Zwei Bemerkungen über die Gleichverteilung

der Idealen und ganzen Punkten

von

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Foreword

The two notes combined in this preprint can be read independently of each other. They are linked together by some general ideas which lie behind the applications of H.Weyl's equidistribution principle as exemplified, for instance, in: J.W.S.Cassels, An introduction to diophantine approximation, Cambridge University Press, 1957 (Chapter VI, §§ 4,5).

S.Lang, Algebraic number theory, Addison-Wesley Publishing Company, 1970 (Chapter VI, § 2).

J-P.Serre, Abelian l-adic representations and elliptic curves, Benjamin, 1968 (Appendix to Chapter I).

Two years ago we proved a rather general theorem concerning equidistribution of integral points on norm-form varieties (see [1], equation (23)). As a simple application of the methods developed in [1] we give here a result much stronger than one can expect to obtain working analytically (cf., e.g., [3]). Let, in notations of [1],

$$U_j(X) = \{a \mid a \in \mathbb{R}^{d_j} ,\ |g_j(a)| < X^{\delta_j}\} ,\ 1 \le j \le r ,$$

and let

$$U(X) = U_1(X) \times \ldots\ldots \times U_r(X) ,$$

where we write, for brevity,

$$|y| = \max_{1 \le i \le n} |y_i| \quad \text{for} \quad y = (y_1,\ldots,y_n) ,\ y_i \in \mathbb{C} .$$

<u>Theorem</u>. Suppose that k_j is a totally complex Galois extension of $\mathbb{Q}$, $1 \le j \le r$, and that the fields $k_1,\ldots,k_r$ are arithmetically independent. Then

$$N_1(U(X) \cap V) = c_1(\vec{k}) X + O(X^{1-c_2(\vec{k})}),\ c_1(\vec{k})>0,\ c_2(\vec{k})>0 . \qquad (1)$$

<u>Notations</u>. We retain the notations introduced in [1]; in what follows $c_j(\vec{k})$, $1 \le j \le 7$, and the 0-constants depend at most on the sequence of fields k_j, $1 \le j \le r$. Let E_2 be the set of all the subsets of V of the form

$$U' = U \times I \ , \ U \in E \ , \ I = \{t | t_1 < t \leq t_2\} \ ,$$

where t_1 and t_2 range over $\mathbb{R}_+$ and satisfy the condition $t_1<t_2$.

Lemma 1. The set $u = U(X) \cap V$, $X > 0$, is (E_2, μ')-smooth and

$$C(u) = O((\log X)^{c_3(\vec{k})}) \ ,$$

where, as always, $C(u)$ denotes the smoothness constant of u .

Lemma 2. We have

$$\mu'(U(X) \cap V) = c_4(\vec{k})X \ , \ c_4(\vec{k})>0 \ .$$

Lemma 3. We have

$$t(U(X) \cap V) = O(X) \ .$$

Proof of the theorem. In view of lemma 1 and lemma 3, it follows from the estimate (19) in [1] that

$$N_1(U(X) \cap V) = b\mu'(U(X) \cap V) + O(X^{1-c_5(\vec{k})}) \ , \ c_5(\vec{k})>0 \ . \qquad (2)$$

Since according to [2, theorem 2] the constant b depends on k_j , $1 \leq j \leq r$, only and since $b > 0$, relation (1) is a consequence of (2) and lemma 2.

Proof of lemma 1. Let

$$\{\varepsilon_{ji} \mid 1 \le i \le \tfrac{1}{2}d_j - 1\} \ , \ 1 \le j \le r \ ,$$

be a system of fundamental units in k_j . Suppose that

$$\alpha \in g_j(V_j(1) \cap U_j(X)) \quad \text{and} \quad \sigma(\varepsilon)\alpha \in g_j(V_j(1) \cap U_j(X))$$

with

$$\varepsilon = \prod_i \varepsilon_{ji}^{n_i} \ , \ n_i \in \mathbb{Z} \ , \ 1 \le i \le \tfrac{1}{2}d_j - 1 \ ,$$

where σ denotes the diagonal embedding of k_j into d_j-dimensional $\mathbb{R}$-algebra

$$\underbrace{\mathbb{C} \oplus \ldots\ldots \oplus \mathbb{C}}_{\frac{1}{2}d_j \text{ times}}$$

defined in [1,§ 2]. It follows from the definitions of σ and g_j that the integers n_i obey the following estimate:

$$n_i = O(\log X) \ .$$

Therefore there is a covering

$$V_0(1) \cap U(X) \subseteq \bigcup_{p=1}^{f} \kappa_p$$

with

$$\kappa_p \in E \ , \ f = O(\log X)^a) \ , \ a := \frac{d}{2} - r \ .$$

The assertion of lemma 1 is an easy consequence of this fact.

Proof of lemma 2. By definition,

$$\mu'(U(X) \cap V) = \int_0^X dt \ \mu(U(X) \ \cap \ V_0(t)) \ . \tag{3}$$

On the other hand,

$$U(X) \cap V_0(h) = U_1(X) \cap V_1(h) \times \ldots\ldots \times U_r(X) \cap V_r(h)$$

and

$$\mu_j(U_j(X) \cap V_j(h)) = \mu(U_j(Xh^{-1}) \cap V_j(1))$$

for $h>0$. An easy computation shows that

$$\mu_j(U_j(y) \cap V_j(1)) = c_6(k_j)(\log y)^{\frac{1}{2}dj - 1} \ , \ c_6(k_j) > 0 \ ,$$

and therefore

$$\mu(U(X) \cap V_0(h)) = c_6(\vec{k})(\log y)^a \ , \ c_6(\vec{k}) > 0 \ , \tag{4}$$

where $y := \frac{X}{h}$. Lemma 2 follows from (3) and (4).

Proof of lemma 3. It follows from the definitions.

Remark. Analogously one can prove that

$$N_1(\widetilde{U}(X)\cap V) = b\mu'(\widetilde{U}(X)\cap V) + O(X^{1-c_7(\vec{k})}) , \quad c_7(\vec{k})>0 ,$$

taking

$$\widetilde{U}(X) = \{\vec{a} \mid \vec{a} = (a_1,\ldots,a_r),\ a_j \in \mathbb{R}^{d_j} ,\ |a_j|<X^{\delta_j} ,\ 1 \le j \le r\}.$$

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Equidistribution of Frobenius classes and the volumes of tubes

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1. Let G be a compact Lie group that fits in an exact sequence

$$1 \longrightarrow T \longrightarrow G \xrightarrow{j} H \longrightarrow 1 , \qquad (1)$$

where T is an n-dimensional real torus and H is a finite group. Given a countable index set P and a set of conjugacy classes

$$\{\sigma_p | p \in P\}$$

in G, we are interested in the following equidistribution problem. Let

$$|.| : P \longrightarrow \mathbb{R}_+$$

be a map satisfying the asymptotic formula (8) below and let $A \subseteq G$. For each x in $\mathbb{R}_+$, let

$$N(A,x) = \text{card } \{p | p \in P,\ \sigma_p \cap A \neq \emptyset,\ |p| < x\} .$$

One studies the asymptotics of $N(A,x)$ as $x \longrightarrow \infty$. Without loss of generality we can assume that A is invariant under conjugation, i.e.

$$\tau^{-1}A\tau = A \quad \text{for} \quad \tau \in G , \tag{2}$$

so that

$$N(A,x) = \text{card}\,\{p|p\in P,\ \sigma_p \subseteq A,\ |p| \le x\} . \tag{3}$$

The manifold G inherits the natural Riemannian structure from T. Let μ be the Haar measure on G normalised by the condition

$$\mu(G) = 1 ,$$

and suppose that A satisfies the following condition:

$$\mu(U_\delta(\partial A)) = O(C(A)\delta^\alpha) \quad \text{with} \quad \alpha > 0 , \tag{4}$$

where ∂A denotes the boundary of A and where $U_\delta(A)$ denotes the δ-neighbourhood of A, i.e. the subset

$$\{x|\, x \in G,\ \rho(x,A) < \delta\} ; \tag{5}$$

here $\delta > 0$ and ρ denotes the Riemannian metrics on G. Consider now the set $\hat{G}$ of all the simple characters of G; let ψ be an irreducible representation of G and let

$$\psi|_T = \text{diag}(\lambda_1,\dots,\lambda_2),\ \chi = \text{tr}\psi,\ \lambda_j\in\hat{T},\ 1\le j\le l . \tag{6}$$

In view of the isomorphism

$$\hat{T} \cong \mathbb{Z}^n ,$$

one can choose a basis

$$\{\mu_j \mid 1 \le j \le n\}$$

of $\hat{T}$. Let

$$\lambda_i = \prod_{j=1}^{n} \mu_j^{m_{ij}}, \; m_{ji} \in \mathbb{Z}, \; 1 \le i \le l , \tag{7}$$

we write then

$$w(\lambda_i) = \prod_{j=1}^{n} (1+|m_{ij}|), w(\chi) = \max_{1 \le i \le l} w(\lambda_i) .$$

<u>Theorem 1</u>. If A satisfies (4) and

$$\sum_{\substack{p \in P \\ |p|<x}} \chi(\sigma_p) = g(\chi)B(x) + O(b(x,w(\chi))), \; \chi \in \hat{G}, \tag{8}$$

where $g(\chi)=1$ if χ is the character of the identical representation and $g(\chi)=0$ for any other character and where

$$\sum_{m=1}^{\infty} b(x,m)m^{-\nu} = b_1(x,\nu) < \infty \tag{9}$$

for some ν in $\mathbb{R}_+$, then (assuming (2) and (3))

$$N(A,x) = \mu(A)B(x)\left(1+O\left(\frac{C(A)}{\mu(A)} \left(\frac{b_1(x,\nu)}{B(x)}\right)^{\frac{\alpha}{\alpha+\nu n}}\right)\right) . \tag{10}$$

<u>Proof</u>. Since, by definition, $\rho(g_1,g_2)=\infty$ when $j(g_1)\neq j(g_2)$, we have

$$\mathcal{U}_\delta(\{1\}) \subseteq \mathcal{T} ,$$

therefore there is a C^∞-function

$$\varphi_\delta : G \longrightarrow [0,1]$$

satisfying the following conditions:

$$\int_G \varphi_\delta(g)\,d\mu(g)=1,\ \varphi_\delta \text{ is H-invariant, } \varphi_\delta(g)=0 \text{ for } g \notin \mathcal{U}_\delta(\{1\}) .$$

Let f_+ and f_- be the characteristic functions of $\mathcal{U}_\delta(A)$ and $A\setminus\mathcal{U}_\delta(G\setminus A)$ respectively, and let

$$g_\pm(\beta) = \int_G f_\pm(\gamma)\varphi_\delta(\gamma^{-1}\beta)\,d\mu(\gamma) .$$

Then $g_\pm \in C^\infty(G)$ and $g_\pm$ is H-invariant (since $f_\pm$ and φ_δ are). Moreover,

$$g_\pm(\beta) = \int_{\mathcal{U}_\delta(\{1\})} f_\pm(\beta\gamma^{-1})\varphi_\delta(\gamma)\,d\mu(\gamma) ,$$

so that

$$g_\pm(\beta)\geq 0 \text{ for } \beta\in G,\ g_+(\beta)=1 \text{ for } \beta\in A,\ g_-(\beta)=0 \text{ for } \beta\notin A .$$

Thus

$$\sum_{|p|<x} g_-(\sigma_p) \le N(A,x) \le \sum_{|p|<x} g_+(\sigma_p) \quad . \tag{11}$$

We write

$$g_{\pm} = \sum_{\chi \in \hat{G}} c_{\pm}(\chi)\chi \tag{12}$$

and substitute (8) in (12) to obtain

$$\sum_{|p|<x} g_{\pm}(\sigma_p) = c_{\pm}(1)B(x) + O\Big(\sum_{\chi \neq 1} |c_{\pm}(\chi)| b(x,w(\chi))\Big) \quad . \tag{13}$$

It follows from (12) that

$$c_{\pm}(1) = \int_G g_{\pm}(\beta)\,d\mu(\beta) \quad ,$$

or recalling the definition of $g_{\pm}$, $f_{\pm}$, and φ_δ ,

$$c_{\pm}(1) = \int_G f_{\pm}(g)\,d\mu(g) = \mu(A) \pm \mu(U_\delta(\partial A)) \quad .$$

Therefore it follows from (4) and (13) that

$$\sum_{|p|<x} g_{\pm}(\sigma_p) = \mu(A)B(x) + O(B(x)\delta^\alpha C(A)) + O\Big(\sum_{\chi \neq 1} |c_{\pm}(\chi)| b(x,w(\chi))\Big) \quad . \tag{14}$$

To estimate $c_{\pm}(\chi)$ let us suppose that χ satisfies (7) and (6) and write

$$G = \bigcup_{\gamma \in H} Th_\gamma, \; j(h_\gamma) = \gamma \quad .$$

Then (12) gives:

$$c_{\pm}(\chi) = \int_T d\mu(\alpha) \sum_{\gamma\in H} g_{\pm}(\alpha h\gamma)\overline{\chi(\alpha h_\gamma)} \quad . \tag{15}$$

In view of (6) ,

$$\chi(\alpha h_\gamma) = \sum_{i=1}^{l} \lambda_i(\alpha)\psi_{ii}(h_\gamma) \quad .$$

Therefore

$$c_{\pm}(\chi) = \sum_{\gamma\in H} \sum_{i=1}^{l} \overline{\psi_{ii}(h_\gamma)} \int_T d\mu(\alpha) g_{\pm}(\alpha h_\gamma)\overline{\lambda_i(\alpha)} \quad . \tag{16}$$

It follows from (7) and the definition of $g_{\pm}$ that (cf., e.g., [2, § 3])

$$\int_T d\mu(\alpha) g_{\pm}(\alpha h_\gamma)\overline{\lambda_i(\alpha)} = O(\delta^{-\nu n}\ w(\lambda_i)^{-\nu}) \tag{17}$$

for each ν in $\mathbb{Z} \cap \mathbb{R}_+$. A classical argument (cf., e.g., [7, § 8.1]) shows that, in fact,

$$w(\chi) = O(w(\lambda_i)),\ 1 \le i \le l \ ,$$

for a simple character χ and that

$$\text{card}\{\chi \mid \chi\in\hat{G},\ w(\chi)=m\} = O(1),\ m \in \mathbb{Z},\ m \ge 1 \ . \tag{18}$$

In view of (9), (14), (17) and (18), we conclude that

$$\sum_{|p|<x^{-}} g_{\pm}(\sigma_p) = \mu(A)B(x) + O(B(x)\delta^{\alpha}C(A)) + O(\delta^{-\nu n} b_1(x,\nu)) \quad . \qquad (19)$$

Taking $\delta = \left(\frac{b_1(x,\nu)}{B(x)}\right)^{\frac{1}{\alpha+\nu n}}$ one deduces (10) from (11) and (19).

This completes the proof of Theorem 1.

Corollary 1. Assume that ∂A is contained in an analytic subset of dimension $n-1$. Then relations (8) and (9) imply (10) with $\alpha=1$.

Proof. By a geometric lemma discussed in the Appendix to this paper, a compact analytic set $\mathcal{B}$ of codimension α satisfies an estimate

$$\mu(\mathcal{U}_{\delta}(\mathcal{B})) = O(C(\mathcal{B})\delta^{\alpha}) \quad .$$

2. To describe an arithmetical application of theorem 1 let k be a finite extension of $\mathbb{Q}$, the field of rational numbers, and let $W(k)$ denote the (absolute) Weil group of k defined as a projective limit of the relative Weil groups $W(K|k)$, where K varies over all the finite Galois extensions of k (cf. [9], [10]). Let us recall that

$$W(K|k) \cong \mathbb{R}_{+}^{*} \times W_1(K|k)$$

with compact $W_1(K|k)$ and that $W(K|k)$ is defined as a group extension

$$1 \longrightarrow C_K \longrightarrow W(K|k) \longrightarrow G(K|k) \longrightarrow 1 \; ,$$

where C_K denotes the idèle-class group of K and where $G(K|k)$ is the Galois group of K over k. Let $S(k)$ be the set of all the prime divisors of k, and let I_p and σ_p be the inertia subgroup and the Frobenius class in $W(k)$ for $p \in S(k)$. Consider a finite dimensional continuous representation

$$\psi : W(k) \longrightarrow GL(V)$$

acting in a <u>complex</u> vector space V; let

$$V_p = \{e | e \in V \ , \ \psi(g)e = e \quad \text{for} \quad g \in I_p\}$$

be the subspace of I_p-invariant vectors and let χ denote the character of ψ. We define $\chi(\sigma_p)$ to be equal to the trace of the operator $\psi(\tau_p)$ on V_p for $\tau_p \in \sigma_p$ and notice that this definition does not depend on the choice of τ_p in σ_p. One can show that the set

$$S_0(\psi) := \{p | p \in S(k), \ V_p \neq V\}$$

is finite and that ψ factors through $W(K|k)$ for a finite extension $K|k$. We say that ψ is <u>normalised</u> if ψ factors through $W_1(K|k)$ for a finite Galois extension $K|k$.

<u>Theorem 2</u>. Let $\mathfrak{M}$ be a finite set of normalised (finite dimensional continuous) representations of $W(k)$, let

$$\check{\mathfrak{M}} = \{\chi | \chi = \mathrm{tr}\psi \quad \text{for some} \quad \psi \quad \text{in} \quad \mathfrak{M}\} ,$$

and choose g_0 in $W(k)$ and ε in the interval $0<\varepsilon<1$. There is a positive constant $a(\mathfrak{M};g_0,\varepsilon)$ such that

$$\operatorname{card}\{p\,|\,p\in S(k),\ |\chi(\sigma_p)-\chi(g_0)|<\varepsilon, N_{k/\mathbb{Q}}p<x\} =$$

$$a(\mathfrak{M};g_0,\varepsilon)\int_2^x \frac{du}{\log u} + O(x\exp(-c_1\sqrt{\log x})),\ c_1>0\ , \tag{20}$$

and

$$a(\mathfrak{M};g_0,\varepsilon) > c_3\varepsilon^{c_2}\ , \tag{21}$$

where $c_j, 1\leq j\leq 3$, and the implied by the O-symbol constant depend at most on $\mathfrak{M}$ (but not on g_0,ε,x).

Proof. Let $K|k$ be a finite Galois extension such that each ψ in $\mathfrak{M}$ factors through $W_1(K|k)$ and let $[K:k] = n+1$. Consider the (closed) subgroup

$$G_0 = \bigcap_{\psi\in\mathfrak{M}} \operatorname{Ker}\psi$$

of $W(k)$ and let $G = W(k)/G_0$. It follows from the definitions that G fits into the exact sequence (1). We let

$$S_0(\mathfrak{M}) = \bigcup_{\psi\in\mathfrak{M}} S_0(\psi)$$

and denote by $\bar{\sigma}_p$ the image of the Frobenius class under the natural homomorphism

$$\varphi : W(k) \longrightarrow G .$$

For $\mathfrak{p} \in S(k) \setminus S_0(\mathfrak{m})$ the set $\bar{\sigma}_{\mathfrak{p}}$ is a conjugacy class in G. Moreover, it can be deduced from the Hecke's Primzahlsatz, [1] (cf. also [5, theorem 4]) that, for each χ in $\hat{G}$, we have:

$$\sum_{|\mathfrak{p}|<x} \chi(\sigma_{\mathfrak{p}}) = g(\chi) \int_2^x \frac{du}{\log u} + O\left(x \exp\left(-c_4 \frac{\log x}{\log w(\chi) + \sqrt{\log x}}\right)\right) \tag{22}$$

with $c_4 > 0$, where $|\mathfrak{p}| := N_{k/\mathbb{Q}}\mathfrak{p}$. Let

$$B = \{g \mid g \in W(k),\ |\chi(g) - \chi(g_0)| < \varepsilon \quad \text{for} \quad \chi \in \check{\mathfrak{m}}\}$$

and let

$$A = \varphi(B) .$$

The set ∂A may be regarded as a semialgebraic set, therefore it satisfies (4) with $C(A)$ and α independent of ε and g_0 (cf. [11, Corollary 4.5]). Estimate (20) follows now from theorem 1, in view of (22). To deduce the inequality (21) we appeal to [3, Proposition 5] (cf. also [4, p.461] and [6, Theorem 2, p.99]).

Remark 1. Theorem 2 may be regarded as a generalization of both Chebotarev's density theorem and the prime number theorem for grossencharacters due to E.Hecke. It confirms our conjecture stated in [3, p.23] and in [6, p.139-140].

Appendix. We reproduce here an argument kindly communicated to the author by Professor J-P.Serre in his letter of April 24th, 1986 (cf. also [8, p.145]).

Lemma. Let $\mathfrak{h}$ be a compact subset of the analytic set

$$\mathfrak{C} = \{x \mid x \in \mathbb{R}^n \ , \ f(x) = 0\} \ ,$$

where $f : \mathbb{R}^N \longrightarrow \mathbb{R}$ is an analytic function, and let d denote the (real) dimension of $\mathfrak{C}$. Then

$$\int_{U_\delta(\mathfrak{h})} dx < C(\mathfrak{h})\,\delta^{n-d} \quad \text{for} \quad 0 < \delta < 1. \tag{23}$$

Sketch of the proof. It follows from the Hironaka's theorem on resolution of singularities that

$$\mathfrak{h} \subseteq \bigcup_{j=1}^{l(\mathfrak{h})} B_j \ , \ B_j = g_j(I^d) \ ,$$

where $I := [0,1]$ and g_j is a continuous map with the Lipschitz property, i.e.

$$|g_j(x+y) - g_j(x)| < C_j |y| \ , \ C_j > 0 \ .$$

Therefore

$$\int_{U_\delta(\mathfrak{h})} dx \leq \sum_{j=1}^{l(\mathfrak{h})} \int_{U_\delta(B_j)} dx \ . \tag{24}$$

Let

$$I(\nu,N) = \left[\frac{\nu}{N}, \frac{\nu+1}{N}\right], \quad 0 \le \nu \le N-1 ,$$

and let

$$B_{j,\vec{\nu}} = g_j(I(\nu_1,N)\times\ldots\times I(\nu_d,N)) , \quad \vec{\nu} := (\nu_1,\ldots,\nu_d) .$$

Then

$$\int_{U_\delta(B_j,\vec{\nu})} dx = O((\delta + \tfrac{1}{N})^n)$$

with an O-constant depending at most on C_j , $1 \le j \le l(h)$, and therefore

$$\int_{U_\delta(B_j)} dx \le \sum_{\vec{\nu}} \int_{U_\delta(B_{j,\vec{\nu}})} dx = O(N^d(\delta + \tfrac{1}{N})^n) .$$

Choosing N to be equal to $\left[\frac{1}{\delta}\right]$ one obtains an estimate

$$\int_{U_\delta(B_j)} dx = O(\delta^{n-d}) . \tag{25}$$

Relation (23) is a consequence of (24) and (25).

Remark 2. As it has been pointed out in [8], one should try to prove this lemma by elementary methods making no use of the theory of resolution of singularities.

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