(G/N)$ and $C(N\backslash G)$ be continuous functions on G/N and $N\backslash G$ respectively. Then one can verify that

$$C(G/N) = A_{G/N} = E_{G/N}(A_G),$$
$$C(N\backslash G) = A_{N\backslash G} = E_{N\backslash G}(A_G).$$

Therefore we will use the symbols $C(G/N)$ and $A_{G/N}$ (resp. $C(N\backslash G)$ and $A_{N\backslash G}$; $C(G)$ and A_G) interchangeably for all quantum groups.

Proposition 2.1. *Let N be a quantum subgroup of a compact quantum group G. Then the following conditions are equivalent:*

(1) *$A_{N\backslash G}$ is a Woronowicz C^*-subalgebra of A_G.*

(2) *$A_{G/N}$ is a Woronowicz C^*-subalgebra of A_G.*

(3) *$A_{G/N} = A_{N\backslash G}$.*

(4) *For every irreducible representation u^λ of G, either $h_N \pi(u^\lambda) = I_{d_\lambda}$ or $h_N \pi(u^\lambda) = 0$, where h_N is the Haar measure on N, d_λ is the dimension of u^λ and I_{d_λ} is the $d_\lambda \times d_\lambda$ identity matrix.*

Proof. We only need to show that (1)⇔(4)⇔(3). The proof of the implications (2)⇔(4)⇔(3) is similar.

(3)⇒(4): In general one has

$$\Delta(\mathcal{A}_{N\backslash G}) \subseteq \mathcal{A}_{N\backslash G} \otimes \mathcal{A}_G, \quad \Delta(\mathcal{A}_{G/N}) \subseteq \mathcal{A}_G \otimes \mathcal{A}_{G/N}.$$

Letting $\mathcal{B} = \mathcal{A}_{N\backslash G} = \mathcal{A}_{G/N}$ then one has

$$\Delta(\mathcal{B}) \subseteq \mathcal{B} \otimes \mathcal{B}.$$

For $\lambda \in \hat{G}$, let n_λ be the multiplicity of the trivial representation of N in the representation $\pi(u^\lambda)$. We claim that either $n_\lambda = d_\lambda$ or $n_\lambda = 0$.

Assume on the contrary that there is a $\lambda \in \hat{G}$ such that $1 < n_\lambda < d_\lambda$. Note that in general

$$E_{N\backslash G}(u^\lambda_{ij}) = (h_N \pi \otimes id)\Delta(u^\lambda_{ij}) = \sum_k h_N \pi(u^\lambda_{ik}) u^\lambda_{kj},$$

$$E_{G/N}(u^\lambda_{ij}) = (id \otimes h_N \pi)\Delta(u^\lambda_{ij}) = \sum_k h_N \pi(u^\lambda_{kj}) u^\lambda_{ik}.$$

Using unitary equivalence if necessary we choose u^λ_{ij} in such a way that the n_λ trivial representations of N appear on the upper left diagonal corner of $\pi(u^\lambda)$. Then

$$E_{N\backslash G}(u^\lambda_{ij}) = \begin{cases} u^\lambda_{ij} & \text{if } \ 1 \le i \le n_\lambda, \ 1 \le j \le d_\lambda, \\ 0 & \text{if } \ n_\lambda < i \le d_\lambda, \ 1 \le j \le d_\lambda, \end{cases}$$

$$E_{G/N}(u^\lambda_{ij}) = \begin{cases} u^\lambda_{ij} & \text{if } \ 1 \le i \le d_\lambda, \ 1 \le j \le n_\lambda, \\ 0 & \text{if } \ 1 \le i \le d_\lambda, \ n_\lambda < j \le d_\lambda. \end{cases}$$

Since $\mathcal{A}_{N\backslash G} = \mathcal{A}_{G/N} = \mathcal{B}$ and both $E_{N\backslash G}$ and $E_{G/N}$ are projections from $\mathcal{A}_G$ onto $\mathcal{B}$, we have

$$E_{N\backslash G} = E_{G/N}.$$

Then for $n_\lambda < j \le d_\lambda$,

$$0 \ne u^\lambda_{ij} = E_{N\backslash G}(u^\lambda_{ij}) = E_{G/N}(u^\lambda_{ij}) = 0,$$

This is a contradiction.

(4)⇒(3): Let $S(N)$ (or $S(N, \pi)$) be the subset of $\hat{G}$ consisting of those λ's such that $h_N \pi(u^\lambda)$ is I_{d_λ}. Then a straightforward calculation using the fact that $E_{N\backslash G}$ and $E_{G/N}$ are projections of $\mathcal{A}_G$ onto $\mathcal{A}_{N\backslash G}$ and $\mathcal{A}_{G/N}$ respectively, one gets

$$\mathcal{A}_{N\backslash G} = \mathcal{A}_{G/N} = \bigoplus \{\mathbb{C} u^\lambda_{ij} \mid \lambda \in S(N), i, j = 1, \cdots, d_\lambda\}.$$

(4)⇒(1): Let $S(N)$ be defined as in the proof of (4)⇒(3). It is clear that $A_{N\backslash G}$ is a Woronowicz C^*-subalgebra of A_G and that

$$\{u^\lambda_{ij} \mid \lambda \in S(N), i, j = 1, \cdots, d_\lambda\}.$$

is a Peter-Weyl basis of the dense $*$-subalgebra $\mathcal{A}_{N\backslash G}$ of $A_{N\backslash G}$.

(1)⇒(4): Let $G_1 = N\backslash G$. Then by Woronowicz's Peter-Weyl theorem for compact quantum groups, every irreducible representation u^λ of G is either an irreducible representation of G_1 or none of the coefficients u^λ_{ij} is in $\mathcal{A}_{G_1}$. That is

$$E_{N\backslash G}(u^\lambda_{ij}) = \begin{cases} u^\lambda_{ij} & \text{if } \lambda \in \hat{G}_1, \\ 0 & \text{if } \lambda \in \hat{G}\backslash\hat{G}_1. \end{cases}$$

By the definition of $E_{N\backslash G}$ and linear independence of the u^λ_{ij}'s, this implies that

$$h_N \pi(u^\lambda_{ik}) = \delta_{ik}, \ \ \lambda \in \hat{G}_1, i, k = 1, \cdots, d_\lambda,$$

$$h_N \pi(u^\lambda_{ik}) = 0, \ \ \lambda \in \hat{G}\backslash\hat{G}_1.$$

This completes the proof of the proposition. □

Definition 2.2. *A quantum subgroup N of a compact quantum group G is said to be* **normal** *if it satisfies the equivalent conditions of Proposition 2.1.*

Remarks. (a) Condition (4) of Proposition 2.1 plays an important role in this paper. It is a reformulation of the following condition for a normal quantum subgroup N that appears near the end of Sect. 2 of [40]: For every irreducible representation v of G, the multiplicity of the trivial representation of N in the representation $\pi(v)$ is either zero or the dimension of v. From the proof of the proposition we see that the counit of $\mathcal{A}_{G/N}$ is equal to the restriction morphism $\pi|_{\mathcal{A}_{G/N}}$.

(b) Note also that on p679 of [40] the following statement is found: "In general, a right quotient quantum group is different from the corresponding left quotient quantum group." Though in the purely algebraic setting of Hopf algebras, one needs to distinguish between left and right normal quantum subgroups, as indicated in 1.5 of Parshall and Wang [24] (see also [1, 32, 36]), however, in view of Proposition 2.1 above, this cannot happen for normal quantum subgroups of compact quantum groups. We will see in Lemma 5.3 later that when applied to compact quantum groups the notion of normality defined in [24] is equivalent to our notion of normality. For compact quantum groups, one can also verify the conditions in 1.5.1.(1) of [24], so the notions of left and right normal quantum subgroups in their sense coincide. We note further that the antipode for compact quantum groups is not continuous and is not defined on the C^*-algebra level in general. As a result one cannot naively carry those definitions involving the antipode in the Hopf algebras context, such as the notion of normality in [24], over to compact quantum groups. (See Lemma 5.2 and Lemma 5.3 later.)

(c) The notion of a normal quantum subgroup depends on the morphism π, which gives the "position" of the quantum group N in G. If (N, π_1) is another quantum subgroup of G with surjection $\pi_1 : A_G \longrightarrow A_N$, (N, π_1) may not be normal even if (N, π) is. This phenomenon already occurs in the group situation. For example a finite group can contain two isomorphic subgroups with one normal but the other not.

Examples. We show in (1) and (2) below that the identity group and the full quantum group G are both normal quantum subgroups of G under natural embeddings. These will be called the **trivial normal quantum subgroups**.

(1) Let $N = \{e\}$ be the one element identity group. Let $\pi = \epsilon =$ counit of A_G be the morphism from A_G to A_N. Then by the counital property, one has

$$A_{G/N} = \{a \in A_G | (id \otimes \epsilon)\Delta(a) = a \otimes 1\} = A_G.$$

That is $(\{e\}, \epsilon)$ is normal and $G/(\{e\}, \epsilon) = G$.

(2) Now let $N = G$ and let $\pi : A_G \to A_N$ be any isomorphism of Woronowicz C^*-algebras [40]. Let h be the Haar measure on G and $a \in A_{G/N}$. Since π is an isomorphism and $(id \otimes \pi)\Delta(a) = a \otimes 1$, one has $\Delta(a) = (id \otimes \pi)^{-1}(a \otimes 1) = a \otimes 1$. From the invariance of h and one has

$$h(a)1 = (1 \otimes h)\Delta(a) = ah(1) = a.$$

Hence

$$A_{G/N} = \{a \in A_G | (id \otimes \pi)\Delta(a) = a \otimes 1\} = \mathbb{C}1.$$

That is (G, π) is normal and $G/(G, \pi) \cong \{e\}$.

(3) We note that besides the embeddings in (2) it is possible to construct examples of compact quantum groups G with non-normal proper embeddings of G into G. In fact this can happen for compact groups already. □

The following is a justification of the above notion of normal quantum subgroups.

Proposition 2.3. *Let $A = C(G)$ with G a compact group. Let N be a closed subgroup of G. Let π be the restriction map from A to $A_N = C(N)$. Then (N, π) is normal in the sense above if and only if N is a normal subgroup of G in the usual sense.*

Proof. Under the Gelfand-Naimark correspondence which associates to every commutative C^*-algebra its spectrum, quotients of G by (ordinary) closed normal subgroups N correspond to Woronowicz C^*-subalgebras of $C(G)$, i.e.,

$$G/N \quad \text{corresponds to} \quad C(G/N),$$

see 2.6 and 2.12 of [40]. Since $A_{G/N} = C(G/N)$ for any closed subgroup N, the proposition follows from Proposition 2.1 above. □

The following result gives a complete description of quantum normal subgroups of the compact quantum group dual of a discrete group Γ, whose proof is straightforward using e.g. [52] and Proposition 2.1.

Proposition 2.4. *Let $A_G = C^*(\Gamma)$. Let N be a quantum subgroup of G with surjection $\pi : A_G \to A_N$. Then N is normal, $L := \pi(\Gamma)$ is a discrete group and $A_N = C^*(L)$. Moreover, $A_{G/N} = C^*(K)$, where $K = \ker(\pi : \Gamma \to L)$.*

To distinguish two different quantum subgroups, we include the following result, which should be known to experts in the theory of C^*-algebras.

Proposition 2.5. *Let $\pi_k : A \to A_k$ be surjections of unital C^*-algebras with kernels I_k $(k = 1, 2)$. Let P_k be the pure state space of A_k. Then the following conditions are equivalent:*

(1) *$\{\phi_1 \circ \pi_1 | \phi_1 \in P_1\} = \{\phi_2 \circ \pi_2 | \phi_2 \in P_2\}$ as subsets of pure states of A.*

(2) *$I_1 = I_2$.*

(3) *There is an isomorphism $\alpha : A_1 \to A_2$ such that $\pi_2 = \alpha \circ \pi_1$.*

Proof. (1) $\Rightarrow$ (2): If $I_1 \neq I_2$, say, there is a $x \in I_1 \backslash I_2$. Then there is a pure state ϕ of A/I_2 such that $\phi(\pi_2(x)) \neq 0$, where we identify A_2 with A/I_2. But $\phi\pi_2$ is a pure state of $A/I_1 \cong A_1$ according to assumption (1). Hence we must have $\phi\pi_2(x) = 0$. This is a contradiction.

(2) $\Rightarrow$ (3): Let $I = I_1 = I_2$. Let π be the quotient map $A \to A/I$. Let $\tilde{\pi}_k$ be the homomorphism from A/I to A_k such that $\pi_k = \tilde{\pi}_k \pi$ $(k = 1, 2)$. Then $\tilde{\pi}_k$ are isomorphisms. Put $\alpha = \tilde{\pi}_2 \circ \tilde{\pi}_1^{-1}$. Then $\pi_2 = \alpha \circ \pi_1$.

(3) $\Rightarrow$ (1): This follows from $P_1 = P_2 \circ \alpha$. □

The following proposition is an easy consequence of Proposition 2.1.

Proposition 2.6. *Let (N_1, π_1) be a normal quantum subgroup of G. Let*

$$\alpha : A_{N_1} \longrightarrow A_{N_2}$$

be an isomorphism of quantum groups. Then $(N_2, \alpha\pi_1)$ is normal.

In view of the above discussions, it is reasonable to have the following definition (cf. also remarks after Proposition 2.3).

Definition 2.7. *Two quantum subgroups (π_1, H_1) and (π_2, H_2) of a quantum group G are called* **the same** *if π_1, π_2 satisfy the equivalent conditions of Proposition 2.5. When this happens, we denote $(\pi_1, H_1) = (\pi_2, H_2)$.*

Geometrically speaking, two "embedded" quantum subgroups (π_1, H_1) and (π_2, H_2) of a quantum group G are called **the same** if their "images" in G are the same.

3. Properties of Normal Quantum Subgroups

Parallel to the three fundamental isomorphism theorems in the theory of groups, one would expect that their analogs are valid in the theory of quantum groups. It is difficult to formulate appropriate quantum analogs of the first two isomorphism theorems, cf. [24, 32, 36, 1]. The following is an analog of the third fundamental isomorphism theorem.

Proposition 3.1. *Let (N, ρ) be a normal quantum subgroup of G. Let (H, π) be a quantum subgroup of G that contains (N, ρ), i.e., there is a morphism π_1 such that (N, π_1) is a quantum subgroup of H with $\rho = \pi_1\pi$. Then (N, π_1) is normal in H. If furthermore H is normal in G, then $(H/N, \pi')$ is normal in G/N and*

$$A_{(G/N)/(H/N)} = A_{G/H},$$

where $\pi' = \pi|_{A_{G/N}}$, the restriction of π to $A_{G/N}$.

Proof. According to the remarks near the beginning of §2, we only need to work with the dense $*$-subalgebras.

Let $z \in \mathcal{A}_H$ be such that $\pi_1(z) = 0$. Assume $z = \pi(x)$ for some $x \in \mathcal{A}_G$. Then $\rho(x) = 0$. Since $\ker(\hat{\rho})$ is normal in the sense of [32] by remark (b) following Lemma 5.3 later, we have

$$\sum \rho(x_{(2)}) \otimes S(x_{(1)})x_{(3)} = 0, \quad \sum \rho(x_{(2)}) \otimes x_{(1)}S(x_{(3)}) = 0,$$

where we use Sweedler's summation convention [35]. Hence

$$\sum \pi_1\pi(x_{(2)}) \otimes \pi(S(x_{(1)})x_{(3)}) = 0, \quad \sum \pi_1\pi(x_{(2)}) \otimes \pi(x_{(1)}S(x_{(3)})) = 0.$$

That is

$$\sum \pi_1(z_{(2)}) \otimes S(z_{(1)})z_{(3)} = 0, \quad \sum \pi_1(z_{(2)}) \otimes z_{(1)}S(z_{(3)}) = 0.$$

This means that (π_1, N) is normal in H.

Let $a \in \mathcal{A}_{G/N}$. Then it is clear that $\pi(a) \in \mathcal{A}_{H/N}$ so $\pi(\mathcal{A}_{G/N})$ is contained in $\mathcal{A}_{H/N}$. Conversely, let $b \in \mathcal{A}_{H/N}$. Assume $b = \pi(a)$ for some $a \in \mathcal{A}_G$. Put $\bar{a} = (1 \otimes h_N\rho)\Delta(a)$. Then one verifies that $\bar{a} \in \mathcal{A}_{G/N}$ (cf. Proposition 2.3 and Sect. 6 of [46]) and $\pi(\bar{a}) = b$. That is $\pi(\mathcal{A}_{G/N}) = \mathcal{A}_{H/N}$.

Now assume (H, π) is normal. For ease of notation, Let $G' = G/N$ and $H' = H/N$. The above shows that (H', π') is a quantum subgroup of G'. If $a \in \mathcal{A}_{G'/H'}$, that is, $(1 \otimes \pi')\Delta(a) = a \otimes 1_{H'}$, then it is clear that $a \in \mathcal{A}_{G/H}$ since $\pi' = \pi|_{A_{G/N}}$ and $1_{H'} = 1_H$. Conversely, if $a \in \mathcal{A}_{G/H}$, that is $(1 \otimes \pi)\Delta(a) = a \otimes 1_H$, then $(1 \otimes \pi_1\pi)\Delta(a) = a \otimes 1_N$. That is $a \in \mathcal{A}_{G'} = \mathcal{A}_{G/N}$. But then we have

$$(1 \otimes \pi')\Delta(a) = (1 \otimes \pi)\Delta(a) = a \otimes 1_H = a \otimes 1_{H'}.$$

Hence $a \in \mathcal{A}_{G'/H'}$ and $A_{G'/H'} = A_{G/H}$.

The result is completely proved. □

Just as in the case of ordinary groups, we have

Proposition 3.2. *Let G_1, G_2 be compact quantum groups. Let π_1 be the natural embedding of G_1 into $G_1 \times G_2$ defined by the surjection*

$$\pi_1 : A_{G_1} \otimes A_{G_2} \to A_{G_1},$$

$$\pi_1 = id_1 \otimes \epsilon_2, \ \ \epsilon_2 \text{ being the counit of } A_{G_2}.$$

Then (G_1, π_1) is a normal quantum subgroup of $G = G_1 \times G_2$ and $G/G_1 \cong G_2$.

Proof. According to [41], every irreducible representation of $G_1 \times G_2$ is of the form

$$u \otimes_{ex} v = \sum_{ijkl} e^u_{ij} \otimes e^v_{kl} \otimes \otimes u_{ij} \otimes v_{kl},$$

where u and v are irreducible representations of G_1 and G_2 respectively, e^u_{ij} and e^v_{kl} are systems of matrix units acting on the Hilbert spaces of u and v respectively. Let h_1 be the Haar state of G_1. Then

$$h_1\pi_1(u \otimes_{ex} v) = \sum_{ijkl} e^u_{ij} \otimes e^v_{kl} h_1(u_{ij})\delta_{kl} = \sum_{ij} e^u_{ij} h_1(u_{ij}) \otimes I_v,$$

which is either $I_u \otimes I_v$ or 0 according as u is trivial or not, where I_u, I_v are the identity matrices acting on the Hilbert spaces of u, v respectively. By Proposition 2.1.(4), G_1 is normal.

Similarly, we have

$$E_{G/G_1}(u_{ij} \otimes v_{kl}) = \begin{cases} 0 & \text{if } u \text{ is non-trivial,} \\ 1 \otimes v_{kl} & \text{if } u \text{ is trivial.} \end{cases}$$

Hence $A_{G/G_1} = 1 \otimes A_{G_2} \cong A_{G_2}$. □

For free products [40], the situation is quite different:

Proposition 3.3. *Let G_1, G_2 be compact quantum groups. Let $G = G_1 \hat{*} G_2$ be the free product compact quantum group underlying $A_{G_1} * A_{G_2}$. Let π_1 be the natural embedding of G_1 into G defined by the surjection*

$$\pi_1 : A_{G_1} * A_{G_2} \to A_{G_1},$$

$$\pi_1 = id_1 * \epsilon_2.$$

If G_1 has at least one irreducible representation of dimension greater than one, then (G_1, π_1) is **not** *a normal quantum subgroup of $G_1 \hat{*} G_2$. Otherwise, (G_1, π_1) is normal in $G_1 \hat{*} G_2$.*

The hat in the symbol $\hat{*}$ above signifies the "Fourier transform" of $*$.

Proof. Let

$$u = \sum_{ij} e^u_{ij} \otimes u_{ij}, \quad v = \sum_{kl} e^v_{kl} \otimes v_{kl}$$

be irreducible representations of G_1, G_2 respectively. Assume that the dimension of u is greater than one. Let $\bar{u} = \sum_{ij} e^u_{ij} \otimes u^*_{ij}$ denote the conjugate representation of u. Then by [40], the interior tensor product representation

$$u \otimes_{in} v \otimes_{in} \bar{u} = \sum_{ijklrs} e^u_{ij} \otimes e^v_{kl} \otimes e^u_{rs} \otimes u_{ij} v_{kl} u^*_{rs}$$

is irreducible. Let h_1 be the Haar state of G_1. Then

$$h_1\pi_1(u \otimes_{in} v \otimes_{in} \bar{u}) = \sum_{ijrs} e^u_{ij} \otimes I_v \otimes e^u_{rs} h_1(u_{ij}u^*_{rs}),$$

where I_v is the identity matrix acting on the Hilbert space of v. Since u is of dimension greater than one, $u \otimes \bar{u}$ properly contains the trivial representation of G_1 (with multiplicity one). Hence

$$h_1(u \otimes_{in} \bar{u}) = \sum_{ijrs} e^u_{ij} \otimes e^u_{rs} h_1(u_{ij}u^*_{rs})$$

is neither the identity matrix, nor the zero matrix on $H_u \otimes H_{\bar{u}}$. (See also Theorem 5.7 of Woronowicz [52].) Therefore

$$h_1\pi_1(u \otimes_{in} v \otimes_{in} \bar{u})$$

is neither the identity matrix, nor the zero matrix on $H_u \otimes H_v \otimes H_{\bar{u}}$. By Proposition 2.1.(4), G_1 is not normal.

If G_1 has no non-trivial irreducible representations of dimension greater than one, then by [52] G_1 is the compact quantum group dual of a discrete group Γ, i.e., $A_{G_1} = C^*(\Gamma)$. By [40], every irreducible representation of G is of the form

$$w^{\lambda_1} \otimes w^{\lambda_2} \otimes \cdots \otimes w^{\lambda_n},$$

where w^{λ_i} is a non-trivial representation belonging to either the set Γ or the set $\hat{G}_2$, w^{λ_i} and $w^{\lambda_{i+1}}$ being in different sets. It is clear that $\pi_1(w^{\lambda_1} \otimes w^{\lambda_2} \otimes \cdots \otimes w^{\lambda_n})$ is a constant diagonal matrix, with diagonal entry (call it $[w^{\lambda_1} \otimes w^{\lambda_2} \otimes \cdots \otimes w^{\lambda_n}]$) equal to the product of those w^{λ_i}'s that belong to Γ. Since $h_{G_1}(\gamma) = 0$ if γ is not the neutral element of Γ, one sees that Proposition 2.1.(4) is fulfilled. Hence (G_1, π_1) is normal in $G = G_1 \hat{*} G_2$ and A_{G/G_1} is equal to the closure of the linear span of entries of the matrix $w^{\lambda_1} \otimes w^{\lambda_2} \otimes \cdots \otimes w^{\lambda_n}$ such that $[w^{\lambda_1} \otimes w^{\lambda_2} \otimes \cdots \otimes w^{\lambda_n}]$ is the neutral element of Γ. $\square$

Similar to the second situation in the proposition above, we have the following

Proposition 3.4. *Consider a crossed product $A \rtimes_\alpha \Gamma$ of a compact quantum group A by a discrete group Γ [41]. Then $C^*(\Gamma)$ is a quantum normal subgroup of $A \rtimes_\alpha \Gamma$ with quotient A via the morphism*

$$\pi : A \rtimes_\alpha \Gamma \longrightarrow C^*(\Gamma), \quad \pi(a\gamma) = \epsilon(a)\gamma,$$

where ϵ is the counit of A and $a \in A$, $\gamma \in \Gamma$.

More generally, one has

Proposition 3.5. *Let (A, Γ, α) be a Woronowicz C^*-dynamical system. Let $I \lhd A$ be an α-invariant Woronowicz C^*-ideal so that A/I is a normal quantum subgroup of A with quotient quantum group B. Let K be the kernel of the action α. Let $\tilde{\alpha}$ be the induced action of Γ/K on A/I. Then $A/I \rtimes_{\tilde{\alpha}} \Gamma/K$ is a normal quantum subgroup of $A \rtimes_\alpha \Gamma$ with quotient $B \rtimes_\alpha K$.*

Proof. Consider the morphisms

$$\pi : A \longrightarrow A/I \rtimes_{\tilde{\alpha}} \Gamma/K,$$

$$U : \Gamma \longrightarrow A/I \rtimes_{\tilde{\alpha}} \Gamma/K,$$

defined by $\pi(a) = \tilde{a}$, $U_\gamma = \tilde{\gamma}$. Here $\tilde{a}$ is the image in A/I of an element $a \in A$ viewed as an element of $A/I \rtimes_{\tilde{\alpha}} \Gamma/K$ via inclusion; $\tilde{\gamma}$ is the image in Γ/K of an element $\gamma \in \Gamma$ viewed as an element of $A/I \rtimes_{\tilde{\alpha}} \Gamma/K$ via inclusion. One can verify that (π, U) is a covariant representation, i.e.,

$$U_\gamma \pi(a) U_\gamma^{-1} = \pi(\alpha_\gamma(a)).$$

Hence there is a surjection

$$\pi \times U : A \rtimes_\alpha \Gamma \longrightarrow A/I \rtimes_{\tilde{\alpha}} \Gamma/K,$$

extending π and U. This morphism preserves the coproducts because its restrictions to A and $C^*(\Gamma)$ do. This shows that $A/I \rtimes_{\tilde{\alpha}} \Gamma/K$ is a quantum subgroup of $A \rtimes_\alpha \Gamma$ (true under only the assumption that A/I is a quantum subgroup of A).

Now assume that A/I is a normal quantum subgroup of A with quotient B. Let θ denote the surjection $\pi \times U$. Let $h = h_{A/I} \rtimes_{\tilde{\alpha}} h_{\Gamma/K}$ be the Haar state on $A/I \rtimes_{\tilde{\alpha}} \Gamma/K$ (cf. [41]). Then every irreducible representation of the quantum group $A \rtimes_\alpha \Gamma$ is of the form $(u_{ij}^\beta \gamma)$, where (u_{ij}^β) is an irreducible representation of dimension d_β of the quantum group A and $\gamma \in \Gamma$ (cf. [41] as well). Let S be as in proof of (4)$\Rightarrow$(3). Then

$$(h\theta(u_{ij}^\beta \gamma)) = (h(\tilde{u}_{ij}^\beta) h(\tilde{\gamma})) = \begin{cases} I_{d_\beta} & \text{if } \beta \in S,\ \gamma \in K, \\ 0 & \text{otherwise.} \end{cases}$$

Hence by Proposition 2.1 and its proof, we conclude that $A/I \rtimes_{\tilde{\alpha}} \Gamma/K$ is a normal quantum subgroup of $A \rtimes_\alpha \Gamma$ with quotient $B \rtimes_\alpha K$. □

We note that A is not a normal quantum subgroup of $A \rtimes_\alpha \Gamma$, unlike the semi-direct product of groups.

4. Simple Compact Quantum Groups

To avoid undue difficulties in the theory of simple compact quantum groups, we assume connectivity as a part of their postulates. The latter concept is given by

Definition 4.1. *We call a compact quantum group G_A* **connected** *if for each non-trivial irreducible representation $u^\lambda \in \hat{G}_A$, the C^*-algebra $C^*(u_{ij}^\lambda)$ generated by the coefficients of u^λ is of infinite dimension.*

In virtue of (28.21) of [17], we have

Proposition 4.2. *Let G_A be an ordinary compact group (i.e. A_G is commutative). Then G_A is connected as a topological space if and only if it is connected in the sense above.*

Definition 4.3. *We call a compact quantum group G_A* **simple** *if it satisfies the following conditions* (1)-(4):

(1) *The Woronowicz C^*-algebra A_G is finitely generated.*
(2) *G_A is connected.*
(3) *G_A has no non-trivial connected normal quantum subgroups.*
(4) *G_A has no non-trivial representations of dimension one.*

A (simple) quantum group is called **absolutely simple** *if it has no non-trivial normal quantum subgroups. Similarly a finite quantum group is called* **simple** *if it has no non-trivial normal quantum subgroups.*

Just as the notion of simple compact Lie groups excludes the torus groups, this definition of simple quantum groups excludes the compact quantum groups coming from group C^*-algebras $C^*(\Gamma)$ of discrete groups Γ (note that $C^*(\Gamma)$ is the algebra of continuous functions on the torus $\mathbb{T}^n$ when Γ is the abelian group $\mathbb{Z}^n$). This is important because it is impossible to have a nice classification theory for discrete groups. However, we do not know if condition (4) in Definition 4.3 (i.e., there is no non-trivial group-like elements) is superfluous, as it is for simple compact Lie groups.

As a justification of this definition, we have the following proposition, which follows from Proposition 4.2 above and Sect. 2 of [40].

Proposition 4.4. *Let G_A be an ordinary compact Lie group (i.e. A_G is commutative). Then G_A is a simple Lie group in the ordinary sense if and only if G_A is simple in the sense above.*

Note that a simple compact Lie group is not a direct product of proper connected subgroups. Also, a simple Lie group is not a semi-direct product. By Proposition 3.2 and Proposition 3.4 (or [41]), similar results hold for quantum groups:

Proposition 4.5. *If G_A is a simple compact quantum group, then A_G is not a tensor product, nor a crossed product by a non-trivial discrete group.*

To put in perspective the examples of simple compact quantum groups to be studied later, we introduce some properties for compact quantum groups. First we recall that the **representation ring** (also called the **fusion ring**) $R(G)$ of a compact quantum group G is an ordered algebra over the integers $\mathbb{Z}$ with positive cone (or semiring, which is also a basis) $R(G)_+ := \{\chi_u\}$ consisting of characters χ_u of irreducible representations $u \in \hat{G}$ of G, and structure constants $c_{uv}^w \in \mathbb{N} \cup \{0\}$ given by the rules

$$\chi_u \chi_v = \sum_{w \in \hat{G}} c_{uv}^w \chi_w,$$

where the product $\chi_u \chi_v$ is taken in the algebra $\mathcal{A}_G$.

Definition 4.6. *Let G be a compact quantum group. We say that G has* **property** *F if each Woronowicz C^*-subalgebra of A_G is of the form $A_{G/N}$ for some normal quantum subgroup N of G. We say that G has* **property** *FD if each quantum subgroup of G is normal.*

We say that G is **almost classical** *if its representation ring $R(G)$ is order isomorphic to the representation ring of a compact group.*

By Proposition 2.3, a compact group trivially has property F. We will give in §5 and §6 non-trivial simple compact quantum groups that are almost classical and have property F. Among compact quantum groups, simple compact quantum groups that are almost classical or have property F are closest to ordinary simple compact Lie groups in regard to noncommutative geometry.

By Proposition 2.4, as the dual of discrete group Γ, a compact quantum group of the form $C^*(\Gamma)$ has property FD. When Γ is finite, $C^*(\Gamma)$ is equal to the dual of the function algebra $C(\Gamma)$. This explains the term FD.

A compact quantum group G is absolutely simple with property F if and only if every non-trivial representation v of G is faithful, i.e., $C^*(v_{ij}) = A_G$, cf. [46].

By a theorem of Handelman [16], the representation ring of a compact connected Lie group is a complete isomorphism invariant. But this fails for compact quantum groups, since the representation rings of a simple compact Lie group K and its standard deformation K_q are order isomorphic.

In [5], Banica uses the positive cone $R_+(G)$ of the representation ring $R(G)$ of a compact quantum group G to define what he calls an R_+ deformation. This is closely related to almost classical quantum groups.

It is clear that a quantum quotient group G/N of an almost classical quantum group G is also almost classical. But a quantum subgroup of an almost classical quantum group need not be almost classical. For example, the quantum permutation groups are almost classical (cf. [4, 45] and remarks preceding Theorem 5.5), but according to of Bichon [8], their quantum subgroups $A_2(\mathbb{Z}/m\mathbb{Z})$ are not almost classical if $m \geq 3$ (see Corollary 2.7 and the paragraph following Corollary 4.3 of [8]). However, for a compact quantum group with property F (resp. FD), we have the following result.

Theorem 4.7. *Let G be a compact quantum group with property F (resp. FD). Then its quantum subgroups and quotient groups G/N (by normal quantum subgroups N) also have property F (resp. FD).*

The assertions in the theorem above will not be needed in this paper. The details of their proofs will appear in [50].

The main goals in the theory of simple compact quantum groups are: (1) to classify up to isomorphism simple quantum groups (we must first find all of them) and (2) their irreducible representations; (3) to analyze the structure of compact quantum groups in terms of simple quantum groups; and (4) to develop applications of simple compact quantum groups in other areas of mathematics and physics. For these purposes, new techniques for compact quantum groups must be developed.

Before solving the classification problem, one should solve the following conceivably easier problems.

Problem 4.8. *Classify simple compact quantum groups with property F.*

Problem 4.9. *Construct simple compact quantum groups with property FD.*

Problem 4.10. *Construct simple compact quantum groups that are not almost classical.*

5. Simplicity of $B_u(Q)$ and $A_{aut}(B, \tau)$

We recall the construction of compact quantum group $B_u(Q)$ associated to a non-singular $n \times n$ complex scalar matrix Q (cf. [2, 40, 38, 44]). The (noncommutative) C^*-algebra of functions on the quantum group $B_u(Q)$ is generated by noncommutative coordinate functions u_{ij} $(i, j = 1, \cdots, n)$ that are subject to the following relations:

$$u^* u = I_n = u u^*, \quad u^t Q u Q^{-1} = I_n = Q u Q^{-1} u^t,$$

where $u = (u_{ij})_{i,j=1}^n$. When $Q\bar{Q}$ is a scalar multiple cI_n of the identity matrix I_n, the quantum group $B_u(Q)$ and the group $SU(2)$ have the same fusion rules for their irreducible representations, as shown by Banica [2], which implies that $B_u(Q)$ is an almost classical quantum group. Under the same condition on Q, the isomorphism classification of $B_u(Q)$ is determined by the author in [45]. For arbitrary Q, $B_u(Q)$ is shown also in [45] to be a free product of its building blocks, involving both $B_u(Q)$'s with $Q\bar{Q}$ being scalars and $A_u(Q)$ with Q being positive (see also later in this section for $A_u(Q)$). For $n = 1$, $B_u(Q) = C(\mathbb{T})$ is the trivial 1×1 unitary group. We will concentrate on the non-trivial case $n \geq 2$. As is shown in [45], the isomorphism class of $B_u(Q)$ depends on the normalized Q only. Hence we assume Q satisfies the normalization condition $Q\bar{Q} = \pm I_n$.

Theorem 5.1. *Let $Q \in GL(n, \mathbb{C})$ be such that $Q\bar{Q} = \pm I_n$. Then $B_u(Q)$ is a simple compact quantum group with property F. In fact it has only one normal subgroup of order* 2.

Proof. Recall that according to Banica [2] irreducible representations of the quantum group $B_u(Q)$ can be parametrized by r_k $(k = 0, 1, 2, \cdots)$ with r_0 trivial and $r_1 = (u_{ij})_{ij=1}^n$, so that the fusion rules for their tensor product representations (i.e., decomposition into irreducible representations) read

$$r_k \otimes r_l = r_{|k-l|} \oplus r_{|k-l|+2} \oplus \cdots \oplus r_{k+l-2} \oplus r_{k+l}, \quad k, l \geq 0.$$

We first show that the quantum group $B_u(Q)$ is connected. If $k = 2m$ is even $(m > 0)$, then let $r_l = r_k$ in the above tensor product decomposition and do the same for the irreducible constituents repeatedly, one sees that the algebra $C^*(r_{2m})$ generated by the coefficients of the representation r_{2m} contains the coefficients of r_{2s} for all s. Hence r_{2m} generates an infinite dimensional algebra:

$$C^*(r_{2m}) = C^*(\{r_{2s} | s \geq 0\}).$$

If $k = 2m + 1$ is odd $(m \geq 0)$, then let $r_l = r_k$ in the above tensor product decomposition, one sees that the representation r_2 appears therein. Apply the decomposition to $r_{2m+1} \otimes r_2$, one sees that $r_1 = (u_{ij})$ appears therein. Hence the algebra generated by the coefficients of r_{2m+1} is the same as the algebra generated by those of $r_1 = (u_{ij})$. We conclude from this analysis that there is only one non-trivial Woronowicz C^*-subalgebra in $B_u(Q)$, the one $C^*(r_{2m})$ generated by coefficients of r_{2m}, which is obviously infinite dimensional as noted above, where m is any nonzero positive number. In particular, the quantum group $B_u(Q)$ is connected.

For rest of the proof, we show that the quantum group $B_u(Q)$ has only one normal quantum subgroup, although it has many quantum subgroups.

Note that the coordinate functions v_{ij} of the matrix group $N = \{I_n, -I_n\}$ satisfy the defining relations of $B_u(Q)$, hence there is a surjection π from the C^*-algebra $B_u(Q)$ to the C^*-algebra A_N of functions on N such that

$$\pi(u_{ij}) = v_{ij}, \quad i, j = 1, 2, \cdots, n.$$

It is clear that π is a morphism of quantum groups, hence (N, π) is a quantum subgroup of the quantum group $B_u(Q)$.

We show that (N, π) is actually a normal quantum subgroup. To see this, it suffices by Proposition 2.1 to show that

$$\pi(r_{2m}) = d_{2m} \cdot v_0, \quad \pi(r_{2m+1}) = d_{2m+1} \cdot v_1,$$

where d_{2m} and d_{2m+1} are dimensions of the representations r_{2m} and r_{2m+1} respectively, v_0 and v_1 are the trivial representation and the non-trivial irreducible representation of N respectively ($v_1(\pm I_n) = \pm 1$). By the definition of π and v_1 the assertion is clearly true for $m = 0$. In general, suppose the assertion is true for m. Then $\pi(r_{2m+1}) \otimes \pi(r_1)$ is a multiple of v_0 since $v_1^2 = v_0$. From the decomposition of $r_{2m+1} \otimes r_1$, we get

$$\pi(r_{2m+1}) \otimes \pi(r_1) = \pi(r_{2m}) \oplus \pi(r_{2m+2}).$$

Hence $\pi(r_{2(m+1)}) = \pi(r_{2m+2})$ is a scalar multiple of v_0. Similarly, from

$$\pi(r_{2m+2}) \otimes \pi(r_1) = \pi(r_{2m+1}) \oplus \pi(r_{2m+3}),$$

we see that $\pi(r_{2(m+1)+1}) = \pi(r_{2m+3})$ is a multiple of v_1. Since v_0 and v_1 are one dimensional representations, the multiples we obtained above must be d_{2m+2} and d_{2m+3} respectively. That is (N, π) is normal and

$$A_{G/N} = C^*(r_2) = C^*(\{r_{2s} | s \geq 0\}),$$

where for simplicity of notation, the symbol G in G/N refers to the quantum group $G_{B_u(Q)}$. The above also shows that this quantum group has property F.

We have to show that $B_u(Q)$ has no other normal quantum subgroups, which will imply that it has no connected normal quantum subgroups and is therefore a simple quantum group.

Let (N_1, π_1) be a non-trivial normal quantum subgroup of $B_u(Q)$. Since $N_1 \neq 1$, by Definition 2.7 and Proposition 2.1 there exists an irreducible representation v of the quantum group $B_u(Q)$ such that $\pi_1(v)$ is not a scalar and therefore $E_{G/N_1}(v) = 0$. Hence by the proof of Proposition 2.1 and Woronowicz's Peter-Weyl theorem [52], $A_{G/N_1} = E_{G/N_1}(A_G) \neq A_G$.

Similarly, we claim that $A_{G/N_1} \neq \mathbb{C}1$, where 1 is the unit of A_G. To prove this, we need two lemmas. It is instructive to compare the first lemma with the ideal theory for C^*-algebras.

Lemma 5.2. (Hopf $*$-ideals vs. Woronowicz C^*-ideals)

(1) *Let G be a compact quantum group. Let $\mathcal{I}$ be a $*$-ideal of $\mathcal{A}_G$ satisfying*

$$\Delta(\mathcal{I}) \subset \mathcal{A}_G \otimes \mathcal{I} + \mathcal{I} \otimes \mathcal{A}_G \ .$$

Then $S(\mathcal{I}) = \mathcal{I}$, $\epsilon(\mathcal{I}) = 0$, hence $\mathcal{I}$ is Hopf $*$-ideal of $\mathcal{A}_G$. The norm closure $\overline{\mathcal{I}}$ in the C^*-algebra A_G is a Woronowicz C^*-ideal. The Hopf $*$-algebra $\mathcal{A}_G/\mathcal{I}$ admits a universal C^*-norm and its completion under this norm is a Woronowicz C^*-algebra isomorphic to $A_G/\overline{\mathcal{I}}$.

(2) *The map $f(\mathcal{I}) = \overline{\mathcal{I}}$ is a bijection from the set of Hopf $*$-ideals $\{\mathcal{I}\}$ of $\mathcal{A}_G$ onto the set of Woronowicz C^*-ideals $\{I\}$ of A_G with full A_G/I. The inverse g of f is given by $g(I) = I \cap \mathcal{A}_G$.*

Remarks. (a) Note that (2) and the last part of (1) in the lemma above are false if the Woronowicz C^*-algebra A_G or A_G/I is not full, as is shown by the following example. Let $A_G = C^*(F_2)$ be the group C^*-algebra of the free group F_2 on two generators. Let I be the kernel of the canonical map $\pi : C^*(F_2) \to C^*_r(F_2)$ where $C^*_r(F_2)$ is the reduced group C^*-algebra of F_2. Then $I \cap \mathcal{A}_G = 0$ but $I \neq \overline{0}$.

(b) Part (1) of the lemma above strengths 2.10 of [40].

Proof of Lemma 5.2. Let $\pi : A_G \to A_G/\overline{\mathcal{I}}$ be the quotient map. Then since

$$\Delta(\mathcal{I}) \subset \mathcal{A}_G \otimes \mathcal{I} + \mathcal{I} \otimes \mathcal{A}_G \subset \ker(\pi \otimes \pi),$$

we have that $\Delta(\overline{\mathcal{I}}) \subset \ker(\pi \otimes \pi)$. That is, $\overline{\mathcal{I}}$ is a Woronowicz C^*-ideal and therefore $A_G/\overline{\mathcal{I}}$ is a Woronowicz C^*-algebra (see [40]). The dense Hopf $*$-subalgebra $\mathcal{B}_0$ of $A_G/\overline{\mathcal{I}}$ is the image under π of the Hopf $*$-algebra $\mathcal{A}_G$ and it is clearly isomorphic to $\mathcal{A}_G/\mathcal{I}$. As the kernel of this morphism $\hat{\pi}$, $\mathcal{I}$ satisfies $S(\mathcal{I}) \subset \mathcal{I}$ and $\epsilon(\mathcal{I}) = 0$, i.e. $\mathcal{I}$ is a Hopf $*$-ideal. The stronger assertion $S(\mathcal{I}) = \mathcal{I}$ follows immediately from the property $S(S(a^*)^*) = a$ [52] (cf. 1.5.1 of [24] for a consequence of this assertion).

For an alternative (algebraic) argument to the one in the last paragraph, one can show that $\mathcal{A}_G/\mathcal{I}$ is a compact quantum group algebra (CQG algebra) in the sense of Dijkhuizen and Koornwinder [13] (See also [19] and [53, 42]). But a better name for compact quantum group (CQG) algebra might be Woronowicz $*$-algebra (or compact Hopf $*$-algebra), since the quantum group C^*-algebra of a compact quantum group G is the C^*-algebra $C^*(G)$ dual to $C(G)$ according to [26, 15].

Let B be the closure of $\mathcal{A}_G/\mathcal{I}$ in the universal C^*-norm. Then B is a Woronowicz C^*-algebra. The composition

$$\mathcal{A}_G \to \mathcal{A}_G/\mathcal{I} \to B$$

is bounded and extends to a morphism of Woronowicz C^*-algebras from A_G to B. Let ρ denotes this morphism from A_G onto B. Since $\overline{\mathcal{I}} \subset \ker(\rho)$, ρ factors through $A_G/\overline{\mathcal{I}}$ via a C^*-algebra morphism ψ:

$$A_G \xrightarrow{\pi} A_G/\overline{\mathcal{I}} \xrightarrow{\psi} B.$$

One can check that ψ is a morphism of Woronowicz C^*-algebras [40]. Then the restriction $\hat{\psi}$ of ψ to a morphism from the dense Hopf $*$-subalgebra $\mathcal{B}_0$ of $A_G/\overline{\mathcal{I}}$ to that of B is easily seen to be an isomorphism. Since B is full, ψ is itself an isomorphism from $A_G/\overline{\mathcal{I}}$ to B, see [42]. (We note in passing that since $A_G/\ker(\rho) \cong B$, we have $\overline{\mathcal{I}} = \ker(\rho)$.) This proves part of (1) of the lemma.

Let $\mathcal{I}$ be as in (2). Let B_1 be the algebra $A_G/\overline{\mathcal{I}}$. Then by (1) and [40] B_1 is a Woronowicz C^*-algebra. Let $\mathcal{B}_1$ be the dense Hopf $*$-algebra of B_1. Let $\hat{\pi} : \mathcal{A}_G \to \mathcal{B}_1$ be the morphism associated with the quotient morphism π. Then clearly

$$\mathcal{I} \subset \overline{\mathcal{I}} \cap \mathcal{A}_G = gf(\mathcal{I}).$$

Conversely if $x \in \overline{\mathcal{I}} \cap \mathcal{A}_G$, then $x \in \ker(\hat{\pi}) = \mathcal{I}$. Hence $gf(\mathcal{I}) = \mathcal{I}$.

Let I be as in (2). We show that $fg(I) = I$. Let $B_2 = A_G/I$ and let π_2 be the quotient morphism from A_G onto B_2. Define $\mathcal{I} = g(I) = I \cap \mathcal{A}_G$. We need to show that $\overline{\mathcal{I}} = I$. The idea of proof is the same as that of the last part in (1).

Using the morphism $\hat{\pi}_2 : \mathcal{A}_G \to \mathcal{B}_2$ associated with π_2, we see that $\mathcal{I} = \ker(\hat{\pi}_2)$. So $\mathcal{I}$ is a Hopf $*$-ideal in $\mathcal{A}_G$. It follows from (1) that $A_G/\overline{\mathcal{I}}$ is a Woronowicz C^*-algebra. Since $\overline{\mathcal{I}} \subset I$, the morphism π_2 factors through $A_G/\overline{\mathcal{I}}$ via a morphism ϕ of Woronowicz C^*-algebras:

$$A_G \xrightarrow{\pi} A_G/\overline{\mathcal{I}} \xrightarrow{\phi} B_2.$$

The restriction $\hat{\phi}$ of ϕ to the dense Hopf $*$-algebras of $A_G/\overline{\mathcal{I}}$ to that of B_2 becomes an isomorphism. Since B_2 is full, ϕ is an isomorphism, which means that $I = \overline{\mathcal{I}}$ (and $B_1 = B_2$). This completes the proof of Lemma 5.2.

Lemma 5.3. (Reconstruct N from G/N)

Let (N, π) be a normal quantum subgroup of a compact quantum group G. Let $\hat{\pi}$ be the associated morphism from $\mathcal{A}_G$ to $\mathcal{A}_N$. Then,

$$\ker(\hat{\pi}) = \mathcal{A}_{G/N}^{+}\mathcal{A}_G = \mathcal{A}_G\mathcal{A}_{G/N}^{+} = \mathcal{A}_G\mathcal{A}_{G/N}^{+}\mathcal{A}_G,$$

where $\mathcal{H}^{+}$ denotes the augmentation ideal (i.e. kernel of the counit) for any Hopf algebra $\mathcal{H}$.

Remarks. (a) In the notation of Schneider [32], the result above can be restated as follows: The map Φ is the left inverse of Ψ, where $\Psi(\ker(\hat{\pi})) := \mathcal{A}_{G/N}$ and $\Phi(\mathcal{A}_{G/N}) := \mathcal{A}_G\mathcal{A}_{G/N}^{+}$. In the language of Andruskiewitsch *et al* [1], the result above implies that the sequence

$$1 \longrightarrow N \longrightarrow G \longrightarrow G/N \longrightarrow 1,$$

or the sequence

$$0 \longrightarrow \mathcal{A}_{G/N} \longrightarrow \mathcal{A}_G \longrightarrow \mathcal{A}_N \longrightarrow 0,$$

is exact. It is instructive to compare this with the purely algebraic situation in [24], where for a given a normal quantum subgroup in their sense the existence of an exact sequence is not known and the uniqueness does not hold in general (cf. 1.6 and 6.3 loc. cit.). Note that the notion of exact sequence of quantum groups in [32] is equivalent to that in [1] under certain faithful (co)flat conditions, which is still not known to be valid in general. However, we have the following

Conjecture: Let G be a compact quantum group. Then the Hopf algebra A_G is faithfully flat over its Hopf subalgebras. An arbitrary Hopf algebra need not be faithfully flat over its Hopf subalgebras.

Similarly, one can formulate a conjecture concerning faithful coflatness.

(b) It follows from Lemma 5.3 and [32] that the notion of normal quantum groups in this paper and the one in [24] are equivalent for compact quantum groups.

Proof of Lemma 5.3. The proof is essentially the same as those of 16.0.2 in Sweedler [35] and (4.21) in Childs [10] for finite dimensional Hopf algebras, though Hopf algebras considered here are infinite dimensional in general. We sketch the main steps here for convenience of the reader.

It suffices to prove $\ker(\hat{\pi}) = \mathcal{A}_{G/N}^{+}\mathcal{A}_G$. The other equality $\ker(\hat{\pi}) = \mathcal{A}_G\mathcal{A}_{G/N}^{+}$ is proved similarly. From these it follows that $\ker(\hat{\pi}) = \mathcal{A}_G\mathcal{A}_{G/N}^{+}\mathcal{A}_G$.

Consider the right $\mathcal{A}_N$-comodule structures on $\mathcal{A}_N$ and $\mathcal{A}_G$ given respectively by

$$\Delta_N : \mathcal{A}_N \to \mathcal{A}_N \otimes \mathcal{A}_N, \text{ and } (id \otimes \hat{\pi})\Delta_G : \mathcal{A}_G \to \mathcal{A}_G \otimes \mathcal{A}_N,$$

where Δ_N and Δ_G are respectively the coproducts of the Hopf algebras $\mathcal{A}_N$ and $\mathcal{A}_G$. Since $\mathcal{A}_N$ is cosemisimple by the fundamental work of Woronowicz [52, 40], it follows from Theorem 3.1.5 of [12] that every $\mathcal{A}_N$-comodule is projective. Furthermore, one checks that the surjection $\hat{\pi} : \mathcal{A}_G \to \mathcal{A}_N$ is a morphism of $\mathcal{A}_N$-comodules. Hence $\hat{\pi}$ has a comodule splitting $s : \mathcal{A}_N \to \mathcal{A}_G$ with $\hat{\pi}s = id_{\mathcal{A}_N}$.

Let $x \in \mathcal{A}_{G/N}^{+}$. By remark (a) following Definition 2.2, $\hat{\pi}(x) = 0$. Hence $\mathcal{A}_{G/N}^{+} \subset \ker(\hat{\pi})$ and therefore $\mathcal{A}_{G/N}^{+}\mathcal{A}_G \subset \ker(\hat{\pi})$.

Define a linear map ϕ on $\mathcal{A}_G$ by $\phi = (s\hat{\pi}) * S = m(s\hat{\pi} \otimes S)\Delta_G$, where m and S are respectively the multiplication map and antipode of $\mathcal{A}_G$. Then using the coassociativity of Δ_G and $\hat{\pi} s = id_{\mathcal{A}_N}$ along with the antipodal property of S, one verifies that $\phi(\mathcal{A}_G) \subset \mathcal{A}_{G/N}$. Since $\ker(\hat{\pi}) \subset Im(id - s\hat{\pi})$, to show $\ker(\hat{\pi}) \subset \mathcal{A}^+_{G/N}\mathcal{A}_G$, it suffices to show that $Im(id - s\hat{\pi}) \subset \mathcal{A}^+_{G/N}\mathcal{A}_G$. Since $(\epsilon - id)\phi(\mathcal{A}_G) \subset \mathcal{A}^+_{G/N}$, the later follows from the identity

$$id - s\hat{\pi} = (\epsilon - id)\phi * id = m((\epsilon - id)\phi \otimes id)\Delta_G,$$

which one verifies using basic properties of the convolution product along with $\epsilon\phi = \epsilon$ and the splitting property of s. This proves Lemma 5.3.

Now we finish the proof of Theorem 5.1. If $A_{G/N_1} = \mathbb{C}1$, we would have $\mathcal{A}_{G/N_1} = \mathbb{C}1$ and $\mathcal{A}^+_{G/N_1} = 0$. Let $\hat{\pi}_1$ be the morphism of Hopf algebras from $\mathcal{A}_G$ to $\mathcal{A}_{N_1}$ associated with π_1. Then by Lemma 5.3,

$$\ker(\hat{\pi}_1) = \mathcal{A}^+_{G/N_1}\mathcal{A}_G = 0.$$

Since $\ker(\hat{\pi}_1)$ is dense in $\ker(\pi_1)$ by Lemma 5.2, we would have $\ker(\pi_1) = 0$. This contradicts the assumption that N_1 is a non-trivial quantum subgroup of G and therefore $A_{G/N_1} \neq \mathbb{C}1$. Then A_{G/N_1} has to be the only non-trivial Woronowicz C^*-subalgebra of $B_u(Q)$, i.e. $A_{G/N_1} = C^*(\{r_{2m} | m \geq 0\})$ as noted near the beginning of the proof of Theorem 5.1. We infer from Proposition 2.1 that $\pi_1(r_{2m})$ is a multiple of the trivial representation of N_1 for any m. From

$$\pi_1(r_1) \otimes \pi_1(r_1) = \pi_1(r_0) \oplus \pi_1(r_2),$$

we see that $\pi_1(r_1) \otimes \pi_1(r_1)$ is a multiple of the trivial representation of N_1. That is

$$\sum_{ijkl} e_{ij} \otimes e_{kl} \otimes \tilde{u}_{ij}\tilde{u}_{kl} = I_n \otimes I_n \otimes 1,$$

where $\tilde{u}_{ij}$ are the (i,j)-entries of $\pi_1(r_1)$ and e_{ij} are matrix units. Hence

$$\tilde{u}_{ij}\tilde{u}_{kl} = 0, \text{ when } i \neq j, \text{ or } k \neq l;$$

$$\tilde{u}_{ii}\tilde{u}_{ll} = 1, \text{ for all } i, l.$$

Therefore $\tilde{u}_{ij} = 0$ for $i \neq j$ and A_{N_1} is commutative. That is, N_1 is an ordinary compact group. Now it is clear that $\tilde{u}_{ii} = \tilde{u}_{ll} = \tilde{u}_{ll}^{-1}$ for all i, l, which we denote by a. Since A_{N_1} is generated by a and N_1 is non-trivial, we conclude that N_1 is a group of order 2. The map α from A_N to A_{N_1} defined by $\alpha(v_{ij}) = \tilde{u}_{ij}$ is clearly an isomorphism such that $\pi_1 = \alpha\pi$. Hence $(N_1, \pi_1) = (N, \pi)$ by Definition 2.7.

For an example of a non-normal quantum subgroup (H, θ) of $B_u(Q)$, take a two-elements group $H = \{I_n, V\}$, where

$$V = \begin{bmatrix} -1 & 0 \\ 0 & I_{n-1} \end{bmatrix},$$

and $\theta(u_{ij}) = w_{ij}$, the coordinate functions on H. $\square$

Let us also recall the construction of the quantum groups $A_u(Q)$ closely related to $B_u(Q)$ [40, 38, 44]. For every non-singular matrix Q, the quantum group $A_u(Q)$ is defined in terms of generators u_{ij} $(i, j = 1, \cdots n)$, and relations:

$$u^*u = I_n = uu^*, \quad u^tQ\bar{u}Q^{-1} = I_n = Q\bar{u}Q^{-1}u^t.$$

According to Banica [3], when $Q > 0$, the irreducible representations of the quantum group $A_u(Q)$ are parameterized by the free monoid $\mathbb{N} * \mathbb{N}$ with generators α and β and anti-multiplicative involution $\bar{\alpha} = \beta$ (the neutral element is e with $\bar{e} = e$). The classes of u and $\bar{u}$ are r_α and r_β respectively. Moreover, for each pair of irreducible representations r_x and r_y ($x, y \in \mathbb{N} * \mathbb{N}$), one has the following direct sum decomposition (fusion) rules:

$$r_x \otimes r_y = \sum_{x = ag, \bar{g}b = y} r_{ab}.$$

In [48], the special $A_u(Q)$'s with $Q > 0$ are classified up to isomorphism and the general $A_u(Q)$'s with arbitrary Q are shown to be free products of the special $A_u(Q)$'s. However, the following problem is open.

Problem 5.4. *Determine whether the quantum groups $A_u(Q)$ are simple for positive $Q \in GL(n, \mathbb{C})$ and $n \geq 2$.*

To prove the simplicity of $A_u(Q)$, one would only need to show that $A_u(Q)$ has no non-trivial connected normal quantum subgroups, since the other conditions are easily verified. If this is indeed true, $A_u(Q)$ will provide the first examples of simple compact quantum groups that are not almost classical.

Now we consider the quantum automorphism group $A_{aut}(B, tr)$ of a finite dimensional C^*-algebra B endowed with a tracial functional tr (cf. [4, 45]). This quantum group is defined to be the universal object in the category of compact quantum transformation groups of B that leave tr invariant. Note that the presence of a tracial functional tr is necessary for the existence of the universal object when B is non-commutative (see Theorem 6.1 of [45]). For an arbitrary finite dimensional C^*-algebra B, the C^*-algebra $A_{aut}(B, tr)$ is described explicitly in [45] in terms of generators and relations. When $B = C(X_n)$ is the commutative C^*-algebra of functions on the space X_n of n points, the quantum automorphism group $A_{aut}(B) = A_{aut}(X_n)$ (also called the **quantum permutation group** on n letters) exists without the presence of a (tracial) functional and its description in terms of generators and relations is surprisingly simple. The C^*-algebra $A_{aut}(X_n)$ is generated by self-adjoint projections a_{ij} such that each row and column of the matrix $(a_{ij})_{i,j=1}^n$ add up to 1. That is,

$$a_{ij}^2 = a_{ij} = a_{ij}^*, \quad i, j = 1, \cdots, n,$$

$$\sum_{j=1}^n a_{ij} = 1, \quad i = 1, \cdots, n,$$

$$\sum_{i=1}^n a_{ij} = 1, \quad j = 1, \cdots, n.$$

For more general finite dimensional C^*-algebras B, the description of $A_{aut}(B, tr)$ in terms of generators and relations is more complicated. We refer the reader to [45] for detailed description.

Assume tr is the canonical trace τ on B (see §1 or p772 of [4] for the definition). Then $A_{aut}(B, \tau)$ is an ordinary permutation group when the dimension of B is less or equal to 3. However, when the dimension of B is greater or equal to 4, $A_{aut}(B, \tau)$ is a non-trivial (noncommutative and noncocommutative) compact quantum group with an infinite dimensional function algebra [45, 46]. Moreover, the representation ring of $A_{aut}(B,\tau)$ is isomorphic to that of $SO(3)$ [4]. Hence $A_{aut}(B,\tau)$ is almost classical for all B. It is easy to see that for $B = C(X_n)$, the canonical trace τ is equal to the unique S_n-invariant state on B, where S_n acts on X_n by permutation. Hence by remark (2) following Theorem 3.1 of [45], $A_{aut}(B,\tau)$ is the same as the quantum permutation group $A_{aut}(X_n)$.

We refer the reader to [4, 45, 46] for more on these quantum groups and [7, 8] for related results. Note that the description in [4] is not exactly as that in [45] but equivalent to it. We now prove

Theorem 5.5. *Let B be a finite dimensional C^*-algebra with* $\dim(B) \geq 4$. *Then $A_{aut}(B,\tau)$ is an absolutely simple compact quantum group with property F.*

Proof. The argument is similar to that of Theorem 5.1. First we claim that there are only two Woronowicz C^*-subalgebras in $A_{aut}(B,\tau)$, namely $\mathbb{C}1$ and $A_{aut}(B,\tau)$.

By Banica [4], the complete set of mutually inequivalent irreducible representations of the quantum group $A_{aut}(B,\tau)$ can be parametrized by r_k ($k \geq 0$, r_0 being the trivial one dimensional representation). Under this parametrization the fusion rules of the irreducible representations are the same as those of $SO(3)$:

$$r_k \otimes r_l = r_{|k-l|} \oplus r_{|k-l|+1} \oplus \cdots \oplus r_{k+l-1} \oplus r_{k+l}, \quad k,l \geq 0.$$

Let $A_1 \neq \mathbb{C}1$ be a Woronowicz C^*-subalgebra of $A_{aut}(B,\tau)$. Let v be a non-trivial irreducible representation of the compact quantum group of A_1. Then $v = r_k$ for some $k \neq 0$ and

$$r_k \otimes r_k = r_0 \oplus r_1 \oplus r_2 \oplus \cdots \oplus r_{2k-1} \oplus r_{2k}, \quad k,l \geq 0.$$

Hence the coefficients of each of the representations r_1, r_2, ..., r_{2k} are in A_1. Similarly, from the decomposition of $r_{2k} \otimes r_{2k}$, we see that the coefficients of each of the representations r_1, r_2, ..., r_{4k} are in A_1. Inductively, the coefficients of each of the representations r_1, r_2, ..., $r_{2^m k}$ are in A_1 ($m > 0$). Hence $A_1 = A_{aut}(B,\tau)$.

Let (π, N) be a normal quantum subgroup of $G = A_{aut}(B,\tau)$ different from the trivial one-element subgroup. Then there is a non-trivial irreducible representation $u^\lambda = (u^\lambda_{ij})$ such that $\pi(u^\lambda)$ is not a multiple of the trivial representation. Using the same argument as in the proof of Theorem 5.1 we have $A_{G/N} = E_{G/N}(A_G) \neq A_G$. Therefore we must have $A_{G/N} = \mathbb{C}1$. Then the argument near the end of the proof of Theorem 5.1 (i.e. the paragraph that follows the proof of Lemma 5.3) shows that $\ker(\pi) = 0$. That is, N is the same quantum group as G. $\square$

Theorem 5.5 applies in particular to quantum permutation groups $A_{aut}(X_n)$ when $n \geq 4$. As Manin pointed out (private communication), the reason that these quantum groups are connected could be that there are so many more quantum symmetries that the originally $n!$ isolated permutations are connected together by them. Note however that their function algebras are generated by orthogonal projections a_{ij}, so these quantum groups are also disconnected, as observed by Bichon [8] recently. It would be interesting to find a satisfactory explanation of this quantum paradox.

The proofs of the main results of this section does not need explicit description (models) of representations of the quantum groups $B_u(Q)$ and $A_{aut}(B,\tau)$. Only the structures of their representation rings (fusion rules) are used. However, explicit constructions of models of irreducible representations of Lie groups are more interesting and have more applications than the structures of their representation rings. In view of this, we believe an answer to the following problem will be even more useful in the theory of quantum groups. (Note that the model for the fundamental representation of the quantum group $A_u(Q)$ is used in [46] to construct ergodic actions on various von Neumann factors.)

Problem 5.6. *Construct explicit models of the irreducible representations of the following quantum groups: $A_u(Q)$ for $Q > 0$; $B_u(Q)$ for $Q\bar{Q} = \pm I_n$; the quantum automorphism group $A_{aut}(B, \tau)$ of a finite dimensional C^*-algebra B endowed with the canonical trace τ.*

6. Simplicity of K_q, K_q^u and K_J

The compact real forms K_q of Drinfeld-Jimbo quantum groups and their twists K_q^u are studied in [33] and [21] respectively. See also [22] for a summary of [33, 21] and [20] for more detailed treatment. Motivated by these works, Rieffel constructs in [28] a deformation K_J of compact Lie group K which contains a torus T and raises the question whether K_q^u can be obtained as a *strict deformation quantization* of K_q. This question is answered in the affirmative by the author in [43]. The purpose of this section is to show that the quantum groups K_q, K_q^u and K_J are simple in the sense of this paper, provided that the compact Lie group K is simple.

We first recall the notation of [33, 21, 22]. Let G be a connected and simply connected simple complex Lie group with Lie algebra $\mathfrak{g}$. Fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, together with the corresponding decomposition $\Delta = \Delta_+ \cup \Delta_-$ of the root system and a fixed basis $\{\alpha_i\}_{i=1}^n$ for Δ_+. For each linear functional λ on $\mathfrak{h}$, H_λ denotes the element in $\mathfrak{h}$ corresponding to λ under the isomorphism $\mathfrak{h} \cong \mathfrak{h}^*$ determined by the Killing form $(\ ,\ )$ on $\mathfrak{g}$. Note that if the reader keeps the context in mind, the symbols α and λ used in this context should not cause confusion with the same symbols used in this paper for other purposes. Let $\{X_\alpha\}_{\alpha \in \Delta} \cup \{H_i\}_{i=1}^n$ be a Weyl basis of $\mathfrak{g}$, where $H_i = H_{\alpha_i}$. This determines a Cartan involution ω_0 on $\mathfrak{g}$ with $\omega_0(X_\alpha) = -X_{-\alpha}$, $\omega_0(H_i) = -H_i$. Let $\mathfrak{k}$ be the compact real form of $\mathfrak{g}$ defined as the fixed points of ω_0 and K the associated compact real form of G. Put $\mathfrak{h}_\mathbb{R} = \oplus_{i=1}^n \mathbb{R} H_i$, $\mathfrak{t} = i\mathfrak{h}_\mathbb{R}$ and $T = \exp(\mathfrak{t})$, the later being the associated maximal torus of K.

Let $q = e^{h/4}$ ($h \in \mathbb{R}\backslash\{0\}$). For $n, k \in \mathbb{N}$, $n \geq k$, define

$$\begin{aligned} [n]_q &= \frac{q^n - q^{-n}}{q - q^{-1}}, \\ \begin{bmatrix} n \\ k \end{bmatrix}_q &= \frac{[n]_q [n-1]_q \ldots [n-k+1]_q}{[k]_q [k-1]_q \ldots [1]_q}. \end{aligned}$$

The quantized universal enveloping algebra $U_q(\mathfrak{g})$ [14, 18] is the complex associative algebra with generators $X_i^{\pm}$, $K_i^{\pm 1}$ $(i = 1, \cdots, n)$ and defining relations:

$$\begin{aligned} &K_i K_i^{-1} = 1 = K_i^{-1} K_i, \quad K_i K_j = K_j K_i, \\ &K_i X_j^{\pm} K_i^{-1} = q^{\pm(\alpha_i, \alpha_j)} X_j^{\pm}, \\ &[X_i^+, X_j^-] = \delta_{ij} \frac{K_i^2 - K_i^{-2}}{q - q^{-1}}, \\ &\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} (X_i^{\pm})^k X_j^{\pm} (X_i^{\pm})^{1-a_{ij}-k} = 0 \ , \ i \neq j, \end{aligned}$$

where $q_i = q^{(\alpha_i, \alpha_i)}$.

On $U_q(\mathfrak{g})$ there is a Hopf algebra structure with coproduct

$$\Delta(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\pm 1}, \quad \Delta(X_i^{\pm}) = X_i^{\pm} \otimes K_i + K_i^{-1} \otimes X_i^{\pm},$$

and counit and antipode respectively

$$\varepsilon(X_i^{\pm}) = 0, \quad \varepsilon(K_i^{\pm 1}) = 1, \quad S(X_i^{\pm}) = -q_i^{\pm 1} X_i^{\pm}, \quad S(K_i^{\pm 1}) = K_i^{\mp 1}.$$

Under the *-structure defined by

$$(X_i^{\pm})^* = X_i^{\mp}, \quad K_i^* = K_i,$$

$U_q(\mathfrak{g})$ is a Hopf *-algebra.

Let $u = \sum_{k,l} c_{kl} H_k \otimes H_l \in \wedge^2 \mathfrak{h}_{\mathbb{R}}$. Then it can be shown (cf. [20]) that the following defines a new coproduct on $U_q(\mathfrak{g})$,

$$\Delta_u(\xi) = \exp(-ihu/2) \Delta(\xi) \exp(ihu/2),$$

where $X \in U_q(\mathfrak{g})$ and Δ is the original coproduct on $U_q(\mathfrak{g})$. The new Hopf *-algebra so obtained is denoted by $U_{q,u}(\mathfrak{g})$.

The function algebra $\mathcal{A}_{K_q}$ of the compact quantum group K_q is defined to be the subalgebra of the dual algebra $U_q(\mathfrak{g})^*$ consisting of matrix elements of finite dimensional representations ρ of $U_q(\mathfrak{g})$ such that eigenvalues of the endomorphisms $\rho(K_i)$ are positive. The function algebra $\mathcal{A}_{K_q^u}$ of the compact quantum group K_q^u is defined to be the subalgebra of the dual algebra $U_{q,u}(\mathfrak{g})^*$ that has the same elements as $\mathcal{A}_{K_q}$, as well as the same $*$-structure, while the product of its elements is defined using Δ_u instead of Δ.

For each (algebraically) dominant integral weight $\lambda \in P_+$ of $(\mathfrak{g}, \mathfrak{h})$, define matrix elements $C^{\lambda}_{\mu,i;\nu,j}$ of the highest weight $U_q(\mathfrak{g})$ module $(L(\lambda), \rho_\lambda)$ as follows. Let $\{v_\nu^{(i)}\}$ be an orthonormal weight basis for the unitary $U_q(\mathfrak{g})$ module $L(\lambda)$. Then $C^{\lambda}_{\mu,i;\nu,j}$ is defined by

$$C^{\lambda}_{\mu,i;\nu,j}(X) = < \rho_\lambda(X) v_\nu^{(j)}, v_\mu^{(i)} >,$$

where $X \in U_q(\mathfrak{g})$ and $< \, , \, >$ is the inner product on $L(\lambda)$. The $C^{\lambda}_{\mu,i;\nu,j}$'s is a linear (Peter-Weyl) basis of both $\mathcal{A}_{K_q}$ and $\mathcal{A}_{K_q^u}$ when λ ranges through the set P_+ of dominant integral weights of $(\mathfrak{g}, \mathfrak{h})$. By [23, 30] it is clear from these definitions that K_q and K_q^u are almost classical quantum groups in the sense of Definition 4.6.

Theorem 6.1. *Let K be a connected and simply connected simple compact Lie group. Then K_q is a simple compact quantum group with property F.*

Proof. First we recall that representations of K and K_q are in one to one correspondence via deformation and the decompositions of tensor products of irreducible representations are not altered under deformation (see Lusztig [23] and Rosso [30] or Chari-Pressley [9]). Let ξ be the map that associates each irreducible representation v of K an irreducible representation $\xi(v)$ of K_q in this correspondence. This map defines an isomorphism of *vector spaces* from $\mathcal{A}_K$ to $\mathcal{A}_{K_q}$, which we also denote by ξ. It follows from this that K_q is connected and has no non-trivial representations of dimension one. Comparing decompositions of tensor products of representations of K and K_q we see that the ξ maps bijectively the set of Hopf subalgebras of $\mathcal{A}_K$ onto the set of Hopf subalgebras of $\mathcal{A}_{K_q}$.

Let ρ_q be the quotient morphism from A_{K_q} to the abelianization $A^{ab}_{K_q}$, which is by definition the quotient of A_{K_q} by the closed two sided ideal of A_{K_q} generated by commutators $[a,b]$, $a, b \in A_{K_q}$. According to [40], $A^{ab}_{K_q}$ is the algebra of continuous functions on the maximal compact subgroup $\hat{A}_{K_q}$ of K_q and ρ_q gives rise to the embedding of the quantum groups from $\hat{A}_{K_q}$ to K_q. It is shown in [33] that the maximal compact subgroup $\hat{A}_{K_q}$ is isomorphic to the maximal torus T of K.

The associated morphism $\hat{\rho}_q$ from $\mathcal{A}_{K_q}$ to $\mathcal{A}_T$ is given by

$$\hat{\rho}_q(C^{\lambda}_{\mu,i;\nu,j})(t) = \delta_{ij}\delta_{\mu\nu}e^{2\pi\mu(x)},$$

where $t = exp(x) \in T$, $x \in \mathfrak{t} = i\mathfrak{h}_{\mathbb{R}}$ (see p438 of [9], but $\sqrt{-1}$ should not appear in the formula there). It is clear that one has the same formula as above for restriction morphism ρ from A_K to A_T:

$$\hat{\rho}(\xi^{-1}(C^{\lambda}_{\mu,i;\nu,j}))(t) = \delta_{ij}\delta_{\mu\nu}e^{2\pi\mu(x)}, \;\; i.e., \;\; \hat{\rho} = \hat{\rho}_q \circ \xi.$$

Let $N \subset K$ be a normal subgroup of K with surjections $\pi : A_K \to A_N$ and $\hat{\pi} : \mathcal{A}_K \to \mathcal{A}_N$. Then N is a finite subgroup of T and $A_N = \mathcal{A}_N$ is a finite dimensional Hopf algebra. It is clear that $\pi = \rho_N \circ \rho$, where ρ_N is the restriction morphism from A_T to A_N. Define

$$\pi_q : A_{K_q} \longrightarrow A_N, \;\; \text{by} \;\; \pi_q := \rho_N \circ \rho_q.$$

We claim that (N, π_q) is a normal subgroup of K_q. This follows immediately from the following identities, which one can easily verify using $\hat{\rho} = \hat{\rho}_q \circ \xi$ and $\hat{\pi} = \hat{\pi}_q \circ \xi$.

$$\mathcal{A}_{K_q/N} = \xi(\mathcal{A}_{K/N}), \;\; i.e.,$$
$$\{a \in \mathcal{A}_{K_q} | (id \otimes \pi_q)\Delta(a) = a \otimes 1\} = \xi(\{a \in \mathcal{A}_K | (id \otimes \pi)\Delta(a) = a \otimes 1\});$$
$$\mathcal{A}_{N\backslash K_q} = \xi(\mathcal{A}_{N\backslash K}), \;\; i.e.,$$
$$\{a \in \mathcal{A}_{K_q} | (\pi_q \otimes id)\Delta(a) = 1 \otimes a\} = \xi(\{a \in \mathcal{A}_K | (\pi \otimes id)\Delta(a) = 1 \otimes a\}).$$

That is, every normal subgroup N of K gives rise to a normal subgroup (N, π_q) of K_q in the manner above.

Conversely, let (N', π') be a quantum normal subgroup of K_q. Then $\mathcal{A}_{K_q/N'}$ is a Hopf subalgebra of $\mathcal{A}_{K_q}$. Since every Hopf subalgebra of $\mathcal{A}_K$ is of the form $\mathcal{A}_{K/N}$ for some normal subgroup N of K (cf. [40]), by the correspondence between Hopf subalgebras of $\mathcal{A}_K$ and those of $\mathcal{A}_{K_q}$ noted near the beginning of the proof we have

$$\mathcal{A}_{K_q/N'} = \xi(\mathcal{A}_{K/N}) = \mathcal{A}_{K_q/N}$$

for some normal subgroup N of K. By Lemma 5.3, we have $\ker(\hat{\pi}) = \ker(\hat{\pi}')$. That is (N', π') and (N, π_q) are the same quantum subgroup of K_q (cf. Definition 2.7

and Lemma 5.2). Since normal subgroups N of K are finite, we conclude from the above that K_q has no non-trivial connected quantum normal subgroups. $\square$

Examining the proof of Theorem 6.1, we formulate the following general result on the invariance of simplicity of compact quantum groups under deformation, which will be used to prove the simplicity of K_q^u and K_J.

Let G be a simple compact quantum group with property F and (H, ρ) a quantum subgroup. Assume all normal quantum subgroups of G are quantum subgroups of H. Let G_v be a family of compact quantum groups ("deformation" of G) indexed by a subset $\{v\}$ of a vector space that includes the origin. Suppose the family G_v satisfy the following conditions:

C1. $G_0 = G$.

C2. There is an isomorphism ξ of vector spaces from $\mathcal{A}_G$ to $\mathcal{A}_{G_v}$.

C3. The coproduct is unchanged under deformation, i.e.,

$$\Delta_v(\xi(a)) = (\xi \otimes \xi)\Delta(a) \text{ for } a \in \mathcal{A}_G.$$

C4. For any pair irreducible representations u^{λ_1} and u^{λ_2}, if

$$u^{\lambda_1} \otimes u^{\lambda_2} \cong u^{\gamma_1} \oplus u^{\gamma_2} \oplus \cdots \oplus u^{\gamma_l}$$

is a decomposition of $u^{\lambda_1} \otimes u^{\lambda_2}$ into direct sum of irreducible subrepresentations u^{γ_j} $(j = 1, 2, \cdots, l)$, then

$$\xi(u^{\lambda_1}) \otimes \xi(u^{\lambda_2}) \cong \xi(u^{\gamma_1}) \oplus \xi(u^{\gamma_2}) \oplus \cdots \oplus \xi(u^{\gamma_l})$$

is a decomposition of $\xi(u^{\lambda_1}) \otimes \xi(u^{\lambda_2})$ into direct sum of irreducible representations, where for instance $\xi(u^{\lambda_1})$ denotes the representation of G_v whose coefficients are images of coefficients of u^{λ_1}.

C5. The quantum subgroup H is undeformed. The latter means that there is a morphism ρ_v of quantum groups from H to G_v such that

$$\rho_v(\xi(a)) = \rho(a) \text{ for } a \in \mathcal{A}_G.$$

Under the assumptions above, we have the following result. The proof is the same as that of Theorem 6.1 (H corresponds to T in Theorem 6.1).

Theorem 6.2. *The compact quantum groups G_v are simple with property F.*

Remarks. (a) Condition C4 above is not the same as the requirement that

$$\xi(u^{\lambda_1} \otimes u^{\lambda_2}) = \xi(u^{\lambda_1}) \otimes \xi(u^{\lambda_2}).$$

The latter requirement together with conditions (2) and (3) imply that ξ is an isomorphism of quantum from G_v to G, which is not the case for the quantum groups under consideration here.

(b) We believe similar results on invariance of simplicity under deformation hold true without the property F assumption on G. But at the moment we do not know of any simple compact quantum groups that do not satisfy this property, though there many non-simple quantum groups without this property.

Next we recall the construction in [28, 43]. Let $A = A_G$ be a compact quantum group with coproduct Δ. Suppose that the quantum group G has a toral subgroup (T, ρ)–to obtain non-trivial deformation we assume that T has rank no less than 2. For any element t in T, denote by E_t the corresponding evaluation functional on $C(T)$. Assume that η is a continuous homomorphism from a vector space Lie group $\mathbb{R}^n$ to T, where n is allowed to be different from the dimension of T. Define an action α of $\mathbb{R}^d := \mathbb{R}^n \times \mathbb{R}^n$ on the C^*-algebra A as follows:

$$\alpha_{(s,v)} = l_{\eta(s)} r_{\eta(v)},$$

where

$$l_{\eta(s)} = (E_{\eta(-s)}\rho \otimes id)\Delta, \quad r_{\eta(v)} = (id \otimes E_{\eta(v)}\rho)\Delta.$$

For any skew-symmetric operator S on $\mathbb{R}^n$, one may apply Rieffel's quantization procedure [27] for the action α above to obtain a deformed C^*-algebra A_J whose product is denoted $\times_J$, where $J = S \oplus (-S)$. The family A_{hJ} ($h \in \mathbb{R}$) is a strict deformation quantization of A (see Chapter 9 of [27]). In [43] the following result is obtained.

Theorem 6.3. *The deformation A_J is a compact quantum group containing T as a (quantum) subgroup; A_J is a compact matrix quantum group if and only if A is.*

We denote by G_J the quantum group for A_J. When G is a compact Lie group, the construction G_J above is the same as Rieffel's construction [28]. By 5.2 of [28], G_J is an almost classical compact quantum group if G is a compact Lie group.

Combining Theorem 6.3 and Theorem 6.2, we obtain

Theorem 6.4. *Let K be a simple compact Lie group with a toral subgroup T. Then the compact quantum group K_J of Rieffel* [28] *is simple with property F.*

We note that unlike in Theorem 6.1, in the result above we do not need to assume K to be simply connected. This is because $\mathcal{A}_{K_q}$ is defined using irreducible representations of $U_q(\mathfrak{g})$ associated with all dominant integral weights P_+ of $(\mathfrak{g}, \mathfrak{h})$, so that $\mathcal{A}_{K_q}$ becomes the algebra of representative functions on a simply connected K when $q \to 1$. One could also start with a non-simply connected K in Theorem 6.1 too, but then one needs to modify the definition of the quantum algebra $\mathcal{A}_{K_q}$ by using irreducible representations of $U_q(\mathfrak{g})$ associated with analytically dominant integral weights only. This newly defined $\mathcal{A}_{K_q}$ is a Hopf subalgebra of the Hopf algebra defined originally. It is clear from the proof of Theorem 6.1 that its conclusion remains valid for this newly defined K_q.

Finally we consider K_q^u. To avoid confusion with the Killing form, we now use $s \oplus v$, instead of (s, v) as is used above, to denote an element of $\mathbb{R}^d = \mathbb{R}^n \times \mathbb{R}^n$. In the present setting, the space $\mathbb{R}^n$ is $\mathfrak{h}_{\mathbb{R}}$, with inner product $< , >= (,)$, where $(,)$ is the Killing form of $\mathfrak{g}$ restricted to $\mathfrak{h}_{\mathbb{R}}$. We will also use $< , >$ to denote the inner product on $\mathfrak{h}_{\mathbb{R}} \oplus \mathfrak{h}_{\mathbb{R}}$. Noting that the compact abelian group T is also a subgroup of both K_q and K_q^u (see [33, 21]). The map η there in this case is defined by $\eta(s) = \exp(2\pi i s)$. We can define as above an action of $\mathbb{R}^d$ on A_{K_q} by

$$\alpha_{s \oplus v} = l_{\exp(-2\pi i s)} r_{\exp(2\pi i v)}.$$

This action may be viewed as an action of $H = T \times T$ in the sense of [27]. For each ν in the weight lattice P of $\mathfrak{g}$, the element H_ν is in $\mathfrak{h}_{\mathbb{R}}$. We use the notation $H_\nu \oplus H_\mu$ to denote $H_\nu + H_\mu$ as an element of $\mathfrak{h}_{\mathbb{R}} \oplus \mathfrak{h}_{\mathbb{R}}$. Keep the notation of [27] for the spectral subspaces of the action α (see 2.22 there).

Let $\check{u}$ be the map on $\mathfrak{h}^*$ determined by u via the Killing form $(,)$ on $\mathfrak{g}$. Let

$$p = -(H_{\nu_1} \oplus H_{\mu_1}), \quad q = -(H_{\nu_2} \oplus H_{\mu_2}),$$
$$J = \frac{h}{4\pi}(S_u \oplus (-S_u)),$$

where S_u is the skew-symmetric operator on $\mathfrak{h}_{\mathbb{R}}$ defined by

$$S_u(H_\nu) = \sum_{k,l} c_{kl}\nu(H_k)H_l.$$

Then one has

$$\begin{aligned}
&C^{\lambda_1}_{\mu_1,i_1;\nu_1,j_1} \circ C^{\lambda_2}_{\mu_2,i_2;\nu_2,j_2} \\
&= \exp(\frac{ih}{2}((\mu_1, \check{u}\mu_2) - (\nu_1, \check{u}\nu_2)))C^{\lambda_1}_{\mu_1,i_1;\nu_1,j_1} C^{\lambda_2}_{\mu_2,i_2;\nu_2,j_2} \\
&= \exp(-2\pi i < p, Jq >)C^{\lambda_1}_{\mu_1,i_1;\nu_1,j_1} C^{\lambda_2}_{\mu_2,i_2;\nu_2,j_2}
\end{aligned}$$

where $\circ$ on the left-hand side is the multiplication in $A_{K^u_q}$ and the right-hand side is the multiplication in A_{K_q}.

On the other hand one has from 2.22 of [27] that

$$C^{\lambda_1}_{\mu_1,i_1;\nu_1,j_1} \times_J C^{\lambda_2}_{\mu_2,i_2;\nu_2,j_2} = \exp(-2\pi i < p, Jq >)C^{\lambda_1}_{\mu_1,i_1;\nu_1,j_1} C^{\lambda_2}_{\mu_2,i_2;\nu_2,j_2}$$

This means that we have the following result [43].

Theorem 6.5. *The Hopf *-algebras $\mathcal{A}_{K^u_q}$ and $(\mathcal{A}_{K_q}, \times_J)$ are isomorphic.*

That is $K^u_q = (K_q)_J$ in the notation of Theorem 6.3, answering Rieffel's question [28] in the affirmative.

Combining Theorem 6.5, Theorem 6.1 and Theorem 6.2, we obtain the following

Theorem 6.6. *Let K be a connected and simply connected simple compact Lie group. Then the compact quantum group K^u_q is simple with property F.*

Recently, Connes and Dubois-Violette [11] found the so called θ-deformation $SO_\theta(n)$ of $SO(n)$. It would be interesting to know if these new quantum groups are simple compact quantum groups also.

Acknowledgments. The research reported here is principally supported by the National Science Foundation grant DMS-0096136. During the writing of this work, the author also received support from the Research Foundation of the University of Georgia in the form of a Faculty Research Grant in the summer of 2000 and the Max Planck Institute of Mathematics at Bonn in the form of a visiting membership for 6 weeks in the summer of 2002. The author is grateful to these agencies and institutions for generous support.

References

[1] Andruskiewitsch, N., Devoto, J.: Extensions of Hopf algebras, *Algebra i Analiz*, **7**:1, 22–61 (1995). translation in *St. Petersburg Math. J.* **7**:1, 17–52 (1996).

[2] Banica, T.: Théorie des représentations du groupe quantique compact libre $O(n)$, *C. R. Acad. Sci. Paris* t. **322**, Serie I, 241-244 (1996).

[3] Banica, T.: Le groupe quantique compact libre $U(n)$, *Commun. Math. Phys.* **190**, 143-172 (1997).

[4] Banica, T.: Symmetries of a generic coaction, math.QA/9811060, *Math. Ann.* **314**, 763-780 (1999).

[5] Banica, T.: Fusion rules for representations of compact quantum groups, *Exposition Math.* **17**, 313-337 (1999).

[6] Banica, T.: Compact Kac algebras and commuting squares, *J. Funct. Anal.* **176**, no. 1, 80–99 (2000).

[7] Bichon, J.: Quantum automorphism groups of finite graphs. Prepublication 22, 1998, Universite Montpellier II. math.QA/9902029

[8] Bichon, J.: Free wreath product by the quantum permutation group. math.QA/0107029

[9] Chari, V. and Pressley, A.: *A Guide to Quantum Groups,* Cambridge University Press, 1994.

[10] Childs, Lindsay N.: *Taming Wild Extensions: Hopf Algebras and Local Galois Module Theory,* American Mathematical Society, 2000.

[11] Connes, A. and Dubois-Violette, M.: Noncommutative finite-dimensional manifolds. I. Spherical manifolds and related examples. 52 pages. math.QA/0107070; LPT-ORSAY 01-01; IHES/M/01/32.

[12] Dăscălescu, S. and Năstăsescu, C. and Raianu, Ş. *Hopf Algebras*, Marcel Dekker, Inc., New York, Basel, 2001.

[13] Dijkhuizen, M. S. and Koornwinder, T.H.: CQG algebras, a direct algebraic approach to compact quantum groups, *Lett. Math. Phys.* **32**, 315-330 (1994).

[14] Drinfeld, V.G., Quantum groups, *in* Proc. of the ICM-1986, Berkeley, Vol I, Amer. Math. Soc., Providence, R.I., 1987, pp798–820.

[15] Effros, Edward G. and Ruan, Zhong-Jin: Discrete quantum groups. I. The Haar measure. *Internat. J. Math.* **5** no. 5, 681–723 (1994).

[16] Handelman, David: Representation ring as invariants for compact groups and limit ratio theorem for them. *Internat. J. Math.* **4**, 59-88 (1994).

[17] Hewitt, E. and Ross, K.: *Abstract Harmonic Analysis* II, Springer–Verlag, 1970.

[18] Jimbo, M., A q-difference analogue of $U\mathfrak{g}$ and the Yang-Baxter equations, *Lett. Math. Phys.* **10**, 63-69 (1985).

[19] Klimyk, A. U. and Schmüdgen, K.: *Quantum Groups and Their Representations*, Springer–Verlag, 1997.

[20] Korogodski, Leonid I. and Soibelman, Yan S., *Algebra of Functions on Compact Quantum Groups*, Part I. Mathematical Surveys and Monographs, 56. AMS, Providence, RI, 1998

[21] Levendorskii, S.: Twisted algebra of functions on compact quantum group and their representations, *Algebra i analiz* **3**:2, 180-198 (1991). *St. Petersburg Math. J.* **3**:2, 405-423 (1992).

[22] Levendorskii, S. and Soibelman, Y.: Algebra of functions on compact quantum groups, Schubert cells, and quantum tori, *Comm. Math. Phys.* **139**, 141-170 (1991).

[23] Lusztig, G.: Quantum deformations of certain simple modules over enveloping algebras, *Adv. in Math.* **70**, 237-249 (1988).

[24] Parshall, B. and Wang, J.: Quantum linear groups, *Memoirs AMS* **439**, 1991.

[25] Podles, P.: Symmetries of quantum spaces. Subgroups and quotient spaces of quantum $SU(2)$ and $SO(3)$ groups, *Commun. Math. Phys.* **170**, 1-20 (1995).

[26] Podles, P. and Woronowicz S. L.: Quantum deformation of Lorentz group, *Commun. Math. Phys.* **130**, 381-431 (1990).

[27] Rieffel, M.: Deformation quantization for actions of $\mathbf{R}^d$, *Memoirs A.M.S.* no. **506**, 1993.

[28] Rieffel, M.: Compact quantum groups associated with toral subgroups, *Contemp. Math.* **145**, 465-491 (1993).

[29] Rosso, M.: Comparaison des groupes $SU(2)$ quantiques de Drinfeld et Woronowicz, *C. R. Acad. Sci. Paris* **304**, 323-326 (1987).

[30] Rosso, M.: Finite dimensional representations of the quantum analog of the enveloping algebra of a complex semisimple Lie algebra, *Commun. Math. Phys.* **117**, 581-593 (1988).
[31] Rosso, M.: Algèbres enveloppantes quantifiées, groupes quantiques compacts de matrices et calcul differentiel non-commutatif, *Duke Math. J.* **61**, 11-40 (1990).
[32] Schneider, H.-J.: Some remarks on exact sequences of quantum groups. *Comm. Algebra* **21**:9, 3337–3357 (1993).
[33] Soibelman, Y., Algebra of functions on compact quantum group and its representations, *Algebra i analiz* **2**:1, 190-212 (1990). *Leningrad Math. J.* **2**:1, 161-178 (1991).
[34] Soibelman, Y. and Vaksman, L.: The algebra of functions on quantum $SU(2)$, *Funct. Anal. ego Pril.* **22**:3, 1-14 (1988). *Funct. Anal. Appl.* **22**:3, 170-181 (1988).
[35] Sweedler, M. E.: *Hopf Algebras,* Benjamin, New York, 1969.
[36] Takeuchi, Mitsuhiro: Quotient spaces for Hopf algebras. *Comm. Algebra* **22**:7, 2503–2523 (1994).
[37] Van Daele, A.: The Haar measure on a compact quantum group, *Proc. Amer. Math. Soc.* **123**, 3125-3128 (1995).
[38] Van Daele, A. and Wang, S. Z.: Universal quantum groups, *International J. Math.* **7**, 255-264 (1996).
[39] Wang, S. Z.: *General Constructions of Compact Quantum Groups,* Ph.D. Thesis, University of California at Berkeley, March, 1993.
[40] Wang, S. Z.: Free products of compact quantum groups, *Commun. Math. Phys.* **167**, 671-692 (1995).
[41] Wang, S. Z.: Tensor products and crossed products of compact quantum groups, *Proc. London Math. Soc.* **71**, 695-720 (1995).
[42] Wang, S. Z.: Krein duality for compact quantum groups, *J. Math. Phys.* **38**:1, 524-534 (1997).
[43] Wang, S. Z.: Deformations of compact quantum groups via Rieffel's quantization, *Commun. Math. Phys.* **178**:3, 747-764 (1996).
[44] Wang, S. Z.: New classes of compact quantum groups, Lecture notes for talks at the University of Amsterdam and the University of Warsaw, January and March, 1995.
[45] Wang, S. Z.: Quantum symmetry groups of finite spaces, math.OA/9807091, *Commun. Math. Phys.* **195**:1, 195-211 (1998).
[46] Wang, S. Z.: Ergodic actions of universal quantum groups on operator algebras, *Commun. Math. Phys.* **203**, 481-498 (1999).
[47] Wang, S. Z.: Quantum $ax+b$ group as quantum automorphism group of $k[x]$, *Communications in Algebra* **30**:4, 1807-1815 (2002).
[48] Wang, S. Z.: Structure and isomorphic classification of compact quantum groups $A_u(Q)$ and $B_u(Q)$, to appear in *J. Operator Theory*
[49] Wang, S. Z.: Equivalence of two notions of normal quantum subgroups and applications, preprint, 2002
[50] Wang, S. Z.: Simple compact quantum groups II, in preparation.
[51] Woronowicz, S. L.: Twisted $SU(2)$ group. An example of noncommutative differential calculus, *Publ. RIMS, Kyoto Univ.* **23**, 117-181 (1987).
[52] Woronowicz, S. L.: Compact matrix pseudogroups, *Commun. Math. Phys.* **111**, 613-665 (1987).
[53] Woronowicz, S. L.: Tannaka-Krein duality for compact matrix pseudogroups. Twisted $SU(N)$ groups, *Invent. Math.* **93**, 35-76 (1988).
[54] Woronowicz, S. L.: A remark on compact matrix quantum groups, *Lett. Math. Phys.* **21**, 35-39 (1991).
[55] Woronowicz, S. L.: Compact quantum groups, in *Quantum Symmetries*, Les Houches Summer School-1995, Session LXIV, A. Connes, K. Gawedzki, J. Zinn-Justin, ed., Elsevier Science, Amsterdam, 1998; pp845-884.

Department of Mathematics, University of Georgia, Athens, GA 30602
Fax: 706-542-2573
E-mail address: szwang@math.uga.edu