Schottky - Landau growth estimates for s-normal families of holomorphic mappings

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Introduction

Recall that a function f holomorphic in the unit disc Δ is called *normal* iff the family of functions $\{f \circ \alpha \mid \alpha \in \mathrm{Aut}\Delta\}$ is normal in Montel's sense. By Montel's Theorem [Mo] this is the case for f omitting two values; in addition, such a function f satisfies the Schottky — Landau growth estimate [Sch, La] :

$$| f(z) | \leq M e^{\frac{\sigma}{1-|z|}} \qquad ,$$

where $M = M(|f(0)|)$, $\sigma = \sigma(f)$. W. K. Hayman [Hay 1 (Theorem 6.8), 2] proved that the same inequality holds for functions in a normal Aut Δ—invariant family F (with $\sigma = \sigma(F)$). In particular, it is valid for any normal function.

Recall also two important criteria of normality. By the Nakano — Lehto —Virtanen metric criterium [Le Vi] a function f is normal iff the mapping $f: \Delta \to \mathbf{C} \subset \mathbf{P}^1$ has a bounded dilatation with respect to the spherical metric on $\mathbf{P}^1$ and Poincaré metric on Δ. This was generalized by W. K. Hayman [Hay 1–2] to normal invariant families of functions. Lange — Gavrilov's P-points criterium [Lan, Ga1] (also [Gau]) states that f is normal iff for any sequence $p = \{z_k\}$ in Δ there exist a subsequence p' of p and $\epsilon > 0$ such that the restriction of f to the union of non-euclidean discs of radius ϵ centered at the points of p' omits two values (in accordance to Bloch's Principle, this is parallel to Milloux's *cercles de remplissage* for entire functions [Mi]).

These facts where generalized in many directions; see, for instance, survey [Ca Wi] for the one-dimensional case and [Do], [Ci Kr], [Al], [Ha 1–2], [Gi Do], [Kr Ma], [Jo Kw], etc. for multidimensional generalizations. Here we give further ones. Following an idea of G. Aladro [Al] we introduced in [Za 2] a notion of *s-normal family of holomorphic mappings of complex spaces* (see Definition 1.1 below), in the non-homogeneous satting. In particular, normal invariant families of holomorphic functions (regarded as C—valued mappings) on hyperbolic homogeneous manifolds are s-normal. Members of s-normal families are called *normal mappings* (this actually coincides with the definition, given in [Ha 2], and turned out to be equivalent to the early known definitions in all particular cases). A metric criterium of s-normality holds in analogy to the Marty — Nakano — Lehto — Virtanen criterium (Theorem 1.6). In fact, for s-normal families one can prove a stronger inequality than the one given by this criterium (Theorem 1.9). This inequality means that dilatations (i.e.

dilatation coeffitients) of mappings in a given s-normal family are uniformly bounded with respect to Kobayashi pseudometric on the preimage and the Kobayashi — Royden — Green-metric (or KRG-metric) on the image (see Definition 1.7). The latter metric is complete in many cases. For instance, in the case of functions (dealing with as mappings into $\mathbf{C}$) it coincides with the Poincaré metric on a punctured disc in a neighborhood of ∞ in $\mathbf{C} \subset \mathbf{P}^1$. This leads to the following generalization of the Schottky — Landau — Hayman growth estimates (see also Corollary 4.15 below):

Theorem. *Let Ψ be an s-normal family of holomorphic functions on a complex manifold X. Then there exists a constant $c = c(\Psi) > 0$ such that for any subset $Q \subset X$ and for any function $\psi \in \Psi$, bounded in modulus on Q by a constant M, the following inequality holds:*

$$\log(c \mid \psi(x) \mid) \leq (\log(cM))\, e^{2k_X(x,Q)} \qquad \forall x \in X$$

(here k_X is the Kobayashi pseudodistance of X). Conversely, if a family Ψ of holomorphic functions on X satisfies the above condition, then it is s-normal.

The original Schottky — Landau — Hayman estimate corresponds to the case when $X = \Delta$ and $Q = \{0\}$. Varying Q one can even get new estimates for normal functions in the upper halfplane $\mathbf{C}_+$ (see Proposition 4.18). A simple consequence is that any normal function in $\mathbf{C}_+$, bounded on some horizontal line in $\mathbf{C}_+$, has an exponential type.

The metric criteria of s-normality are discussed in § 1. In § 4 we prove, for families of holomorphic functions, another criteria of s-normality including the one in the Theorem above. In § 3 we consider a generalization of the notion of P-points sequence and of the Lange — Gavrilov normality criterium for a holomorphic mapping (Theorem 3.15). Closely related results were obtained in [Ha 2, Jo Kw].

Hyperbolic metrics play an essential role in the theory of normal functions since the pioneer work of O. Lehto and K. I. Virtanen [Le Vi]. We would like to emphasize here that there is a deep interaction of this theory with hyperbolic analysis. Namely, normal mappings into arbitrary complex manifolds inherit the most important properties of holomorphic mappings into compact hyperbolic manifolds (or hyperbolically embedded manifolds). In particular, they satisfy the Kiernan — Kobayashi — Kwack analog of the Big Picard Theorem [Fu, Jä, Jo Kw] and Kiernan's analog of Montel's Theorem (see Corollary 1.14 below). In both cases the proofs are mainly based on some versions of the Schwarz — Pick Lemma or on a distance-decreasing property. This contraction property holds with respect to Kobayashi pseudometrics for general holomorphic mappings or with respect to some Hermitian metric on the image and Kobayashi pseudometric on the preimage for normal mappings (via the metric criterium of normality).

In § 2 we proceed further along this line. An easy remark is that for a relatively compact subspace of a compact space, being hyperbolically embedded is equivalent to the normality of the inclusion mapping (Theorem 2.1; this is a reformulation of Kiernan's hyperbolic embedding criterium [Ki]). Eastwood's criterium for hyperbolicity of preimages [Ea] is extended to normal mappings into non-hyperbolic spaces (Theorems 2.7, 2.8). Finally, we obtain a criterium of s-normality in terms of the absence of certain entire curves (Theorem 2.14). This generalizes both Brody's hyperbolicity criterium [Br] and Hahn's normality criterium [Ha 1].

In § 5 the latter criterium of s-normality is applied to extend some results due to V. I. Ostrovskii [Os 1–2] on growth estimates of solutions of polynomial identities $p(x(z), y(z)) \equiv 0$ of the special forms in functions $x(z), y(z)$ holomorphic in the upper halfplane, to solutions of a wide class of polynomial identities or inequalities in two variables in functions holomorphic on a complex manifold. We establish that they are normal and therefore satisfy the inequalities in the Theorem above (Corollary 5.4; Theorem 5.6).

Most of the results of this paper were announced in [Za 2–3], some of them in a slightly weaker form. The inequality in Corollary 5.4 was proved also in [Za 5] in a different way.

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§ 1. s-normal families of holomorphic mappings. Metric criteria of s-normality

Let X and $\overline{Y}$ be complex spaces, and let Y be a relatively compact subspace of $\overline{Y}$. As usual, $\mathrm{Hol}(X, Y)$ denotes the space of all holomorphic mappings $X \to Y$, endowed with the compact-open topology; Δ denotes the unit disc in $\mathbf{C}$.

1.1 Definition. *A family $\mathcal{F}$ of holomorphic mappings $X \to Y$ will be called* <u>*s − normal*</u> *iff the family of compositions*

$$\mathcal{F} \circ \mathrm{Hol}(\Delta, X) := \{f \circ \varphi \mid f \in \mathcal{F}, \varphi \in \mathrm{Hol}(\Delta, \mathrm{X})\}$$

is a relatively compact subspace of the space $\mathrm{Hol}(\Delta, \overline{Y})$.

It is obvious that a subfamily of an s-normal family is also s-normal. Later on we will show that an s-normal family is normal in Montel's sense (Corollary 1.14). The converse is not true in general, even for a family consisting of a single mapping.

1.2 Definition. *A mapping $f : X \to Y$ will be called* <u>*normal*</u> *iff the family $\mathcal{F} = \{f\}$ is s-normal.*

1.3 Remarks. It is easily seen that the restriction $f \mid Z : Z \to Y$ of the normal mapping $f : X \to Y$, where Z is a subspace of X, is also a normal mapping. Moreover, if $\varphi : Z \to X$ is a holomorphic mapping of complex spaces and the mapping $f : X \to Y$ is normal, then $f \circ \varphi : Z \to Y$ is also normal. Furthermore, the direct product $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ of two normal mappings $f_i : X_i \to Y_i$ $(i = 1, 2)$ is normal. The same is true for restrictions, compositions with a given mapping and direct products of s-normal families.

1.4 Notations. Let k_X be the Kobayashi pseudodistance on X. If X is a complex manifold, let K_X denote the Kobayashi-Royden differential pseudometric on TX (see [Ro 1]). For an arbitrary metric λ on Y we denote by $\mathrm{Hol}_c(X, Y, \lambda)$ the family of all those mappings $f \in \mathrm{Hol}(X, Y)$, which satisfy the inequality

$$f^* \lambda \leq c k_X .$$

If X and Y are smooth and λ is generated by an upper semicontinuous differential metric Λ on TY, the latter inequality has the following equivalent form:

$$f^*\Lambda \leq cK_X .$$

The next lemma easily follows from the Arzela-Ascoli Theorem.

1.5 Lemma. *Let ρ be a Hermitian metric on $\overline{Y}$ and λ be a metric on Y. If $\lambda \geq \rho \mid Y$, then the family $\mathrm{Hol}_c(X, Y, \lambda)$ as defined above is s-normal for any $c > 0$.*

The following metric criteria of s-normality is an easy consequence of Theorem 1.9 below. It is simpler and therefore often more convenient to use than the one contained in Theorem 1.9.

1.6 Theorem. *A family $\mathcal{F} \subset \mathrm{Hol}(X, Y)$ is s-normal if and only if $\mathcal{F} \subset \mathrm{Hol}_c(X, Y, \rho \mid Y)$ for some constant $c > 0$, where ρ is a Hermitian metric on $\overline{Y}$.*

To formulate Theorem 1.9 we need the notion of a KRG-metric, introduced below. From now on we assume that X and $\overline{Y}$ are smooth.

1.7 Definition. *Fix an arbitrary finite covering $\{U_i\}_{i=1,\ldots,k}$ of the closure $\mathrm{Cl}(Y)$ of Y in $\overline{Y}$. Let $U_i^* := U_i \cap Y$, $i = 1, \ldots, k$. Consider the envelope $H = \min\limits_{1\leq i\leq k} \{K_{U_i^*}\}$ of the local Kobayashi-Royden pseudometrics. Let $G = \max(H, \rho \mid Y)$. The metric G and the corresponding distance g on Y will be called a Kobayashi-Royden-Green metric, or a KRG-metric for short.*

1.8 Remark. KRG—metrics were first used in [Gr] and later in [Za 1]. It is evident that a KRG-metric depends on the choice of a covering. For sufficiently small coverings $\{U_i\}$, the metrics G and H in (1.7) are equivalent. G is a complete metric for an appropriate covering iff Y is a locally complete hyperbolic subvariety of $\overline{Y}$ (this is the case for complements of divisors, analytic polyhedra and strictly pseudoconvex domains; see [Ki Ko, Za 1]).

1.9 Theorem. *Fix a KRG-metric g on Y. A family $\mathcal{F} \subset \mathrm{Hol}(X, Y)$ is s-normal if and only if $\mathcal{F} \subset \mathrm{Hol}_c(X, Y, g)$ for some constant $c > 0$.*

Proof: The "if" part follows from Lemma 1.5. To prove the "only if" part assume that there is an s-normal family $\mathcal{F} \subset \mathrm{Hol}(X, Y)$ such that $\mathcal{F} \not\subset \mathrm{Hol}_c(X, Y, g)$ for any $c > 0$. It follows that there exist sequences $\{f_n\} \subset \mathcal{F}$, $\{v_n\} \subset TX$ such that $K_X(v_n) < \frac{1}{n}$ and $\mid df_n(v_n) \mid_G = 1$ for every $n \in \mathbf{N}$.

By the definition of the pseudometric K_X, there exist sequences $\{\varphi_n\} \subset \mathrm{Hol}(\Delta, X)$, $\{u_n\} \subset T_0\Delta$ such that $\mid u_n \mid < 1/n$ and $d\varphi_n(u_n) = v_n$, $n \in \mathbf{N}$. Let $r_n := \mid u_n \mid^{-1}$ and $\psi_n := \varphi_n(u_n z)$, $\psi_n \in \mathrm{Hol}(\Delta_{r_n}, X)$, where $\Delta_r := \{z \in \mathbf{C} \mid |z| < r\}$. Consider two sequences of holomorphic discs

$$\Phi_n := f_n \circ \varphi_n \in \mathrm{Hol}(\Delta, Y)$$

and

$$\Psi_n := f_n \circ \psi_n \in \mathrm{Hol}(\Delta_{r_n}, Y) .$$

Since $\mathcal{F}$ is an s-normal family, both sequences can be assumed to be convergent: $\Phi_n \underset{n}{\rightarrow} \Phi$, $\Psi_n \underset{n}{\rightarrow} \Psi$, where $\Phi \in \mathrm{Hol}(\Delta, \overline{Y})$ and $\Psi \in \mathrm{Hol}(\mathbf{C}, \overline{Y})$. Let us show that $\Psi \equiv const.$

Fix an arbitrary point $z_o \in \Delta$. Put $p_0 = \Psi(0)$ and $q_0 = \Psi(z_0)$ (here $p_0, q_0 \in \mathrm{Cl}(Y)$). Consider the sequences

$$z_n := u_n z_0 \in \Delta, \quad z_n \underset{n}{\to} 0 ,$$

$$q_n := \Phi_n(z_n) \in Y, \;\; p_n = \Phi_n(0) \in Y .$$

We have:

$$q_n = \Psi_n(z_0) \underset{n}{\to} \Psi(z_0) = q_0$$

and

$$p_n = \Psi_n(0) \underset{n}{\to} \Psi(0) = p_0 .$$

Since $\Phi_n \underset{n}{\to} \Phi$, for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\rho(\Phi_n(z),\, \Phi(0)) < \epsilon \;\; \forall z \in \Delta_\delta ,$$

when n is sufficiently large. In particular, $\rho(q_n, p_0) < \epsilon$ for $n >> 1$. Thus $q_n \underset{n}{\to} p_0$ and therefore $q_0 = p_0$, i.e. $\Psi(z_0) = \Psi(0)$. So $\Psi \equiv const = p_0 \in \overline{Y}$.

Let the KRG-metric g correspond to the covering $\{U_i\}_{i=1,\dots,k}$ of $\mathrm{Cl}(Y)$. Let $p_0 \in U_i$. Since $\Psi_n \underset{n}{\to} \Psi \equiv p_0$, we have: $\Psi_n(\Delta_2) \subset U_i^* = U_i \cap Y$, when n is large enough. For such values of n one has: $K_{U_i^*}(d\Psi_n(\frac{d}{dz} \mid_0)) < \frac{1}{2}$. Since $d\Psi_n(\frac{d}{dz} \mid_0) = df_n(v_n)$, we have that $\mid df_n(v_n) \mid_H < 1/2$. But $\mid df_n(v_n) \mid_G = 1$ by construction and hence $\mid df_n(v_n) \mid_\rho = 1$ (see Definition 1.6). It follows that

$$\mid d\Psi\left(\frac{d}{dz} \mid_0\right) \mid_\rho = \lim_n \mid d\Psi_n\left(\frac{d}{dz} \mid_0\right) \mid_\rho = 1 .$$

This contradicts the constancy of Ψ . Q.E.D.

1.10 Corollary. *Let ρ be a Hermitian metric on $\overline{Y}$ and G be a KRG-metric on Y. Let $\mathcal{F} \subset \mathrm{Hol}(X,Y)$ be a family such that*

$$\mid df(v) \mid_\rho \leq cK_X(v) \;\; \forall f \in \mathcal{F}, \;\; \forall v \in TX$$

for some constant $c > 0$. Then there exists a constant $c_1 = c_1(G, \rho, c) > 0$ such that

$$\mid df(v) \mid_G \leq c_1 K_X(v) \;\; \forall f \in \mathcal{F}, \;\; \forall v \in TX .$$

1.11 Example. Let $\overline{Y} = \mathbf{P}^1 = \mathbf{P}^1_{\mathbf{C}}, \;\; Y = \mathbf{C} = \mathbf{P}^1 \setminus \{(1:0)\}$, ρ be the spherical metric on $\mathbf{P}^1$. Let a KRG-metric G on $Y = \mathbf{C}$ be defined as follows:

$$\mid u \mid_G = \frac{\mid u \mid}{\mid z \mid_+ \log(\mid z \mid_+)} ,$$

where $u \in T_z\mathbf{C}$, $|z|_+ := \max\{e, \mid z \mid\}$. Then, by (1.10), any holomorphic function f on a complex space X, which satisfies the inequality

$$\frac{\mid df(v) \mid}{1 + \mid f(x) \mid^2} \leq cK_X(v) \;\; \forall (x,v) \in TX ,$$

actually satisfies the stronger inequality

$$\frac{|\, df(v)\,|}{|\, f(v)\,|_{+} \log(|\, f(v)\,|_{+})} \le c_1 K_X(v) \quad \forall (x,v) \in TX$$

with a constant $c_1 = c_1(c) > 0$. In fact, the last inequality can also be strengthened; see (4.9) below.

1.12 Corollary. *The set* $N(X,Y)$ *of all normal mappings* $X \to Y$ *coincides with the unions*

$$\bigcup_{c>0} \mathrm{Hol}_c(X,Y,g) = \bigcup_{c>0} \mathrm{Hol}_c(X,Y,\rho)\,.$$

(In general, the family $N(X,Y)$ is not s-normal.)

1.13 Corollary. *Let* Z *be a complex manifold and* $\mathcal{F} \subset \mathrm{Hol}(X,Y)$ *be an s-normal family. Consider the family*

$$\mathcal{F}_Z := \{ f \circ \varphi \mid f \in \mathcal{F},\ \ \varphi \in \mathrm{Hol}(Z,X)\}\,.$$

Then $\mathcal{F}_Z \subset \mathrm{Hol}(Z,Y)$ *is an s-normal family; moreover, it is a relatively compact subspace of the space* $\mathrm{Hol}(Z,\overline{Y})$.

Indeed, by Theorem 1.6 $\mathcal{F} \subset \mathrm{Hol}_c(X,Y,\rho)$ for some $c > 0$. Since holomorphic mappings $Z \to X$ are contractions with respect to the pseudometrics K_Z and K_X , we have that $\mathcal{F}_Z \subset \mathrm{Hol}_c(X,Y,\rho)$. Now the assertion follows from the Arzela-Ascoli Theorem. Q.E.D.

Applying this to the case when $X = Z$, we get the following:

1.14 Corollary. *An s-normal family* $\mathcal{F} \subset \mathrm{Hol}(X,Y)$ *is normal in Montel' sense.*

§ 2. Normality and hyperbolicity

Here we show that, to some extent, the notion of normality is a relative version of the notion of hyperbolicity. The next theorem is a rewording of the criterium of hyperbolic embedding, due to P. Kiernan [Ki].

2.1 Theorem. *Let* Y *be a relatively compact subspace of a complex space* $\overline{Y}$. *The following conditions are equivalent:*

i) *The family* $\mathcal{F} := \mathrm{Hol}(\Delta, Y)$ *is s-normal;*
ii) $f := id_Y$ *is a normal mapping;*
iii) $\mathrm{Hol}(\Delta, Y)$ *is a relatively compact subspace of the space* $\mathrm{Hol}(\Delta, \overline{Y})$;
iv) Y *is hyperbolically embedded in* $\overline{Y}$.

Proof: Conditions i), ii), iii) are actually tautologically equivalent. By Kiernan's Theorem [Ki] iii) and iv) are equivalent. Q.E.D.

2.2 Remark. By Kiernan's theorem [Ki] iii) and iv) are equivalent to the condition

ii′) $cK_Y \ \ge\ \rho \mid Y$,

where ρ is a Hermitian metric on $\overline{Y}$ and $c = c(\rho, Y) > 0$ (see also [Za 4]). The equivalence of ii) and ii′) follows from Theorem 1.6 (in the smooth case).

2.3 Notations. Let (X, ρ) be a metric space. For a subset Q of X we denote by $Q^{(r,\rho)}$ the r-neighborhood of Q , i.e. the union $\bigcup_{x \in Q} B_\rho(x, r)$ of $\rho-$balls of radius r with centers in points of Q. For a complex space X the neighborhood $Q^{(r,k_X)}$ with respect to the Kobayashi pseudodistance will be denoted simply by $Q^{(r)}$.

2.4 Lemma. *Let $X, Y, \overline{Y}$ be as in § 1. Let ρ be a Hermitian metric on $\overline{Y}$ and g be a KRG-metric on Y. Let $\mathcal{F} \subset \mathrm{Hol}(X, Y)$ be an s-normal family. Then there exist constants $c > 0,\ c_1 > 0$ such that for any $f \in \mathcal{F},\quad r > 0$ and $Q \subset X$ the following inclusions hold:*

$$f\left(Q^{(r)}\right) \subset [f(Q)]^{(cr,\rho)}$$

and

$$f\left(Q^{(r)}\right) \subset [f(Q)]^{(c_1 r, g)} .$$

Proof: Indeed, by Theorems 1.6 and 1.8 $\mathcal{F} \subset \mathrm{Hol}_c(X, Y, \rho)$ and $\mathcal{F} \subset \mathrm{Hol}_{c_1}(X, Y, g)$ for some $c, c_1 > 0$. Q.E.D.

2.5 Remark. Let us call $Q \subset X$ <u>$r-dense$</u> iff $Q^{(r)} = X$, and <u>$holomorphically\ r-dense$</u> iff the hull of $Q^{(r)}$ with respect to the algebra $\mathfrak{H}(X)$ of holomorphic functions on X coincides with X. The next statement easily follows from Lemma 2.4 and the maximum modulus principle:

A normal holomorphic function f in X, which is bounded on a holomorphically r-dense subset $Q \subset X$, is bounded on X.

In fact, applying Lemma 2.4 one can estimate $\| f \|_\infty$ by a function of $\| f \mid Q \|_\infty$.

A holomorphically r-dense real curve γ in the unit disc Δ is called <u>$a\ spiral\ of\ density\ r$</u> (see [Ga 2]); that means that $\gamma^{(r)}$ contains some annulus $K_\epsilon = \{z \in \Delta \mid\ 1 - \epsilon <\mid z \mid < 1\},\quad \epsilon > 0$. As an application one can obtain the following fact from [Ga 2]:

A normal function in the unit disc Δ, which is bounded on a spiral of finite density, is bounded in Δ .

The next lemma is well-known; for the sake of completeness we provide a proof.

2.6 Lemma. *Let X be a complex manifold and Q be a subset of X. Then for any $r > 0$ the following inequalities hold:*

$$(\tanh(r))K_{Q^{(r)}} \mid Q \leq K_X \mid Q \leq K_{Q^{(r)}} \mid Q .$$

Proof: The second inequality follows from the contracting property of the Kobayashi-Royden pseudometrics. To prove the first one, fix $x \in Q$ and $v \in T_x X$ arbitrarily. By the definition of K_X , for any $\epsilon > 0$ there exist $s \geq (K_X(v) + \epsilon)^{-1}$ and $\varphi \in \mathrm{Hol}(\Delta_s, X)$ such that $d_\varphi(\frac{d}{dz} \mid_0) = v$. Denote by w_r the hyperbolic disc in Δ_s of radius r centered at the origin. By the contracting property of Kobayashi metrics we have: $f(w_r) \subset Q^{(r)}$. Hence $K_{Q^{(r)}}(v) \leq t^{-1}$, where $t = t(s, r)$ is the Euclidean radius of $w_r = \Delta_t$. Here $r =$ arctanh $\left(\frac{t}{s}\right)$, or $t = s \cdot \tanh(r)$. Therefore

$$K_{Q^{(r)}}(v) \leq \frac{1}{\tanh(r)}\, (K_X(v) + \epsilon) ,$$

and the inequality follows. Q.E.D.

The following statement extends Eastwood's Theorem [Ea] to normal holomorphic mappings into non-hyperbolic spaces.

2.7 Theorem. *Let $f : X \to Y$ be a normal mapping. If for some covering $\{U_\alpha\}$ of Y all preimages $f^{-1}(U_\alpha)$ are hyperbolic, then X is hyperbolic. If these preimages are complete hyperbolic and Y is locally complete hyperbolic in $\overline{Y}$, then X is complete hyperbolic.*

Proof: For an arbitrary point $y \in Y$ fix a $\rho-$ball $B := B_\rho(y, r)$ such that the preimage $P := f^{-1}(B)$ is hyperbolic. Put $Q := f^{-1}(B_\rho(y, \frac{r}{2}))$. By Theorem 1.6, $f \in \mathrm{Hol}_c(X, Y, \rho)$ for some $c > 0$. By Lemma 2.4 we have:

$$f\Big(Q^{(r/2c)}\Big) \subset B_\rho(y, r) = B ,$$

or $Q^{(r/2c)} \subset P$. The hyperbolicity of P implies that the metric K_P is strictly nondegenerate in the following sense: it locally majorises some Hermitian metrics on P (see [Ro 1]) . By Lemma 2.6

$$K_X \mid Q \geq \Big(\tanh\Big(\frac{r}{2c}\Big)\Big) K_P \mid Q ,$$

hence the pseudometric K_X is strictly nondegenerate, too. This in turn implies the hyperbolicity of X [Ro 1].

To prove the second statement, first of all fix a complete KRG-metric g on Y (such a metric does exist since Y is locally complete hyperbolic; see (1.8)). Let $D := B_{k_X}(x, R)$ be an arbitrary Kobayashi ball in X, where $x \in X$ and $R > 0$. By Theorem 1.9 $f \in \mathrm{Hol}_{c_1}(X, Y, g)$ for some $c_1 > 0$, and hence $f(D) \subset B_g(y, c_1 R)$, where $y := f(x)$ (see Lemma 2.4). Since g is a complete metric the closure $K := \mathrm{Cl}(B_g(y, c_1 R))$ is compact in Y. Hence $K \subset \bigcup_{i=1}^{k} U_{\alpha_i}$ for some finite subfamily of $\{U_\alpha\}$. Let $r > 0$ be the Lebesgue number of this finite covering of K with respect to g .

Put $W_i = f^{-1}(U_{\alpha_i})$ and $W = \bigcup_{i=1}^{k} W_i$. Then $\mathrm{Cl}(D) \subset f^{-1}(K) \subset W$. The envelope $H := \min\{K_{W_i}\}$ of complete metrics is a complete differential metric on W. The covering $\{w_y\}_{y \in K}$ of the compact K by the $g-$balls $w_y := B_g(y, r/2)$ induces the covering $\{Q_y\}$ of $f^{-1}(K)$ by open sets $Q_y := f^{-1}(w_y)$. Put $P_y := f^{-1}(B_g(y, r))$. Then

$$K_X \mid Q_y \geq \Big(\tanh\Big(\frac{r}{2c_1}\Big)\Big) K_{P_y} \mid Q_y$$

as above. Therefore

$$K_X \mid Q_y \geq \lambda\, K_{W_i} \mid Q_y ,$$

where $\lambda := \tanh\Big(\frac{r}{2c_1}\Big)$ and W_i is such that $P_y \subset W_i$. It follows that

$$K_X \mid f^{-1}(K) \geq \lambda\, H \mid f^{-1}(K) .$$

Hence the Kobayashi ball $D \subset f^{-1}(K)$ is contained in the $H-$ball $B_h(x, R/\lambda)$. By the completeness of the metric H, the closure $\mathrm{Cl}(B_h(x, R/\lambda))$ is compact, and so $\mathrm{Cl}(D) \subset X$ is compact. This implies that k_X is a complete metric [Ko]. Q.E.D.

Similarly, a version of Brody's Theorem [Br] in [Za 1, Proposition 4.6] can be generalized as follows:

2.8 Theorem. *Let $f : X \to Y$ be a proper normal mapping. If every fibre $X_y := f^{-1}(y)$, $y \in Y$, is hyperbolic, then X is hyperbolic. If, in addition, Y is locally complete hyperbolic, then X is complete hyperbolic.*

Proof: By Proposition 4.6 in [Za 1] any point $y \in Y$ has a neighborhood w_y in Y such that $f^{-1}(w_y)$ is hyperbolic. Now the hyperbolicity of X follows by applying of Theorem 2.6.

If Y is locally complete hyperbolic in $\overline{Y}$, then the image of the Kobayashi ball $D = B_{k_X}(x, R)$ under f is contained in a compact set $K \subset Y$ (see the proof of Theorem 2.6). Since f is proper, $f^{-1}(K)$ is a compact in X, and hence $\mathrm{Cl}(D) \subset f^{-1}(K)$ is compact. As before, this implies the completeness of the metric k_X. Q.E.D.

Next we give a normality criterium 2.14 analogous to Brody's hyperbolicity criterium [Br] as generalized by K. T. Hahn [Ha 1]. For the reader's convenience we recall first these facts.

2.9 Definitions. *By an <u>entire curve</u> in a complex space $\overline{Y}$ we mean a non-constant mapping $\varphi : \mathbf{C} \to \overline{Y}$. If Y is a subspace of $\overline{Y}$ and φ can be approximated by holomorphic mappings $\varphi_n : \Delta_n \to Y$, we call φ a <u>Y − limiting entire curve</u> [Za 1]. If a family $\mathcal{F} \subset \mathrm{Hol}(X, Y)$ is given, we say that φ is an <u>$\mathcal{F}$ − limiting entire curve in $\overline{Y}$</u> iff it can be approximated by holomorphic curves $f_n \circ \varphi_n : \Delta_n \to Y$, where $\varphi_n \in \mathrm{Hol}(\Delta_n, X)$ and $f_n \in \mathcal{F}$.*

Recall [Za 1, Lemma 2.9] that when Y is locally complete hyperbolic in $\overline{Y}$, every Y-limiting holomorphic curve in $\overline{Y}$ is contained either in Y or in ∂Y.

2.10 Theorem (R. Brody [Br]). *A compact complex manifold Y is hyperbolic iff it does not contain entire curves.*

2.11 Theorem (see [Za 1]). *A relatively compact subspace Y of a complex manifold $\overline{Y}$ is hyperbollically embedded in $\overline{Y}$ iff $\overline{Y}$ contains no Y-limiting entire curve.*

2.12 Theorem (K. T. Hahn [Ha 1, Theorem 6.5]). *A family $\mathcal{F} \subset \mathrm{Hol}(\Delta, Y)$ is relatively compact in $\mathrm{Hol}(\Delta, \overline{Y})$ iff there exist no sequences $\{b_n\} \subset \Delta$, $\{r_n\}$, $r_n \downarrow 0$, and $\{\varphi_n\} \subset \mathcal{F}$ such that $\{\varphi_n(r_n z + b_n)\}$ converges to an $\mathcal{F}$—limiting entire curve $\mathbf{C} \to \overline{Y}$.*

2.13 Remarks. In fact, Y is assumed to be compact in Theorem 6.5 in [Ha 1], but the proof works also in this slightly more general setting (following the line of proof of Theorem 6.3 in [Ha 1]). Note also that if the family $\mathcal{F} \subset \mathrm{Hol}(\Delta, Y)$ is Aut Δ—invariant and non-normal, then we may assume above that $b_n = 0$ for all n, i.e. that $\{\varphi_n(r_n z)\}$ converges to an entire curve $\mathbf{C} \to \overline{Y}$ for some sequences $\{\varphi_n\} \subset \mathcal{F}$, $\{r_n \in (0, 1/n)\}$.

The next theorem generalizes these facts and Corollary 6.7 in [Ha 1].

2.14 Theorem. *A family $\mathcal{F} \subset \mathrm{Hol}(X, Y)$ is s−normal iff there exists no $\mathcal{F}$−limiting entire curve $\mathbf{C} \to \overline{Y}$.*

Proof: Let $\mathcal{F}$ be an s−normal family. Assume that a sequence $\Phi_n = f_n \circ \varphi_n$, converges to a mapping $\Phi : \mathbf{C} \to \overline{Y}$, where $f_n \in \mathcal{F}$ and $\varphi_n \in \mathrm{Hol}(\Delta_n, X)$. By Theorem 1.6 $\mathcal{F} \subset \mathrm{Hol}_c(X, Y, \rho)$ for some $c > 0$. Hence for any pair of points $u, v \in \mathbf{C}$ we have:

$$\rho(\Phi_n(u), \Phi_n(v)) \le c\, k_X(\varphi_n(u), \varphi_n(v)) \le c\, k_{\Delta_n}(u, v) \underset{n}{\to} 0 .$$

Thus $\Phi(u) = \Phi(v)$ and so Φ is constant. If $\mathcal{F}$ is not an $s-$normal family, then $\mathcal{F}_\Delta := \{f \circ \varphi \mid f \in \mathcal{F},\ \varphi \in \mathrm{Hol}(\Delta, X)\}$ is not a relatively compact subspace of the space $\mathrm{Hol}(\Delta, \overline{Y})$ (see (1.1)). Applying Theorem 2.12 to the family $\mathcal{F}_\Delta \subset \mathrm{Hol}(\Delta, Y)$ we easily get the existence of an $\mathcal{F}-$limiting entire curve $\mathbf{C} \to \overline{Y}$. Q.E.D.

2.15 Corollary. *A mapping $f \in \mathrm{Hol}(X, Y)$ is normal iff the only $f-$limiting mappings $\mathbf{C} \to \overline{Y}$ are constants.*

§ 3. Normality and P-sequences

Here we discuss a generalization of the $P-$points normality criterium for holomorphic functions, due to L. H. Lange [Lan] and V. I. Gavrilov [Ga 1], in a more general setting of holomorphic mappings.

3.1 Definition. *Let X be a complex space and λ be a metric on X. Two sequences of points $p = (x_k) \subset X$ and $q = \left(x_k^{'}\right) \subset X$ will be called confinal (resp. $\lambda-$confinal) iff $k_X\left(x_k, x_k^{'}\right) \underset{k}{\to} 0$ (resp. $\lambda\left(x_k, x_k^{'}\right) \underset{k}{\to} 0$). A sequence $\overline{p} = (x_k)_{k \geq n}$ will be called a shortening of the sequence $p = (x_k)$.*

Fix a mapping $f \in \mathrm{Hol}(X, Y)$, where Y is a relatively compact subspace of a complex space $\overline{Y}$. Let ρ be a fixed Hermitian metric on $\overline{Y}$ and $p = (x_k) \subset X$ be a fixed sequence.

3.2 Definition. *The sequence p will be called an $s-$sequence of f iff*

$$f\left(p^{(r)}\right) \subset U$$

for some $r > 0$ and for some domain U in Y, which is hyperbolically embedded in $\overline{Y}$.

3.3 Definition. *A sequence of points in X, which does not contain any $s-$subsequence of f, will be called a $P-$sequence of f.*

3.4 Definition. *Let Λ be a differential pseudometric on TY and λ be the corresponding pseudodistance on Y. The quantity*

$$\mathrm{dil}_{\Lambda, x_0}(f) := \sup_{v \in T_{x_0} X} \frac{\mid df(v) \mid_{\Lambda}}{K_X(v)}$$

will be called a $\lambda-$dilatation of f at the point $x_0 \in X$.

It is clear that $f \in \mathrm{Hol}_c(X, Y, \lambda)$ iff $\mathrm{dil}_{\Lambda, x}(f) \leq c$ for every $x \in X$.

3.5 Definition. *The sequence p will be called a $d-$sequence of f iff*

$$\sup_{x \in \overline{p}^{(r)}} \mathrm{dil}_{\rho, x}(f) < \infty$$

for some shortening $\overline{p}$ of p and for some $r > 0$.

3.6 Definition. *The sequence p will be called a $g-$sequence of f iff for any other sequence $q = \left(x_k^{'}\right) \subset X$, confinal with p, the sequences $f(p) = \{f(x_k)\}$ and $f(q) = \left\{f\left(x_k^{'}\right)\right\}$ are $\rho-$confinal, i.e. iff $\rho\left(f(x_k), f\left(x_k^{'}\right)\right) \underset{k}{\to} 0$ when $k_X\left(x_k, x_k^{'}\right) \underset{k}{\to} 0$.*

It is easily seen that any subsequence of an $s-$sequence of f is itself an $s-$sequence of f; the same is true for d —, g — or P —sequences. Moreover:

3.7 Lemma. *Let sequences* $p = (x_k) \subset X$ *and* $q = \left(x'_k\right) \subset X$ *be confinal. If* p *is an* $s-$ (resp., $d-$, $g-$, $P-$)*sequence of* f, *then the same is true for* q.

Proof: For a given $r > 0$ and for some shortening $\overline{p}$ (resp. $\overline{q}$) of p (resp. q) we have: $\overline{q}^{(r)} \subset \overline{p}^{(2r)}$. This implies that if p is an $s-$ or $d-$sequence of f, then the same is true for q. By the transitivity of confinality q is a $g-$sequence of f, if p has this property. If q contains an $s-$subsequence of f, then the corresponding subsequence of p will be also $s-$sequence of f, as just has been proved. Thus, q would be a $P-$sequence of f, if p is such a sequence. Q.E.D.

3.8 Lemma. *Every* $s-$*sequence of* f *is a* $d-$*sequence of* f; *every* $d-$*sequence of* f *is a* $g-$*sequence of* f.

Proof: Let $p = (x_k) \subset X$ be an $s-$sequence of f, i.e. $f\left(p^{(r)}\right) \subset U$ for some $r > 0$, where U is a hyperbolically embedded in $\overline{Y}$ domain in Y. The latter means that $K_U \geq c\rho \mid U$ for some $c > 0$, and so

$$f^*\rho \mid p^{(r)} \leq c^{-1} f^* K_U \mid p^{(r)} \leq c^{-1} K_{p^{(r)}}$$

by the contracting property of Kobayashi-Royden pseudometrics. From Lemma 2.5 one gets:

$$f^*\rho \mid p^{(rk)} \leq c^{-1} K_{p^{(r)}} \mid p^{(r/2)} \leq [c \cdot \tanh(r/2)]^{-1} K_X \mid p^{(r/2)}$$

and hence

$$\sup_{x \in p^{(r/2)}} \mathrm{dil}_{\rho,x}(f) \leq [c \cdot \tanh(r/2)]^{-1}.$$

Thus p is a $d-$sequence of f.

Let p be a $d-$sequence of f. Then for some shortening $\overline{p}$ of p and for some $c, r > 0$ the mapping $f \mid \overline{p}^{(r)} : \overline{p}^{(r)} \to Y$ is a contraction with respect to the pseudometric $K_X \mid \overline{p}^{(r)}$ and the metric $c\rho \mid Y$. This implies that p is a $g-$sequence of f. Q.E.D.

3.9 Lemma. *Every* $d-$ or $g-$*sequence* p *of* f *contains an* $s-$*sequence of* f.

Proof: Since Y is relatively compact in $\overline{Y}$, there exist a subsequence $q = (x_{n_k}) \subset p$ such that $f(x_{n_k}) \xrightarrow[k]{} y_0$, where $y_0 \in \overline{Y}$. Let $r > 0$ be small enough, so that the ball $U := B_\rho(y_0, r)$ is hyperbolically embedded in $\overline{Y}$. Let p be a $d-$sequence of f, i.e.

$$\sup_{x \in \overline{p}^{(t)}} \mathrm{dil}_{\rho,x}(f) \leq M,$$

for some constants $t, M > 0$ and for some shortening $\overline{p}$ of p. Then the restriction $f \mid \overline{p}^{(t)} : \overline{p}^{(t)} \to Y$ is a contraction with respect to the pseudometrics $k_X \mid \overline{p}^{(\epsilon)}$ and $M^{-1}\rho$. Let $\overline{q}$ be a shortening of q such that $f(\overline{q}) \subset B_\rho(y_0, r/2) \subset U$. Then $f\left(\overline{q}^{(\tau)}\right) \subset U$ for $0 < \tau < \min(t, r/2M)$. Hence $\overline{q} \subset p$ is an $s-$sequence of f.

Assume further that p is a $g-$sequence of f. Let U and $\overline{q} \subset p$ be the same as above. If $\overline{q} = (x_{n_k})$ is not an $s-$sequence of f, then $f\left(\overline{q}^{(\epsilon)}\right) \not\subset U$ for any $\epsilon > 0$. Thus for every $n \in \mathbb{N}$ large enough there exist $k = k(n)$ and $x'_k \in X$ such that

$$k_X\left(x_{n_k}, x'_k\right) < 1/n$$

and $\rho\left(f\left(x_k'\right), y_0\right) > r$. For $m \neq n_k$ put $x_m'' = x_m$ and for $m = n_k$ put $x_m'' = x_k'$. Then the sequences $p = (x_m)$ and $p'' = \left(x_m''\right)$ are confinal, but the sequences $f(p)$ and $f(p^n)$ are not $\rho-$confinal. This contradicts the assumption that p is a $g-$sequence of f.. Q.E.D.

From Lemmas 3.8 and 3.9 we find

3.10 Corollary. *The sequence p is a $P-$sequence of f iff it does not contain any $d-$* (resp., $g-$)*subsequence of f.*

3.11 Corollary. *If* $\mathrm{dil}_{\rho,x_k}(f) \underset{k}{\to} \infty$, *then $p = (x_k)$ is a $P-$sequence of f.*

Indeed, in this case p does not contain $d-$subsequences of f.

3.12 Remark. The converse to Corollary 3.11 is not true in general, as the following example ((4.3) in [Ca Wi]) shows. Let $X = \Delta$, $Y = \mathbf{C}$, $\overline{Y} = \mathbf{P}^1$, $f(z) = \exp \frac{i}{1-z}$, $x_n = \frac{n^2}{1+n^2} - \frac{i}{n+n^3}$, $x_n' = \frac{n^2}{1+n^2}$. Here $\mathrm{dil}_{\rho,x_n'}(f) \underset{n}{\to} \infty$ and hence by Corollary 3.11 $q := \left(x_n'\right)$ is a $P-$sequence of f. By Lemma 3.7 sequence $p := (x_n)$, confinal to q, is a $P-$sequence of f, too, while the sequence $\{\mathrm{dil}_{\rho,x_n}(f)\}$ is bounded.

Nevertheless, the following criterium for $P-$sequences holds (it is a generalization of Theorem 4.4 in [Ca Wi]).

3.13 Proposition. *The following conditions are equivalent:*

i) *p is a $P-$sequence of f;*

ii) *there exists a sequence $q = \left(x_n'\right) \subset X$, confinal to p, such that*

$$\mathrm{dil}_{\rho,x_n'}(f) \underset{n}{\to} \infty ;$$

iii) *there exists a sequence $\epsilon_n \to 0$ such that*

$$\sup_{x \in B_{k_X}(x_n, \epsilon_n)} \{\mathrm{dil}_{\rho,x}(f)\} \underset{n}{\to} \infty .$$

Proof: The equivalence of ii) and iii) is easy. The implication ii) $\Rightarrow$ i) follows from Corollary 3.11 and Lemma 3.7. Thus, it is enough to prove i) $\Rightarrow$ iii). Let $p = (x_n)$ be a $P-$sequence of f. Put

$$\varphi_n(t) = \sup_{x \in B_{k_X}(x_n, t)} \{\mathrm{dil}_{\rho,\mathrm{x}}(\mathrm{f})\} .$$

By Corollary 3.10 p contains no $d-$subsequence of f and so $\varphi_n(t) \underset{n}{\to} \infty$ for any $t > 0$. Now the assertion follows from the next simple lemma:

3.14 Lemma. *Let $\{\varphi_n(t)\}$ be a sequence of real-valued functions on the segment [0,1] such that $\varphi_n(t) \underset{n}{\to} \infty$ for any $t > 0$. Then there exists a sequence $\epsilon_n \to 0$ such that $\varphi_n(\epsilon_n) \underset{n}{\to} \infty$.*

Proof: Fix n_1 such that $\varphi_n\left(\frac{1}{2}\right) > 2$ for all $n > n_1$; then fix $n_2 > n_1$ such that $\varphi_n\left(\frac{1}{3}\right) > 3$ for all $n > n_2$, and so on. Put $\epsilon_1 = \epsilon_2 = \ldots = \epsilon_{n_1} = 1$, $\epsilon_{n_1+1} = \ldots = \epsilon_{n_2} = \frac{1}{2}$, $\epsilon_{n_2+1} = \ldots = \epsilon_{n_3} = \frac{1}{3}$, etc. It is easily seen that $\varphi_n(\epsilon_n) \underset{n}{\to} \infty$. Q.E.D.

The next theorem is a generalization of the Lange-Gavrilov normality criterium for functions, holomorphic in the unit disc [Lan, Ga 1, Gau].

3.15 Theorem. *A mapping $f \in \mathrm{Hol}(X,Y)$ is normal if and only if there is no $P-$sequence of f, i.e. iff any sequence $p = \{x_k\} \subset X$ contains an $s-$subsequence of f.*

Proof: If f is normal, then by Theorem 1.6, $f \in \mathrm{Hol}_c(X,Y,\rho)$ for some $c > 0$. Hence $\sup_{x \in X} \mathrm{dil}_{\rho,x}(f) \leq c$ and therefore any sequence p in X is a $d-$sequence of f. By Lemma 3.9 p contains an $s-$subsequence of f. Conversely, if f is not normal, then its $\rho-$dilatation is unbounded on X and hence there exists a sequence $p = (x_k)$ in X such that $\lim_{k\to\infty} \mathrm{dil}_{\rho,x_k}(f) = \infty$. By Corollary 3.11 p is a $P-$sequence of f. Q.E.D.

3.16 Example. Let D_{2n+1} be a union of $(2n+1)$ hyperplanes in $\mathbf{P}^n$ in general position. It is well known that $\mathbf{P}^n \setminus D_{2n+1}$ is hyperbolically imbedded in $\mathbf{P}^n$. Let $f \in \mathrm{Hol}(X, \mathbf{P}^n)$ be a non-normal mapping and $p = (x_n)$ be a $P-$sequence of f. Then for any subsequence $q = (x_{n_k})$ of p and any $\epsilon > 0$ the function $f \mid q^{(\epsilon)}$ infinitely often takes values from D_{2n+1}. In the case of non-normal meromorphic functions $(n = 1)$ this result due to V. I. Gavrilov [Ga 1].

§ 4. The Schottky-Landau growth estimates for normal functions

4.1 Let X be a complex space. By *a normal holomorphic (resp., meromorphic) function on* X we mean a normal mapping $X \to \mathbf{C} \hookrightarrow \mathbf{P}^1$ (resp., $X \to \mathbf{P}^1$) in sense of Definition 1.2 (see Example 1.11). In the same manner we understand the terme *an $s-$normal family of holomorphic (meromorphic) functions on* X. In the case of functions holomorphic (meromorphic) in the unit disc $X = \Delta$ the metric criterium of normality given in Theorem 1.6 coincides with the well-known Nakano-Lehto-Virtanen criterium: *f is normal iff the inequality*

$$\frac{|f'(z)|}{1+|f(z)|^2} \leq \frac{c}{1-|z|^2}$$

holds for some $c > 0$ *and for all* $z \in \Delta$. Therefore the above definition of normality for functions holomorphic in the unit disc is equivalent to the classical one. The same arguments shows the equivalence of our definition of normality to the ones given in [Ha 1] for holomorphic functions on homogeneous hyperbolic complex manifolds, and in [Do, Ci Kr] for holomorphic (meromorphic) functions on bounded strictly pseudoconvex domains in $\mathbf{C}^n$.

4.2 Every normal function f in the unit disc belongs to *a normal* $\mathrm{Aut}\,\Delta-$*invariant family of functions* (indeed, by the classical definition of normality the family $\mathcal{F} = \{f \circ \alpha \mid \alpha \in \mathrm{Aut}\,\Delta\}$ is normal in Montel' sense). Moreover, by Theorem 1.6, every $s-$normal family $\mathcal{F} \subset \mathrm{Hol}(\Delta, \mathbf{C})$ is also contained in some normal invariant family $\mathrm{Hol}_c(\Delta, \mathbf{C})$. On the other hand, a normal invariant family $\mathcal{F} \subset \mathrm{Hol}(\Delta, \mathbf{C})$ is $s-$normal. This follows from the theorem of Hayman, cited below. One may consider the notion of an $s-$normal family of holomorphic functions on a complex space as a generalization of the notion of normal invariant family of holomorphic functions in the unit disc, which was treated by Hayman [Hay 1–2].

4.3 F. Schottky [Sch] and E. Landau [La] proved that every function f, holomorphic in the unit disc and omitting two values, has at most exponential growth, i.e. it satisfies the inequality

$$|f(z)| \leq A \exp(\sigma/(1-|z|))$$

for some constants $A,\ \sigma > 0$. This theorem was extended by W. K. Hayman [Hay 1–2] to normal invariant families of functions, holomorphic in the unit disc. Here we prove a

Schottky-Landau' type inequality for $s-$normal families of holomorphic functions on complex manifolds. As a consequence, we obtain some new estimates of growth of normal functions in the upper halfplane.

4.4 Notation. By $\mathfrak{H}(X)$ we denote the space of holomorphic functions on X. For a given positive function $\theta(s,t)$ in $\mathbf{R}^2_+ := \{(s,t) \in \mathbf{R}^2 \mid s,t \geq 0\}$, non-decreasing in each of the arguments, we denote by $\mathfrak{H}(X,\theta)$ the subspace of $\mathfrak{H}(X)$, consisting of all those functions $f \in \mathfrak{H}(X)$ which satisfy the inequality

$$| f(x) | \leq \theta(| f(x_0) | , \ k_X(x,x_0)) \qquad \forall x, \ x_0 \in X .$$

4.5 Lemma. *The family $\mathfrak{H}(X,\theta)$ is $s-$normal and* Aut X$-$*invariant.*

Proof: Consider the family $\mathcal{F}_\Delta \subset \mathfrak{H}(X)$, where

$$\mathcal{F}_\Delta := \{f \circ \varphi \mid f \in \mathfrak{H}(X,\theta), \ \varphi \in \mathrm{Hol}(\Delta, X)\} .$$

It is easily seen that $\mathcal{F}_\Delta \subset \mathfrak{H}(\Delta,\theta)$. The family $\mathfrak{H}(\Delta,\theta)$ is normal; indeed, the family $\{g(z) - g(0) \mid g \in \mathfrak{H}(\Delta,\theta)\}$ is uniformly bounded in any disc Δ_r , $r < 1$. This proves the $s-$normality of $\mathfrak{H}(X,\theta)$. The Aut $X-$invariance follows from the invariance of the Kobayashi pseudodistance k_X . Q.E.D.

4.6 Notations. On $\mathbf{C} \subset \mathbf{P}^1$ consider a family of KRG-metrics

$$G_c(z,v) := \left(\frac{1}{2} \mid z \mid_+ \log\left(c \mid z \mid_+\right)\right)^{-1} \mid v \mid ,$$

where $c > 0$ and $\mid z \mid_+ := \max\{e, \mid z \mid\}$. In the neighborhood $\Omega_e = \{z \in \mathbf{C} \mid |z| > e\}$ of the puncture $\infty \in \mathbf{P}^1$, the metric $G_c \mid \Omega_e$ coincides with the restriction to Ω_e of the Poincare metric of the punctured disc Ω_r , where $r = c^{-1}$. Remark that the corresponding complete distances g_c in $\mathbf{C}$ lie between the Euclidean metric in $\mathbf{C}$ and the spherical (incomplete) metric of $\mathbf{P}^1$ restricted to $\mathbf{C}$. Put $\mathfrak{H}_c(X) := \mathrm{Hol}_1(X, \mathbf{C}, g_c)$. Put also

$$\theta_c(s,t) := c^{-1} \exp\left[\left(\log\left(c \mid s \mid_+\right)\right) \exp 2t\right] .$$

4.7 Definition. *Let X be a complex space. We say that a family $\mathcal{F} \subset \mathfrak{H}(X)$ <u>has exponential type</u> $c > 1$ iff*

$$\mathcal{F} \subset \mathfrak{H}(X,\theta_c) ,$$

i.e. iff

$$\log\left(c \mid f(x) \mid_+\right) \leq \left(\log\left(c \mid f(x_0) \mid_+\right)\right) \exp\left(2k_X(x,x_0)\right)$$

for every $f \in \mathcal{F}$ and for any pair of points $x, x_0 \in X$.

4.8 It is easily seen that a holomorphic function in the unit disc, having exponential type, has exponential growth in the sense of (4.3), i.e., it satisfies the Schottky-Landau estimate.

4.9 Hayman's Theorem ([Hay 1, Theorem 6.8]).

(a) Let $\mathcal{F} \subset \mathfrak{H}(\Delta)$ be a normal invariant family. Then $\mathcal{F} \subset \mathfrak{H}_c(\Delta)$ for some $c > 1$.

(b) $\mathfrak{H}_c(\Delta) \subset \mathfrak{H}(\Delta,\theta_c)$ for each $c > 1$.

4.10 Corollary. *Any normal invariant family $\mathcal{F} \subset \mathfrak{H}(\Delta)$ has exponential type.*

4.11 Remark. In fact, the inequalities in Theorem 6.8 in [Hay 1] contain more precise constants (see also [Je 1–2, He]) .

4.12 Definition. *A family $\mathcal{F} \subset \mathfrak{H}(X)$ has set-exponential type $c > 1$ iff for any subset $Q \subset X$, any $f \in \mathcal{F}$ and any $x \in X$ the following inequality holds:*

$$\log \left(c \mid f(x) \mid _+ \right) \leq \left(\sup_{x' \in Q} \log \left(c \mid f(x') \mid _+ \right) \right) \exp \left(2k_X(x, Q) \right) .$$

4.13 Remark. For $Q = \{x_0\}$ the above inequality coincides with the inequality in (4.7). Hence a family $\mathcal{F}$ of set-exponential type $c > 0$ has exponential type c. The next theorem, which is the main result of this section, shows in particular that the converse is true, i.e. that a family $\mathcal{F}$ of exponential type c is also of set-exponential type c.

4.14 Theorem. *Let X be a complex manifold. For a family $\mathcal{F} \subset \mathfrak{H}(X)$ the following conditions are equivalent:*

a) $\mathcal{F}$ is an $s-$normal family.

b) $\mathcal{F} \subset \mathrm{Hol}_1(X, \mathbf{C}, \lambda)$ for some metric λ in $\mathbf{C}$ with the linear element $\Lambda(z, v) = q(\mid z \mid) \mid v \mid$, where q is a non-increasing positive function in $\mathbf{R}_+$.

c) $\mathcal{F} \subset \mathrm{Hol}\left(X, \mathbf{P}^1, \rho\right)$ for some $c > 0$, where ρ is the spherical metric in $\mathbf{P}^1$.

d) $\mathcal{F} \subset \mathfrak{H}_c(X)$ for some $c > 1$ (see (4.6)).

e) $\mathcal{F} \subset \mathfrak{H}(X, \theta)$ for some positive function θ in $\mathbf{R}_+^2$, non-decreasing in each argument (see (4.4)).

f) $\mathcal{F}$ is of set-exponential type.

Proof: The logical scheme of the proof is the following:

$$\begin{array}{ccccc} e) & \Rightarrow & a) & \Leftarrow & b) \\ \Uparrow & & \Downarrow & & \Uparrow \\ f) & \Leftarrow & d) & \Rightarrow & c) \end{array}$$

Implications d) $\Rightarrow$ c) $\Rightarrow$ b) are evident ; for f) $\Rightarrow$ e) see (4.13) ; e) $\Rightarrow$ a) is proved in Lemma 4.5 above.

Proof of b) $\Rightarrow$ a). Put $\mathcal{F}_\Delta := \mathcal{F} \circ \mathrm{Hol}(\Delta, X)$, $\mathcal{F}_\Delta \subset \mathfrak{H}(\Delta)$. Let $\mathcal{F} \subset \mathrm{Hol}_1(X, \mathbf{C}, \lambda)$. Then $\mathcal{F}_\Delta \subset \mathrm{Hol}(\Delta, \mathbf{C}, \lambda)$ and hence for any $r \in (0,1)$ and any $\varphi \in \mathcal{F}_\Delta$ the following inequality holds:

$$\mid f'(z) \mid \leq \frac{1}{h(r)(1-r^2)} \qquad \forall z \in \Delta_r .$$

Therefore $\mathcal{F}_\Delta$ is a normal family, and so $\mathcal{F}$ is an $s-$normal family.

Proof of a) $\Rightarrow$ d). Let $\mathcal{F}$ be $s-$normal. Then the family $\mathcal{F}_\Delta$ as above is a normal invariant family and by Hayman's Theorem 4.9 $\mathcal{F}_\Delta \subset \mathfrak{H}_c(\Delta)$ for some $c > 1$. Fix $f \in \mathcal{F}$, $\varphi \in \mathrm{Hol}(\Delta, X)$ and $z \in \Delta$ arbitrarily. Denote $\tilde{f} = f \circ \varphi$ $\left(\tilde{f} \in \mathcal{F}_\Delta\right)$, $x_0 = \varphi(0)$ and $x_1 = \varphi(z)$. Since $\tilde{f} \in \mathfrak{H}_c(\Delta)$ we have: $\tilde{f}^*(g_c) \leq k_\Delta$ (see (4.6)) and therefore

$$f^* g_c(x_0, x_1) \leq k_\Delta(0, z) .$$

Let x', x'' be a pair of points in X and $\lambda = \left(\{\varphi_i\}_{i=1}^N \subset \mathrm{Hol}(\Delta, X),\ \{z_i\}_{i=1}^N \subset \Delta\right)$ be a chain of holomorphic discs between x' and x''. From the above inequality it follows that

$$f^*g_c(x', x'') \leq \sum_{i=1}^{N} (f^*g_c)(x_{i-1}, x_i) \leq \sum_{i=1}^{N} k_{\Delta}(0, z_i) = \mathrm{length}_{k_X}(\gamma),$$

where $x_0 := x'$, $x_i := \varphi_{i+1}(0) = \varphi_i(z_i)$, $x_N := x''$. Hence $f^*g_c \leq k_X$, and so $\mathcal{F} \subset \mathfrak{H}_c(X)$.

Proof of d) $\Rightarrow$ f). Fix a subset $Q \subset X$. Let $f \in \mathfrak{H}_c(X)$ be such that

$$R := \sup_{x \in Q} |f(x)|_+ < \infty .$$

Put

$$L_f(R) := \{x \in X \mid |f(x)| \leq R\} .$$

Remark that $Q \subset L_f(R)$ and the inequality in (4.12) is valid for points $x \in L_f(R)$. Fix an arbitrary point $x \in X \setminus L_f(R)$. Since k_X is an inner pseudometric [Ko], for each $\epsilon > 0$ there exist a point $x_1 \in L_f(R)$ and a piecewise smooth path $\gamma : [0,1] \to X$, which connects x and x_1, such that

$$\mathrm{length}_{k_X}(\gamma) \leq k_X(x, L_f(R)) + \epsilon \leq k_X(x, Q) + \epsilon .$$

We may suppose that $|f(x_1)| = R$ and $|f \circ \gamma(t)| \geq R$ for $t \in [0,1]$. Since $R \geq e$, we have: $f \circ \gamma([0,1]) \subset \overline{\Omega}_e$ (see (4.6)). From the identity $g_c \mid \overline{\Omega}_e \equiv k_{\Omega_{c^{-1}}} \mid \overline{\Omega}_e$ (4.6) it follows that

$$g_c(|f(x)|, |f(x_1)|) = k_{\Omega_{c^{-1}}}(|f(x)|, R) = \frac{1}{2} \log \left(\frac{\log(|cf(x)|)}{\log(cR)} \right).$$

Since $f \in \mathfrak{H}_c(X)$, i.e. $f^*g_c \leq k_X$, we have:

$$\begin{aligned} g_c(|f(x)|, |f(x_1)|) &\leq g_c(f(x), f(x_1)) \leq \\ &\leq \mathrm{length}_{g_c}(f \circ \gamma) = \mathrm{length}_{f^*g_c}(\gamma) \leq \\ &\leq \mathrm{length}_{k_X}(\gamma) \leq k_X(x, Q) + \epsilon . \end{aligned}$$

Now the inequality in (4.12) easily follows.

This completes the proof of Theorem 4.14.

4.15 Corollary. *Every s–normal family $\mathcal{F} \subset \mathfrak{H}(X)$ is of set-exponential type.*

4.16 Remarks. 1. As follows from the proof of implication d) $\Rightarrow$ f), a family $\mathcal{F} \subset \mathfrak{H}_c(X)$ is of set-exponential type c.

2. In the case of s–normal families of functions Theorem 1.9 gives weaker estimates of growth. Indeed, for any fixed $a \in (0;1)$ the scale of metrics $\{g_c\}$ (see (4.6)) majorises the scale $\{c\, g_a\}$. In particular, Theorem 4.14 leads to the conclusion that normal functions in the unit disc are of exponential type, whereas from Theorem 1.9 it follows only that they are of finite order.

Next we give some applications of Theorem 4.14 to the one-dimensional case.

4.17 Corollary. *If a normal function in the unit disc is bounded on some geodesic (resp., horocycle), then it is also bounded in any strip between two branches of an equi-distant to this geodesic (resp., between two parallel horocycles).*

We turn now to normal holomorphic functions in the upper halfplane $\mathbf{C}_+ = \{z \in \mathbf{C} \mid \operatorname{Im} z > 0\}$. Let

$$L_a = \{z \in \mathbf{C} \mid \operatorname{Im} z = a\}, \quad \Lambda_0 = \{z \in \mathbf{C} \mid \operatorname{Re} z = 0, \ \operatorname{Im} z > 0\} .$$

4.18 Proposition. *Let a normal function $f \in \mathfrak{H}(\mathbf{C}_+)$ belong to the class $\mathfrak{H}_c(\mathbf{C}_+)$. Then:*

a) For every $a > 0$ the following inequality holds:

$$\log\left(c \mid f(z) \mid_+\right) \le \left(\log\left(c \mid f(ia) \mid_+\right)\right) \frac{\mid z \mid^2 + a^2}{a \operatorname{Im} z}, \qquad z \in \mathbf{C}_+ .$$

b) If $(\mid f \mid_+) \mid L_a \le R$ for some $a > 0$, $R > 0$, then

$$\log\left(c \mid f(z) \mid_+\right) \le (\log(cR))\left(\operatorname{Im}\frac{z}{a}\right)^{\operatorname{sign}(\operatorname{Im} z - a)}, \qquad z \in \mathbf{C}_+ .$$

In particular, f is bounded in any horizontal strip $\{z \in \mathbf{C} \mid 0 < a_1 < z < a_2\}$ in $\mathbf{C}_+$.

c) If $(\mid f \mid_+) \mid \Lambda_0 \le R$, then

$$\log\left(c \mid f \mid_+\right) \le (\log(cR)) \cot\left(\frac{1}{2}\arg z\right), \qquad z \in \mathbf{C}_+ .$$

In particular, f is bounded in any Stolz' angle $\prod_\alpha = \{z \in \mathbf{C} \mid \alpha < \arg z < \pi - \alpha\}$, where $0 < \alpha < \pi/2$.

The proof is easy and may be omitted.

4.19 Remark. The last statement in c) is known in a stronger form for f bounded on an asymptotic curve γ , contained in some Stolz' angle $\prod_{\alpha_0}$ [Ba, Theorem 4]. We can supplement this result of F. Bagemihl by the inequality in c) with the constant $\log(cR)$ being substituted by the constant $(\cot(\alpha_0/2))\log(cM)$, where $M = \sup\{(\mid f \mid_+) \mid \gamma\}$. In the same way we can make Theorem 8 in [Ga 2] more precise. Namely, if $p = (z_n) \subset \mathbf{C}_+$ is a sequence such that $k_{\mathbf{C}_+}(z, p) \le r < \infty$ for any $z \in \Lambda_0$ (i.e. $\Lambda_0 \subset p^{(r)}$) and $(\mid f \mid_+) \mid p \le M$, then the inequality in c) is valid with the constant $(\log(cM))e^{2r}$ instead of $\log(cR)$.

§ 5. Normality of solutions of polynomial equations

Let us start this section with the next simple facts.

5.1 Lemma. *Every regular function f on a quasiprojective hyperbolic curve Γ is a normal function.*

Proof: In order to use Corollary 2.15 remark that Γ is hyperbolically embedded in its projective completion $\overline{\Gamma}$ and f can be extended to a regular mapping $\overline{f} : \overline{\Gamma} \to \mathbf{P}^1$. Hence in any sequence $\{\varphi_n \in \operatorname{Hol}(\Delta_n, \Gamma)\}$ such that $\{f \circ \varphi_n\}$ converges to an entire function g_0 , there is a subsequence $\{\varphi_{n_k}\}$ which converges to a constant mapping $\varphi_0 : \mathbf{C} \to p \in \overline{\Gamma}$. Thus $g_0 = \overline{f} \circ \varphi_0$ is constant and by Corollary 2.15, f is normal. Q.E.D.

From Lemma 5.1 and Corollary 1.3 we have:

5.2 Proposition. *Let an algebraic curve Γ be hyperbolic. Then for any regular function f on Γ and for any complex space X the family*

$$\mathcal{F}_X := \{f \circ \varphi \mid \varphi \in \mathrm{Hol}(X, \Gamma)\} \subset \mathfrak{H}(\mathrm{X})$$

is $s-$normal.

5.3 Corollary. *Let $\Gamma \subset \mathbf{C}^n$ be a hyperbolic affine algebraic curve. Then for any complex space X the family of coordinate functions of all holomorphic mappings $X \to \Gamma$ is $s-$normal. Hence it is of set-exponential type for smooth X. .*

5.4 Corollary. *Let a curve $\Gamma = \{(x,y) \in \mathbf{C}^2 \mid p(x,y) = 0\}$ be hyperbolic, where $p \in \mathbf{C}\{x,y\}$. Let X be a complex space. If a pair of functions $f, g \in \mathfrak{H}(X)$ satisfies the polynomial identity $p(f,g) \equiv 0$, then f and g are normal functions. In particular, if X is smooth, they satisfy the Schottky-Landau' type growth estimates.*

5.5 Remark. For some special polynomial identities growth estimates of such type in the upper halfplane were earlier obtained by V. I. Ostrovskii [Os 1–2].

The next statement deals with polynomial inequalities instead of polynomial identities (compare [Os 1 (Lemma 1), 2]) .

5.6 Theorem. *Let X be a complex space, and $p \in \mathbf{C}[x,y]$ be a polynomial such that for every $c \in \mathbf{C}$ the curve $\Gamma_c := p^{-1}(c) \subset \mathbf{C}^2$ is hyperbolic.*

a) For a pair of functions $f, g \in \mathfrak{H}(X)$, let the function $h := p(f,g) \in \mathfrak{H}(X)$ be bounded. Then the functions f and g are normal.

b) The same conclusion is true if one assumes h only to be normal, but additionally assumes that the closure of the curve $\Gamma_0 = p^{-1}(0) \subset \mathbf{C}^2$ in $\mathbf{P}^1 \times \mathbf{P}^1$ intersects with each of two projective lines at infinity in at least three distinct points.

Proof of a): Put $Y = \mathbf{C}^2$ and $\overline{Y} = \mathbf{P}^1 \times \mathbf{P}^1 = \mathbf{C}^2 \cup D$, where D is a union of two projective lines in the quadric $\mathbf{P}^1 \times \mathbf{P}^1$. We must prove that the mapping $F := (f,g)$, $F : X \to Y \hookrightarrow \overline{Y}$, is normal. Fix a sequence $\{\varphi_n \in \mathrm{Hol}(\Delta_n, X)\}$ such that the sequence $\{F \circ \varphi_n\}$ converges to a holomorphic mapping $\Phi : \mathbf{C} \to \overline{Y}$. Then $\Phi(\mathbf{C}) \subset \mathrm{Cl}(P)$, where P is the polyhedron $P = \{(x,y) \in \mathbf{C}^2 \mid \mid p(x,y) \mid < R\}$ and $R = \| h \|_\infty$.

By Corollary 2.15 it is enough to check that $\Phi = \mathrm{const}$. Consider two cases: 1) $\Phi(\mathbf{C}) \subset \mathbf{C}^2$; 2) $\Phi(\mathbf{C}) \cap D \neq \emptyset$. By Hurwith's Theorem, in the second case $\Phi(\mathbf{C}) \subset D$, and so $\Phi(\mathbf{C}) \subset D \cap \mathrm{Cl}(P)$. It is easily seen that the latter set is finite, and therefore Φ is constant. In the first case $p \circ \Phi$ is a bounded entire function and hence is constant. This means that $\Phi(\mathbf{C}) \subset \Gamma_c$ for some $c \in \mathbf{C}$. Since Γ_c is a hyperbolic curve, Φ is constant. Q.E.D.

Proof of b): Let Y, $\overline{Y}$, F, $\{\varphi_n\}$ and Φ be as above. Since h is a normal function, the sequence $\{p \circ F \circ \varphi_n = h \circ \varphi_n\}$ contains a subsequence, which converges to a constant mapping $\mathbf{C} \to c_0 \in \mathbf{P}^1$. If $c_0 \in \mathbf{C}$, then $\Phi(\mathbf{C}) \subset \mathrm{Cl}(\Gamma_{c_0})$ and by Hurwith's Theorem either $\Phi(\mathbf{C}) \subset \Gamma_{c_0}$ or $\Phi(\mathbf{C}) \subset (\Gamma_{c_0} \cap D)$. In both cases $\Phi \equiv \mathrm{const}$. If $c_0 = \infty$, then $\Phi(\mathbf{C}) \subset D$. Furthermore, in this case $F \circ \varphi_n(\Delta) \cap \Gamma_0 = \emptyset$, if n is large enough and hence by Hurwith's Theorem either $\Phi(\mathbf{C}) \cap \mathrm{Cl}(\Gamma_0) = \emptyset$, i.e. $\Phi(\mathbf{C}) \subset (D \setminus \mathrm{Cl}(\Gamma_0))$, or $\Phi(\mathbf{C}) \subset (D \cap \mathrm{Cl}(\Gamma_0))$. By the assumption in b), the curve $D \setminus \mathrm{Cl}(\Gamma_0)$ is hyperbolic, while $D \cap \mathrm{Cl}(\Gamma_0)$ is a finite set. Thus in both cases $\Phi \equiv \mathrm{const}$. Q.E.D.

5.7 Remark. It would be interesting to learn whether b) is true without the additional assumption on p.

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